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On new relations in dispersive wave motion

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Abstract

Based on Whitham's variational approach and employing the 4×4 formalism for dispersive wave motion, new balance and conservation laws were established. The general relations are illustrated with a specific example.

Keywords: Dispersive waves; Conservation laws; Balance laws

1. Introduction

Whitham [1,2] has developed a variational approach to study linear and also non-linear wavetrains and its many ramifications and applications in a variety of fields, including modulation theory.

The essence of Whitham's approach consists in postulating a Lagrangian function for the system under consideration, specializing this function for a slowly varying wavetrain, averaging the Lagrangian over one period and, finally, to derive variational equations for this averaged Lagrangian. Since the average variational principle is invariant with respect to a translation in time, the corresponding energy equation was derived, and since it is also invariant to a translation in space, the "wave momentum" equation was also established. Kienzler and Herrmann [3] have shown that the two relations may be derived also by calculating the time rate of change of the average Lagrangian and the spatial gradient of the same function. It is also possible to obtain the energy equation and the three "wave momentum" equations through a simple operation by applying the *grad* operator in four dimensions of space-time. This has been carried out for elastodynamics by Kienzler and Herrmann [4].

The purpose of this contribution is to consider not only the *grad* operator as applied to the average Lagrangian, but additionally also the *div* and *curl* operator, which has not been done before.

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Section 2 summarizes the basic relations of the problem at hand as presented in Whitham, Kienzler and Herrmann, while Section 3 presents new results stemming from the application of the *div* and *curl* operators. Section 4 concludes with a brief summary and some general comments.

2. Whitham's variational approach

Whitham's variational approach begins with postulating a Lagrangian $L = L\left(\phi, -\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x_i}\right)$ for any system governed by a dependent variable $\phi = \phi(t, x_i)$ where t is the time and x_i are Cartesian coordinates. For linear systems, L is a quadratic function of ϕ and its derivatives.

Next a slowly varying wavetrain is considered

$$\phi \sim a\omega(\theta + \eta), \tag{2.1}$$

where a is the amplitude, η the phase shifting angle and θ is the phase

$$\theta(x_i, t) = x_i k_i - \omega t, \tag{2.2}$$

where k_i is the wave number, $k_i = \frac{\partial\theta}{\partial x_i}$ and ω the frequency, $\omega = -\frac{\partial\theta}{\partial t}$. This form is substituted into the Lagrangian L, derivatives of a, η, ω and k_i are all neglected as being small and the result is averaged over one period

$$\mathcal{L} = \frac{1}{2\pi}\int_0^{2\pi} L \,\mathrm{d}\theta. \tag{2.3}$$

For any linear system, the resulting $\mathcal{L}$ is a function

$$\mathcal{L} = \mathcal{L}(\omega, k_i, a), \tag{2.4}$$

or, more specifically,

$$\mathcal{L} = G(\omega, k_i)a^2, \tag{2.5}$$

where

$$G(\omega, k_i) = 0 \tag{2.6}$$

is the dispersion relation.

Whitham proposes then an "average variational principle"

$$\delta \int \mathcal{L}\left(-\frac{\partial\theta}{\partial t}, \frac{\partial\theta}{\partial x_i}, a\right) \,\mathrm{d}t\,\mathrm{d}x_i = 0 \tag{2.7}$$

for the functions $a(x_i, t)$ and $\theta(x_i, t)$.

Since derivatives of a do not occur, the Euler–Lagrange variational equation for this variable is merely

$$\frac{\partial\mathcal{L}}{\partial a} = 0, \tag{2.8}$$

while the variational equation for θ is

$$\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\theta}{\partial t}\right)}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\theta}{\partial x_i}\right)}\right) = 0, \tag{2.9}$$

or

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \omega}\right)-\frac{\partial}{\partial x_i}\left(\frac{\partial \mathcal{L}}{\partial k_i}\right)=0. \tag{2.10}$$

This conservation equation has become known as the conservation of “wave action”. According to Whitham, it plays a more fundamental role than energy. Additionally, the consistency equations for the existence of θ (integrability conditions) require

$$\frac{\partial k_i}{\partial t}+\frac{\partial \omega}{\partial x_i}=0, \tag{2.11}$$

$$\frac{\partial k_i}{\partial x_j}-\frac{\partial k_j}{\partial x_i}=0. \tag{2.12}$$

Whitham then invokes Noether’s theorem, which briefly states that there exists a conservation equation corresponding to any group of transformations for which the variational principle is invariant. Invariance with respect to a translation in time t leads to the energy equation in the form

$$\frac{\partial}{\partial t}\left(\omega\frac{\partial \mathcal{L}}{\partial \omega}-\mathcal{L}\right)+\frac{\partial}{\partial x_j}\left(-\omega\frac{\partial \mathcal{L}}{\partial k_j}\right)=0. \tag{2.13}$$

Here $\omega\frac{\partial \mathcal{L}}{\partial \omega}-\mathcal{L}$ is the total energy (Hamiltonian) and $\omega\frac{\partial \mathcal{L}}{\partial k_j}$ is the flux.

Invariance with respect to space x_i results in

$$\frac{\partial}{\partial t}\left(k_i\frac{\partial \mathcal{L}}{\partial \omega}\right)+\frac{\partial}{\partial x_j}\left(-k_i\frac{\partial \mathcal{L}}{\partial k_j}+\mathcal{L}\delta_{ij}\right)=0, \tag{2.14}$$

where δ_{ij} is Kronnecker’s tensor of unity. This is called the “wave momentum” equation.

In [3] it was shown, that the energy Eq. (2.13) may be established by considering the time rate of change of $\mathcal{L}$, i.e., $\partial\mathcal{L}/\partial t$ and the “wave momentum” Eq. (2.14) may be derived by considering the spatial gradient $\partial\mathcal{L}/\partial x_i$.

The complete set of four conservation equations may be obtained by the application of the *grad* operator in space-time, as it was shown in [4].

It is to be noted that for non-uniform and non-linear waves neither the energy equation nor the wave momentum equation remain conservation laws, since both have to be supplemented by source terms and thus become balance laws. By contrast, the wave action relation always remains a conservation law.

3. 4 × 4 Formalism, application of *grad*

In order to introduce the 4×4 formalism we agree upon that greek indices have the range 0, 1, 2, 3 whereas latin indices have the range 1, 2, 3 as before. Further we introduce the four coordinates x_μ as

$$x_\mu=\begin{cases} c_0 t,\\ x_j.\end{cases} \tag{3.1}$$

The velocity c_0 is used to render the dimension of the coordinates equal. It may be chosen arbitrarily, $c_0 = 1$ m/s, c_0 = velocity sound, or a characteristic wave speed, etc. In the theory of relativity c_0 must be equal to c, the velocity of light. Partial derivatives of a dependent variable with respect to the independent variables (coordinates) are abbreviated by a comma followed by the index of the coordinate

$$\frac{\partial()}{\partial x_\mu} = (),_\mu = \begin{cases} \dfrac{\partial()}{\partial x_0} = \dfrac{1}{c_0}\dfrac{\partial()}{\partial t} = \dfrac{1}{c_0}()^{\cdot}, \\ \dfrac{\partial()}{\partial x_j} = (),_j \end{cases} \tag{3.2}$$

Accordingly, we introduce the phase θ as

$$\theta = k_\mu x_\mu \tag{3.3}$$

with the four-vector of wave numbers

$$k_\mu = \begin{cases} k_0 = -\dfrac{\omega}{c_0}, \\ k_j. \end{cases} \tag{3.4}$$

With (3.4), Eqs. (3.3) and (2.2) coincide. Finally, the derivatives of the phase are

$$\theta,_\mu = \begin{cases} \theta,_0 = -\dfrac{\omega}{c_0}, \\ \theta,_j = k_j. \end{cases} \tag{3.5}$$

With (3.5), the averaged Lagrangian $\mathcal{L}$ (2.3), (2.4) is a function of $\theta,_\mu$

$$\mathcal{L} = \mathcal{L}(\theta,_\mu). \tag{3.6}$$

The partial derivatives of $\mathcal{L}$ with respect to $\theta,_\mu$ are then calculated to be

$$\frac{\partial \mathcal{L}}{\partial \theta,_\mu} = \mathcal{L},_{\theta,_\mu} = \begin{cases} -c_0\mathcal{L},_\omega, \\ \mathcal{L},_k. \end{cases} \tag{3.7}$$

The Euler–Lagrange equation appertaining to (3.6) is thus written in a very compact form

$$(\mathcal{L},_{\theta,_\mu}),_\mu = 0. \tag{3.8}$$

With (3.7) and (3.2) it is easily seen that the Euler–Lagrange Equation (3.8) coincides with the conservation of wave action (2.10).

Next, we examine the four-gradient of the Lagrangian

$$\mathrm{grad}\,\mathcal{L} = \frac{\partial \mathcal{L}}{\partial x_\nu} = \frac{\partial}{\partial x_\mu}(\delta_{\mu\nu}\mathcal{L}). \tag{3.9}$$

Using (3.6) and integration by parts leads to

$$\mathcal{L},_\nu = \frac{\partial \mathcal{L}}{\partial \theta,_\mu}\frac{\partial \theta,_\mu}{\partial x_\nu} = L,_\theta\theta,_{\nu\mu} = (\mathcal{L},_\theta\theta,_\nu),_\mu - (\mathcal{L},_\theta),_\mu\theta,_\nu. \tag{3.10}$$

Due to (3.8), the last term on the right-hand side of (3.10) vanishes and (3.10) can be rearranged with (3.9) to the form

$$(\mathcal{L}\delta_{\mu\nu} - \mathcal{L},_\theta\theta,_\nu),_\mu = 0, \tag{3.11}$$

which is a conservation law in space-time. The quantity in parentheses is usually designated in field theory as the energy–momentum “tensor” or material-momentum, cf. e.g., [3,5]

$$\mathcal{T}_{\mu\nu} = \mathcal{L}\delta_{\mu\nu} - \mathcal{L},_{\theta,_\mu}\theta,_\nu. \tag{3.12}$$

It may be mentioned that this "tensor" is not invariant in space-time, i.e., Lorentz invariant but rather only Galileo invariant.

Due to (3.11), $\mathcal{T}_{\mu v}$ is divergence-free.

$$\mathcal{T}_{\mu\nu,\mu} = 0. \tag{3.13}$$

Equation (3.11) is valid only as long as $\mathcal{L}$ depends on the parameters x_ν through the functions $\theta,_\mu$ only. If $\mathcal{L}$ depends on x_ν explicitly, (3.11) differs from zero by terms involving the explicit derivatives of $\mathcal{L}$ with respect to x_ν. The material behaviour of the system under consideration is then either time-dependent or inhomogeneous.

Exploring the physical interpretation of the conservation law (3.13) we consider first the component $\nu = 0$. Using (3.1)–(3.5) and (3.7) leads to

$$\mathcal{T}_{\mu 0,\mu} = -\frac{1}{c_0}\left[\frac{\partial}{\partial t}(\omega\mathcal{L},_\omega - \mathcal{L}) - \frac{\partial}{\partial x_j}(\omega\mathcal{L},_{k_j})\right] = 0, \tag{3.14}$$

which is identical with the conservation-of-energy equation given in (2.13). Proceeding similarly with $\nu = i$ we arrive at the wave momentum Eq. (2.14)

$$\mathcal{T}_{\mu i,\mu} = \left[\frac{\partial}{\partial t}(k_i\mathcal{L},_\omega) + \frac{\partial}{\partial x_j}(\mathcal{L}\delta_{ji} - k_i\mathcal{L},_{k_j})\right] = 0. \tag{3.15}$$

Thus, the application of *grad* in space-time delivers both conservation of energy and conservation of wave momentum.

With (3.14) and (3.15) the components of the material-momentum tensor are [5]

$$\begin{aligned}
&\mathcal{T}_{00} = \mathcal{L} - \omega\mathcal{L},_\omega = -\mathcal{H} && \text{Hamiltonian or total energy,}\\
&\mathcal{T}_{j0} = \frac{1}{c_0}\omega\mathcal{L},_{k_j} = \frac{1}{c_0}\mathcal{S}_j && \text{field intensity or energy flux,}\\
&\mathcal{T}_{0i} = c_0 k_i\mathcal{L},_\omega = -c_0\mathcal{R}_i && \text{field-momentum or wave-momentum density,}\\
&\mathcal{T}_{ji} = \mathcal{L}\delta_{ji} - k_i\mathcal{L},_{k_j} = -\mathcal{B}_{ji} && \text{material-momentum or Eshelby tensor.}
\end{aligned} \tag{3.16}$$

All quantities are understood as average quantities in the spirit of (2.3). With (3.16), Eq. (3.13) can be rewritten as

$$\mathcal{T}_{\mu\nu,\mu} = \begin{cases} -\dfrac{1}{c_0}(\mathcal{H}^{\cdot} - \mathcal{S}_{i,i}) = 0, \\ -(\mathcal{R}_j^{\cdot} + \mathcal{B}_{ji,j}) = 0. \end{cases} \tag{3.17}$$

Concluding this section it may be mentioned that Eq. (3.10) reveals a further possibility to derive the conservation laws treated so far. By multiplying the Euler–Lagrange Eq. (3.8), with the phase derivative $\theta,_\nu$ leads after rearranging to (3.11) along solutions of (3.8), i.e., multiplying conservation of wave action (2.10) with ω delivers conservation of energy whereas multiplying with k_j delivers conservation of wave momentum. Obviously, ω and k_j play the role of integrating factors and the resulting conservation laws are first integrals of the wave-action equation. A deeper discussion of this matter and its connection to the Neutral–Action Method introduced by Honein et al. [6] may be found in [3].

4. New relations

Since the application of the standard vector operation *grad* to the Lagrangian $\mathcal{L}$ delivers conservation laws, it seems to be intriguing to explore whether or not the application of the vector operations *div* and *curl* lead to further conservation laws. The operators *div* and *curl* are to be applied to vectors (or tensors). Following [4], the appropriate vector is expected to be $x_\nu\mathcal{L}$.

4.1. Application of the div operator

First let us consider the divergence of the vector $x_\nu\mathcal{L}$. We have

$$\frac{\partial(x_\nu\mathcal{L})}{\partial x_\nu} = \alpha\mathcal{L} + x_\nu\frac{\partial\mathcal{L}}{\partial x_\nu}, \tag{4.1}$$

where $\alpha = \delta_{\nu\nu}$ relates to the dimensionality of the problem. If we consider three spatial coordinates and the time coordinate, α is equal to four. With (3.10) and (3.8) the quantity $\partial\mathcal{L}/\partial x_\nu$ may be replaced by $(\mathcal{L},_{\theta,_\mu}\theta,_\nu),_\mu$ and instead of $\partial(x_\nu\mathcal{L})/\partial x_\nu$ we use $\partial(x_\nu\delta_{\nu\mu}\mathcal{L})/\partial x_\mu$. Rearranging (4.1) in this way and integration by parts where appropriate leads with (3.12) to

$$\begin{aligned} x_\nu(\mathcal{L},_{\theta,_\mu}\theta,_\nu),_\mu - (x_\nu\delta_{\nu\mu}\mathcal{L}),_\mu + \alpha\mathcal{L} &= \left[x_\nu(\mathcal{L},_{\theta,_\mu}\theta,_\nu - \mathcal{L}\delta_{\nu\mu})\right],_\mu - \mathcal{L},_{\theta,_\mu}\theta,_\nu\delta_{\nu\mu} + \alpha\mathcal{L} \\ &= \left[-x_\nu\mathcal{T}_{\mu\nu}\right],_\mu - \mathscr{L},_{\theta,_\mu}\theta,_\mu + \alpha\mathscr{L} = 0 \end{aligned} \tag{4.2}$$

If the Lagrangian is a homogeneous function of degree n in $\theta,_\mu$, i.e., if

$$n\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\theta,_\mu}\theta,_\mu \tag{4.3}$$

holds, and since

$$(\mathcal{L},_{\theta,_\mu})\theta,_\mu = (\mathcal{L},_{\theta,_\mu}\theta),_\mu$$

along solutions of the Euler–Lagrange Eq. (3.8), Eq. (4.2) is rewritten as

$$\left[x_\nu\mathcal{T}_{\mu\nu} - \frac{n-\alpha}{n}\theta\mathcal{L},_{\theta,_\mu}\right],_\mu = 0 \tag{4.4}$$

The vector

$$\mathcal{V}_\mu = x_\nu\mathcal{T}_{\mu\nu} - \frac{n-\alpha}{n}\theta\mathcal{L},_{\theta,_\mu} \tag{4.5}$$

might be called the virial of the system [7,8]and its divergence vanishes

$$\mathcal{V}_{\mu,\mu} = 0. \tag{4.6}$$

The resulting conservation law can be transformed into the more usual form by emplying (3.1)–(3.5), (3.7) and (3.12). In addition, we assume linear elastic behaviour, $n = 2$ and $\alpha = 4$. The result is

$$\frac{\partial}{\partial t}\left[t(\omega\mathcal{L},_\omega - \mathcal{L}) - (x_jk_j + \theta)\mathcal{L},\omega\right] - \frac{\partial}{\partial x_i}\left[(t\omega - \theta)\mathcal{L},_{k_i} + x_j(\mathcal{L}\delta_{ij} - k_j\mathcal{L},_{k_i})\right] = 0. \tag{4.7}$$

On checking Eq. (4.7) we use (3.5) and the product rule where appropriate, and arrive at

$$\begin{aligned} &-2(2\mathcal{L} - \omega\mathcal{L},_\omega - k_i\mathcal{L},_{k_i}) + t\left[\frac{\partial}{\partial t}(\omega\mathcal{L},_\omega - \mathcal{L}) - \frac{\partial}{\partial x_i}(\omega\mathcal{L},_{k_i})\right] - x_j\left[\frac{\partial}{\partial t}(k_j\mathcal{L},_\omega) + \frac{\partial}{\partial x_i}(\mathcal{L}\delta_{ij} - k_j\mathcal{L},_{k_i})\right] \\ &\quad -\theta\left[\frac{\partial}{\partial t}(\mathcal{L},_\omega) - \frac{\partial}{\partial x_i}(\mathcal{L},_{k_i})\right] = 0. \end{aligned} \tag{4.8}$$

Since it was assumed that the Lagrangian is homogeneous of grade 2, the first bracket of (4.8) vanishes due to (4.3). With the remainder of Eq. (4.8) we arrive at the statement (S.1):

$$\begin{array}{ll} & t \text{ times conservation of energy} \\ \text{minus} & x_j \text{ times conservation of wave momentum} \\ \text{minus} & \theta \text{ times conservation of wave action} \\ \text{equals} & a \text{ divergence-free expression.} \end{array} \tag{S.1}$$

4.2. Application of the curl operator

The *curl* operator in space-time makes use of the completely skew-symmetric permutation tensor $\varepsilon_{\alpha\beta\mu\nu}$ of rank 4.

Applying *curl* to the vector $x_\mu\mathcal{L}$ and use of (3.10) with (3.8) results in

$$\begin{aligned} \varepsilon_{\alpha\beta\mu\nu}(x_\mu\mathcal{L}),_\nu &= \varepsilon_{\alpha\beta\mu\nu}(\delta_{\nu\lambda}x_\mu\mathcal{L}),_\lambda = \varepsilon_{\alpha\beta\mu\nu}(\delta_{\mu\nu}\mathcal{L} + x_\mu\mathcal{L},_\nu) = \varepsilon_{\alpha\beta\mu\nu}\left[x_\mu(\mathcal{L},_{\theta,_\lambda}\theta,_\nu),_\lambda + \mathcal{L}\delta_{\mu\nu}\right] \\ &= \varepsilon_{\alpha\beta\mu\nu}\left[(x_\mu\mathcal{L},_{\theta,_\lambda}\theta,_\nu),_\lambda - (\mathcal{L},_{\theta,_\mu}\theta,_\nu - \mathcal{L}\delta_{\mu\nu}\right] \end{aligned} \tag{4.9}$$

Rearrangment leads to

$$\varepsilon_{\alpha\beta\mu\nu}\left[x_\mu(\mathcal{L}\delta_{\lambda\nu} - \mathcal{L},_{\theta,_\lambda}\theta,_\nu)\right],_\lambda = \varepsilon_{\alpha\beta\mu\nu}(\mathcal{L}\delta_{\mu\nu} - \mathcal{L},_{\theta,_\mu}\theta,_\nu), \tag{4.10}$$

or, with (3.12), to a more compact form

$$\varepsilon_{\alpha\beta\mu\nu}(x_\mu\mathcal{T}_{\lambda\nu}),_\lambda = \varepsilon_{\alpha\beta\mu\nu}\mathcal{T}_{\mu\nu}. \tag{4.11}$$

It turns out that the application of the *curl* operator leads to a conservation law only if the material-momentum tensor $\mathcal{T}_{\mu\nu}$ is symmetric, i.e., $\mathcal{T}_{\mu\nu} = \mathcal{T}_{\nu\mu}$, which is generally not satisfied. Exploring the physical significance of (4.10) and (4.11) further, we return to the usual notation in the same manner as shown above. We have to distinguish between two cases, namely

Case a : $\alpha = k, \beta = l$

$$\begin{aligned} &\frac{\partial}{\partial t}\left[c_0tk_j\mathcal{L},_\omega + \frac{1}{c_0}x_j(\omega\mathcal{L},_\omega - \mathcal{L})\right] + \frac{\partial}{\partial x_i}\left[c_0t(\mathcal{L}\delta_{ij} - k_j\mathcal{L},_{k_i}) - \frac{1}{c_0}x_j(\omega\mathcal{L},_{k_i})\right] \\ &\quad = c_0k_j\mathcal{L},_\omega - \frac{1}{c_0}\omega\mathcal{L},_{k_j} = -(c_0\mathcal{R}_j + \frac{1}{c_0}\mathcal{S}_j). \end{aligned} \tag{4.12}$$

Thus, rotation in space and time results in a balance rather than a conservation laws. The right-hand side is equal to the negative sum of field momentum and field intensity appropriately scaled by the characteristic velocity c_0.

Applying the product rule of differentiation where appropriate, (4.12) is transformed to

$$\begin{aligned} &c_0t\left[\frac{\partial}{\partial t}(k_j\mathcal{L},_\omega) + \frac{\partial}{\partial x_i}(\mathcal{L}\delta_{ij} - k_j\mathcal{L},_{k_i})\right] + \frac{1}{c_0}x_j\left[\frac{\partial}{\partial t}(\omega\mathcal{L},_\omega - \mathcal{L}) - \frac{\partial}{\partial x_i}(\omega\mathcal{L},_{k_i})\right] \\ &\quad - c_0\mathcal{R}_j - \frac{1}{c_0}\mathcal{S}_j = \text{divergence}, \end{aligned} \tag{4.13}$$

which leads to the statement (S.2):

$$\begin{array}{ll} & c_0 t \text{ times conservation of wave momentum} \\ \text{plus} & x_j \text{ times conservation of energy} \\ \text{minus} & \text{the sum of field momentum and field intensity} \\ \text{equals} & a \text{ divergence} \end{array} \tag{S.2}$$

This divergence generally does not vanish. Note the interchange of conservation of wave momentum and energy and the change in sign between the statements (S.1) and (S.2).

Case b : $\alpha = 0, \beta = k$

$$\frac{\partial}{\partial t}\left[(x_j k_i - x_i k_j)\mathcal{L},_{\omega}\right] + \frac{\partial}{\partial x_l}\left[x_j(\mathcal{L}\delta_{li} - k_i\mathcal{L},_{k_l}) - x_i(\mathcal{L}\delta_{lj} - k_j\mathcal{L},_{k_l})\right] = \mathcal{L},_{k_i} k_j - \mathcal{L},_{k_j} k_i. \tag{4.14}$$

The right-hand side of (4.14) vanishes for isotropic materials, i.e., if the wave speed is direction-independent [2] and it turns out that rotation in space (while keeping the time axis fixed) leads to a conservation law for isotropic materials. On checking (4.14) by differentiation as above yields the statement (S.3):

$$\begin{array}{ll} & x_j \text{ times conservation of wave action in } x_i - \text{direction} \\ \text{minus} & x_i \text{ times conservation of wave action in } x_j\text{-direction} \\ \text{equals} & a \text{ divergence-free expression} \end{array} \tag{S.3}$$

4.3. Example

As a first simple example we consider wave motion in a one-dimensional, linearly elastic bar on a linearly elastic foundation, with Young's modulus E, cross-sectional area A, density ρ, mass per unit of length $\mu = \rho A$, spring-stiffness per unit of length β and axial displacement u. A prime indicates differentiation with respect to the axial coordinate x and a dot indicates differentiation with respect to time. The Lagrangian L is given by [2]

$$L = \frac{1}{2}\mu\dot{u}^2 - \frac{1}{2}EAu'^2 - \beta u^2. \tag{4.15}$$

After proper adjustment of the constant, (4.15) may also be identified as the Lagrangian of the Klein–Gordon equation of quantum mechanics.

The averaged Lagrangian $\mathcal{L}$ follows to be

$$\mathcal{L} = \frac{1}{4}a^2(\mu\omega^2 - EAk^2 - \beta), \tag{4.16}$$

the term in parentheses being the dispersion relation of the system. With elastic foundation ($\beta \neq 0$) the system is dispersive, without it ($\beta = 0$) the system is non-dispersive. The Euler–Lagrange equation or conservation of wave action is

$$\mu\dot{\omega} + EAk' = 0. \tag{4.17}$$

As a second example we consider wave motion in a one-dimensional, linerarly elastic beam with E and μ as before, second moment of inertia I and transverse displacement w. As mentioned a prime indicates differentiation with respect to the axial coordinate x and a dot indicates differentiation with respect to time. The Lagrangian L is given by [2]

$$L = \frac{1}{2}\mu\dot{w}^2 - \frac{1}{2}EI(w'')^2 \tag{4.18}$$

and the averaged Lagrangian $\mathcal{L}$ follows to be

$$\mathcal{L} = \frac{1}{4}a^2(\mu\omega^2 - EIk^4), \tag{4.19}$$

the term in parentheses being again, the dispersion relation of the system. The Euler–Lagrange equation or conservation of action is

$$\mu\dot{\omega} + 6EIk^2k' = 0. \tag{4.20}$$

The components of the material-momentum tensor (3.16) for this two-dimensional (x, t) setting are identified as

$$\begin{aligned} \mathcal{T}_{00} &= \mathcal{L} - \omega\mathcal{L},_{\omega} = -\mathcal{H}, \\ \mathcal{T}_{10} &= \frac{1}{c_0}\omega\mathcal{L},_k = \frac{1}{c_0}\mathcal{S}, \\ \mathcal{T}_{01} &= c_0 k\mathcal{L},_{\omega} = -c_0\mathcal{R}, \\ \mathcal{T}_{11} &= \mathcal{L} - k\mathcal{L},_k = -\mathcal{B}. \end{aligned} \tag{4.21}$$

The *grad* operator leads to two equations that correspond to (3.14) and (3.15), namely $\nu = 0$:

$$\frac{1}{c_0}\mathcal{T}_{00}{}^{\cdot} + \mathcal{T}_{10}{}^{/} = -\frac{1}{c_0}\left[(\omega\mathcal{L},_{\omega} - \mathcal{L})^{\cdot} - (\omega\mathcal{L},_k)^{/}\right] = 0, \tag{4.22}$$

or

$$\mathcal{H}^{\cdot} - \mathcal{S}^{/} = 0, \tag{4.23}$$

which states that energy is conserved, and $\nu = 1$:

$$\frac{1}{c_0}\mathcal{T}_{01}{}^{\cdot} + \mathcal{T}'_{11} = (k\mathcal{L},_{\omega})^{\cdot} + (\mathcal{L} - k\mathcal{L},_k)^{/} = 0, \tag{4.24}$$

or

$$\mathcal{H}^{\cdot} + \mathcal{B}^{/} = 0, \tag{4.25}$$

which states that wave momentum is conserved.

The dimensionality of the problem under consideration is $\alpha = 2$ and the degree of homogeneity is $n = 2$. Thus $\alpha - n$ in (4.4) is equal to zero and the *div* operator yields

$$[t(\omega\mathcal{L},_{\omega} - \mathcal{L}) - xk\mathcal{L},_{\omega}]^{\cdot} - [t\omega\mathcal{L},_k + x(\mathcal{L} - k\mathcal{L},_k)]^{/} = 0, \tag{4.26}$$

or

$$[t\mathcal{H} + x\mathcal{R}]^{\cdot} - [t\mathcal{S} - x\mathcal{B}]^{/} = 0. \tag{4.27}$$

The term in the first bracket might be identified as the energy–field momentum virial, $\mathcal{V}_0$ (scalar moment), while the term in the second brackets is the material momentum-energy flux (field intensity) virial, $\mathcal{V}_1$, e.g.,

$$\overset{\bullet}{\mathcal{V}}_0 + \mathcal{V}_1^{/} = 0. \tag{4.28}$$

Thus, the time rate of change of the virial $\mathcal{V}_0$ is balanced by the spatial rate of change of the virial $\mathcal{V}_1$.

The application of the product rule of differentiation verifies the well-known fact that the Hamiltonian $\mathcal{H}$ equals the negative of the material force $\mathcal{B}$ in one-dimensional problems of elasticity [3].

The *curl* operator involves the permutation tensor $\varepsilon_{\mu\nu}$ of rank two and only rotation in space-time is possible. The corresponding equation to (4.12) is

$$\left[-c_0 tk\mathcal{L},_\omega - \frac{1}{c_0}x(\omega\mathcal{L},_\omega - \mathcal{L})\right]^{\cdot} + \left[-c_0 t(\mathcal{L} - k\mathcal{L},_k) + \frac{1}{c_0}x(\omega\mathcal{L},_k)\right]^{/} = -c_0 k\mathcal{L},_\omega + \frac{1}{c_0}\omega\mathcal{L},_k, \tag{4.29}$$

or

$$\left[c_0 t\mathcal{R} - \frac{1}{c_0}x\mathcal{H}\right]^{\cdot} + \left[c_0 t\mathcal{B} + \frac{1}{c_0 x}\mathcal{S}\right]^{/} = c_0\mathcal{R} + \frac{1}{c_0}\mathcal{S}, \tag{4.30}$$

which is, as mentioned above, a balance rather than a conservation law. The first term in brackets might be labelled as the energy–field momentum *curl* $-\mathcal{C}_0$, as the second term in brackets is the material momentum energy flux *curl* $\mathcal{C}_1$. The right hand side represents the negative of the sum c of material momentum density and the energy flux. Thus

$$\mathcal{C}_0^{\bullet} + \mathcal{C}_1^{/} = -c, \tag{4.31}$$

which states that the sum of the time rate of change of the curl $\mathcal{C}_0$ and the spatial rate of change of the curl $\mathcal{C}_1$, is not vanishing, but is rather balanced by $-c$. Note again the chiasmus between $\mathcal{R} \leftrightarrow \mathcal{H}$ and $\mathcal{S} \leftrightarrow \mathcal{B}$ and the change in sign between (4.27) and (4.30).

5. Conclusions

Based on Whitham's variational formulation of dispersive wave motion, it has become possible to extend his conservation laws of "wave motion", "energy" and "wave momentum". While Whitham's approach to establish the latter two laws was based on the application of Noether's theorem, it was shown here that they in fact can be obtained by subjecting the average Lagrangian to the operator of *grad* in four-dimensions of time and three-dimensional space as this is done in the theory of relativity. Developments were restricted to linear problems and a four-dimensional "Lagrangian vector" was subjected to two further standard operators of *div* and *curl*. In the first of these two cases a conservation law for the "wave virial" was derived, while in the second case merely a balance equation for the "wave curl" was obtained because it did not appear possible to remove a non-vanishing source term, when rotation in space and time was considered. Rotation in space, while keeping the time axis fixed, led to a conservation law for isotropic materials. To illustrate the general relations, two two-dimensional (in t, x) examples were presented.

It is recalled that the *grad* operator (translation) leads in fracture mechanics to the J-integral, the *div* operator (self-similar expansion) yields the M-integral and the *curl* operator (rotation) results in the L-integral, as discussed in [3,4].

Whitham has shown that his variational formulation of dispersive wave motion for linear uniform problems may be extended to non-uniform (non-homogeneous and/or time-dependent) media and also to non-linear problems. It would indeed be a tempting task to extend the essential contents of the present contribution along those two directions and the authors intend to tackle this task in the near future.

As regards the value and usefulness of conservation and balance laws in a general way, reference may be made to an evaluation of such laws by Olver [9]. It may suffice to mention here the applicability of conservation (and balance) laws in numerics. Being incorporated into various algorithms, the accuracy of the numerical results can be validated by checking whether or not the conservation laws are satisfied identically. If the equations are not satisfied, so-called spurious material nodel forces occur in finite-element calculations, which can be used to improve the finite-element mesh by shifting the nodes in such a way as to eliminate the spurious forces, cf. Braun [10], Müller and Maugin [11], Steinmann et al. [12].

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