The inverse scattering problem for *LCRG* transmission lines[a)]

M. Jaulent[b)]

Département de Physique Mathématique, Université des Sciences et Techniques du Languedoc, 34060 Montpellier Cedex, France

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The inverse scattering problem for one-dimensional nonuniform transmission lines with inductance $L(z)$, capacitance $C(z)$, series resistance $R(z)$ and shunt conductance $G(z)$ per unit length ($z\in\mathbb{R}$) is considered. It is reduced to the inverse scattering problem for the Zakharov–Shabat system. It is found that one can construct from the data the following functions of the travel time x:

$$\tilde{q}^{\pm}(x)=\left[\frac{1}{4}\frac{d}{dx}\left(\ln\frac{L}{C}\right)\pm\frac{1}{2}\left(\frac{R}{L}-\frac{G}{C}\right)\right]\exp\left(\mp\int_{\infty}^{x}\left(\frac{R}{L}+\frac{G}{C}\right)dy\right).$$

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I. INTRODUCTION

In this paper we consider the inverse scattering problem (ISP) for transmission lines extending in a z direction from $z=-\infty$ to $z=\infty$, with inductance $L(z)$, capacitance $C(z)$, series resistance $R(z)$, and shunt conductance $G(z)$ per unit length. We suppose that

—$L(z)$, $C(z)$, $R(z)$, and $G(z)$ ($z\in\mathbb{R}$) are sufficiently regular real functions;

—$L(z)>0$, $C(z)>0$, $R(z)\geqslant 0$, $G(z)\geqslant 0$;

—$L(z)$ and $C(z)$ have strictly positive finite limits $L(\infty)$ and $C(\infty)$, [resp. $L(-\infty)$ and $C(-\infty)$], as $z\rightarrow\infty$ (resp. $z\rightarrow-\infty$).

$I(z,t)$ and $U(z,t)$ being, respectively, the intensity of the current and the voltage at position z and time t, we use the transmission lines equation ($z\in\mathbb{R}$):

$$\frac{\partial I}{\partial z}+C(z)\frac{\partial U}{\partial t}+G(z)U=0,$$
$$\frac{\partial U}{\partial z}+L(z)\frac{\partial I}{\partial t}+R(z)I=0. \tag{1.1}$$

For a wave of frequency k, i.e., for

$$I(z,t)=I(k,z)e^{-ikt}, \tag{1.2}$$
$$U(z,t)=U(k,z)e^{-ikt}, \tag{1.3}$$

Eq. (1.1) may be written in the form

$$\frac{dI}{dz}-ikC(z)U+G(z)U=0,$$
$$\frac{dU}{dz}-ikL(z)I+R(z)I=0. \tag{1.4}$$

In the following instead of z, we obtain the variable x defined by

$$x(z)=\int_0^z(L(u)C(u))^{1/2}\,du. \tag{1.5}$$

We also use the convention $I(k,z(x))=I(k,x)$, $L(z(x))=L(x)$, etc., justified by the one-to-one correspondence between z and x, $x(z)$ varying from $x(-\infty)=-\infty$ to $x(\infty)=\infty$. We shall see below that $x(z)$ is the travel time of waves from the origin to the position z.

The data of the ISP are the reflection coefficients to the right and to the left, $r(k)$ and $\tilde{r}(k)$, and the transmission coefficient $t(k)$, for $k>0$, and also the quantities $L(-\infty)$, $L(\infty)$, $C(-\infty)$, and $C(\infty)$. The ISP can be stated thus: what information can be obtained on L, R, C, and G from the data?, i.e., what quantities connecting L, C, R, and G can be constructed from the data?

In the lossless case, i.e., $R=G=0$, it is well known—see the survey by Kay[1]—that this ISP can be solved by reduction to the ISP for the one-dimensional Schrödinger equation

$$\text{(S):}\quad \frac{d^2y}{dx^2}+[k^2-V(x)]y=0,\quad x\in\mathbb{R}. \tag{1.6}$$

For the solution of the ISP for (S) see Kay[2], Kay and Moses[3], and Faddeev[4]. In the lossless case, it is then found that the quantity which can be constructed from the data, is the quotient L/C as a function of the travel time x.

The lossy case with only one kind of absorption, i.e., $R=0$ or $G=0$, has been studied by Jaulent[5,6] and independently by Schmidt.[7] This ISP can be solved by reduction to the ISP for the one-dimensional Schrödinger equation with an energy-dependent potential

$$\text{(S')}:\quad \frac{d^2y}{dx^2}+[k^2-V(k,x)]y=0, \tag{1.7}$$
$$V(k,x)=V(x)+kQ(x). \tag{1.8}$$

There also exists a radial version of the ISP for the lines (i.e., $z\geqslant 0$ instead of $z\in\mathbb{R}$) which can be solved using the radial version of the ISP for (S') (i.e., $x\geqslant 0$ instead of $x\in\mathbb{R}$) (see Ref. 6). For the solution of the ISP for (S') see Jaulent and Jean,[8] Jaulent,[9,10] for the radial case ($x\geqslant 0$), and Jaulent and Jean[11] for the one-dimensional case ($x\in\mathbb{R}$). In the lossy case with $R=0$ (resp. $G=0$) it is then found that the quantities which can be constructed from the data are the quotients L/C and G/C (resp. L/C and R/L) as functions of the travel time x. In Sec. II of this paper we give some additional indications on the lossless case and the lossy case with $R=0$ or $G=0$.

In this paper we consider the general lossy case. In Sec.

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III we prove that the lines equation (1.4) can be put into the form of a generalized Zakharov–Shabat system $(Z)[q^+, q^-, q_3]$:

$$\frac{dY}{dx} + ik\sigma_3 Y = \begin{pmatrix} iq_3 & q^+ \\ q^- & -iq_3 \end{pmatrix} Y, \tag{1.9}$$

with

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.10}$$

Then through $(Z)[q^+, q^-, q_3]$ we introduce the scattering data associated to the lines equation (1.4). In Sec. IV we reduce the ISP for $(Z)[q^+, q^-, q_3]$ to the well-known ISP for the Zakharov–Shabat system $(Z)[\tilde{q}^+, \tilde{q}^-, \tilde{q}_3 = 0]$. The motivation to study this ISP was first to solve nonlinear evolution equations. See Zakharov-Shabat[12], Ablowitz, Kaup, Newell, Segur,[13] and Calogero and Degasperis.[14] In Sec. V we briefly reproduce the solution of this ISP.

As a result we find that the following quantities, $\tilde{q}^+$ and $\tilde{q}^-$, can be constructed from the ISP data for the lines in the general lossy case, as functions of the travel time x:

$$\tilde{q}^{\pm}(x) = \left[\frac{1}{4} \frac{d}{dx}\left(\ln \frac{L}{C}\right) \pm \frac{1}{2}\left(\frac{R}{L} - \frac{G}{C}\right)\right] \times \exp\left(\mp \int_\infty^x \left(\frac{R}{L} + \frac{G}{C}\right) dy\right) \tag{1.11}$$

where the indices + and − correspond to each other. Indeed $\tilde{q}^+$ and $\tilde{q}^-$ data are equivalent to the ISP data for the lines, so that we can conclude that, although it is widely underdetermined, the ISP for the lines is theoretically solved. $\tilde{q}^+$ and $\tilde{q}^-$ represent two functional relations between L/C, R/L and G/C. In order to determine the quotients L/C, R/L and G/C (as functions of x) we need another relation between L/C, R/L and G/C. Such is the case if we are given $R/L + G/C$ or R/L or L/C. We notice that if $R = 0$ (resp. $G = 0$) we find again the result of Ref. 6, i.e., L/C and G/C (resp. L/C and R/L) are determined from the data. Indeed these two approaches are equivalent since it has been proved by Jaulent and Miodek[15] that the ISP for the Schrödinger equation $(S')[V,Q]$ and the Zakharov–Shabat system $(Z)[\tilde{q}^+, \tilde{q}^-, \tilde{q}_3 = 0]$ are equivalent. The keystone of the proof consists in introducing the generalized Zakharov–Shabat equation $(Z)[q^+, q^-, q_3]$ and noticing that $(S')[V,Q]$ and $(Z)[\tilde{q}^+, \tilde{q}^-, \tilde{q}_3 = 0]$ are in some way "particular cases" of this equation. Furthermore, it is possible to go easily from one inversion procedure to the other.

II. THE LOSSLESS CASE AND THE CASE $R = 0$ OR $G = 0$

If $R = G = 0$ it is easy from the lines equation (1.4) to obtain

$$\frac{d}{dz}\left(\frac{1}{L(z)} \frac{dU}{dz}\right) + k^2 C(z) U = 0. \tag{2.1}$$

Using the Liouville transformation, i.e., using the variable x defined by (1.5) and setting

$$y(k,x) = \left[\frac{C(x)}{L(x)}\right]^{1/4} U(k,x), \tag{2.2}$$

we find that $y(k,x)$ satisfies the Schrödinger equation (S) with the potential

$$V(x) = \left[\frac{C(x)}{L(x)}\right]^{-1/4} \frac{d^2}{dx^2}\left[\frac{C(x)}{L(x)}\right]^{1/4}. \tag{2.3}$$

It is assumed that $V(x)$ is a sufficiently regular function going to 0 fast enough as $|x| \to \infty$. The solution of the ISP for (S) allows to construct $V(x)$ and therefore $C(x)/L(x)$.

If $R = 0$ we obtain from (1.4) the equation

$$\frac{d}{dz}\left(\frac{1}{L(z)} \frac{dU}{dz}\right) + k^2 C(z) U + ikG(z) U = 0. \tag{2.4}$$

Using the Liouville transformation defined by (1.5) and (2.2) we find that $y(k,x)$ satisfies the Schrödinger equation (S′) with the potentials

$$V(x) = \left[\frac{C(x)}{L(x)}\right]^{-1/4} \frac{d^2}{dx^2}\left[\frac{C(x)}{L(x)}\right]^{1/4}, \tag{2.5}$$

$$Q(x) = -i \frac{G(x)}{C(x)}. \tag{2.6}$$

It is assumed that $V(x)$ and $Q(x)$ are sufficiently regular functions going to 0 fast enough as $|x| \to \infty$. The solution of the ISP for (S′) allows one to construct $V(x)$ and $Q(x)$ and therefore $C(x)/L(x)$ and $G(x)/C(x)$. The case $G = 0$ is treated exactly in the same way by replacing $U(k,z)$ by $I(k,z)$, $L(z)$ by $C(z)$, $C(z)$ by $L(z)$, and $G(z)$ by $R(z)$.

III. REDUCTION OF THE LINES EQUATION (1.4) TO (Z) $[q^+, q^-, q_3]$ AND DEFINITION OF THE SCATTERING DATA

We use the variable x defined by (1.5) and we set

$$w_1(k,x) = \left[\frac{L(x)}{C(x)}\right]^{1/4} I(k,x), \tag{3.1}$$

$$w_2(k,x) = -\left[\frac{C(x)}{L(x)}\right]^{1/4} U(k,x), \tag{3.2}$$

$$W(k,x) = \begin{pmatrix} w_1(k,x) \\ w_2(k,x) \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.3}$$

Then we find that the lines equation (1.4) can be put into the form

$$\frac{dW}{dx} + ik\sigma_1 W = \begin{pmatrix} \frac{1}{4} \frac{d}{dx} \ln \frac{L}{C} & \frac{G}{C} \\ \frac{R}{L} & -\frac{1}{4} \frac{d}{dx} \ln \frac{L}{C} \end{pmatrix} W, \quad x \in \mathbb{R}. \tag{3.4}$$

One may readily put Eq. (3.4) into the form (Z) $[q^+, q^-, q_3]$ by setting

$$Y = NW, \tag{3.5}$$

$$N = N^{-1} = \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_3) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{3.6}$$

$$q^{\pm}(x) = \frac{1}{4} \frac{d}{dx}\left(\ln \frac{L}{C}\right) \pm \frac{1}{2}\left(\frac{R}{L} - \frac{G}{C}\right), \tag{3.7}$$

$$iq_3(x) = \frac{1}{2}\left(\frac{R}{L} + \frac{G}{C}\right). \tag{3.8}$$

(Note that $N\sigma_3 = \sigma_1 N$).

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We assume that $q^+(x)$, $q^-(x)$, and $q_3(x)$ are sufficiently regular functions going to 0 fast enough as $|x|\to\infty$. Since the trace of the matrix

$$\begin{pmatrix} iq_3 & q^+ \\ q^- & -iq_3 \end{pmatrix}$$

is 0, it is possible to introduce the scattering data for (Z) $[q^+, q^-, q_3]$ in the same way as in the well-known case $q_3 = 0$. Instead of (Z) $[q^+, q^-, q_3]$ it is technically convenient to consider both systems $(Z)^\pm[q^+, q^-, q_3]$:

$$\frac{dY^\pm}{dx} + ik\sigma_3 Y^\pm = \begin{pmatrix} \pm iq_3 & q^\pm \\ q^\mp & \mp iq_3 \end{pmatrix} Y^\pm. \quad (3.9)$$

If $Y^-(k,x)$ is a solution of $(Z)^-$ then $\sigma_1 Y^-(-k,x)$ is a solution of $(Z)^+$. This symmetry property allows one to reduce the study of two types of Jost solutions at $+\infty$ (or at $-\infty$) to only one.

The right and left Jost solutions of $(Z)^\pm$, $F^\pm(k,x)$ and $\overleftarrow{F}^\pm(k,x)$, are defined as

$$F^\pm(k,x) \underset{x\to\infty}{\sim} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}, \quad \overleftarrow{F}^\pm(k,x) \underset{x\to-\infty}{\sim} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}. \quad (3.10)$$

$\sigma_1 F^\mp(-k,x)$ and $\sigma_1 \overleftarrow{F}^\mp(-k,x)$ are also Jost solutions of $(Z)^\pm$ with

$$\sigma_1 F^\mp(-k,x) \underset{x\to\infty}{\sim} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, \quad \sigma_1 \overleftarrow{F}^\mp(-k,x) \underset{x\to-\infty}{\sim} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}. \quad (3.11)$$

Using standard arguments (see Ref. 13 for example) one can prove that $F^\pm(k,x)$ and $\overleftarrow{F}^\pm(k,x)$ are analytic in k for $\operatorname{Im} k>0$ and continuous for $\operatorname{Im} k\geqslant 0$. $F^\pm(k,x)$ and $\sigma_1 F^\mp(-k,x)$ [resp. $\overleftarrow{F}^\pm(k,x)$ and $\sigma_1 \overleftarrow{F}^\mp(k,x)$] form a fundamental system of solutions of $(Z)^\pm$ for $k\in\mathbb{R}$. The reflection coefficients to the right and to the left, $r^\pm(k)$ and $\overleftarrow{r}^\pm(k)$, and the transmission coefficient $t^\pm(k)$ associated with $(Z)^\pm$ are defined for $k\in\mathbb{R}$ by

$$\overleftarrow{F}^\pm(k,x) = \frac{r^\pm(k)}{t^\pm(k)} F^\pm(k,x) + \frac{1}{t^\pm(k)} \sigma_1 F^\mp(-k,x), \quad (3.12)$$

$$F^\pm(k,x) = \frac{\overleftarrow{r}^\pm(k)}{t^\pm(k)} \overleftarrow{F}^\pm(k,x) + \frac{1}{t^\pm(k)} \sigma_1 \overleftarrow{F}^\mp(-k,x). \quad (3.13)$$

It follows from (3.10)–(3.13) that there exist two solutions of $(Z)^\pm$: $\overleftarrow{\Psi}^\pm(k,x)$ $[=t^\pm(k)\overleftarrow{F}^\pm(k,x)]$ and $\Psi^\pm(k,x)$ $[=t^\pm(k)F^\pm(k,x)]$ such that $(k\in\mathbb{R})$

$$\overleftarrow{\Psi}^\pm(k,x) \underset{x\to-\infty}{\sim} t^\pm(k)\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}$$
$$\underset{x\to\infty}{\sim} r^\pm(k)\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, \quad (3.14)$$

$$\Psi^\pm(k,x) \underset{x\to-\infty}{\sim} \overleftarrow{r}^\pm(k)\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}$$
$$\underset{x\to\infty}{\sim} t^\pm(k)\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}. \quad (3.15)$$

The scattering matrix associated with $(Z)^\pm$ is defined by

$$S^\pm(k) = \begin{pmatrix} t^\pm(k) & r^\pm(k) \\ \overleftarrow{r}^\pm(k) & t^\pm(k) \end{pmatrix}, \quad k\in\mathbb{R}. \quad (3.16)$$

The function $1/t^\pm(k)$ is analytic for $\operatorname{Im} k>0$ and continuous for $\operatorname{Im} k\geqslant 0$. We assume that this function has no zero for $\operatorname{Im} k\geqslant 0$, i.e., $(Z)^\pm$ has no bound state (square integrable solution). This point should be studied thoroughly. Indeed, in the case $R=G=0$ one can prove a similar but weaker result for Eq. (1.7) (see Ref. 6).

We deduce from (3.5), (3.6), (3.14), and (3.15) that there exist two solutions of (3.4), $\overleftarrow{w}(k,x)$ and $W(k,x)$ such that

$$\overleftarrow{W}(k,x) \underset{x\to-\infty}{\sim} t^+(k)\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-ikx}$$
$$\underset{x\to\infty}{\sim} r^+(k)\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{ikx} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-ikx}, \quad (3.17)$$

$$W(k,x) \underset{x\to-\infty}{\sim} \overleftarrow{r}^+(k)\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-ikx} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{ikx}$$
$$\underset{x\to\infty}{\sim} t^+(k)\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{ikx}. \quad (3.18)$$

Therefore [use (1.5), (3.1)–(3.3)] there exist two solutions of the lines equation (1.4),

$$\begin{pmatrix} \overleftarrow{I}(k,z) \\ \overleftarrow{U}(k,z) \end{pmatrix} \text{ and } \begin{pmatrix} I(k,z) \\ U(k,z) \end{pmatrix},$$

such that

$$\overleftarrow{I}(k,z) \underset{z\to-\infty}{\sim} \left[\frac{C(-\infty)}{L(-\infty)}\right]^{1/4} t^+(k) e^{-ikx(z)}$$
$$\underset{z\to\infty}{\sim} \left[\frac{C(\infty)}{L(\infty)}\right]^{1/4} \{r^+(k)e^{ikx(z)} + e^{-ikx(z)}\}, \quad (3.19)$$

$$\overleftarrow{U}(k,z) \underset{z\to-\infty}{\sim} -\left[\frac{L(-\infty)}{C(-\infty)}\right]^{1/4} t^+(k) e^{-ikx(z)}$$
$$\underset{z\to\infty}{\sim} -\left[\frac{L(\infty)}{C(\infty)}\right]^{1/4} \{-r^+(k)e^{ikx(z)} + e^{-ikx(z)}\}, \quad (3.20)$$

$$I(k,z) \underset{z\to-\infty}{\sim} \left[\frac{C(-\infty)}{L(-\infty)}\right]^{1/4} \{\overleftarrow{r}^+(k)e^{-ikx(z)} + e^{ikx(z)}\}$$
$$\underset{z\to\infty}{\sim} \left[\frac{C(\infty)}{L(\infty)}\right]^{1/4} t^+(k)e^{ikx(z)}, \quad (3.21)$$

$$U(k,z) \underset{z\to-\infty}{\sim} -\left[\frac{L(-\infty)}{C(-\infty)}\right]^{1/4} \{\overleftarrow{r}^+(k)e^{-ikx(z)} - e^{ikx(z)}\}$$
$$\underset{z\to\infty}{\sim} \left[\frac{L(\infty)}{C(\infty)}\right]^{1/4} t^+(k)e^{ikx(z)}. \quad (3.22)$$

We see on Eqs. (3.19)–(3.22) that $r^+(k)$, $\overleftarrow{r}^+(k)$, and $t^+(k)$ represent also for the lines equation (1.4) the reflection coefficients to the right and to the left and the transmission coefficient for the frequency k $(k>0)$. Furthermore, recalling (1.2), (1.3), and (1.5), it is clear that

$$v(z) = (L(z)C(z))^{-1/2} \quad (3.23)$$

is the local wave velocity at point z and that $x(z)$ is the travel time of waves from the origin to the position z.

The ISP for the line is the construction of quantities connecting L, C, R, and G from the data of $S^+(k)$ $(k>0)$, $L(\infty)$, $L(-\infty)$, $C(-\infty)$, and $C(\infty)$. q^+, q^-, and iq_3 being real [see Eqs. (3.7) and (3.8)], one can prove that

$$\overline{S^+(k)} = S^+(-k), \quad k\in\mathbb{R}, \quad (3.24)$$

where $\overline{S^+(k)}$ is the complex conjugate matrix of $S^+(k)$. Therefore $S^+(k)$ $(k\in\mathbb{R})$ is determined by $S^+(k)$ $(k>0)$. $S^-(k)$

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($k\in\mathbb{R}$) is also determined because of the general identity

$$S^{\pm}(k)^t S^{\mp}(-k)=I, \tag{3.25}$$

where "t" means "transposed" and I is the 2×2 identity matrix. So the data $S^{+}(k)$ ($k>0$) imply the data $r^{+}(k)$ and $r^{-}(k)$ ($k\in\mathbb{R}$). For this reason we consider in what follows the ISP associated with (Z) $[q^{+}, q^{-}, q_3]$ from the data of $r^{+}(k)$ and $r^{-}(k)$ ($k\in\mathbb{R}$).

IV. REDUCTION OF THE ISP FOR (Z) $[q^+, q^-, q_3]$ TO THE ISP FOR (Z) $[\tilde q^+, \tilde q^-, \tilde q_3 = 0]$

Let us write the generalized Zakharov–Shabat equation $(Z)^{\pm}[q^{+}, q^{-}, q_3]$ [id. Eq. (3.9)] in the form

$$\left(\frac{d}{dx}\mp iq_3\sigma_3\right)Y^{\pm}+ik\sigma_3 Y^{\pm}=\begin{pmatrix}0 & q^{\pm}\\ q^{\mp} & 0\end{pmatrix}Y^{\pm}, \tag{4.1}$$

and let us notice that

$$\frac{d}{dx}\mp iq_3\sigma_3=M^{\mp}(x)\left(\frac{d}{dx}\right)\circ M^{\pm}(x), \tag{4.2}$$

$$\sigma_3=M^{\mp}(x)\sigma_3 M^{\pm}(x), \tag{4.3}$$

where

$$M^{\pm}(x)=\exp\left(\mp i\int_{\infty}^{x}q_3(y)\sigma_3\,dy\right)=\begin{pmatrix}\exp\left(\mp i\int_{\infty}^{x}q_3(y)\,dy\right) & 0\\ 0 & \exp\left(\pm i\int_{\infty}^{x}q_3(y)\,dy\right)\end{pmatrix}. \tag{4.4}$$

It is then easy to see that $\tilde Y^{\pm}$ defined by

$$\tilde Y^{\pm}=M^{\pm}(x)Y^{\pm}, \tag{4.5}$$

is a solution of the Zakharov–Shabat system $(Z)^{\pm}$ $[\tilde q^{+}, \tilde q^{-}, \tilde q_3=0]$ with

$$\tilde q^{\pm}=q^{\pm}\exp\left(\mp 2i\int_{\infty}^{x}q_3(y)\,dy\right). \tag{4.6}$$

The Jost solutions of $(Z)^{\pm}[q^{+}, q^{-}, q_3]$ and $(Z)^{\pm}[\tilde q^{+}, \tilde q^{-}, \tilde q_3=0]$ are connected by

$$\tilde F^{\pm}(k,x)=M^{\pm}(x)F^{\pm}(k,x), \tag{4.7}$$

$$\overleftarrow{\tilde F}^{\pm}(k,x)=\exp\left(\mp i\int_{-\infty}^{\infty}q_3(y)\,dy\right)M^{\pm}(x)\overleftarrow{F}^{\pm}(k,x). \tag{4.8}$$

From (3.12), (3.13), (4.7), and (4.8) it is easy to obtain the connection between the scattering data ($k\in\mathbb{R}$)

$$\tilde r^{\pm}(k)=r^{\pm}(k), \tag{4.9}$$

$$\overleftarrow{\tilde r}^{\pm}(k)=\exp\left(\pm 2i\int_{-\infty}^{\infty}q_3(y)\,dy\right)\overleftarrow{r}^{\pm}(k), \tag{4.10}$$

$$\tilde t^{\pm}(k)=\exp\left(\pm i\int_{-\infty}^{\infty}q_3(y)\,dy\right)t^{\pm}(k). \tag{4.11}$$

Note that this result is a particular case of a lemma used in Ref. 15. So we are led to solve the ISP for the Zakharov–Shabat system $(Z)^{\pm}[\tilde q^{+}, \tilde q^{-}, \tilde q_3=0]$ from the data of $\tilde r^{+}(k)=r^{+}(k)$ and $\tilde r^{-}(k)=r^{-}(k)$ for $k\in\mathbb{R}$. Because of Eqs. (4.6), (3.7), and (3.8), the potentials $\tilde q^{+}$ and $\tilde q^{-}$, solutions of this ISP, are connected to L, C, R, and G through Eq. (1.11).

We remark that it would be quite possible to introduce the ISP data for the lines [see formulas (3.19)–(3.22)] by using the Zakharov–Shabat system (Z)$[\tilde q^{+}, \tilde q^{-}, \tilde q_3=0]$ instead of the generalized Zakharov–Shabat system (Z) $[q^{+}, q^{-}, q_3]$ as done in Sec. III. The choice of the intermediate step (Z)$[q^{+}, q^{-}, q_3]$ may be justified by the following remarks:

—The equation $[q^{+}, q^{-}, q_3]$ is more directly connected with the lines equation (1.4) than the equation (Z) $[\tilde q^{+}, \tilde q^{-}, \tilde q_3=0]$ and is no more difficult to investigate;

—The potentials q^{+}, q^{-}, and iq_3 have the nice property of being real and thus lead to the nice relation (3.24).

V. SOLUTION OF THE ISP FOR (Z) $[\tilde q^+, \tilde q^-, \tilde q_3 = 0]$

The starting point is the following representation formula for the Jost solution $\tilde F^{\pm}(k,x)$ of $(Z)^{\pm}[\tilde q^{+}, \tilde q^{-}, \tilde q_3=0]$:

$$\tilde F^{\pm}(k,x)=\begin{pmatrix}0\\1\end{pmatrix}e^{ikx}+\int_{x}^{\infty}K^{\pm}(x,y)e^{iky}dy, \tag{5.1}$$

where the kernel

$$K^{\pm}(x,y)=\begin{pmatrix}K_1^{\pm}(x,y)\\ K_2^{\pm}(x,y)\end{pmatrix}$$

is such that

$$K_1^{\pm}(x,x)=-\frac{1}{2}\tilde q^{\pm}(x), \tag{5.2}$$

$$K_2^{\pm}(x,x)=\frac{1}{2}\int_{x}^{\infty}\tilde q^{+}(y)\tilde q^{-}(y)\,dy. \tag{5.3}$$

The insertion of (5.1) into (3.12) and the use of contour integration in the complex k plane yield the inversion equations

$$\sigma_1 K^{\pm}(x,y)=\begin{pmatrix}0\\1\end{pmatrix}p^{\mp}(x+y)+\int_{x}^{\infty}p^{\mp}(y+t)K^{\mp}(x,t)\,dt,\quad y\geqslant x,\quad x\in\mathbb{R}, \tag{5.4}$$

where $p^{+}(x)$ and $p^{-}(x)$ are the scalar functions connected to the data $\tilde r^{+}(k)$ and $\tilde r^{-}(k)$ ($k\in\mathbb{R}$) by

$$p^{\pm}(x)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde r^{\pm}(k)e^{ikx}dk \tag{5.5}$$

(we recall that we have assumed that there is no bound state). Therefore the steps of the solution of this ISP are:

(a) construct p^{+} and p^{-} from $\tilde r^{+}(k)$ and $\tilde r^{-}(k)$ ($k\in\mathbb{R}$) using Eq. (5.5);

(b) find the solution (K^{+}, K^{-}) of the system of integral equations (5.4);

(c) obtain $[\tilde q^{+}, \tilde q^{-}]$ from (K^{+}, K^{-}) using Eq. (5.2).

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