WAVE PROPAGATION THROUGH A NEWTONIAN FLUID CONTAINED WITHIN A THICK-WALLED, VISCOELASTIC TUBE

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ABSTRACT The propagation of harmonic pressure waves through a Newtonian fluid contained within a thick-walled, viscoelastic tube is considered as a model of arterial blood flow. The fluid is assumed to be homogeneous and Newtonian, and its motion to be laminar and axisymmetric. The wall is assumed to be isotropic, incompressible, linear, and viscoelastic. It is also assumed that the motion is such that the convective acceleration is negligible. The motion of the fluid is described by the linearized form of the Navier-Stokes equations and the motion of the wall by classical elasticity theory. The frequency dependence of the wall mechanical properties are represented by a three parameter, relaxation-type model. Using boundary conditions describing the continuity of stress and velocity components in the fluid and the wall, explicit solutions for the system of equations of the model have been obtained. The longitudinal fluid impedance has been expressed in terms of frequency and the system parameters. The frequency equation has been solved and the propagation constant also expressed in terms of frequency and system parameters. The results indicate that the fluid impedance is smaller than predicted by the rigid tube model or by Womersley's constrained elastic tube model. Also, the velocity of propagation is generally slower and the transmission per wavelength less than predicted by Womersley's elastic tube model. The propagation constant is very sensitive to changes in the degree of wall viscoelasticity.

INTRODUCTION

The problem of blood flow and wave propagation in the arterial system has stimulated the interest of physiologists and mathematicians for years. Its fundamental importance for the complete understanding of the control and regulation of cardiovascular function is well-known. Much of the impetus for present day research on these problems can be traced to the pioneering work of Witzig (1914) and later to the work of Womersley (1955, 1957). Other significant contributions to the theory of "oscillatory blood flow" have been made by Morgan and his coworkers (1954, 1955), Taylor (1959*a*), Atabek and his coworkers (1961, 1966), Whirlow and Rouleau (1965), and Mirsky (1967). Generally, most of these workers dealt with models that were either extensions or modifications of Womersley's original work (1955). Thus, the contributions of Womersley represent the best attempt to date in developing a complete, practical, unified theory of arterial blood flow and pressure propagation.

There are, however, several significant points of discrepancy between the predictions of Womersley's theory and experimental data published in the literature (Taylor, 1959*b*; Fry et al., 1964; Klip, 1962). Because of these contradications, this work was undertaken in order to develop, as fully as possible, a complete and realistic hemodynamic model, which could be tested experimentally.

ANALYTICAL MODEL

The model we have developed considers the propagation of harmonic traveling waves through a Newtonian fluid contained within a cylindrical, thick-walled, viscoelastic tube. The motion of the fluid is assumed to be described by the linearized, axisymmetric form of the Navier-Stokes equations, and the motion of the tube is assumed to be given by classical elasticity theory. Using the boundary conditions coupling the motion of the fluid and tube, this system of equations has been solved explicitly.

Equations of Motion of the Fluid

For the case of a viscous, incompressible fluid the equations of motion are governed by the momentum equations and the equation of continuity. For axisymmetric motion, these equations are given in cylindrical coordinates by (Pai, 1956):

$$\frac{\partial v_r}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left\{\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial x^2}\right\} \tag{1}$$

$$\frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left\{\frac{\partial^2 v_x}{\partial r^2} + \frac{1}{r}\frac{\partial v_x}{\partial r} + \frac{\partial^2 v_x}{\partial x^2}\right\} \tag{2}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r} = 0 \tag{3}$$

where ρ is the fluid density, ν the fluid kinematic viscosity, p the pressure, v_r and v_x the radial and axial components of velocity, respectively, r the radial coordinate, x the axial coordinate, and t the time.

In the development of these equations it has been assumed that the fluid motion is (*a*) laminar, (*b*) axially symmetric, and (*c*) contains negligible convective acceleration. Also, it has been assumed that the fluid is incompressible, homogeneous, and Newtonian.

Equations of Motion of the Tube

The tube, representing the arterial wall, is assumed to be thick-walled; linear, viscoelastic; and isotropic, incompressible. The equations of motion of an isotropic material are given in cylindrical coordinates by (Love, 1944):

$$\frac{\rho_w}{\mu^*}\frac{\partial^2 u_r}{\partial t^2} = \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_r}{\partial x^2} - \frac{1}{\mu^*}\frac{\partial \Omega}{\partial r} \tag{4}$$

$$\frac{\rho_w}{\mu^*}\frac{\partial^2 u_x}{\partial t^2} = \frac{\partial^2 u_x}{\partial r^2} + \frac{1}{r}\frac{\partial u_x}{\partial r} + \frac{\partial^2 u_x}{\partial x^2} - \frac{1}{\mu^*}\frac{\partial \Omega}{\partial r} \tag{5}$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_r}{\partial r} + \frac{u_r}{r} = 0 \tag{6}$$

where ρ_w is the density of the wall material, μ^* is the modulus of rigidity of the wall, Ω is a finite pressure, and u_x and u_r are the axial and radial components of the wall displacement, respectively. Equation 6 represents the condition of incompressibility.

Boundary Conditions

The boundary conditions, coupling the fluid and tube motions, involve the continuity of stress and velocity components at the fluid-tube interface and may be specified as follows: (*a*) the fluid velocity components are finite at $r = 0$; (*b*) the axial and radial velocity components of the fluid and wall are continuous at the boundary $r = a$; (*c*) the stress components of the fluid and wall are continuous at the boundary $r = a$; and (*d*) the stress components in the wall at $r = b$ are zero. Stated mathematically these conditions are equivalent to (Klip, 1962)

$$v_r = 0 \qquad @ \quad r = 0 \tag{7}$$

$$\frac{\partial v_x}{\partial r} = 0 \qquad @ \quad r = 0 \tag{8}$$

$$v_r = \frac{\partial u_r}{\partial t} \qquad @ \quad r = a \tag{9}$$

$$v_x = \frac{\partial u_x}{\partial t} \qquad @ \quad r = a \tag{10}$$

$$\mu\left\{\frac{\partial v_r}{\partial x} + \frac{\partial v_x}{\partial r}\right\} = \mu^*\left\{\frac{\partial u_r}{\partial x} + \frac{\partial u_x}{\partial r}\right\} \qquad @ \quad r = a \tag{11}$$

$$-p + 2\mu\frac{\partial v_r}{\partial r} = -\Omega + 2\mu^*\frac{\partial u_r}{\partial r} \qquad @ \quad r = a \tag{12}$$

$$\mu^* \left\{\frac{\partial u_r}{\partial x} + \frac{\partial u_x}{\partial r}\right\} = 0 \qquad @ \quad r = b \tag{13}$$

$$-\Omega + 2\mu^* \frac{\partial u_r}{\partial r} = 0 \qquad @ \quad r = b. \tag{14}$$

Solution of the System of Equations

We seek solutions of this system of equations for the case of the propagation of forced pressure waves in the form of a finite series of harmonic terms. The pressure and velocities are assumed to be of the form

$$p(r, x, t) = \sum_{n=0}^{N} {}^{n}P(r) \exp j(n\omega t - \gamma_n x) \tag{15}$$

$$v_r(r, x, t) = \sum_{n=0}^{N} {}^{n}V_r(r) \exp j(n\omega t - \gamma_n x) \tag{16}$$

$$v_x(r, x, t) = \sum_{n=0}^{N} {}^{n}V_x(r) \exp j(n\omega t - \gamma_n x) \qquad (1)7$$

where ω is the angular frequency, n the harmonic number, γ_n the propagation constant of the nth harmonic, and N a constant.

The solutions to equations 1–3, subject to the conditions 15–17 and satisfying the boundary conditions in equations 7 and 8, have been given by Witzig (1914) and may be expressed as

$$v_x = -\sum_{n=0}^{N} j\{A_1\gamma_n J_0(j\gamma_n r) + A_2\kappa_n J_0(j\kappa_n r)\} \exp j(n\omega t - \gamma_n x) \tag{18}$$

$$v_r = -\sum_{n=0}^{N} j\gamma_n\{A_1 J_1(j\gamma_n r) + A_2 J_1(j\kappa_n r)\} \exp j(n\omega t - \gamma_n x) \tag{19}$$

$$p = \sum_{n=0}^{N} A_3 J_0(j\gamma_n r) \exp j(n\omega t - \gamma_n x) \tag{20}$$

where A_1, A_2, and A_3 are complex constants of integration, J_0 and J_1 are Bessel functions of the first kind, and κ_n is defined by

$$\kappa_n^2 = \frac{jn\omega}{\nu} + \gamma_n^2. \tag{21}$$

Also A_1 and A_3 are related according to (Karreman, 1952)

$$A_1 = \frac{j}{n\omega\rho} A_3 . \tag{22}$$

The solutions for the displacement components are assumed to be of the form

$$u_r(r, x, t) = \sum_{n=0}^{N} {}^nU_r(r) \exp j(n\omega t - \gamma_n x) \tag{23}$$

$$u_x(r, x, t) = \sum_{n=0}^{N} {}^nU_x(r) \exp j(n\omega t - \gamma_n x). \tag{24}$$

The solution of equations 4–6 are obtained in a manner similar to that used for the velocity components and may be expressed as

$$u_r = \sum_{n=0}^{N} - j\gamma_n\{A_4J_1(k_nr) + B_4Y_1(k_nr) + A_5J_1(j\gamma_nr) + B_5Y_1(j\gamma_nr)\}$$
$$\times \exp j(n\omega t - \gamma_n x) \tag{25}$$

$$u_x = \sum_{n=0}^{N} - \{k_nA_4J_0(k_nr) + k_nB_4Y_0(k_nr) + j\gamma_nA_5J_0(j\gamma_nr) + j\gamma_nB_5Y_0(j\gamma_nr)\}$$
$$\times \exp j(n\omega t - \gamma_n x) \tag{26}$$

$$\Omega = \sum_{n=0}^{N} \{A_6J_0(j\gamma_nr) + B_6Y_0(j\gamma_nr)\} \exp j(n\omega t - \gamma_n x) \tag{27}$$

where the A's and B's are complex constants, J_0 and J_1 are Bessel functions of the first kind, Y_0 and Y_1 are Bessel functions of the second kind, and k_n is defined as

$$k_n^2 = \frac{n^2\omega^2\rho_w}{\mu^*} - \gamma_n^2. \tag{28}$$

Also A_5 and B_5 are related to A_6 and B_6 according to

$$A_6 = n^2\omega^2\rho_wA_5 ; \qquad B_6 = n^2\omega^2\rho_wB_6 . \tag{29}$$

In order to complete the solutions of the equations of motion, they must be substituted into the boundary conditions. Doing this we obtain six simultaneous equations in the six unknown coefficients A_1, A_2, A_4, A_5, B_4, and B_5. In order that the solutions to equations 18–20 and 25–27 be nontrivial, it is required that the determinant of the coefficients of the unknown constants be zero (Wylie, 1960). By expanding the

$$\begin{vmatrix} -2\mu\gamma_n^2 J_1(j\gamma_n a) & -\mu(\gamma_n^2+\kappa_n^2)J_1(j\kappa_n a) & (\gamma_n^2-k_n^2)J_1(k_n a) & (\gamma_n^2-k_n^2)Y_1(k_n a) & 2\mu^*\gamma_n^2 J_1(j\gamma_n a) & 2\mu^*\gamma_n^2 Y_1(j\gamma_n a) \\ \begin{matrix} 2\mu\gamma_n^2 J_1'(j\gamma_n a) \\ +jn\omega\rho J_0(j\gamma_n a) \end{matrix} & 2\mu\gamma_n\kappa_n J_1'(j\kappa_n a) & 2jk_n\gamma_n J_1'(k_n a) & 2jk_n\gamma_n Y_1'(k_n a) & \begin{matrix} -2\mu^*\gamma_n^2 J_1'(j\gamma_n a) \\ +n^2\omega^2\rho_w J_0(j\gamma_n a) \end{matrix} & \begin{matrix} -2\mu^*\gamma_n^2 Y_1'(j\gamma_n a) \\ +n^2\omega^2\rho_w Y_0(j\gamma_n a) \end{matrix} \\ 0 & 0 & (\gamma_n^2-k_n^2)J_1(k_n b) & (\gamma_n^2-k_n^2)Y_1(k_n b) & 2\mu^*\gamma_n^2 J_1(j\gamma_n b) & 2\mu^*\gamma_n^2 Y_1(j\gamma_n b) \\ 0 & 0 & 2jk_n\gamma_n J_1'(k_n b) & 2jk_n\gamma_n Y_1'(k_n b) & \begin{matrix} -2\mu^*\gamma_n^2 J_1'(i\gamma_n b) \\ +n^2\omega^2\rho_w J_0(i\gamma_n b) \end{matrix} & \begin{matrix} -2\mu^*\gamma_n^2 Y_1'(j\gamma_n b) \\ +n^2\omega^2\rho_w Y_0(j\gamma_n b) \end{matrix} \\ -\gamma_n J_0(j\gamma_n a) & -\kappa_n J_0(j\kappa_n a) & \frac{n\omega k_n}{\mu^*} J_0(k_n a) & \frac{n\omega k_n}{\mu^*} Y_0(k_n a) & jn\omega\gamma_n J_0(j\gamma_n a) & jn\omega\gamma_n Y_0(j\gamma_n a) \\ jJ_1(j\gamma_n a) & iJ_1(j\kappa_n a) & \frac{n\omega}{\mu^*} J_1(k_n a) & \frac{n\omega}{\mu^*} Y_1(k_n a) & n\omega J_1(i\gamma_n a) & n\omega Y_1(j\gamma_n a) \end{vmatrix} \qquad (30)$$

determinant, given by equation 30 (see facing page), it is possible to obtain a relationship expressing the propagation constant as a function of frequency and system parameters, the so-called frequency equation. The roots of the propagation constant from equation 30 are complex and can be assumed to be of the form $\gamma = \beta + j\alpha$. Here β is the wave number, α the attenuation coefficient, and the phase velocity is given by $c_p = \omega/\beta$.

Since the system of equations obtained from the boundary condition is homogeneous, the unknown constants cannot be calculated independently. Instead, they may be interrelated according to the relations (Wylie, 1960)

$$\frac{A_1}{\Delta_{11}} = -\frac{A_2}{\Delta_{12}} = \frac{A_4}{\Delta_{13}} = -\frac{B_4}{\Delta_{14}} = \frac{A_5}{\Delta_{15}} = -\frac{B_5}{\Delta_{16}} \tag{31}$$

where Δ_{kl} is the cofactor obtained by eliminating the kth row and lth column of equation 30. Using these relations, the nth harmonic of the fluid velocity components becomes

$${}^{n}V_x = \frac{j\gamma_n A_3}{jn\omega\rho}\left\{J_0(j\gamma_n r) - \frac{\kappa_n}{\gamma_n}\frac{\Delta_{12}}{\Delta_{11}}J_0(j\kappa_n r)\right\} \tag{32}$$

$${}^{n}V_r = \frac{j\gamma_n A_3}{jn\omega\rho}\left\{J_1(j\gamma_n r) - \frac{\Delta_{12}}{\Delta_{11}}J_1(j\kappa_n r)\right\} \tag{33}$$

where use has been made of equation 22. The axial flow rate is obtained by integrating the axial velocity over the cross-section of the tube and is given by

$$Q_n = \pi a^2 \frac{j\gamma_n A_3 J_0(j\gamma_n a)}{jn\omega\rho}\left\{\frac{2J_1(j\gamma_n a)}{j\gamma_n a J_0(j\gamma_n a)} - \frac{2J_1(j\kappa_n a)}{j\gamma_n a J_0(j\gamma_n a)}\frac{\Delta_{12}}{\Delta_{11}}\right\}. \tag{34}$$

The term in the numerator outside the bracket is equal to the negative of the axial pressure gradient, so that the nth harmonic of the hydraulic fluid impedance is given by

$$Z_n = \frac{jn\omega\rho}{\pi a^2\left\{\frac{2J_1(j\gamma_n a)}{j\gamma_n a J_0(j\gamma_n a)} - \frac{2J_1(j\kappa_n a)}{j\gamma_n a J_0(j\gamma_n a)}\frac{\Delta_{12}}{\Delta_{11}}\right\}}. \tag{35}$$

As previously mentioned, the frequency equation expresses the dependence of the propagation constant on frequency and the system parameters and is obtained by determining the value of γ_n that makes the determinant in equation 30 zero. As a first approximation of the solution of equation 30, series expansions were used to approximate the Bessel functions of arguments $j\gamma_n a$, $j\gamma_n b$, $k_n a$, and $k_n b$, since all of these values are generally much less than unity, about 10^{-3}. Using these approximations, solutions of the frequency equation were obtained. Later, when the complete frequency equation was solved using an iterative approach, it was found that these approximations resulted in a substantial error in the propagation constant in some cases. The approximation method was then abandoned and the iterative approach used thereafter.

RESULTS

The frequency variations of the theoretical fluid impedance given by equation 35 is shown in Fig. 1. The fluid impedance has been nondimensionalized by dividing by the Poiseuille resistance, and the abscissa is the amplitude of the nondimensional parameter z. Also shown in the figure for comparison is the nondimensional fluid

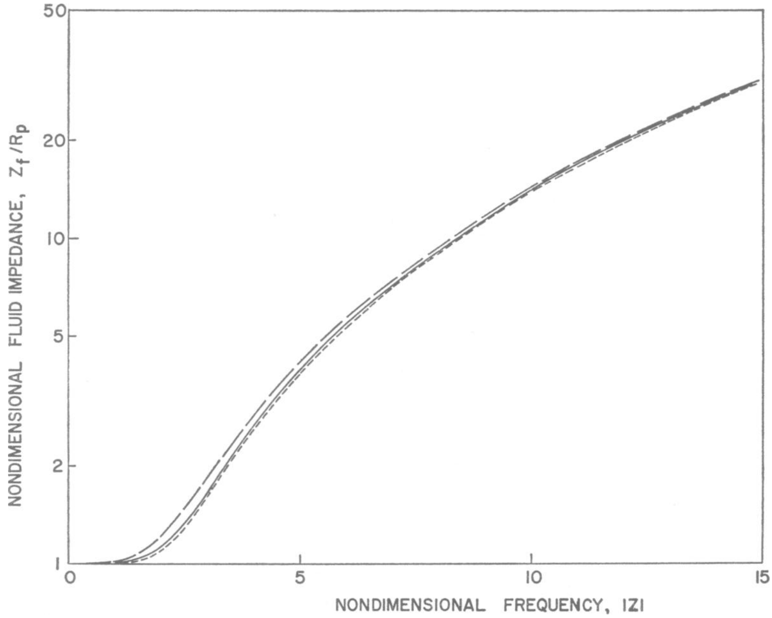

FIGURE 1 Comparison of the dependence of theoretical nondimensional fluid impedance from present model (continuous line) with that of rigid tube model (long dashed line) and Womersley's elastic tube model (short dashed line) on z.

impedance of the rigid tube model (Crandall, 1927) and of Womersley's elastic tube model (1955). Fig. 2 shows the frequency variation of the nondimensional fluid resistance and inductance. Similar values from the rigid tube and Womersley's elastic tube model are shown for comparison. The theoretical impedance functions in Figs. 1 and 2 as well as in subsequent figures were derived from equation 35 using the following "typical" values for the various parameters: a = 0.2 cm, b/a = 1.15, μ = 3.3 centipoise, ρ = 1.056, and ρ_w = 1.1 g/cm^3, μ_0 = 6 $\times$ 10^6 dynes/cm^2, λ_1 = 35 msec, and τ = 1.3. The last three parameters are viscoelastic constants and their significance will be discussed more fully in succeeding paragraphs. The de-

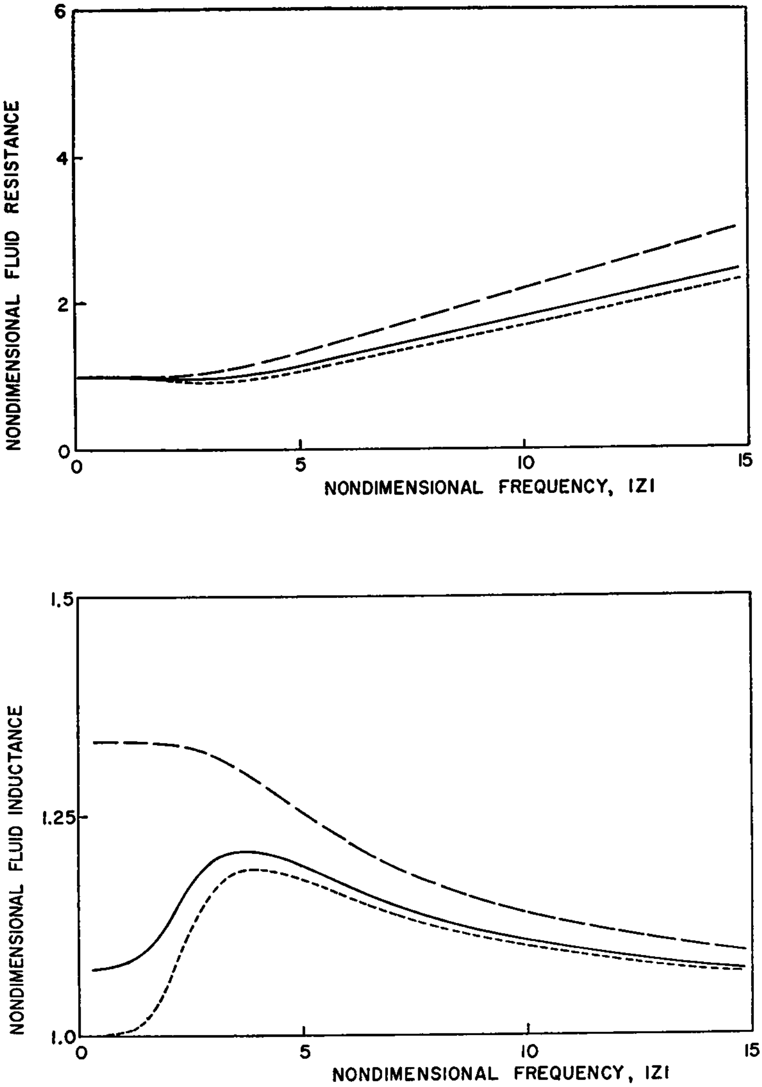

FIGURE 2 Comparison of the dependence of theoretical nondimensional fluid resistance (upper half) and fluid inductance (lower half) from present model from rigid tube model, and Womersley's elastic tube model on z.

pendence of the fluid impedance on the inner tube radius, the fluid viscosity and density, and frequency are eliminated by nondimensionalizing the fluid impedance by the Poiseuille resistance and considering its variations with respect to the parameter z.[1] The fluid impedance is not significantly altered for values of the static

[1] While z is actually proportional to the square root of frequency, it will be referred to herein as the nondimensional frequency as a matter of convenience.

modulus of rigidity, μ_0, between 10^6 and 10^8 dynes/cm^2 (typical range for the femoral artery) or for any nominal values of the two time constants. There is a small dependence of the nondimensional fluid impedance on the ratio of the external to the internal radius (b/a) at low values of z. Table I shows the value of the nondimensional fluid impedance at six different values of z, for $b/a = 1.1$ and 1.2. These two levels represent approximately the limits of the radius ratio for the femoral artery.

The amplitude and phase of the nondimensional fluid impedance for values of z from 0.1 to 20 in steps of 0.1 are given in Table II. The modulus of rigidity was taken as 10^7 dynes/cm^2 and the radius ratio as 1.15. Linear interpolation for intermediate values of z is accurate to better than 0.1 %.

TABLE I

DEPENDENCE OF THE NONDIMENSIONAL FLUID IMPEDANCE ON THE RADIUS RATIO (b/a)

Z	$b/a = 1.1$		$b/a = 1.2$	
	Amplitude	Phase	Amplitude	Phase
2	1.11	30.0	1.13	30.4
3	1.63	54.9	1.66	54.7
5	3.87	73.3	3.90	73.0
7	7.15	78.9	7.19	78.7
10	13.94	82.7	13.99	82.5
15	30.26	85.4	30.32	85.3

The amplitudes are nondimensionalized using the Poiseuille resistance; the phase angles are given in degrees.

It has been shown elsewhere (Cox, 1968)[2] that the frequency-dependent mechanical properties of the femoral artery can be represented approximately by a three parameter model given by the following equation,

$$\mu^* = \mu_0 \frac{1 + j\omega\lambda_2}{1 + j\omega\lambda_1} \tag{36}$$

where μ_0 is the static modulus of rigidity, μ^* the dynamic modulus, and λ_1 and λ_2 are the relaxation and retardation time constants, respectively. This model is based upon experimental evidence which has shown that the dynamic elasticity of arteries increases with frequency from the static modulus, at first moderately and then more slowly, finally attaining a high frequency asymptotic value above 10 Hz (Bergel, 1960). Also, it has been shown that the viscous modulus of arteries decreases with increasing frequency. The model given by equation 36 exhibits this sort of behavior.

[2] Cox, R. H. 1968. Submitted for publication.

TABLE II

VARIATION OF THE AMPLITUDE AND PHASE OF THE THEORETICAL NONDIMENSIONAL FLUID IMPEDANCE WITH Z

Z	Amplitude	Phase	Z	Amplitude	Phase	Z	Amplitude	Phase	Z	Amplitude	Phase
0.1	1.000	0.07	5.1	4.032	73.53	10.1	14.231	82.71	15.1	30.680	85.34
0.2	1.000	0.31	5.2	4.175	73.93	10.2	14.499	82.79	15.2	31.073	85.38
0.3	1.000	0.70	5.3	4.320	74.31	10.3	14.769	82.87	15.3	31.468	85.41
0.4	1.000	1.24	5.4	4.468	74.67	10.4	15.042	82.95	15.4	31.866	85.44
0.5	1.000	1.94	5.5	4.619	75.02	10.5	15.317	83.02	15.5	32.266	85.47
0.6	1.001	2.79	5.6	4.772	75.35	10.6	15.595	83.10	15.6	32.669	85.51
0.7	1.002	3.79	5.7	4.927	75.66	10.7	15.875	83.17	15.7	33.074	85.54
0.8	1.003	4.96	5.8	5.085	75.97	10.8	16.158	83.24	15.8	33.482	85.57
0.9	1.004	6.27	5.9	5.245	76.26	10.9	16.443	83.31	15.9	33.892	85.60
1.0	1.006	7.74	6.0	5.408	76.54	11.0	16.731	83.38	16.0	34.305	85.63
1.1	1.010	9.36	6.1	5.573	76.80	11.1	17.021	83.45	16.1	34.721	85.66
1.2	1.014	11.14	6.2	5.741	77.06	11.2	17.313	83.52	16.2	35.138	85.69
1.3	1.019	13.07	6.3	5.911	77.31	11.3	17.609	83.58	16.3	35.559	85.72
1.4	1.026	15.14	6.4	6.084	77.55	11.4	17.906	83.64	16.4	35.982	85.74
1.5	1.034	17.36	6.5	6.259	77.81	11.5	18.210	83.70	16.5	36.407	85.77
1.6	1.045	19.72	6.6	6.437	78.00	11.6	18.509	83.76	16.6	36.835	85.80
1.7	1.059	22.20	6.7	6.617	78.22	11.7	18.815	83.82	16.7	37.265	85.83
1.8	1.075	24.80	6.8	6.800	78.43	11.8	19.122	83.88	16.8	37.698	85.85
1.9	1.095	27.49	6.9	6.985	78.63	11.9	19.432	83.94	16.9	38.133	85.88
2.0	1.119	30.24	7.0	7.173	78.82	12.0	19.745	84.00	17.0	38.571	85.91
2.1	1.148	33.04	7.1	7.363	79.01	12.1	20.060	84.05	17.1	39.011	85.93
2.2	1.182	35.83	7.2	7.555	79.19	12.2	20.378	84.11	17.2	39.454	85.96
2.3	1.221	38.59	7.3	7.751	79.37	12.3	20.698	84.16	17.3	39.900	85.98
2.4	1.266	41.28	7.4	7.948	79.54	12.4	21.021	84.21	17.4	40.347	86.01
2.5	1.316	43.88	7.5	8.148	79.70	12.5	21.346	84.26	17.5	40.798	86.03
2.6	1.372	46.35	7.6	8.351	79.86	12.6	21.674	84.31	17.6	41.251	86.06
2.7	1.434	48.68	7.7	8.556	80.02	12.7	22.004	84.36	17.7	41.706	86.08
2.8	1.501	50.86	7.8	8.764	80.17	12.8	22.337	84.41	17.8	42.164	86.11
2.9	1.572	52.89	7.9	8.974	80.31	12.9	22.672	84.46	17.9	42.624	86.13
3.0	1.649	54.78	8.0	9.187	80.45	13.0	23.010	84.51	18.0	43.087	86.15
3.1	1.730	56.52	8.1	9.402	80.59	13.1	23.350	84.55	18.1	43.553	86.18
3.2	1.815	58.12	8.2	9.620	80.73	13.2	23.693	84.60	18.2	44.020	86.20
3.3	1.904	59.60	8.3	9.840	80.86	13.3	24.038	84.65	18.3	44.491	86.22
3.4	1.996	60.96	8.4	10.063	80.98	13.4	24.386	84.69	18.4	44.964	86.24
3.5	2.093	62.22	8.5	10.288	81.11	13.5	24.736	84.73	18.5	45.439	86.26
3.6	2.192	63.37	8.6	10.516	81.23	13.6	25.089	84.78	18.6	45.917	86.29
3.7	2.295	64.44	8.7	10.746	81.34	13.7	25.444	84.82	18.7	46.398	86.31
3.8	2.402	65.42	8.8	10.979	81.46	13.8	25.802	84.86	18.8	46.880	86.33
3.9	2.511	66.34	8.9	11.214	81.57	13.9	26.162	84.90	18.9	47.366	86.35
4.0	2.623	67.18	9.0	11.451	81.68	14.0	26.525	84.94	19.0	47.854	86.37
4.1	2.738	67.97	9.1	11.692	81.78	14.1	26.890	84.98	19.1	48.344	86.39
4.2	2.856	68.70	9.2	11.934	81.89	14.2	27.258	85.02	19.2	48.837	86.41
4.3	2.976	69.38	9.3	12.180	81.99	14.3	27.628	85.06	19.3	49.333	86.43
4.4	3.099	70.02	9.4	12.427	82.08	14.4	28.001	85.09	19.4	49.831	86.45
4.5	3.225	70.61	9.5	12.677	82.18	14.5	28.376	85.13	19.5	50.331	86.47
4.6	3.353	71.17	9.6	12.930	82.27	14.6	28.754	85.17	19.6	50.834	86.49
4.7	3.484	71.70	9.7	13.185	82.36	14.7	29.134	85.20	19.7	51.340	86.51
4.8	3.617	72.19	9.8	13.443	82.45	14.8	29.517	85.24	19.8	51.848	86.53
4.9	3.753	72.66	9.9	13.703	82.54	14.9	29.902	85.27	19.9	52.358	86.55
5.0	3.891	73.11	10.0	13.966	82.62	15.0	30.290	85.31	20.1	52.871	86.56

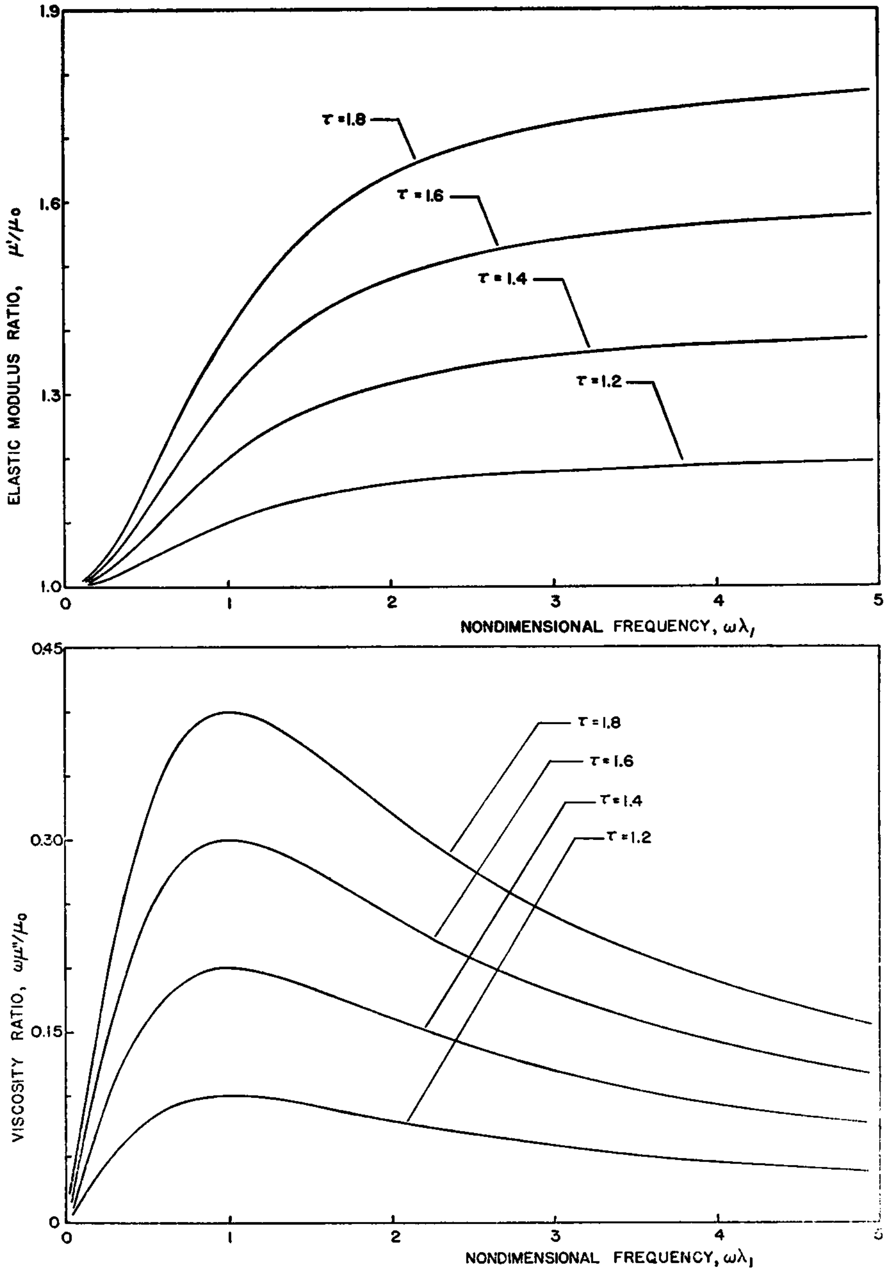

FIGURE 3 Dependence of the dynamic elastic modulus ratio (μ'/μ_0) and dynamic viscosity ratio ($\omega\mu''/\mu_0$) on nondimensional frequency ($\omega\lambda_1$) for various values of time constant ratio, τ.

The frequency variation of the real and imaginary parts of the theoretical modulus of rigidity are shown in Fig. 3. Defining $\mu^* = \mu' + j\omega\mu''$, the upper panel shows the dependence of the nondimensional elastic modulus (μ'/μ_0) on the nondimensional frequency ($\omega\lambda_1$) at four different values of the ratio of time constants, $\tau = \lambda_2/\lambda_1$. The dependence of the wall viscosity ratio ($\omega\mu''/\mu_0$) on nondimensional frequency is shown in the lower panel for the same values of τ. It should be noted that the frequency variation of the real and imaginary parts of the complex modulus are not independent but are coupled according to the definition of μ^* in equation 36.

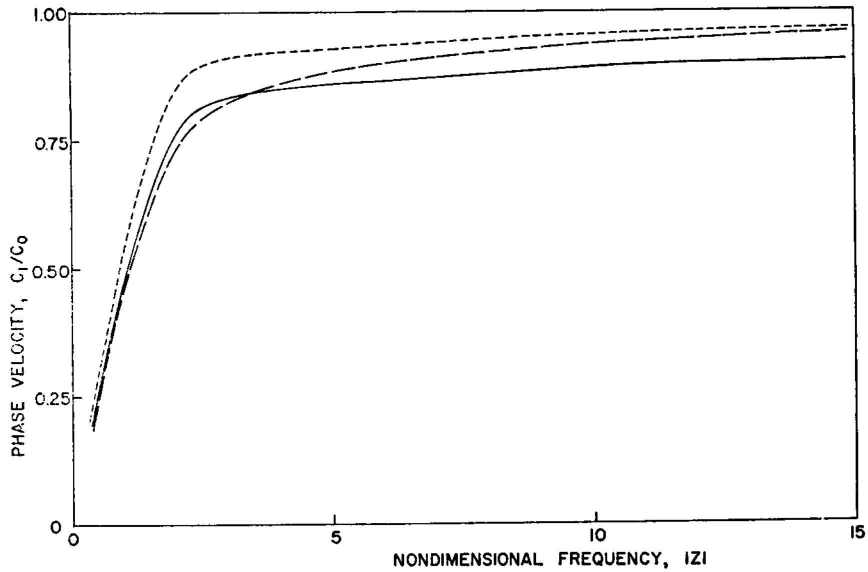

FIGURE 4 Comparison of the dependence of the phase velocity (c_1/c_0) of the first root from present model (continuous line) and from Womersley's constrained tube (long dashed line) and elastic tube models (short dashed line) on z.

The frequency equation is quadratic in γ^2, having four solutions for γ, two of which represent forward propagating waves. The frequency dependence of the phase velocity of the first root is shown in Fig. 4 for the case of an elastic tube ($\tau = 1.0$). Both the phase velocity and frequency are nondimensionalized in the graph, the former by dividing by the inviscid phase velocity, c_0. The inviscid phase velocity is obtained by setting μ equal to a very small number, about 10^{-8} cp, and then solving the frequency equation from 30. Also shown in the figure for comparison are the phase velocities predicted by Womersley's constrained elastic tube model (1957) and Womersley's freely moving elastic tube model (1955). The frequency dependence of the transmission per wavelength for the first root is shown in Fig. 5. The transmission per wavelength is given by $e^{-\alpha\lambda}$, where α is the attenuation coefficient for the

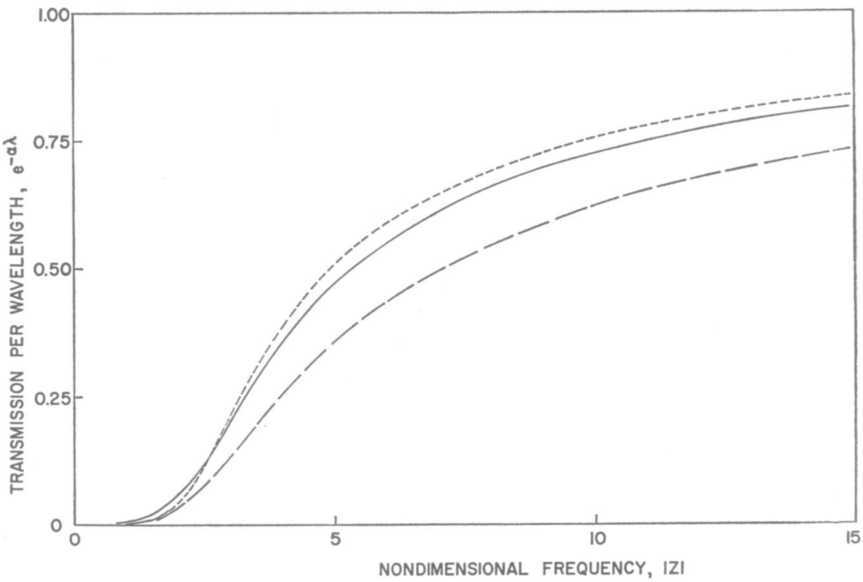

FIGURE 5 Comparison of the dependence of transmission per wavelength of first root of present model (continuous line), Womersley's constrained tube (long dashed line), and elastic tube models (short dashed line) on z.

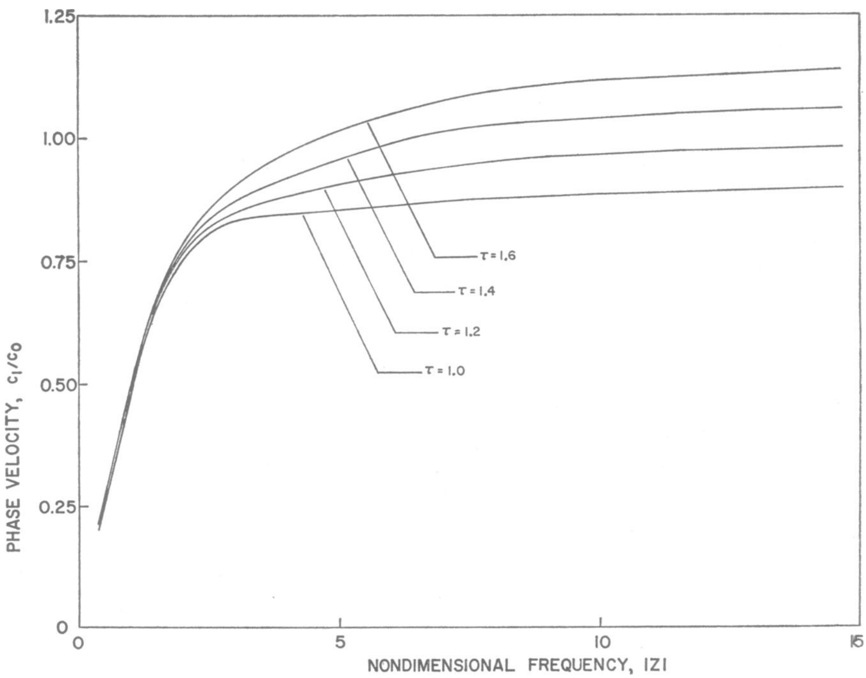

FIGURE 6 Dependence of phase velocity of first root (c_1/c_0) on z for various values of time constant ratio, τ.

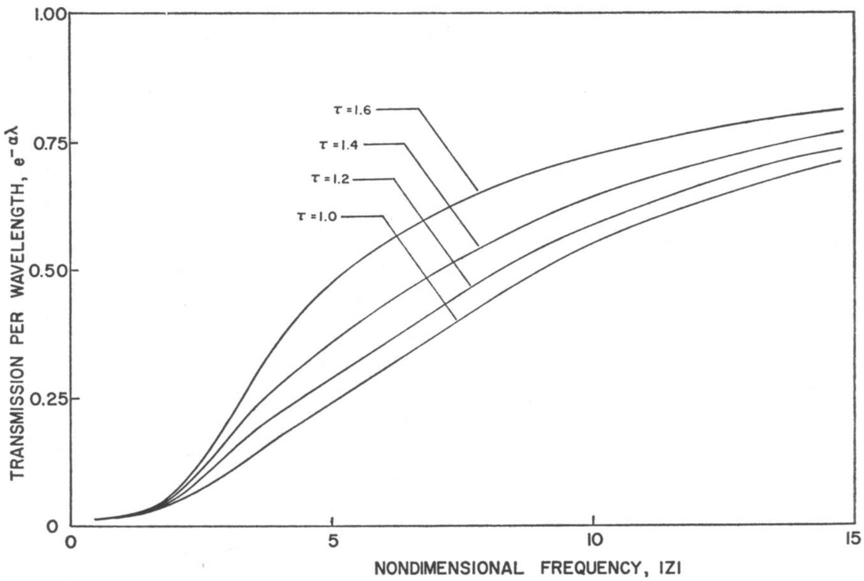

FIGURE 7 Dependence of transmission per wavelength of first root on z for various values of time constant ratio, τ.

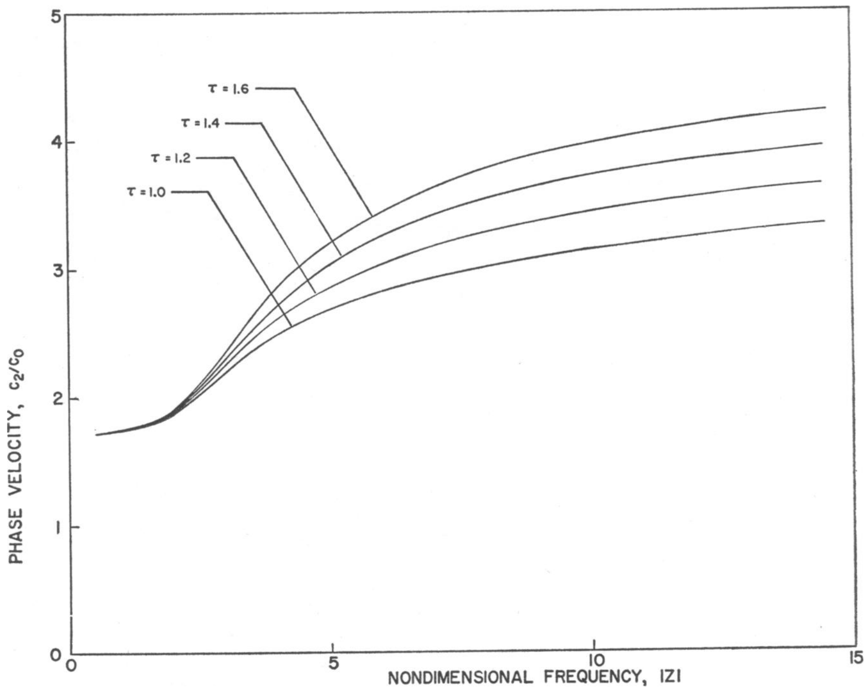

FIGURE 8 Dependence of phase velocity of second root (c_2/c_0) on z for various values of time constant ratio, τ.

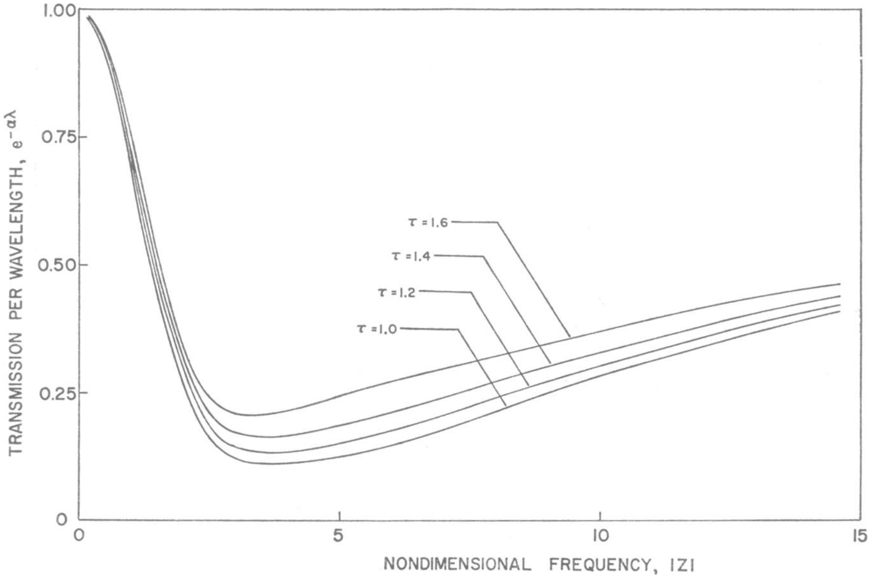

FIGURE 9 Dependence of transmission per wavelength of second root on z for various values of time constant ratio, τ.

first root of the frequency equation. Corresponding values of the transmission per wavelength from Womersley's two models are also shown for comparison.

The variation of the phase velocity of the first root with the frequency for the case of different degrees of wall viscoelasticity is shown in Fig. 6. The variation of the transmission per wavelength of the first type of wave with frequency, for various levels of wall viscosity are shown in Fig. 7. Again, increasing the amount of wall viscosity (τ) affects both the phase velocity and the attenuation because of the coupling between the real and imaginary parts of the complex modulus of rigidity. The phase velocity and transmission per wavelength of the second type of waves are shown in Figs. 8 and 9, respectively, as a function of frequency and wall viscoelasticity.

DISCUSSION

In the formulation of hemodynamic models for the purpose of representing the pulsatile flow of blood in arteries, it is necessary to apply certain simplifying assumptions to obtain solutions of the resulting system of differential equations. The assumptions that have been employed in the derivation of the model presented herein have been listed briefly in the analytical section, and some comments on their validity seem appropriate. Concerning the Navier-Stokes equations, it has been assumed that blood flow is linear and that the convective acceleration terms can be ignored. Morgan and Kiely (1954) and others have shown that the nonlinear terms in the Navier-Stokes equations can be neglected with respect to the linear terms provided $V_z \ll c$ and $a \ll \lambda$, where V_z is the average axial velocity, c the phase velocity, a the inner tube radius, and λ the wavelength of the pressure wave. Using representative values of these parameters for the aorta (Attinger, 1965) (V_z = 25 cm/sec, a = 1 cm, and c = 500 cm/sec), V_z/c = 0.05 and a/λ = 0.02 at a frequency of 10 Hz. For the case of the femoral artery (V_z = 5 cm/sec, a = 0.2 cm, and c = 1000), V_z/c = 0.005 and a/λ = 0.002 at a frequency of 10 Hz. Thus, both ratios decrease peripherally and the values for the femoral (at least) seem small enough for the assumption to be valid.

Implicit in the use of the Navier-Stokes equation is the assumption that blood is a Newtonian fluid. It has been shown directly by several authors (Haynes and Burton, 1959; Replogle et al., 1967) that over a wide range of shear rate, blood is not Newtonian. However, these authors have concluded that at the normal, physiologic levels of wall shear stress existing in larger arteries the shear stress-shear rate curves are linear, and blood behaves as a Newtonian fluid.

The boundary conditions, equations 7–14, were applied at the mean rather than at the instantaneous values of the inner and outer radii. This amounts to a linearization of the boundary conditions. Morgan and Kiely (1954) have shown that the error introduced by this approximation is of the same magnitude as that associated with the linearization of the Navier-Stokes equations. Again this effect will be small provided $a \ll \lambda$ and $u_r \ll a$, where u_r is the radial displacement. The former has already been shown to be small and the latter has been shown experimentally to be small (Peterson et al., 1960). Thus, the assumption also appears to be valid.

Despite the fact that a large number of different hemodynamic models have been published in the literature, only two quantitatively different models for the fluid impedance have been derived explicitly. One is the rigid tube model first derived by Witzig (1914), and the other is the elastic tube model of Womersley (1955). In a number of models, the fluid impedance relationships were not derived (Morgan and Kiely, 1954; Klip, 1962), in some cases numerical solutions were indicated (Streeter et al., 1963; Barnard et al., 1966; Mirsky, 1967), while in other cases the assumptions employed were such that the fluid impedance reduced to the same form as the rigid tube model (Womersley, 1957; Jager, 1965; Whirlow and Rouleau, 1965).

There are some areas in which there are significant differences between these former two explicit fluid impedance models, as well as with the model derived herein. Womersley's elastic tube model predicts a fluid impedance that is lower than that of the rigid tube, while the model we have derived predicts a fluid impedance greater than Womersley's elastic tube model but less than the rigid tube model at all values of z. It is worth noting that for an infinite elastic modulus (rigid wall), the fluid impedance of Womersley's model and our model are identical with that of the rigid tube model.

The theoretical values of the fluid resistance and inductance (shown in Fig. 2) also indicate significant differences between the three models.

The elastic tube phase velocity from the present model shows significant differences compared with the phase velocity predicted by Womersley's models (1955, 1957), being lower than the latter over most of the frequency range of interest. The transmission per wavelength from our model for the same case also possesses a significantly different behavior from the Womersley models.

The effects of viscoelasticity on the phase velocity and transmission presented in Figs. 6 and 7 are in agreement with intuition; viscoelasticity increases the phase velocity and decreases the transmission. The variations of the phase velocity and transmission per wavelength of the second type of waves shown in Figs. 8 and 9 are similar to those found by Atabek and Lew (1966) and Mirsky (1967).

While the field of theoretical hemodynamics has witnessed the presentation of a large variety of mathematical models, little experimental data exist in the literature for use in the verification of such models. The validity of the rigid tube model has been demonstrated several times by experiments utilizing flow in rigid tubes (Thurston, 1952; Richardson and Tyler, 1929), but the applicability of this model to the arterial system is still to be proven. In a subsequent paper, the model developed in this paper will be tested on measurements of fluid impedance and propagation characteristics from the femoral artery of anesthetized dogs.

SYMBOLS

a, b	Inner and outer tube radii.
A_i, B_i	Complex integration constants.
c_1, c_2	Phase velocities of first and second roots of frequency equations.
c_0	Inviscid fluid phase velocity.
j	Complex number, $\sqrt{-1}$.
J_i, Y_i	Bessel functions of first and second kind of order i.
k_n	Complex tube parameter.
n	Harmonic number.
p	Intra-arterial fluid pressure.
Q_n	Nth harmonic of axial flow.
r, x	Radial and axial coordinate.
R_p	Poiseuille resistance, $R_p = 8\,\mu/\pi a^4$.
t	Time.
u_r, u_x	Radial and axial components of tube displacement.
V_x	Average axial velocity.
v_r, v_x	Radial and axial components of fluid velocity.
z	Nondimensional fluid parameter, $z = j\kappa_n a$.
Z_n	Nth harmonic of fluid impedance.
α	Attenuation coefficient.
β	Phase constant.
γ	Propagation constant.
κ_n	Complex fluid parameter.
λ	Wavelength.
λ_1, λ_2	Time constants of viscoelastic wall behavior.
μ	Fluid viscosity.

μ^* Complex modulus of rigidity, $\mu^* = \mu' + j\omega\mu''$.
μ_0 Static modulus of rigidity.
ν Kinematic viscosity, $\nu = \mu/\rho$.
ρ, ρ_w Fluid and tube densities.
τ Time constant ratio, $\tau = \lambda_2/\lambda_1$.
ω Angular frequency.
Ω Tube wall pressure.

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