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Velocity profiles in laminar oscillatory flow in tubes

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Abstract The oscillatory laminar flow of a Newtonian fluid in a circular tube has been investigated when the fluid is subjected to a periodic pressure gradient. A technique has been developed which allows the velocity amplitude to be determined as a function of radius and the results agree closely with theoretical predictions. The work is being continued with a view to employing the technique for the determination of the rheological properties of viscoelastic fluids.

1 Introduction

This investigation is concerned with the oscillatory laminar flow of a Newtonian fluid in a circular tube about a mean position when the fluid is subjected to a periodic pressure gradient. The equations which govern this system are the equation of motion

$$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(r\tau) \tag{1}$$

together with the constitutive equation for a Newtonian fluid,

$$\tau = -\mu \frac{\partial v}{\partial r} \tag{2}$$

where v is the axial velocity at radius r, τ is the corresponding shear stress, $\partial p/\partial z$ is the axial pressure gradient and ρ and μ are the density and viscosity of the fluid.

The periodic pressure gradient can be represented in complex notation by the equation

$$-\frac{\partial p}{\partial z} = p_0\, e^{i\omega t} \tag{3}$$

where p_0 is the amplitude of the pressure gradient and ω is the frequency.

The shear stress τ, the axial velocity at radius r, v, and the volumetric flow Q will also be periodic and can be represented by the equations

$$\tau = \tau_0\, e^{i\omega t} \tag{4}$$

$$v = v_0\, e^{i\omega t} \tag{5}$$

$$Q = Q_0\, e^{i\omega t} \tag{6}$$

where τ_0, v_0 and Q_0 are the respective amplitudes.

The solution to this problem has been given by Richardson and Tyler (1929) and also by Sexl (1930), the equations giving the amplitudes of the velocity at radius r, v_0 and the volumetric flow rate Q_0 being as follows,

$$v_0(r) = \frac{p_0}{i\omega\rho}\left[1 - \frac{J_0(\lambda r)}{J_0(\lambda R)}\right] \tag{7}$$

$$Q_0 = \frac{\pi R^2 p_0}{i\omega\rho}\left[1 - \frac{2J_1(\lambda R)}{\lambda R J_0(\lambda R)}\right] \tag{8}$$

where $\lambda^2 = -i\omega\rho/\mu$ and R is the radius of the tube.

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This analysis predicts an approximately parabolic profile for the velocity amplitude v_0 at low frequencies. At higher frequencies a peak appears near the wall and the amplitude in the central part of the tube is approximately uniform. Using a hot wire anemometer to investigate the oscillatory flow of air Richardson and Tyler (1929) proved experimentally the existence of an amplitude peak. Sexl (1930) has shown that the data given by Richardson and Tyler for the position of the peak agree closely with his theoretical predictions.

These experiments were not a complete verification of the solution since the work did not include a comparison between the experimental and theoretically predicted values oi the velocity profile over the radius of the tube.

2 Experimental method

The length S of the track travelled by a fluid element at radius r during the oscillatory motion will be given by the equation

$$S = \int_0^{\pi/\omega} v\, dt = \int_0^{\pi/\omega} v_0\, e^{i\omega t}\, dt.$$

This gives

$$S = \frac{2p_0\{J_0(\lambda R) - J_0(\lambda r)\}}{\omega^2 \rho J_0(\lambda R)}. \tag{9}$$

Therefore by a visualization of the fluid flow and by measuring the tracks of the fluid elements one can test the theoretical predictions by comparing the measured track lengths with those derived from equation (9).

The sketch of the equipment used is shown in figure 1. The perspex tube 1 with a length of 1·20 m and an internal diameter of 25·2 mm was connected with the bellows 3, the end of which was fixed on the heavy metal plates 4. The follower 10 was pressed on the cam 5 by means of two springs. The speed was changed by means of the gear box 6 and measured by means of a tachometer. To maintain a constant temperature and to avoid optical distortion the tube 1 had a perspex jacket 2 of square section through which water was circulated from a thermostatic bath. The experiments were carried out at room temperature.

The tube was filled from a pressure tank through the tap 9. By means of the syringe 11, guided by the tube 12 to traverse a diameter of the tube 1, a slurry of polystyrene spheres with a diameter less than 250 μm was introduced. The tracks of the particles were photographed by means of the camera 7.

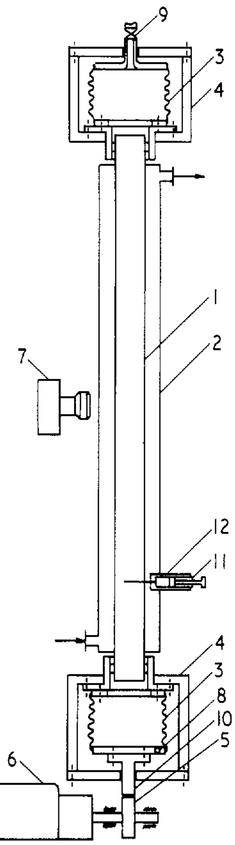


Figure 1 Diagram of the equipment

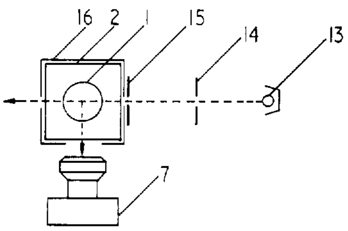


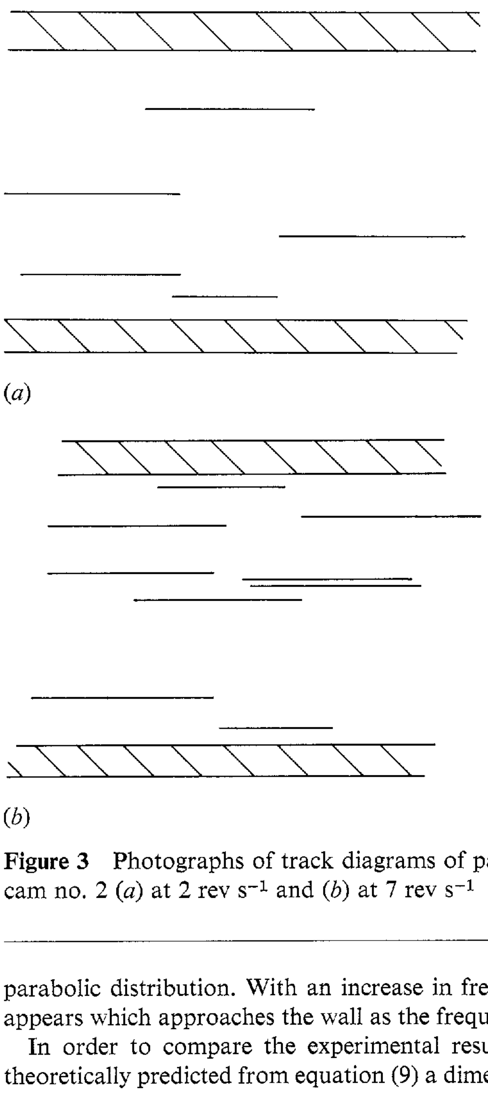
Figure 2 The optical arrangement

The fluid was a mixture of 10% by volume of glycerol and 90% by volume of propane 1·2 diol. This mixture has a viscosity of 55·2 cP at 24°C and a density of 1·057 g cm^{-3}, that is approximately the same as that of the tracer particles, 1·055 g cm^{-3}. It also has excellent optical properties. The optical arrangement is shown in figure 2. It consisted of source of light 13, a slit 14 with a width of 2 mm and a slit 15 with a width of 0·5 mm. The time of exposure was 6 s.

The investigations were made at 1, 2, 5 and 7 rev s^{-1} using two cams with different eccentricities. Cam no. 1 gave at low speeds of rotation a volumetric displacement of 9 cm^3 per half cycle and cam no. 2 gave 6·15 cm^3. Track diagrams are shown in figure 3.

3 Treatment of the experimental results

The particle track lengths and the corresponding distances from the tube wall were measured from magnified photographs. These are plotted in figure 4(*a*) for the two cams at 1 and 5 rev s^{-1}, and in figure 4(*b*) for 2 and 7 rev s^{-1}. It can be seen that at 1 rev s^{-1} the track lengths have a nearly parabolic distribution. With an increase in frequency a peak appears which approaches the wall as the frequency increases.



(*a*)

(*b*)

Figure 3 Photographs of track diagrams of particles with cam no. 2 (*a*) at 2 rev s^{-1} and (*b*) at 7 rev s^{-1}

In order to compare the experimental results with those theoretically predicted from equation (9) a dimensionless form of treatment was adopted. If one uses the Kelvin functions which are tabulated (Dwight 1961) then

$$J_0(\lambda r) = \mathrm{ber}\,(\lambda r) + \mathrm{i}\,\mathrm{bei}\,(\lambda r).$$

The modulus of the track lengths is given by

$$\bar{S} = \{(\mathrm{Re}\,S)^2 + (\mathrm{Im}\,S)^2\}^{1/2}.$$

From these equations one finally obtains the following expression for $\bar{S}$

$$\bar{S} = \frac{2p_0}{\omega^2\rho}\left[\left\{1 - \frac{\mathrm{ber}(\lambda r)\,\mathrm{ber}(\lambda R) + \mathrm{bei}(\lambda r)\,\mathrm{bei}(\lambda R)}{\mathrm{ber}^2(\lambda R) + \mathrm{bei}^2(\lambda R)}\right\}^2 + \left\{\frac{\mathrm{ber}(\lambda r)\,\mathrm{bei}(\lambda R) - \mathrm{bei}(\lambda r)\,\mathrm{ber}(\lambda R)}{\mathrm{ber}^2(\lambda R) + \mathrm{bei}^2(\lambda R)}\right\}^2\right]^{1/2}.$$

The ratio of the track length at radius r to the maximum length is given by the expression (at radius r_m)

$$\frac{\bar{S}}{\bar{S}_{\max}} = \left[\frac{\{\mathrm{ber}^2(\lambda R) + \mathrm{bei}^2(\lambda R) - \mathrm{ber}(\lambda r)\,\mathrm{ber}(\lambda R) - \mathrm{bei}(\lambda r)\,\mathrm{bei}(\lambda R)\}^2 + \{\mathrm{ber}(\lambda r)\,\mathrm{bei}(\lambda R) - \mathrm{bei}(\lambda r)\,\mathrm{ber}(\lambda R)\}^2}{\{\mathrm{ber}^2(\lambda R) + \mathrm{bei}^2(\lambda R) - \mathrm{ber}(\lambda r_m)\,\mathrm{ber}(\lambda R) - \mathrm{bei}(\lambda r_m)\,\mathrm{bei}(\lambda R)\}^2 + \{\mathrm{ber}(\lambda r_m)\,\mathrm{bei}(\lambda R) - \mathrm{bei}(\lambda r_m)\,\mathrm{ber}(\lambda R)\}^2}\right]^{1/2}.$$

At high frequencies an asymptotic expansion of the Bessel function can be used, i.e.

$$J_0(z) \simeq \frac{2}{\pi z} \exp\{i(z-\tfrac{1}{4}\pi)\}.$$

In this case the track length will be given by the equation

$$S=\frac{2p_0}{\omega^2}\left[1-\left(\frac{R}{r}\right)^{1/2}\exp\{-(1-i)\,\gamma(R-r)\}\right]$$

where $\gamma=(\omega\rho/2\mu)^{1/2}$. Thus for the modulus $\bar{S}$ one obtains the equation

$$\bar{S}=\frac{2p_0}{\omega^2}\left[1-2\left(\frac{R}{r}\right)^{1/2}\frac{\cos\gamma(R-r)}{\exp\{\gamma(R-r)\}}+\frac{R}{r\exp\{2\gamma(R-r)\}}\right]^{1/2}$$

and the ratio $\bar{S}/\bar{S}_{\max}$ will be given by

$$\frac{\bar{S}}{\bar{S}_{\max}}=\left[\frac{r\exp\{2\gamma(R-r)\}-2(Rr)^{1/2}\exp\{\gamma(R-r)\}\cos\gamma(R-r)+R}{r_m\exp\{2\gamma(R-r_m)\}-2(Rr_m)^{1/2}\exp\{\gamma(R-r_m)\}\cos\gamma(R-r_m)+R}\right]^{1/2}$$

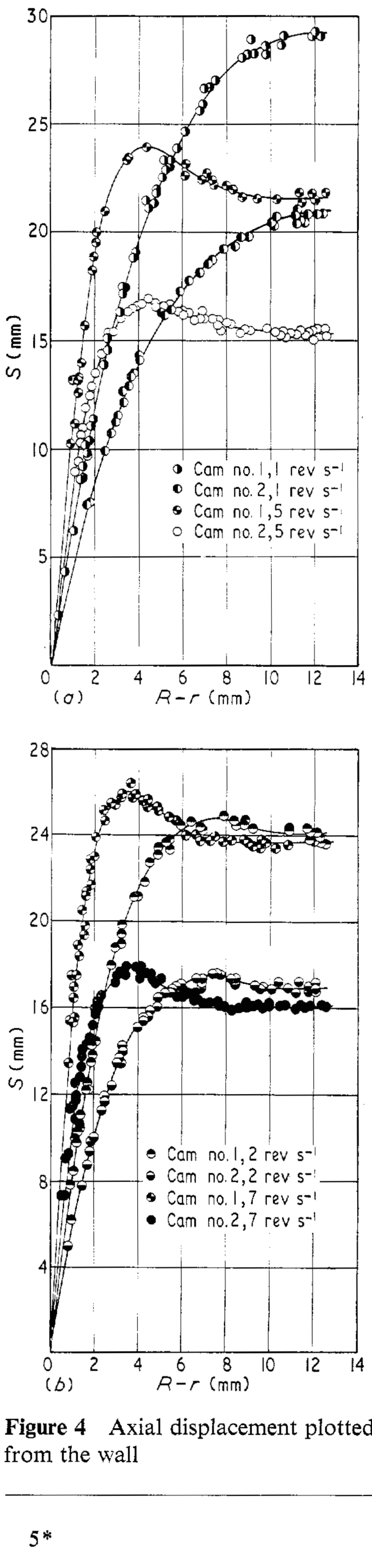


Figure 4 Axial displacement plotted against the distance from the wall

The calculations using these equations were made by means of a digital computer and the theoretical curves obtained are shown in figure 5(*a*) for 1 and 5 rev s^{-1} and in figure 5(*b*) for 2 and 7 rev s^{-1}. It can be seen that there is excellent agreement between the experimental data and the theoretical predictions.

4 Conclusions

An experimental method has been developed which can be used to test the validity of theoretical predictions for the velocity profile of a fluid which is undergoing oscillatory laminar flow in a long circular tube under the influence of a periodic pressure gradient. The method has been demonstrated for the case of a Newtonian fluid and the experimental

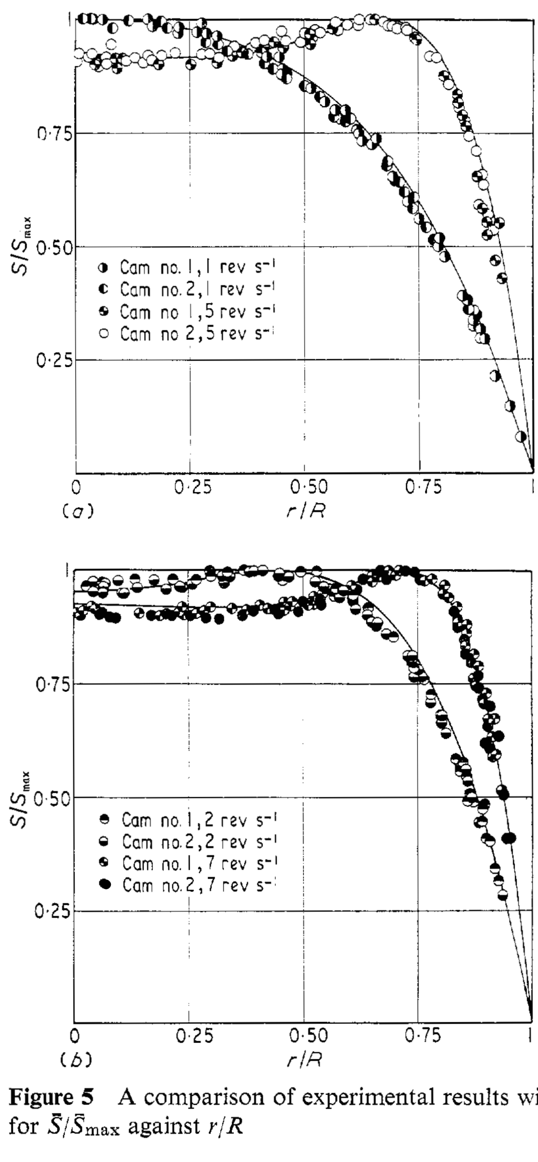


Figure 5 A comparison of experimental results with theory for $\bar{S}/\bar{S}_{\max}$ against r/R

results show that for such a fluid the amplitude of the periodic axial velocity as a function of radial position and frequency agrees closely with that derived theoretically.

At low frequencies the amplitude of the velocity is approximately parabolic, with a maximum at the tube axis. At higher frequencies a peak in the amplitude occurs between the axis and the wall and this peak moves towards the wall as the frequency increases.

The method is being developed to study the behaviour of viscoelastic liquids in oscillatory flow. Following the work of Jones and Walters (1967), this will enable the rheological properties of such fluids to be determined from velocity profile measurements.

References

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