On the Propagation of Sound Waves in a Cylindrical Conduit

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The characteristic impedance and propagation constant of a cylindrical conduit are calculated on the basis of an equivalent electrical T-section. Numerical values of the results are plotted for air at 20°C, for a range of values of the independent variable which includes the region of transition from isothermal to adiabatic conditions.

I. INTRODUCTION

In a previous paper[1] the author calculated the acoustical impedance of three types of enclosures, the shapes of which permitted an exact solution. Mawardi[2] noticed that, if the results of the calculations are plotted against $f^{\frac{1}{2}}S/P$ (P being the perimeter and S the sectional area of a conduit), the curves are nearly independent of the shape of the conduit. It is therefore possible to choose an intermediate curve to represent a conduit of arbitrary shape, and Mawardi has used such a curve in calculating the characteristic impedance of a tube filled with a bundle of wires. The results obtained by this method are only approximate, but are probably sufficiently accurate for most practical applications. However, for the important particular case of a cylindrical conduit, an exact solution for the characteristic impedance and propagation constant in terms of Bessel functions can be obtained. The present paper deals with this solution for the particular case of a conduit filled with air at 20°C.

II. NOTATION

The following symbols have been adopted, c.g.s. units being used throughout:

- ρ—density
- f—frequency
- ω—$2\pi f$
- μ—viscosity
- σ—$(\omega\rho/2\mu)^{\frac{1}{2}}=4.57f^{\frac{1}{2}}$ for air at 20°C
- η—$(\omega\rho C_p/2K)^{\frac{1}{2}}=4.06f^{\frac{1}{2}}$ for air at 20°C
- K—thermal conductivity
- C_p—specific heat at constant pressure
- a—radius of conduit
- v—velocity of sound in conduit
- c—velocity of sound in free space.
- κ—ratio of specific heats.

III. OUTLINE OF METHOD

In the Helmholtz-Kirchhoff treatment of propagation through tubes,[3] the dynamical equations are modified to include the effects of viscosity and heat conduction and a solution of these equations is obtained in terms of Bessel functions. Limiting expressions which apply to narrow and wide tubes are obtained from series expansions of the Bessel functions which are valid for small and large values of the independent variable, respectively.

In this paper the conduit will be represented by an electrical transmission line and the effects of viscosity and heat conduction will be introduced separately by modifying the series and shunt elements of the equivalent T-section. Numerical results will be computed, using tabulated values of the Bessel functions.

IV. EQUIVALENT T-SECTION

The equivalent T-section of a short length of conduit is shown in Fig. 1 in which inertance and frictional resistance are represented by the series elements L and R and acoustical capacitance and loss due to heat conduction to the walls are represented by the shunt elements C' and G. For sufficiently large values of the variable $af^{\frac{1}{2}}$, L approaches the inertance per unit length, C' approaches the adiabatic capacitance per unit length, and R and G approach zero. For the general case, L and R can be obtained from an expression for particle velocity given by Crandall,[4] and C' and G from the expression for the impedance of a cylindrical enclosure derived in the author's earlier paper.[1] In applying Crandall's result, the ratio of pressure to particle

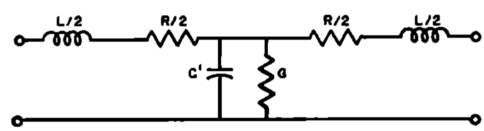

FIG. 1. Equivalent T-section of a short length of conduit.

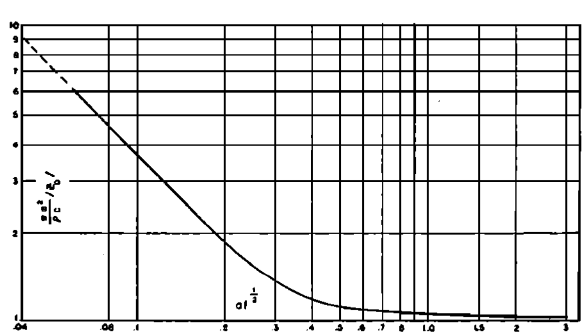

FIG. 2. Absolute value of the characteristic impedance.

[1] Fred B. Daniels, J. Acous. Soc. Am. **19**, 569 (1947).
[2] Osman K. Mawardi, J. Acous. Soc. Am. **21**, 482 (1949).
[3] Lord Rayleigh, *Theory of Sound* (Dover Publications, New York, 1945), Vol. 2, pp. 319–328.
[4] I. B. Crandall, *Theory of Vibrating Systems and Sound* (D. Van Nostrand Company, Inc., New York, 1927), p. 234.

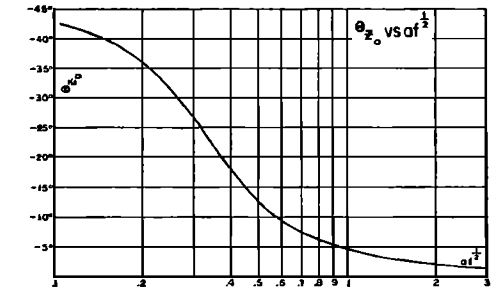

FIG. 3. Phase angle of the characteristic impedance.

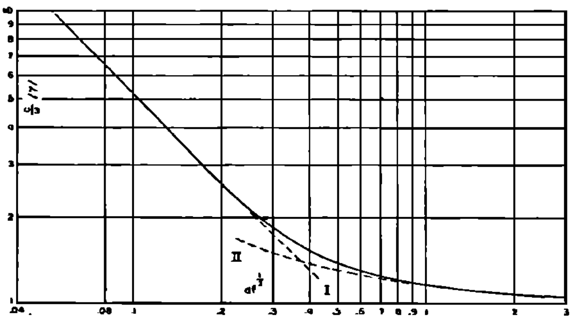

FIG. 4. Absolute value of the propagation constant. Solid curve: Exact value. Curve I: Rayleigh, narrow tube. Curve II: Rayleigh, wide tube.

velocity is divided by the cross-sectional area in order to express impedance in terms of volume velocity. It should be noted that the above derivations are based upon the assumption of uniform pressure across the tube, and that the final result will therefore be predicated upon this assumption.

V. EVALUATION OF CHARACTERISTIC IMPEDANCE AND PROPAGATION CONSTANT

The impedance and admittance of the series and shunt arms of the T-section are, respectively,

$$Z=R+j\omega L=\frac{\rho\omega}{\pi a^2}(A^2+B^2)^{-\frac{1}{2}}\exp j\psi \tag{1}$$

an

$$Y=G+j\omega C'=\frac{\pi a^2\omega}{\rho c^2}(C^2+D^2)^{\frac{1}{2}}\exp -j\phi, \tag{2}$$

where $A=1+(1/\sigma a)(F+G)$, $B=(1/\sigma a)(G-F)$, $F=(VX-UY)/(U^2+V^2)$, $G=(UX+VY)/(U^2+V^2)$, $U=$real part of $J_0[\sigma a(-2j)^{\frac{1}{2}}]$, $V=$imaginary part of $J_0[\sigma a(-2j)^{\frac{1}{2}}]$, $X=$real part of $J_1[\sigma a(-2j)^{\frac{1}{2}}]$, $Y=$imaginary part of $J_1[\sigma a(-2j)^{\frac{1}{2}}]$, and $\psi=\tan^{-1}A/B$. $C=A'(K-1)-\kappa$ and $D=B'(K-1)$ where A' and B' are calculated from the same equations as A and B, respectively, with σa replaced by ηa as the independent variable. $\phi=\tan^{-1}D/C$.

From electrical network theory, the characteristic impedance Z_0 and propagation constant γ are given by

$$Z_0=(Z/Y)^{\frac{1}{2}} \quad\text{and}\quad \gamma=\alpha+j\beta=(ZY)^{\frac{1}{2}}.$$

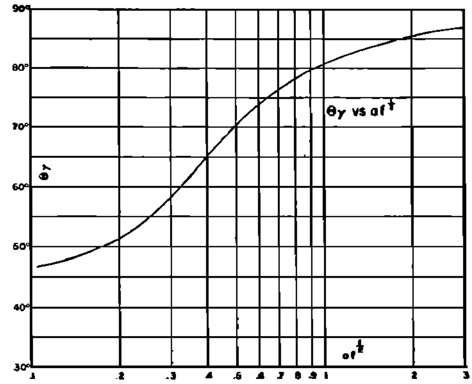

FIG. 5. Phase angle of the propagation constant.

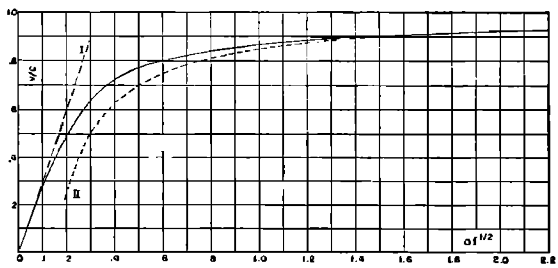

FIG. 6. Ratio of the velocity in the conduit to the velocity in free space. Solid curve: Exact value. Curve I: Rayleigh, narrow tube. Curve II: Rayleigh, wide tube.

Substituting the values of Z and Y from (1) and (2),

$$Z_0=(\rho c/\pi a^2)(C^2+D^2)^{-\frac{1}{4}}(A^2+B^2)^{-\frac{1}{4}}\exp j\tfrac{1}{2}(\psi+\phi), \tag{3}$$

and

$$\gamma=(\omega/c)(C^2+D^2)^{\frac{1}{4}}(A^2+B^2)^{-\frac{1}{4}}\exp j\tfrac{1}{2}(\psi-\phi), \tag{4}$$

whence

$$\alpha=|\gamma|\cos\tfrac{1}{2}(\psi-\phi), \tag{5}$$

$$\beta=|\gamma|\sin\tfrac{1}{2}(\psi-\phi), \tag{6}$$

and

$$v=\omega/\beta. \tag{7}$$

From the above results, numerical values of Z_0 and γ have been computed as functions of $af^{\frac{1}{2}}$, assuming the conduit to be filled with air at 20°C. The absolute value and phase angle of Z_0 and γ and the value of v/c are plotted in Figs. 2 to 6, inclusive, for a range of values of $af^{\frac{1}{2}}$ which includes the region of transition from isothermal to adiabatic conditions. For the purpose of comparison, values of $c/\omega|\gamma|$ and v/c were computed from Rayleigh's limiting expressions. The results of these calculations are given by the dotted curves in Fig. 4 and Fig. 6.

VI. ACKNOWLEDGMENT

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