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An extended Huygens' principle for modelling scattering from general discontinuities within hollow waveguides

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SUMMARY

The modal fields, generalized scattering matrix (GSM) theory and dyadic Green's functions relating to a general uniform hollow waveguide are briefly reviewed in a mutually consistent normalization. By means of an analysis linking these three concepts, an extended version of the mathematical expression of Huygens' principle is derived, applying to scattering from an arbitrary object within a hollow waveguide. The integral-equation result expresses the total field in terms of the incident waveguide modal fields, the dyadic Green's functions and the tangential electromagnetic field on the surface of the object. It is shown how the extended principle may be applied in turn to perfect conductor, uniform material and inhomogeneous material objects using a quasi method of moments (MM) approach, coupled in the last case with the finite element method. The work reported, which indicates how the GSM of the object may be recovered, is entirely theoretical but displays a close similarity with established MM procedures. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: dyadic Green's functions; scattering matrix; modal fields; hollow waveguide; Huygens' principle; finite element method; method of moments

1. INTRODUCTION

General forms for the modal vector wavefields in uniform hollow waveguides are well known [1,2] and are the basis for generalized scattering-matrix (GSM) theories [2]. There are many instances of successful numerical modelling of low-order GSM coefficients for simple waveguide discontinuities [3]. The most general numerical procedure for such modelling currently available is mode-matching at a pair of planes normal to the waveguide axis and isolating the discontinuity; the modal fields are coupled, over the planes, to fields in the inhomogeneous space between them, the latter fields being modelled numerically say by the finite element method (FEM) [4]. However, there are situations in which parallel planes conform poorly with the scattering object, defining a largely empty region, so that the numerical procedure becomes inefficient. Indeed when the final objective is to model, by FEM, a multimode cavity such as a short-circuit loaded waveguide, the planar-boundary FEM region may itself represent a high-Q cavity, its matrices becoming ill-conditioned even though the object itself is non-resonant [5].

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Green's functions may be used in scattering analysis to conform closely with an irregular object. Bunger and Arndt [6] use free-space Green's functions to model a metallic object in a general shaped cavity coupled to a waveguide system, thus forming a waveguide N-port, the corresponding scattering matrix being determined by mode-matching at planar apertures in the cavity walls. The scattering of pressure waves in an acoustic waveguide has been modelled by applying the scalar variable Kirchhoff–Huygens' principle [7]. As an alternative to transverse-plane mode matching, dyadic Green's functions satisfying the waveguide boundary constraints may be employed to model a general object in an otherwise continuous waveguide. The dyadics corresponding to standard empty waveguide cross-sections have been worked out and are summarized, separately for various cross-sections, by Tai [8]. In 1978, Wang [9] modelled the dominant-mode reflection and transmission for an arbitrary dielectric object in rectangular waveguide using such dyadics, replacing the object by equivalent volume-dispersed sources. The resulting volumetrically dense matrices precluded use of any but the coarsest discretization of the object. Yakovlev et al. [10] deal with electromagnetic scattering from infinitely thin planar layers in rectangular waveguide using dyadic Green's functions in a method of moments (MM) treatment. The waveguide dyadic Green's functions are closely related to the corresponding modal wavevectors. Pathak [11] has given the hollow waveguide dyadics in terms of such general modal wavefunctions, however without considering their relationship with the scattering matrix of an arbitrary waveguide discontinuity.

It is shown here how scattering from an arbitrary-shaped object in a general electromagnetic hollow waveguide may be accounted for by an integral equation involving the waveguide dyadic Green's functions; the equation derived is a generalization of the mathematical expression of the vector variable Huygens' principle for free space [12] and is closely related to MM scattering formulations in simpler situations [13,10]. The Huygens equation is discretized, for mode-by-mode excitation, in terms of a triangular patch representation of the surface of the arbitrary object. Simultaneous expansion of the tangential **E** and **H** fields in terms of the Rao–Wilton–Glisson patch-functions [14] and the related Whitney-edge elements [15,16] is required. The fields inside the object are to be dealt with separately, either by uniform space dyadics or by FEM analysis. Thus there is no need to invoke volume-dispersed sources as in Wang [9], so that the dense matrices which will arise have dimensions corresponding to a surface discretization of the object. This paper is entirely theoretical and no numerical results are presented. However, it is shown how the discretized integral equation may be applied to a number of different scattering-in-waveguide problems, to establish the object's GSM. A similarity with corresponding free-space scattering problems, solvable by applying 'equivalence' theory [17] and MM [18] or hybrid FEM methods [19] may be noted, so that the established numerical techniques applicable there would carry over directly to the present problem.

2. REVIEW OF HOLLOW WAVEGUIDE MODAL FIELDS AND DYADIC GREEN'S FUNCTIONS

The modal fields in empty, arbitrary cross-section uniform waveguide, applicable to GSM theory, and the corresponding dyadic Green's functions are briefly reviewed below. A consistent normalization scheme relating to both is selected so as to allow the Green's theorem application which is critical to the principle result of this paper.

2.1. Modal fields

The configuration considered is a pair of similar, perfectly conducting, arbitrary cross-section empty waveguides Ω_1 and Ω_2 separated by a contiguous waveguide Ω containing an inhomogeneous object Ω_L (see Figure 1a). With $k_0 = \omega\sqrt{\mu_0\varepsilon_0}$, $Z_0 = 1/Y_0 = \sqrt{\mu_0/\varepsilon_0}$, $\mathbf{R} = (x, y, z)$, $\mathbf{r} = (x, y)$ and omitting a common $e^{j\omega t}$ factor, the general solution of Maxwell's equations valid in waveguides Ω_i, $i = 1, 2$, may be written as modal vector eigenfunction expansions [1,2]:

$$\mathbf{E}^{(i)}(\mathbf{R}) = \sum_{m=1}^{\infty} \sqrt{Z_m}\{a_m^{(i)} e^{-\gamma_m z}[\mathbf{e}_{tm}(\mathbf{r}) + \mathbf{e}_{zm}(\mathbf{r})] + b_m^{(i)} e^{\gamma_m z}[\mathbf{e}_{tm}(\mathbf{r}) - \mathbf{e}_{zm}(\mathbf{r})]\} \tag{1}$$

$$\mathbf{e}_{zm}(\mathbf{r}) = \frac{1}{\gamma_m} \nabla_t \times \mathbf{h}_{tm} (\text{TM modes}), = 0 \ (\text{TE modes}) \tag{2}$$

$$\mathbf{H}^{(i)}(\mathbf{R}) = \sum_{m=1}^{\infty} \sqrt{Y_m}\{a_m^{(i)} e^{-\gamma_m z}[\mathbf{h}_{tm}(\mathbf{r}) + \mathbf{h}_{zm}(\mathbf{r})] - b_m^{(i)} e^{\gamma_m z}[\mathbf{h}_{tm}(\mathbf{r}) - \mathbf{h}_{zm}(\mathbf{r})]\} \tag{3}$$

$$\mathbf{h}_{zm}(\mathbf{r}) = -\frac{1}{\gamma_m} \nabla_t \times \mathbf{e}_{tm} (\text{TE modes}), \ = 0 \ (\text{TM modes}) \tag{4}$$

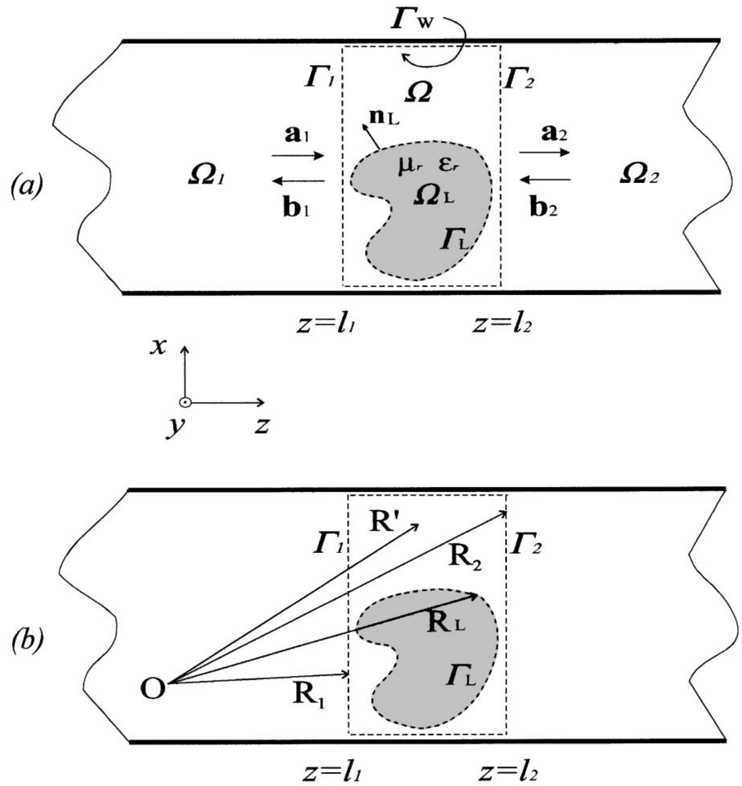

Figure 1. (a) Arbitrary cross-section hollow waveguide enclosing a load object Ω_L. The dotted line represents the notional doubly-connected surface $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_L \cup \Gamma_W$. (b) The same, showing typical positioning of vectors $\mathbf{R}$ (as $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_L$) and $\mathbf{R}'$.

where subscripts t and z denote transverse and axial components, respectively. Here $\gamma_m = (\alpha_m + \mathrm{j}\beta_m) = \mathrm{j}\sqrt{k_0^2 - k_{cm}^2}$, $k_0 > k_{cm}$ or $\sqrt{k_{cm}^2 - k_0^2}$, $k_0 < k_{cm}$ while k_{cm} is the cutoff wavenumber for mode m. Equations (1) and (3) each may represent both TE and TM types. The multiplying factors there, $\sqrt{Z_m}$ and $\sqrt{Y_m}$, follow a common normalization convention [20], widely (but not universally) adopted in GSM manipulations for defining $a_m^{(i)}$ and $b_m^{(i)}$ uniquely, Z_m (and $Y_m = 1/Z_m$) being the empty waveguide modal impedences (admittances) where $Z_m = Z_0\gamma_m/\mathrm{j}k_0$ (TM modes) or $Z_0\mathrm{j}k_0/\gamma_m$ (TE modes). The modal vectors above are mutually orthogonal when integrated over any transverse waveguide cross-section [1]. In the normalization selected

$$\mathbf{h}_{tm} = \mathbf{1}_z \times \mathbf{e}_{tm} \tag{5}$$

$$\int_\Gamma \mathbf{e}_{tm} \cdot \mathbf{e}_{tn}\, \mathrm{d}\Gamma = \int_\Gamma \mathbf{h}_{tm} \cdot \mathbf{h}_{tn}\, \mathrm{d}\Gamma = \delta_{mn} \tag{6}$$

where δ_{mn} is the Kroenecker delta, so that the real power carried by any propagating mode m turns out to be

$$P_m^{(i)} = \tfrac{1}{2}(a_m^{(i)} a_m^{(i)*} - b_m^{(i)} b_m^{(i)*}) \tag{7}$$

where * denotes complex conjugation. For a waveguide discontinuity separating identical waveguides, such as illustrated in Figure 1(a), it is well known [2] that the associated wave amplitude coefficients $a_m^{(1)}$, $a_m^{(2)}$, $b_m^{(1)}$ and $b_m^{(2)}$ may be related by the GSM, say $[\mathbf{S}]$ where

$$\begin{bmatrix} \mathbf{b}^{(1)} \\ \mathbf{a}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{S}^{(11)} & \mathbf{S}^{(12)} \\ \mathbf{S}^{(21)} & \mathbf{S}^{(22)} \end{bmatrix} \begin{bmatrix} \mathbf{a}^{(1)} \\ \mathbf{b}^{(2)} \end{bmatrix}, \quad \mathbf{a}^{(1)} = [a_1^{(1)}, a_2^{(1)}, \ldots, a_M^{(1)}]^{\mathrm{T}}, \text{ etc.,} \tag{8}$$

the infinite series Equations (1) and (3) having been truncated to M terms. Note that, for clarity in the following analysis, there is a departure here from the commonly adopted notation where **a** and **b**, respectively, denote incident and scattered waves relative to the scatterer.

2.2. *Dyadic Green's functions*

Dyadic Green's functions applied to hollow waveguide are discussed by a number of authors and enlarged upon in Tai's monograph [8], whereas Pathak [11] was the first author clearly to set out such dyadics for a *general* hollow waveguide in terms of the corresponding modal functions. However, the relationship between the GSM and dyadic Green's function waveguide formalisms has not been worked out before. As a basis for establishing this link, Pathak's work is briefly reviewed in terms of the normalization selected here for the modal functions.

The electric and magnetic dyadic Green's functions, $\mathbb{G}_{\mathrm{E}}$ and $\mathbb{G}_{\mathrm{H}}$, respectively, for a hollow waveguide are defined, subject to $\mathbf{n} \times \mathbb{G}_{\mathrm{E}} = 0$, $\mathbf{n} \times \nabla \times \mathbb{G}_{\mathrm{H}} = 0$ on the waveguide walls, by

$$\nabla \times \nabla \times \mathbb{G}_{\mathrm{E}}(\mathbf{R}, \mathbf{R}') - k_0^2 \mathbb{G}_{\mathrm{E}}(\mathbf{R}, \mathbf{R}') = \mathbb{I}\delta(\mathbf{R} - \mathbf{R}') \tag{9}$$

$$\nabla \times \nabla \times \mathbb{G}_{\mathrm{H}}(\mathbf{R}, \mathbf{R}') - k_0^2 \mathbb{G}_{\mathrm{H}}(\mathbf{R}, \mathbf{R}') = \nabla \times \mathbb{I}\delta(\mathbf{R} - \mathbf{R}') \tag{10}$$

where $\mathbb{I}$ is the unit dyadic, $\delta(\mathbf{R}-\mathbf{R}')$ is the three-dimensional Dirac delta function centred about $\mathbf{R}=\mathbf{R}'$ while $\mathbb{G}_E$ and $\mathbb{G}_H$ are related by

$$\mathbb{G}_H = \nabla \times \mathbb{G}_E \tag{11}$$

$$\mathbb{G}_E = \frac{1}{k_0^2}[\nabla \times \mathbb{G}_H - \mathbb{I}\delta(\mathbf{R}-\mathbf{R}')] \tag{12}$$

Pathak gives solutions to Equations (9) and (10) for hollow waveguides in terms of the **e** and **h** eigenfunctions. When translated to the modal normalization chosen in Section 2.1, the dyadic results written out fully referred to Cartesian axes are

$$\mathbb{G}_E^{\pm}(\mathbf{R},\mathbf{R}') = \frac{1}{jk_0 Z_0}\sum_{n=1}^{\infty}\frac{Z_n}{2}\begin{bmatrix} e_{xn}(\mathbf{r})e_{xn}(\mathbf{r}') & e_{xn}(\mathbf{r})e_{yn}(\mathbf{r}') & \mp e_{xn}(\mathbf{r})e_{zn}(\mathbf{r}') \\ e_{yn}(\mathbf{r})e_{xn}(\mathbf{r}') & e_{yn}(\mathbf{r})e_{yn}(\mathbf{r}') & \mp e_{yn}(\mathbf{r})e_{zn}(\mathbf{r}') \\ \pm e_{zn}(\mathbf{r})e_{xn}(\mathbf{r}') & \pm e_{zn}(\mathbf{r})e_{yn}(\mathbf{r}') & -e_{zn}(\mathbf{r})e_{zn}(\mathbf{r}') \end{bmatrix} e^{\mp\gamma_n(z-z')}$$

$$-\frac{1}{k_0^2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \delta(\mathbf{R}-\mathbf{R}') \end{bmatrix} \tag{13}$$

$$\mathbb{G}_H^{\pm}(\mathbf{R},\mathbf{R}') = \sum_{n=1}^{\infty}\frac{1}{2}\begin{bmatrix} \mp h_{xn}(\mathbf{r})\mathrm{e}_{xn}(\mathbf{r}') & \mp h_{xn}(\mathbf{r})\mathrm{e}_{yn}(\mathbf{r}') & h_{xn}(\mathbf{r})\mathrm{e}_{zn}(\mathbf{r}') \\ \mp h_{yn}(\mathbf{r})\mathrm{e}_{xn}(\mathbf{r}') & \mp h_{yn}(\mathbf{r})\mathrm{e}_{yn}(\mathbf{r}') & h_{yn}(\mathbf{r})\mathrm{e}_{zn}(\mathbf{r}') \\ -h_{zn}(\mathbf{r})\mathrm{e}_{xn}(\mathbf{r}') & -h_{zn}(\mathbf{r})\mathrm{e}_{yn}(\mathbf{r}') & 0 \end{bmatrix} e^{\mp\gamma_n(z-z')} \tag{14}$$

where the upper sign corresponds to $z > z'$ and the lower sign to $z < z'$. Evidently, some of the elements of both $\mathbb{G}_E$ and $\mathbb{G}_H$ are either discontinuous or singular at $\mathbf{R}=\mathbf{R}'$.

3. DERIVATION OF AN EXTENDED HUYGENS' PRINCIPLE

Consider the configuration of Figure 1(a), a uniform hollow waveguide containing an inhomogeneous object Ω_L bounded by the closed surface Γ_L and lying between, but not touching, a pair of notional planes Γ_1 and Γ_2 filling the waveguide cross-section at $z=l_1$ and $z=l_2$, respectively. In the empty regions Ω_1 and Ω_2, to the left of Γ_1 and to the right of Γ_2, the waveguide carries (differing) multimode fields consisting, to sufficient precision, of the first M modes of the unobstructed waveguide represented, respectively, by the column vectors $\mathbf{a}^{(1)}$, $\mathbf{b}^{(1)}$ and $\mathbf{a}^{(2)}$, $\mathbf{b}^{(2)}$ and given in full detail by truncated field expansions of the form Equations (1) and (3). Within the region Ω, which is outside Ω_L bounded by the closed doubly-connected surface $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_W \cup \Gamma_L$ (where Γ_W is an appropriate segment of the waveguide walls), the fields are, as elsewhere in the empty waveguide, subject to

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{R}) - k_0^2\mathbf{E}(\mathbf{R}) = 0 \tag{15}$$

$$\nabla \times \nabla \times \mathbf{H}(\mathbf{R}) - k_0^2\mathbf{H}(\mathbf{R}) = 0 \tag{16}$$

Apply Green's theorem in its vector/dyadic form [8] to $\mathbf{E}(\mathbf{R})$ and $\mathbb{G}_{\mathrm{E}}(\mathbf{R}, \mathbf{R}')$ with respect to the unprimed space variable, the curl operator also corresponding to the unprimed space variable, giving

$$\int_{\Omega} [\mathbf{E}(\mathbf{R}) \cdot \nabla \times \nabla \times \mathbb{G}_{\mathrm{E}}(\mathbf{R}, \mathbf{R}') - \nabla \times \nabla \times \mathbf{E}(\mathbf{R}) \cdot \mathbb{G}_{\mathrm{E}}(\mathbf{R}, \mathbf{R}')]\, \mathrm{d}\Omega$$

$$= -\oint_{\Gamma} \{[\mathbf{n}_{\Gamma} \times \nabla \times \mathbf{E}(\mathbf{R})] \cdot \mathbb{G}_{\mathrm{E}}(\mathbf{R}, \mathbf{R}') + [\mathbf{n}_{\Gamma} \times \mathbf{E}(\mathbf{R})] \cdot \nabla \times \mathbb{G}_{\mathrm{E}}(\mathbf{R}, \mathbf{R}')\}\, \mathrm{d}\Gamma \qquad (17)$$

Note that $\mathbf{R}$ and $\mathbf{R}'$ here represent the *closed* set of points $\mathbf{R}, \mathbf{R}' \in \Omega \cup \Gamma$ although obviously $\mathbf{R} \in \Gamma$, $\mathbf{R} \notin \Omega$ on the right-hand side of Equation (17). Substituting Equations (9), (11) and (15) into Equation (17), also using Maxwell's curl equations and noting that the right-hand side surface integral over the Γ_{W}-part of Γ vanishes because of the waveguide wall boundary constraints, gives

$$\int_{\Omega} [\mathbf{E}(\mathbf{R}) \cdot \mathbb{I}\, \delta(\mathbf{R} - \mathbf{R}')]\, \mathrm{d}\Omega$$

$$= \int_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_{\mathrm{L}}} \{\, \mathrm{j} k_0 Z_0 [\mathbf{n}_{\Gamma} \times \mathbf{H}(\mathbf{R})] \cdot \mathbb{G}_{\mathrm{E}}(\mathbf{R}, \mathbf{R}') - [\mathbf{n}_{\Gamma} \times \mathbf{E}(\mathbf{R})] \cdot \mathbb{G}_{\mathrm{H}}(\mathbf{R}, \mathbf{R}')\}\, \mathrm{d}\Gamma \qquad (18)$$

The left-hand side of Equation (18) reduces to $\mathbf{E}(\mathbf{R}')$, thus from here onwards the unprimed $\mathbf{R}$ is only to be met as $\mathbf{R} \in \Gamma$. Substituting the Green's functions given explicitly in Equations (13) and (14) into the right-hand side of Equation (18) it is noted that, by inspection, the singular Dirac delta function appearing in Equation (13) plays no part in the integrations over either Γ_1 or Γ_2. Nevertheless, it is wished to make the substitution so as to include the possibility of $\mathbf{R}'$ being placed in turn upon each of Γ_1 and Γ_2, in which case the discontinuities of the dyadic elements in Equations (13) and (14) at these planes need carefully to be considered.

First, examine the substitution with respect to the Equation (18) integration over Γ_1. It is seen from Figure 1(b) that $z \leqslant z'$ for $z (= l_1)$ on Γ_1 for all z' belonging to $\Omega \cup \Gamma$. However if z' happens to be placed on Γ_1, the inequality sign $z < z'$ may still be used so as consistently to select $\mathbb{G}^{-}_{\mathrm{E,H}}$ from the alternative Green's functions expressions in each of Equations (13) and (14). The justification for this assertion is that since by definition the object Γ_{L} does not touch Γ_1, the unobstructed waveguide fields valid in Ω_1, $i = 1$ in Equation (1), remain completely continuous on passing into the unobstructed hollow waveguide portions of the notional region Ω (see Figure 1(a)). Thus for the integral over Γ_1, in a standard limit procedure, it is permissible to put $z' = l_1 + \varepsilon$ where the positive number ε is finite but as small as is pleased, so that the equality sign may be ignored. A similar argument applies to the integration region over Γ_2 in Equation (18).

Substituting the truncated Equations (1), (3) and appropriate choices from Equations (13) and (14) into the right-hand side of Equation (18), while using Equation (5), gives

$$\mathbf{E}(\mathbf{R}') = \mathrm{j} k_0 Z_0 \int_{\Gamma_1} \sum_{m=1}^{M} \sqrt{Y_m} [a_m^{(1)} \mathrm{e}^{-\gamma_m l_1} - b_m^{(1)} \mathrm{e}^{\gamma_m l_1}] \mathbf{e}_{tm}(\mathbf{r}) \cdot \mathbb{G}_{\mathrm{E}}^{-}(\mathbf{R}, \mathbf{R}')|_{z = l_1}\, \mathrm{d}x\, \mathrm{d}y$$

$$+ \int_{\Gamma_1} \sum_{m=1}^{M} \sqrt{Z_m}[a_m^{(1)} e^{-\gamma_m l_1} + b_m^{(1)} e^{\gamma_m l_1}] \mathbf{h}_{tm}(\mathbf{r}) \cdot \mathbb{G}_H^-(\mathbf{R}, \mathbf{R}')|_{z=l_1} \, dx \, dy$$

$$- jk_0 Z_0 \int_{\Gamma_2} \sum_{m=1}^{M} \sqrt{Y_m}[a_m^{(2)} e^{-\gamma_m l_2} - b_m^{(2)} e^{\gamma_m l_2}] \mathbf{e}_{tm}(\mathbf{r}) \cdot \mathbb{G}_E^+(\mathbf{R}, \mathbf{R}')|_{z=l_2} \, dx \, dy$$

$$- \int_{\Gamma_2} \sum_{m=1}^{M} \sqrt{Z_m}[a_m^{(2)} e^{-\gamma_m l_2} + b_m^{(2)} e^{\gamma_m l_2}] \mathbf{h}_{tm}(\mathbf{r}) \cdot \mathbb{G}_H^+(\mathbf{R}, \mathbf{R}')|_{z=l_2} \, dx \, dy$$

$$+ \oint_{\Gamma_L} \{- jk_0 Z_0 [\mathbf{n}_L \times \mathbf{H}(\mathbf{R})] \cdot \mathbb{G}_E(\mathbf{R}, \mathbf{R}') + [\mathbf{n}_L \times \mathbf{E}(\mathbf{R})] \cdot \mathbb{G}_H(\mathbf{R}, \mathbf{R}')\} \, d\Gamma_L \qquad (19)$$

(Note that $\mathbf{n}_L = -\mathbf{n}_\Gamma$ has been used here.) Explicitly putting in the expressions for $\mathbb{G}_E^{\pm}$, $\mathbb{G}_H^{\pm}$, after some algebra, employing the orthogonality relations Equation (6) and recalling that the Dirac delta function appearing in Equation (13) is of no significance in the first four integrals of Equation (19), a substantial cancellation results, removing terms involving $b_m^{(1)}$ and $a_m^{(2)}$, to give

$$\mathbf{E}(\mathbf{R}') = \sum_{m=1}^{M} \sqrt{Z_m} \{a_m^{(1)} [\mathbf{e}_{tm}(\mathbf{r}') + \mathbf{e}_{zm}(\mathbf{r}')] e^{-\gamma_m z'} + b_m^{(2)} [\mathbf{e}_{tm}(\mathbf{r}') - \mathbf{e}_{zm}(\mathbf{r}')] e^{\gamma_m z'}\}$$

$$+ \oint_{\Gamma_L} \{- jk_0 Z_0 [\mathbf{n}_L \times \mathbf{H}(\mathbf{R})] \cdot \mathbb{G}_E(\mathbf{R}, \mathbf{R}') + [\mathbf{n}_L \times \mathbf{E}(\mathbf{R})] \cdot \mathbb{G}_H(\mathbf{R}, \mathbf{R}')\} \, d\Gamma_L, \quad \mathbf{R}' \in \Omega \cup \Gamma \qquad (20)$$

In being explicitly free from terms involving the unknown scattered modal wave amplitudes $b_m^{(1)}$ and $a_m^{(2)}$ while nevertheless including $a_m^{(1)}$ and $b_m^{(2)}$, which latter may be considered as given incident wave amplitudes, Equation (20) exhibits the causality principle, as must be expected. In Equation (20) $\mathbf{R}'$ may validly be placed upon any of Γ_1, Γ_2, and Γ_L, so that provided there is sufficient information to establish the tangential components of $\mathbf{H}$ and $\mathbf{E}$ on the obstacle boundary Γ_L, the *total* field becomes known at Γ_1 and Γ_2, both of which are cross-sections of the waveguide unencumbered with the obstacle. Thus implicitly the scattering matrix becomes known. Equation (20) will be recognized as a generalization, applicable inside a hollow waveguide, of the mathematical representation of Huygens' principle for scattering in free space [12].

4. QUASI-MM DISCRETIZATION OF THE HUYGENS EQUATION

The field terms embodied in the surface integral of Equation (20) are seen to appear as the notional equivalent electric and magnetic surface current densities of the MM, respectively $\mathbf{J}_L = \mathbf{n}_L(\mathbf{R}) \times \mathbf{H}(\mathbf{R})$, $\mathbf{K}_L = -\mathbf{n}_L(\mathbf{R}) \times \mathbf{E}(\mathbf{R})$, $\mathbf{R} \in \Gamma_L$. However, because of the left-hand side term of Equation (20), it is more convenient to keep $\mathbf{E}$ and $\mathbf{H}$ as working variables. In the truncated eigenmode space relating to an $M \times M$ GSM array, consider the excitation represented by the $\mathbf{a}^{(1)}$, $\mathbf{b}^{(2)}$ terms in Equation (20) to occur sequentially on a unit mode-by-mode basis, $n^{\pm} = 1, 2, \ldots, M$, so that an n^{+}-excitation is defined as requiring all coefficients of $\mathbf{a}^{(1)}$, $\mathbf{b}^{(2)}$ save $a_n^{(1)} = 1$ to be put to zero, while the n^{-}-excitation similarly has just $b_n^{(2)} = 1$ non-zero. Let the corresponding fields in Ω be $\mathbf{E}_n^{\pm}(\mathbf{R})$, $\mathbf{H}_n^{\pm}(\mathbf{R})$. Then placing $\mathbf{R}'$ upon Γ_{L} gives

$$\mathbf{E}_n^{\pm}(\mathbf{R}') = \sqrt{Z_n}[\mathbf{e}_{tn}(\mathbf{r}') \pm \mathbf{e}_{zn}(\mathbf{r}')]\mathrm{e}^{\mp \gamma_n z'} + \oint_{\Gamma_{\mathrm{L}}} \{-\mathrm{j}k_0 Z_0[\mathbf{n}_{\mathrm{L}} \times \mathbf{H}_n^{\pm}(\mathbf{R})] \cdot \mathbb{G}_{\mathrm{E}}(\mathbf{R}, \mathbf{R}') + [\mathbf{n}_{\mathrm{L}} \times \mathbf{E}_n^{\pm}(\mathbf{R})] \cdot \mathbb{G}_{\mathrm{H}}(\mathbf{R}, \mathbf{R}')\} \, \mathrm{d}\Gamma_{\mathrm{L}}, \quad \mathbf{R}' \in \Gamma_{\mathrm{L}} \tag{21}$$

Suppose Γ_{L} is approximated by M_{T} triangular patches T_k, $k = 1, 2, \ldots, M_{\mathrm{T}}$, having a total of M_{L} edges of length l_j, $j = 1, 2, \ldots, M_{\mathrm{L}}$. Expand the triangular patch tangential fields, $\mathbf{E}_{\mathrm{L}n}^{\pm}(\mathbf{R})$ and $\mathbf{H}_{\mathrm{L}n}^{\pm}(\mathbf{R})$ say, in terms of Whitney-edge elements [15] in the dimensionless form $\mathbf{N}_j$ (see Reference [18]) as

$$\mathbf{E}_{\mathrm{L}n}^{\pm}(\mathbf{R}) = \sum_{j=1}^{M_{\mathrm{L}}} \mathscr{E}_{nj}^{\pm} \mathbf{N}_j(\mathbf{R}), \quad \mathbf{H}_{\mathrm{L}n}^{\pm}(\mathbf{R}) = \sum_{j=1}^{M_{\mathrm{L}}} \mathscr{H}_{nj}^{\pm} \mathbf{N}_j(\mathbf{R}), \quad \mathbf{R} \in \Gamma_{\mathrm{L}} \tag{22}$$

$$\begin{aligned} \mathbf{N}_j(\mathbf{R}) &= l_j[\zeta_{j1} \nabla \zeta_{j2} - \zeta_{j2} \nabla \zeta_i], \quad \mathbf{R} \in T_k \\ &= 0, \quad \mathbf{R} \notin T_k \end{aligned} \tag{23}$$

where ζ_{j1}, ζ_{j2} are the barycentric co-ordinates associated with the nodes forming this edge (a third co-ordinate ζ_{j3} is associated with the remaining patch node opposite to edge j, with $\zeta_{j1} + \zeta_{j2} + \zeta_{j3} = 1$). The procedure now is to substitute Equations (22) into Equation (21), at the same time applying the Galerkin operation

$$-\oint_{\Gamma_{\mathrm{L}}} (\ldots) \cdot [\mathbf{n}_{\mathrm{L}}(\mathbf{R}') \times \mathbf{N}_i(\mathbf{R}')] \, \mathrm{d}\Gamma'_{\mathrm{L}} = \oint_{\Gamma_{\mathrm{L}}} (\ldots) \cdot \mathbf{f}_i(\mathbf{R}'), \text{ (say)}, \quad i = 1, 2, \ldots, M_{\mathrm{L}} \tag{24}$$

Before writing down the result it should be observed that it may be shown [14,21] that $\mathbf{f}_i$ is precisely the Rao–Wilton–Glisson (RWG) patch function used in expanding MM surface currents $\mathbf{J}_{\mathrm{L}}$ and $\mathbf{K}_{\mathrm{L}}$,

$$\begin{aligned} \mathbf{f}_i(\mathbf{R}) &= \frac{\rho_i l_i}{2A_k}, \quad \mathbf{R} \in T_k \\ &= 0, \quad \mathbf{R} \notin T_k \end{aligned} \tag{25}$$

where A_k is the area of T_k and ρ_i the patch-plane position vector with its origin the vertex opposite edge i. The fact that each edge is associated with two adjoining patches is accountable routinely in

MM [14] and edge-element procedures [16]. Thus from Equation (21) is obtained

$$\sum_{j=1}^{M_L} \mathscr{E}_j^{(\pm n)} \oint_{\Gamma_L} \mathbf{N}_j(\mathbf{R}')\cdot\mathbf{f}_i(\mathbf{R}')\,\mathrm{d}\Gamma'_L = \sqrt{Z_n}\oint_{\Gamma_L} \mathrm{e}^{\mp\gamma_n z'}[\mathbf{e}_{tn}(r') \pm \mathbf{e}_{zn}(\mathbf{r}')]\cdot\mathbf{f}_i(\mathbf{R}')\,\mathrm{d}\Gamma'_L$$
$$+ \sum_{j=1}^{M_L} \mathscr{H}_j^{(\pm n)} \oint_{\Gamma_L}\oint_{\Gamma_L} \mathrm{j}k_0 Z_0 \mathbf{f}_j(\mathbf{R})\cdot\mathbb{G}_E(\mathbf{R},\mathbf{R}')\cdot\mathbf{f}_i(\mathbf{R}')\,\mathrm{d}\Gamma_L\,\mathrm{d}\Gamma'_L$$
$$- \sum_{j=1}^{M_L} \mathscr{E}_j^{(\pm n)} \oint_{\Gamma_L}\oint_{\Gamma_L} \mathbf{f}_j(\mathbf{R})\cdot\mathbb{G}_H(\mathbf{R},\mathbf{R}')\cdot\mathbf{f}_i(\mathbf{R}')\,\mathrm{d}\Gamma_L\,\mathrm{d}\Gamma'_L \quad (26)$$

which may be written in array form as

$$[\mathscr{B}_E]\{\mathscr{E}_{Ln}^{\pm}\} = \{\mathbf{e}_n^{\pm}\} + [\mathscr{M}_E]\{\mathscr{H}_{Ln}^{\pm}\} + [\mathscr{M}_H]\{\mathscr{E}_{Ln}^{\pm}\} \quad (27)$$

where $\{\mathbf{e}_n^{\pm}\}$ is a driving vector-array pertaining separately to each possible incident single mode, $\{\mathscr{E}_{Ln}^{\pm}\}$ and $\{\mathscr{H}_{Ln}^{\pm}\}$ are vector arrays representing the resulting edge element expansion coefficients of Equation (22), $[\mathscr{B}_E]$ is a sparse matrix which is simple to assemble from the Whitney and RWG functions while $[\mathscr{M}_E]$ and $[\mathscr{M}_H]$ are dense matrices, similar to those occurring in MM formulations, involving integrations (some singular) of $\mathbf{f}_j(\mathbf{R})\cdot\mathbb{G}_{E,H}(\mathbf{R},\mathbf{R}')\cdot\mathbf{f}_i(\mathbf{R}')$ over the patch-approximated Γ_L. The latter integrals, although rather complicated, may be evaluated by standard MM procedures [18].

5. APPLICATIONS OF THE DISCRETISED HUYGENS EQUATION

Equation (27) may be solved in a number of different cases. The simplest of these occurs when the scattering object Ω_L is a perfect electrical conductor (PEC), in which case $\{\mathscr{E}_{Ln}^{\pm}\}$ representing tangential electric field on Γ_L vanishes, so that Equation (27) is directly soluble for $\{\mathscr{H}_{Ln}^{\pm}\}$. The tangential **H**-field on Γ_L can be recovered from Equation (22) and substituted back, as $[\mathbf{n}_L\times\mathbf{H}(\mathbf{R})]$, into Equation (20). Since the latter equation is valid everywhere within Ω and on its boundary Γ, the full incident plus scattered field **E** becomes available on Γ_1 and Γ_2, for comparison with that expressed by Equation (1) for the empty waveguide. Exploiting the orthogonal properties of the transverse modal waveguide fields, using appropriately weighted numerical quadratures, the *a*- and *b*-coefficients can be recovered from the scattered field corresponding to each unit $n^{\pm}$ single-mode excitation. Thus the GSM of the PEC object Ω_L can thus be reconstructed, provided the MM procedure is repeated over the appropriately truncated modal spectrum.

The next simplest case is when the object is comprised of a single homogeneous material. Then a second quasi-MM matrix equation in the same unknowns $\{\mathscr{E}_{Ln}^{\pm}\}$ and $\{\mathscr{H}_{Ln}^{\pm}\}$, representing the tangential fields continuous into Ω_L, may be set up in the form

$$[\mathscr{B}_E]\{\mathscr{E}_{Ln}^{\pm}\} = [\mathscr{M}_E^L]\{\mathscr{H}_{Ln}^{\pm}\} + [\mathscr{M}_H^L]\{\mathscr{E}_{Ln}^{\pm}\} \quad (28)$$

but without any source term and relating to the now-uniform region Ω_L. Equation (28) is constructed in the same way as was Equation (27) but from a modified Equation (26) employing the Green's functions $\mathbb{G}^{\mathrm{L}}_{\mathrm{E,H}}$ representing the simpler, uniform space dyadics [12] of the material comprising Ω_{L}. The same patch discretization of Γ_{L} must be used, but now there is no source term corresponding to $\{\mathbf{e}_n^{\pm}\}$ while there are appropriate sign changes due to the involvement of opposite-signed normal vectors on Γ_{L}. Equations (27) and (28) may be solved simultaneously for $\{\mathscr{H}^{\pm}_{\mathrm{L}n}\}$ and $\{\mathscr{E}^{\pm}_{\mathrm{L}n}\}$, after which the procedure already described for the PEC object can similarly be applied.

If the material of Ω_{L} is inhomogeneous, a relationship equivalent to Equation (28) can be set up through discretizing Ω_{L} into a finite element mesh of tetrahedra matching with the triangular patches of Γ_{L}. The governing equation for the unknown $\mathbf{H}$ in Ω_{L} is

$$\nabla \times \boldsymbol{\varepsilon}_r^{-1} \nabla \times \mathbf{H}(\mathbf{R}) - \boldsymbol{\mu}_r k_0^2 \mathbf{H}(\mathbf{R}) = 0 \tag{29}$$

subject to appropriate boundary conditions on Γ_{L}. A standard FEM procedure [22] applied to Equation (29) gives the weak-form equivalent

$$-\mathrm{j} k_0 Y_0 \oint_{\Gamma_{\mathrm{L}}} [\mathbf{n}_{\mathrm{L}}(\mathbf{R}) \times \mathbf{E}(\mathbf{R})] \cdot \mathbf{W}(\mathbf{R}) \, \mathrm{d}\Gamma_{\mathrm{L}} = \int_{\Omega_{\mathrm{L}}} \{\boldsymbol{\varepsilon}_r^{-1} (\nabla \times \mathbf{W}) \cdot (\nabla \times \mathbf{H}) - \boldsymbol{\mu}_r k_0^2 \mathbf{W} \cdot \mathbf{H}\} \, \mathrm{d}\Omega \tag{30}$$

where $\mathbf{W}$ is an arbitrary vector weighting function. Corresponding to the $n^{\pm}$ unit modal excitation and with the edge-element/Galerkin choice $\mathbf{W} = \mathbf{N}_j$ in Equation (30), this FEM procedure gives

$$\begin{aligned} &\mathrm{j} k_0 Y_0 \sum_{j=1}^{M_{\mathrm{L}}} \mathscr{E}_j^{(\pm n)} \oint_{\Gamma_{\mathrm{L}}} \mathbf{N}_j(\mathbf{R}) \cdot \mathbf{f}_i(\mathbf{R}) \, \mathrm{d}\Gamma_{\mathrm{L}} \\ &\quad = \int_{\Omega_{\mathrm{L}}} \sum_{j=1}^{M_{\mathrm{V}}} \mathscr{H}_j^{(\pm n)} \{\boldsymbol{\varepsilon}_r^{-1} [\nabla \times \mathbf{N}_i(\mathbf{R})] \cdot [\nabla \times \mathbf{N}_j(\mathbf{R})] - \boldsymbol{\mu}_r k_0^2 \mathbf{N}_i(\mathbf{R}) \cdot \mathbf{N}_j(\mathbf{R})\} \, \mathrm{d}\Omega \end{aligned} \tag{31}$$

where M_{V} is the total number of tetrahedron edges in the volume discretisation of Ω_{L} while $\sum_{j=1}^{M_{\mathrm{V}}} \mathscr{H}_j^{(\pm n)} \mathrm{N}_j(\mathbf{R})$ is the FEM edge element approximation for $\mathbf{H}$ applying to Ω_{L}, including its surface Γ_{L}. Equation (31) leads to a matrix equation

$$\mathrm{j} k_0 Y_0 [\mathscr{B}_{\mathrm{E}}] \{\mathscr{E}^{\pm}_{\mathrm{L}n}\} = ([\mathscr{S}] - k_0^2 [\mathscr{T}]) \{\mathscr{H}^{\pm}_n\} \tag{32}$$

where the matrices $[\mathscr{S}]$ and $[\mathscr{T}]$ appearing in Equation (32) are standard finite element sparse-arrays while $[\mathscr{B}_{\mathrm{E}}]$ has already appeared in the quasi-MM treatment leading to Equation (27). It is to be noted that the unknown vector array $\{\mathscr{H}^{\pm}_{\mathrm{L}n}\}$ which appears in Equation (27) is a subset, relating to the edges in Γ_{L}, of the array $\{\mathscr{H}^{\pm}_n\}$ corresponding to *all* of the edges in the discretization of Ω_{L} appearing in Equation (32). Note that although edge elements $\mathbf{N}_i$ relating to the triangular patch expansions are nominally the two-dimensional versions, while here the three-dimensional (3-D) ones are required, it is easily seen that the 3-D $\mathbf{N}_i$ would serve perfectly well throughout. It is clear that Equations (27) and (32) together represent a simultaneous pair of matrix equations which can be solved for $\{\mathscr{E}^{\pm}_{\mathrm{L}n}\}$ and $\{\mathscr{H}^{\pm}_n\}$. Once this solution has been achieved it is possible to recreate the scattering matrix of the object-in-waveguide in exactly the same way as indicated for the simpler situations described. It may be noted that at the frequencies of interest in most practical cases Ω_{L} itself will be non-resonant, thus the problem of ill-conditioned finite element matrices [5] should not arise while, in any case, the volume of the FEM discretization is reduced to an absolute minimum.

6. CONCLUSIONS

The vector/dyadic form of Green's theorem has been applied to a doubly-connected surface in a hollow waveguide containing a general object, the latter constituting a waveguide discontinuity. The double connection embraces both the closed surface of the object and a notional pair of transverse planes filling the waveguide cross-section. The result is an integral equation which is a generalization of the free-space mathematical expression of Huygens' principle and is consistent with the notions of causality. It allows the specification of an arbitrary excitation, which may be resolved into its modal vector components, originating from either or both axial directions but without prior knowledge of the forward or backward scattering due to the discontinuity. Providing the tangential components of **E** and **H** on the surface of the object can be suitably interrelated from the internal nature of the object itself, the Huygens equation provides a means for determining the scattered fields, while in unobstructed cross-sections the latter fields can be resolved into their empty waveguide modal components. Such a resolution, if carried out on a mode-by-mode basis, provides all of the necessary information to reconstruct the generalized scattering matrix of the object-in-waveguide. A discretization of the Huygens equation in terms of Whitney-edge elements and RWG patch functions (both of these closely related vector interpolation functions are required) has been suggested so as to apply to a triangular patch representation of the scattering object's surface. A corresponding matrix equation, connecting the unknown tangential electromagnetic fields on the surface of the object and the given external excitation, has been written down. The equation becomes immediately soluble for a PEC object while it has been shown how a second simultaneous matrix equation, representing the unknown surface tangential fields in relation to the internal constitution of the object, arises and allows the solution to be effected for more general objects. In the case of a uniform object the second matrix equation derives from a further Green's function approach while for the inhomogeneous case a finite element treatment of the object volume is required. The field-variable approach adopted here, rather than the more usual method of moments one expressed in terms of notional surface currents, is dictated by the necessity to employ both Whitney and RWG elements. A close relationship with the techniques employed in MM analyses, applied to simpler situations, is established. No numerical confirmation of the theory is presently available while the complexity of the realization of such confirmation should not be underestimated. However, in view of the relationship shown with the MM it can confidently be predicted that such confirmation will be forthcoming. Moreover, one of the chief difficulties hitherto found concerning MM implementation in comparison with alternative methods, e.g. FEM or the finite differences method (when available), namely the occurrence of dense matrices in the numerical processing, has recently been ameliorated by the use of wavelet packet transformation [23].

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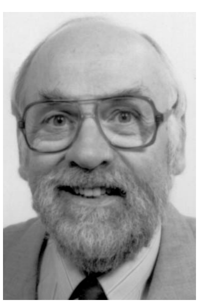

Ronald Ferrari was born in England. He graduated in mathematics at Imperial College of Science and Technology, London University in 1950 and was awarded the Diploma of Imperial College in 1951. After service in the U.K. Royal Air Force as a technical signals officer doing operational research in radar, he joined the Hirst Research Centre of GEC Ltd., Wembley, U.K. in 1956, researching into semiconductors and microwaves. He was appointed Lecturer in Engineering at Cambridge University in 1965, Fellow of Trinity College, Cambridge in 1966 and was awarded the Cambridge University ScD doctorate in 1994. In addition to numerous papers on semiconductors, microwaves, radio-echo sounding, numerical electromagnetics and numerical quantum mechanics, his publications include a book on finite element methods (with P.P. Silvester) and a text book on electromagnetics. He has held visiting academic appointments at Cornell, McGill and Florence Universities. Although now retired from teaching duties, he is still active in research, Dr Ferrari is a Fellow of the IEE and a chartered engineer.

