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On the advancements of conformal transformations and their associated symmetries in geometry and theoretical physics

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Dedicated to the memory of Julius Wess (1934–2007), colleague and friend for many years.

The historical developments of conformal transformations and symmetries are sketched: Their origin from stereographic projections of the globe, their blossoming in two dimensions within the field of analytic complex functions, the generic role of transformations by reciprocal radii in dimensions higher than two and their linearization in terms of polyspherical coordinates by Darboux, Weyl's attempt to extend General Relativity, the slow rise of finite dimensional conformal transformations in classical field theories and the problem of their interpretation, then since about 1970 the rapid spread of their acceptance for asymptotic and structural problems in quantum field theories and beyond, up to the current AdS/CFT conjecture.
The occasion for the present article: hundred years ago Bateman and Cunningham discovered the form invariance of Maxwell's equations for electromagnetism with respect to conformal space-time transformations.

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1 Introduction

1.1 The occasion

Hundred years ago, on September 21 of 1908, Hermann Minkowski (1864–1909) gave his famous talk on "Space and Time" at a congress in Cologne [1] in which he proposed to unify the traditionally independent notions of space and time in view of Einstein's (and Lorentz's) work to a 4-dimensional space-time with a corresponding metric

$$(x,x) = (ct)^2 - x^2 - y^2 - z^2 \equiv (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2, \;\; x = (x^0, x^1, x^2, x^3), \tag{1}$$

to what nowadays is called "Minkowski Space" M^4.

Only a few days later, on October 9, the London Mathematical Society received a paper [2] by Harry Bateman (1882–1946) in which he showed – among others – that the wave equation

$$\frac{1}{c^2}\partial_t^2 f(t,\vec{x}) - \Delta f(t,\vec{x}) = 0, \quad \Delta \equiv \partial_x^2 + \partial_y^2 + \partial_z^2\,, \;\; \vec{x} = (x,y,z), \tag{2}$$

is invariant under the (conformal) "inversion"

$$R: \qquad x^\mu \to (Rx)^\mu \equiv \hat{x}^\mu = \frac{x^\mu}{(x,x)}\,,\; \mu = 0,1,2,3, \tag{3}$$

in the following sense: If $f(x)$ is a solution of Eq. (2), then

$$\hat{f}(x) = \frac{1}{(x,x)}\, f(Rx), \;\; (x,x) \neq 0, \tag{4}$$

is a solution of the wave equation, too. Bateman generalized an important result from 1847 by William Thomson (Lord Kelvin) (1824–1907) (more details in Sect. 2.3 below) which said: If $h(\vec{x})$ is a solution of the Laplace equation

$$\Delta h(\vec{x}) = 0, \tag{5}$$

then

$$\hat{h}(\vec{x}) = \frac{1}{r}\, h(\vec{x}/r),\ r = (x^2 + y^2 + z^2)^{1/2}, \tag{6}$$

is a solution, too. In doing so Bateman introduced $w = ict$ and $r^2 = x^2 + y^2 + z^2 + w^2$. In a footnote on pp. 75–76 of his paper he pointed out that Maxwell's equations, as formulated by H.A. Lorentz (1853–1928), take a more symmetrical form if the variable ict is used. He does not mention Minkowski's earlier introduction of $x_4 = ict$ in his fundamental treatise on the electrodynamics of moving bodies [3], following the previous work by Lorentz, Poincaré (1854–1912) and Einstein (1879–1955), nor does Bateman mention Einstein's work. But he discusses "hexaspherical" coordinates as introduced by Darboux (see Sect. 2.4 below).

Bateman's paper led, after a few months, to two more by himself [4,5] and one by his colleague Ebenezer Cunningham (1881–1977) [6] in which the form (structure) invariance of Maxwell's electrodynamical equations – including non-vanishing charge and current densities and even special "ponderable bodies" – under conformal space-time transformations is established, as a generalization of the invariances previously discussed by Lorentz, Einstein and Minkowski.

Bateman's paper is more modern and more elegant in that he uses efficiently a precursor of differential forms (from 1-forms up to 4-forms) for his arguments.

In both papers there is no discussion of possible connections of the newly discovered additional form invariance of Maxwell's equation to new conservation laws. Here the remark is important that form invariance of differential equations with respect to certain transformations in general leads to new solutions (see, e.g. Eqs. (5) and (6)), but not necessarily to new conservation laws! See Sect. 4 for more details.

Bateman also speculated [4] that the conformal transformations may be related to accelerated motions, an issue we shall encounter again below (Sect. 5.2).

The "correlations" between the two authors of the papers [4] and [6] are not obvious, but the initiative appears to have been on Bateman's side: In a footnote on the first page of his paper Cunningham says: "This paper contains in an abbreviated form the chief parts of the work contributed by the author to a joint paper by Mr. Bateman and himself read at the meeting held on February 11th, 1909, and also the work of the paper by the author read at the meeting held on March 11th, 1909." And in a footnote on the third page Cunningham remarks: "This was pointed out to me by Mr. Bateman, a conversation with whom suggested the present investigation." Here Cunningham is refering to invariance of the wave equation under the transformation by reciprocal radii Bateman had investigated before [2]. In the essential part II of his paper Cunningham first gives the transformation formulae for the electric and magnetic *fields* with respect to the inversion (3) and says in a footnote on p. 89 of [6] that the corresponding formulae for the scalar and vector *potentials* were suggested to him by Bateman.

Bateman does not mention a joint paper with Cunningham which, as far as I know, was never published. He also read his paper [4] at the meeting of the Mathematical Society on March 11th. On the second page of his article [4] he says: "I have great pleasure in thanking Mr. E. Cunningham for the stimulus which he gave to this research by the discovery of the formulae of transformation in the case of an inversion in the four-dimensional space." And in his third paper [5] when he discusses transformation by reciprocal radii Bateman says: "Cunningham [6] has shown that any electrodynamical field may be transformed into another by means of this transformation."

So it is not clear who of the two – after Bateman's first paper [2] on the wave equation – had the idea or suggested to look for conformal invariance of Maxwell's electrodynamics, and why the initial joint paper was not published. Perhaps the archives of the London Mathematical Society can shed more light on this. From the publications one may conclude that Bateman found the transformations with respect to the inversion (3) for the potentials and Cunningham – independently – those for the fields!

Those papers by Bateman and Cunningham were the beginning of discussing and applying conformal transformations in modern physical field theories. But it took more than 50 years till the physical meaning of those conformal transformations became finally clarified and its general role in theoretical physics fully established. From about 1965/70 on conformal symmetries have been creatively and successfully exploited for many physical systems or their more or less strong idealizations. The emphasis of these notes – which are not complete at all – will be on different stages till about 1970 of that period and they will mention more recent developments more superficially, because there are many modern reviews on the topics of those activities.

1.2 The issue

Conformal transformations of geometrical spaces with a metric may appear in two different ways:

1.2.1 Conformal mappings as point transformations

Let $\mathcal{M}^n, n \geq 2$, be an n-dimensional Riemannian or pseudo-Riemannian manifold with local coordinates $x = (x^1, \dots, x^n)$ and endowed with a (pseudo)-Riemannian non-degenerate metric

$$g_x = \sum_{\mu,\nu=1}^{n} g_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu, \tag{7}$$

i.e. if

$$a = \sum_{\mu=1}^{n} a^\mu(x)\partial_\mu, \qquad b = \sum_{\nu=1}^{n} b^\nu(x)\partial_\nu, \tag{8}$$

are two tangent vectors at the point x, then they have the scalar product

$$g_x(a,b) = \sum_{\mu,\nu=1}^{n} g_{\mu\nu}(x)\, a^\mu\, b^\nu, \tag{9}$$

and the cosine of the angle between them is given by

$$\frac{g_x(a,b)}{\sqrt{g_x(a,a)}\,\sqrt{g_x(b,b)}}\,. \tag{10}$$

Let $\hat{\mathcal{M}}^n$ be a second corresponding manifold with local coordinates $\hat{x}^\mu$ and metric $\hat{g}_{\hat{x}}$. Then a mapping

$$x \in G \subset \mathcal{M}^n \to \hat{x} \in \hat{G} \subset \hat{\mathcal{M}}^n \tag{11}$$

is said to be *conformal* if

$$\hat{g}_{\hat{x}} = C(x)\, g_x, \quad C(x) \neq 0, \infty, \tag{12}$$

where the function $C(x)$ depends on the mapping. The last equation means that the angle between two smooth curves which meet at x is the same as the angle between the corresponding image curves meeting at the image point $\hat{x}$. Note that the mapping (11) does not have to be defined on the whole $\mathcal{M}^n$.

Two important examples:

I. Transformation by reciprocal radii
For the inversion (3) (mapping by "reciprocal radii" of the Minkowski space into itself) we have

$$(\hat{x}, \hat{y}) = \frac{(x,y)}{(x,x)\,(y,y)}, \tag{13}$$

and

$$\hat{g}_{\hat{x}} \equiv (d\hat{x}^0)^2 - (d\hat{x}^1)^2 - (d\hat{x}^2)^2 - (d\hat{x}^3)^2 = \frac{1}{(x,x)^2}\, g_x, \tag{14}$$
$$g_x = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2\,.$$

These equations show again that the mapping (3) is not defined on the light cone $(x,x) = 0$. We shall later see how this problem can be cured by adding points at infinity, i.e. by extending the domain of definition for the mapping (3).

It will be discussed in the next Sect. that there is an important qualitative difference as to conformal mappings of Euclidean or pseudo-Euclidean spaces $\mathbb{R}^n$ with a metric

$$(x,x) = \sum_{\mu,\nu=1}^{n} \eta_{\mu\nu}\, x^\mu x^\nu, \quad \eta_{\mu\nu} = \pm\delta_{\mu\nu}, \tag{15}$$

for $n = 2$ and for $n > 2$:
For $n = 2$ any holomorphic or meromorphic function

$$w = u + i\,v = f(z),\; z = x + iy \tag{16}$$

provides a conformal map of regions of the complex plane:

$$(du)^2 + (dv)^2 = |f'(z)|^2\,[(dx)^2 + (dy)^2]. \tag{17}$$

Here it is assumed that $f'(z) = df/dz$ does not vanish at z and that the Cauchy-Riemann eqs. hold (see Eq. (36) below). A map by such a holomorphic or meromorphic function also preserves the orientation of the angle. On the other hand, a corresponding antiholomorphic function $g(z^*),\; z^* = x - iy$, does preserve angles, too, but reverses their orientations.
One here can, of course, go beyond the complex plane to Riemann surfaces with more complicated structures.

For $n > 2$, however, conformal mappings constitute "merely" a $[(n+1)(n+2)/2]$-dimensional Lie transformation group which may be generated by the inversion

$$R:\; x^\mu \to (Rx)^\mu \equiv \hat{x}^\mu = \frac{x^\mu}{(x,x)},\;\; \mu = 1,\ldots,n > 2,\;\; R^2 = 1, \tag{18}$$

and the translations

$$T_n(b): \qquad x^\mu \to \hat{x}^\mu = x^\mu + b^\mu,\; b^\mu \in \mathbb{R},\;\; \mu = 1,\ldots,n. \tag{19}$$

II. Stereographic projections
A historically very important example is the stereographic projection of the surface S^2 of a sphere with radius a in $\mathbb{R}^3$ onto the plane (see Fig. 1):

Let the south pole of the sphere coincide with the origin $(x,y) = (0,0)$ of the plane and its north pole with the point $(x = 0, y = 0, \zeta = 2a) \in \mathbb{R}^3$. The projection is implemented by connecting the north pole

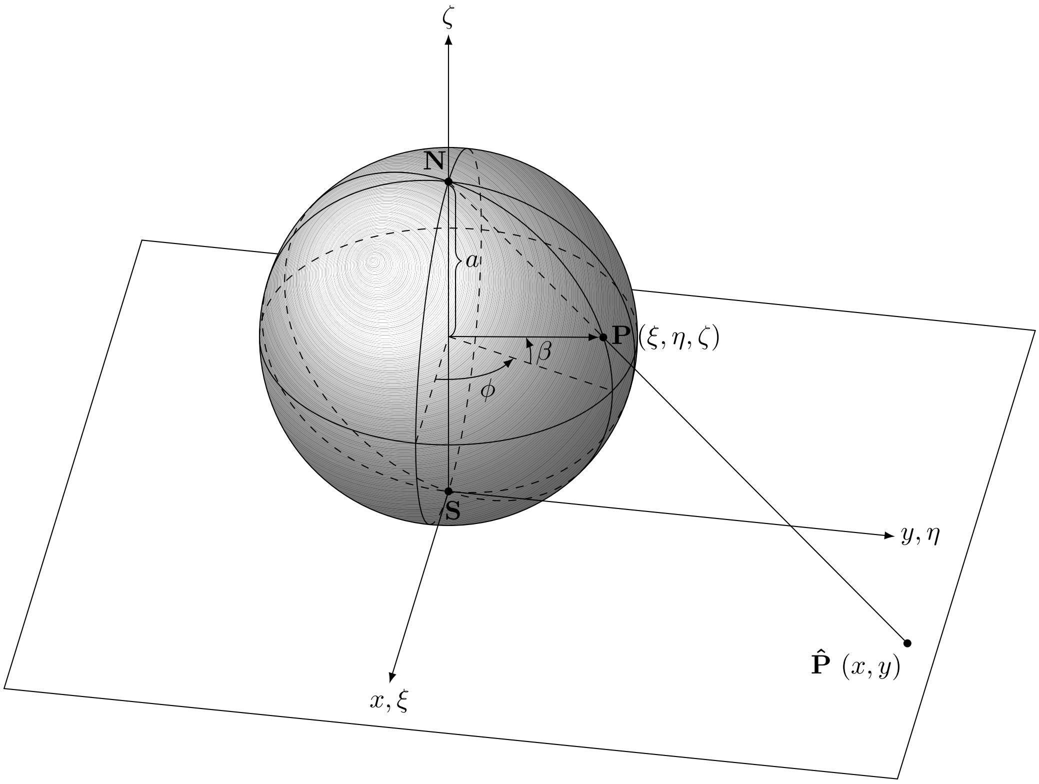

Fig. 1 Stereographic projection: The points P on the surface of a sphere with radius a are mapped onto points $\hat{P}$ in the plane – and vice versa – by drawing a straight line from the north pole N of the sphere through P towards $\hat{P}$. The mapping is conformal and arbitrary circles on the sphere are mapped onto circles or straight lines in the plane.

with points $\hat{P}(x,y)$ in the plane by straight lines which intersect the surface of the sphere in the points $P(\xi,\eta,\zeta)$ with $\xi^2+\eta^2+(\zeta-a)^2=a^2$. In this way the point $P(\xi,\eta,\zeta)$ of the sphere is mapped into the point $\hat{P}(x,y)$ of the plane.

Analytically the mapping is given by

$$x=\frac{2a\,\xi}{2a-\zeta}, \quad y=\frac{2a\,\eta}{2a-\zeta}, \quad \xi^2+\eta^2+\zeta^2-2a\,\zeta=0, \tag{20}$$

with the inverse map

$$\xi=\frac{4a^2\,x}{4a^2+x^2+y^2}, \quad \eta=\frac{4a^2\,y}{4a^2+x^2+y^2}, \quad \zeta=\frac{2a\,(x^2+y^2)}{4a^2+x^2+y^2}. \tag{21}$$

Note that the north pole of the sphere is mapped to “infinity” of the plane which has to added as a “point” in order to make the mapping one-to-one!

Parametrizing the spherical surface by an azimutal angle ϕ (“longitude”) in its equatorial plane $\zeta=a$ parallel to the (x,y)-plane, with the initial meridian ($\phi=0$) given by the plane $y=\eta=0$, and the angle β (“latitude”) between that plane and the position vector of the point (ξ,η,ζ), with respect to the centre of the sphere, with β positive on the northern half and negative on the southern half. We then have

$$\xi=a\,\cos\phi\,\cos\beta, \quad \eta=a\,\sin\phi\,\cos\beta, \quad \zeta-a=a\,\sin\beta, \tag{22}$$

and

$$x = \frac{2a\,\cos\phi\,\cos\beta}{1-\sin\beta}, \quad y = \frac{2a\,\sin\phi\,\cos\beta}{1-\sin\beta}, \quad \frac{\cos\beta}{1-\sin\beta} = \tan(\beta/2+\pi/4). \tag{23}$$

The last equations imply

$$g_{(x,y)} = (dx)^2 + (dy)^2 = \frac{4}{(1-\sin\beta)^2}\, g_{(\phi,\beta)}, \quad g_{(\phi,\beta)} = a^2[\cos^2\beta\,(d\phi)^2 + (d\beta)^2]. \tag{24}$$

Here $g_{(\phi,\beta)}$ is the standard metric on a sphere of radius a. Eq. (24) shows that the stereographic projection (23) is a conformal one, with the least distortions of lengths from around the south pole ($\sin\beta \geq -1$).

Besides being conformal, the stereographic projection given by the Eqs. (20) and (21) has the second important property that circles on the sphere are mapped onto the circles on the plane (where straight lines are interpreted as circles of infinite radii) and vice versa. This may be seen as follows: Any circle on the sphere can be generated by the intersection of the sphere with a plane

$$c_1\,\xi + c_2\,\eta + c_3\,\zeta + c_0 = 0. \tag{25}$$

Inserting the relations (21) with $2a = 1$ into this equation yields

$$(c_0 + c_3)(x^2 + y^2) + c_1\,x + c_2 y + c_0 = 0, \tag{26}$$

which for $c_0 + c_3 \neq 0$ describes the circle

$$(x + \tilde{c}_1/2)^2 + (y + \tilde{c}_2/2)^2 = \rho^2, \quad \tilde{c}_j = \frac{c_j}{c_0 + c_3},\ j = 0,1,2\,; \quad \rho^2 = (\tilde{c}_1^2 + \tilde{c}_2^2)/4 - \tilde{c}_0. \tag{27}$$

The coefficients c_j in Eq. (25) have to be such that the plane actually intersects or touches the plane. This means that $\rho^2 \geq 0$ in Eq. (27).

If $c_0 + c_3 = 0,\ c_0 \neq 0$, then the Eqs. (25) and (26) can be reduced to

$$\hat{c}_1\,\xi + \hat{c}_2\,\eta - \zeta + 1 = 0 \tag{28}$$

and

$$\hat{c}_1\,x + \hat{c}_2\,y + 1 = 0. \tag{29}$$

Here the plane (28) passes through the north pole $(0,0,1)$ and the image of the associated circle on the sphere is the straight line (29).

If $c_3 = c_0 = 0$ then the plane (25) contains a meridian and Eq. (26) becomes a straight line through the origin.

On the other hand the inverse image of the circle

$$(x-\alpha)^2 + (y-\beta)^2 = \rho^2 \tag{30}$$

is, according to the Eqs. (20), associated with the plane

$$2\alpha\,\xi + 2\beta\,\eta + (\alpha^2 + \beta^2 - \rho^2 - 1)\,\zeta + \rho^2 - \alpha^2 - \beta^2 = 0, \tag{31}$$

where the relation $\xi^2 + \eta^2 = \zeta\,(1-\zeta)$ has been used.

As the stereographic projection plays a very crucial role in the long history of conformal transformations, up to the newest developments, a few historical remarks are appropriate:

The early interest in stereographic projections was strongly influenced by its applications to the construction of the astrolabe – also called planisphaerium –, an important (nautical) instrument [7] which used a stereographic projection for describing properties of the celestial (half-) sphere in a plane. It may have been known already at the time of Hipparchos (ca. 185 – ca. 120 B.C.) [8]. It was definitely used for that purpose by Claudius Ptolemaeus (after 80 – about 160 A.D.) [9]. Ptolemaeus knew that circles are mapped onto circles or straight lines by that projection, but it is not clear whether he knew that *any* circle on the sphere is mapped onto a circle or a straight line. That property was proven by the astronomer and engineer Al-Farghānī (who lived in Bagdad and Cairo in the first half of the 9th century) [10] and independently briefly after 1200 by the European mathematician "Jordanus de Nemore", the identity of which appears to be unclear [11].

That the stereographic projection is also conformal was explicitly realized considerably later: In his book on the "Astrolabium" from 1593 the mathematician and Jesuit Christopher Clavius (1537–1612) showed how to determine the angle at the intersection of two great circles on the sphere by merely measuring the corresponding angle of their images on the plane [12]. This is equivalent to the assertion that the projection is conformal [13].

Then there is Thomas Harriot (1560–1621) who about the same time also showed – in unpublished and undated notes – that the stereographic projection is conformal. Several remarkable mathematical, cartographical and physical discoveries of this ingenious nautical adviser of Sir Walter Raleigh (ca. 1552–1618) were rediscovered and published between 1950 and 1980 [14]. During his lifetime Harriot published none of his mathematical insights and physical experiments [15]. His notes on the conformality of stereographic projections have been dated (not conclusively) between 1594 and 1613/14 [16], the latter date appearing more likely. So in principle Harriot could have known Clavius' Astrolabium [17].

In 1696 Edmond Halley (1656–1742) presented a paper to the Royal Society of London in which he proves the stereographic projection to be conformal, saying that Abraham de Moivre (1667–1754) told him the result and that Robert Hooke (1635–1703) had presented it before to the Royal Society, but that the present proof was his own [18].

1.2.2 Weyl's geometrical gauge transformation

A second way of implementing a conformal transformation for a Riemannian or pseudo-Riemannian manifold is the possibility of merely multiplying the metric form (7) by a non-vanishing positive smooth function $\omega(x) > 0$:

$$g_x \to \hat{g}_x = \omega(x)\, g_x. \tag{32}$$

More details for this type of conformal transformations, introduced by Hermann Weyl (1885–1955), are discussed below (Sect. 3).

The Eqs. (12) and (32) show that the corresponding conformal mappings change the length scales of the systems involved. As many *physical* systems have inherent fixed lengths (e.g. Compton wave lengths (masses) of particles, coupling constants with non-vanishing length dimensions etc.), applying the above conformal transformations to them in many cases cannot lead to genuine symmetry operations, like, e.g. translations or rotations. As discussed in more detail below, these limitations are one of the reasons for the slow advance of conformal symmetries in physics!

Here it is very important to emphasize the difference between transformations which merely change the coordinate frame and the analytical description of a system and those mappings where the coordinate system is kept fixed: in the former case the system under consideration, e.g. a hydrogen atom with its discrete and continuous spectrum, remains the same, only the description changes; here one may choose any macroscopic unit of energy or an equivalent unit of length in order to describe the system. However, in the case of mappings one asks whether there are other systems than the given one which can be considered as images of that initial system for the mapping under consideration. But now, in the case of dilatations, there is no continuous set of hydrogen atoms the energy spectra of which differ from the the original one by arbitrary scale transformations! For the existence of conservation laws the invariance with respect to mappings is crucial (see Sect. 4 below). These two types of transformations more recently have also been called “passive” and “active” ones [19].

2 Conformal mappings till the end of the 19th century

2.1 Conformal mappings of 2-dimensional surfaces

With the realization that the earth is indeed a sphere and the discoveries of faraway continents the need for maps of its surface became urgent, especially for ship navigation. Very important progress in cartography [20] was made by Gerhardus Mercator (1512–1594), particularly with his world map from 1569 for which he employed a conformal “cylindrical” projection [21], now named after him [22].

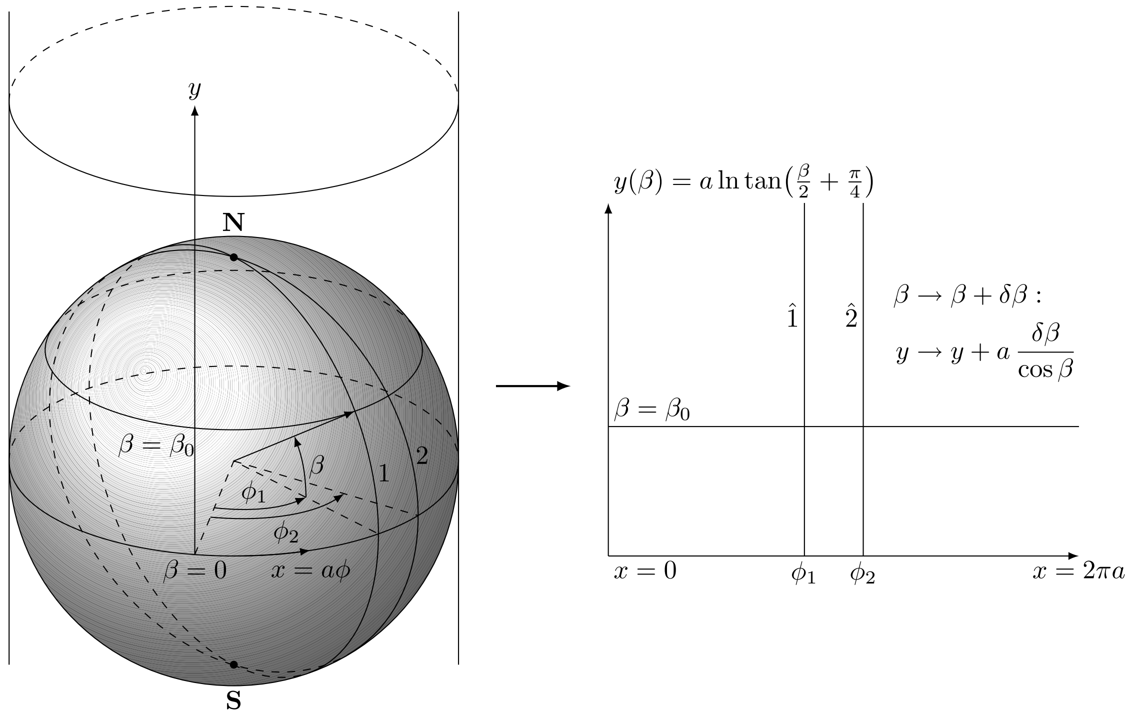

Fig. 2 Mercator projection: The points (longitude ϕ, latitude β) on the surface of a sphere with radius a are mapped on the mantle of a cylinder which touches the equator and is then unrolled onto the plane. The circles of fixed latitude β are mapped onto straight lines parallel to the x-axis, different meridians are mapped onto parallel lines along the y-axis, itself parallel to the cylinder axis. The mapping is characterized by the property that an increase $\delta\beta$ in latitude implies an increase $\delta y = a\,\delta\beta/\cos\beta$ on the cylinder mantle. This makes it a conformal one.

Here the meridians are projected onto parallel lines on the mantle of a cylinder which touches the equator of the sphere and which is unrolled onto a plane afterwards (see Fig. 2): Let ϕ and β have the same meaning as in example 2 of Sect. 1.2 above (longitude and latitude). If a again is the radius of the sphere, x the coordinate around the cylinder where it touches the equator, then $x = a\,\phi$. The orthogonal y-axis on the mantle of the cylinder, parallel to its axis, meets the equator at $\phi = 0 = x$, $\beta = 0 = y$. The mapping of the meridians onto parallels of the y-axis on the mantle is determined by the requirement that $\cos\beta\,\delta y = a\,\delta\beta$ for a small increment $\delta\beta$.

Thus the Mercator map is characterized by

$$\delta x = a\,\delta\phi, \quad \delta y = \frac{a}{\cos\beta}\,\delta\beta, \tag{33}$$

yielding

$$(dx)^2 + (dy)^2 = \frac{1}{\cos^2\beta}\, g_{(\phi,\beta)}, \quad g_{(\phi,\beta)} = a^2[\cos^2\beta\,(d\phi)^2 + (d\beta)^2], \tag{34}$$

which shows the mapping to be conformal, with strong length distortions near the north pole! (Integrating the differential equation $dy/d\beta = 1/\cos\beta$ gives $y(\beta) = \ln\tan(\beta/2 + \pi/4)$ with $y(\beta = 0) = 0$.)

A first pioneering explicit mathematical "differential" analysis of Mercator's projection, the stereographic projection and the more general problem of mapping the surface of a sphere onto a plane was published in 1772 [23] by Johann Heinrich Lambert (1728–1777): Lambert posed the problem which "global" projections of a spherical surface onto the plane are compatible with local (infinitesimal) requirements like angle-preserving or area-preserving, noting that both properties cannot be realized simultaneously! He showed that his differential conditions for angle preservation are fullfilled by Mercator's and the stereographic projection. In addition he presented a new "conical" – also conformal – solution, still known and used as "Lambert's projection" [24].

The term "stereographic projection" was introduced by the Belgian Jesuit and mathematician (of Spanish origin) François d'Aiguillon (1567–1617) in 1613 in the sixth and last part – dealing with projections ("Opticorum liber sextus de proiectionibus") – of his book on optics [25] which became also well-known for its engravings by the painter Peter Paul Rubens (1577–1640) at the beginning of each of the six parts and on the title page [26]! When introducing the 3 types of projections he is going to discuss (orthographic, stereographic and scenographic ones) d'Aiguillon says almost jokingly [27]: "... Second," [projection] "from a point of contact" [on the surface of the sphere], " which not improperly could be called stereographic: a term that might come into use freely, as long as no better one occurs, if you, Reader, allows for it". The readers did allow for it!

Only three years after the publication of Lambert's work Leonhard Euler (1707–1783) in 1775 presented three communications to the Academy of St. Petersburg (Russia) on problems concerning (cartographical) mappings from the surface of a sphere onto a plane, the first two being mainly mathematical. The papers were published in 1777 [28]. Euler approached the problem Lambert had posed from a more general point of view by looking for a larger class of solutions of the differential equations by using methodes he had employed previously in 1769 [29]. In his "Hypothesis 2" (first communication) Euler formulated the differential equations for the condition that small parts on the earth are mapped on similar figures on the plane ("Qua regiones minimae in Terra per similes figuras in plano exhibentur"), i.e. the mapping should be conformal. For obtaining the general solution of those differential equations Euler used complex coordinates $z = x + iy$ in the plane. This appears to be the first time that such a use of complex variables was made [30]. Euler further observed (second communication) that the mapping

$$z \to \frac{a\,z + b}{c\,z + d}, \quad z = x + iy, \tag{35}$$

which connects the different projections in the plane is a conformal one! Euler does not mention Lambert's work nor did Lambert mention Euler's earlier paper from 1769 on the construction of a family of curves which are orthogonal to the curves of a given family [29]. As Euler had supported a position for Lambert in Berlin in 1764 before he – Euler – left for St. Petersburg in 1766, this mutual silence is somewhat surprising.

Lambert mentions in his article that he informed Joseph-Louis de Lagrange (1736–1813) about the cartographical problems he was investigating. In 1779 Lagrange, who was in Berlin since 1766 as president of the Academy, presented two longer Memoires on the construction of geographical maps to the Berlin Academy of Sciences [31]. Lagrange says that he wants to generalize the work of Lambert and Euler and look for all projections which map circles on the sphere onto circles in the plane.

The more general problem of mapping a 2-dimensional (simply-connected) surface onto another one while preserving angles locally was finally solved completely in 1822 by Carl Friedrich Gauss (1777–1855) in a very elegantly written paper [32] in which he showed that the general solution is given by functions of complex numbers $q + ip$ or $q - ip$. As he assumes differentiability of the functions with respect to their complex arguments he – implicitly – assumes the validity of the Cauchy-Riemann differential Eqs. (36)!

Gauss does not use the term "conform" for the mapping in his 1822 paper, but he introduces it in a later one from 1844 [33]. The expression "conformal projection " appears for the first time as "proiectio conformis" in an article written in Latin and presented in 1788 to the St. Petersburg (Russia) Academy of Sciences by the German-born astronomer and mathematician Friedrich Theodor Schubert (1758–1825) [34]. It was probably the authority of Gauss which finally made that term "canonical"!

The development was brought to a certain culmination by Gauss' student Georg Friedrich Bernhard Riemann (1826–1866) who in his Ph.D. Thesis [35] from 1851 emphasized the important difference between global and local properties of 2-dimensional surfaces described by functions of complex variables and who formulated his famous version of Gauss' result (he quotes Gauss' article from 1822 at the beginning of his paper; except for a mentioning of Gauss' paper from 1827 [39] at the end, this is the only reference Riemann gives!), namely that every simply-connected region of the complex plane can be mapped (conformally) into the interior of the unit circle $|z| < 1$ by a holomorphic function [36].

After writing down the conditions (Cauchy – Riemann equations [37])

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{36}$$

for the uniqueness of differentiating a complex function $w = u + iv = f(z = x + iy)$ with respect to z, Riemann notes that they imply the second order [Laplace] equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{37}$$

Conformal mappings, of course, still play a central role for finding solutions of 2-dimensional Laplace equations which obey given boundary conditions in a vast variety of applications [38]. This brings us – slowly – back to the history of Eqs. (5) and (6):

2.2 On circles, spheres, straight lines and reciprocal radii

The history of conformal mappings described in the last Sect. represents the beginning of modern differential geometry – strongly induced by new cartographical challenges – which culminated in Gauss' famous paper from 1827 on the general theory of curved surfaces [39]. Another pillar of that development was the influential work of the French mathematician Gaspard Monge (1746–1818), especially by his book on the application of analysis to geometry [40]. Through his students he also influenced the subject to be discussed now:

About the time of 1825 several of those mathematicians which were more interested in the global purely *geometrical* relationships between circles and lines, spheres and planes and more complicated geometrical objects (so–called "synthetic" or "descriptive geometry" [41, 42], as contrasted to the – more modern – analytical geometry) discovered the mapping by reciprocal radii (then also called "inversion"):

A seemingly thorough, balanced and informative account of that period and the questions of priorities involved was given in 1933 by Patterson [43]. He missed, however, a crucial paper from 1820 by a 22 years old self-educated mathematician, who died only 5 years later. In view of the general scope of Patterson's work I can confine myself to a few additional illustrating, but crucial, remarks on the rather complicated and bewildering beginning of the concept "transformation by reciprocal radii":

There is the conjecture that the Swiss mathematician Jakob Steiner (1796–1863) was the first to know the mapping around the end of 1823 or the beginning of 1824. The corresponding notes and a long manuscript were found long after his death and not known when his collected papers where published [44]. In a first publication of notes from Steiner's literary estate in 1913 by Bützberger [45] it was argued that

Steiner knew the inversion at least in February 1824 and that he was the first one. In 1931 a long manuscript by Steiner on circles and spheres from 1825/1826 was finally published [46] which also shows Steiner's vast knowledge of the subject.
However, the editors of that manuscript, Fueter and Gonseth, say in their introduction that in January 1824 Steiner made extensive excerpts from a long paper by the young French mathematician J.B. Durrande (1798–1825), published in July 1820 [47]. From that they draw the totally unconvincing conlusion that Steiner knew already what he extracted from the journal! A look at Durrande's paper shows immediately that he definitely deserves the credit for priority [48]! Not much is known about this self-educated mathematician who died at the age of 27 years: From March 1815 till October 1825 twenty eight papers by Durrande were published in Gergonne's Annales [49], the last one after his death [50]. In a footnote on the title page of Durrande's first paper Gergonne remarks that the author is a 17 years old geometer who learnt mathematics only with the help of books [51].

Many of Durrande's contributions present solutions of problems which had been posed in the Journal previously, most of them by Gergonne himself. The important paper of July 1820 originated, however, from Durrande's own conceptions. In it he appears with the title " professeur de mathématiques, spéciales et de physique au collége royal de Cahors" and in his second last paper from November 1824 [52] as " professeur de physique au collége royal de Marseille" [53]. At the end of that paper and in his very last one, [50], Durrande again used the inversion, he had introduced before in his important paper from 1820.

The Annales de Gergonne were full of articles dealing with related geometrical problems. Steiner himself published 8 papers in volumes 18 (1827/28) and 19 (1828/29) of that journal.

Around 1825 the two Belgian mathematicians and friends Germinal Pierre Dandelin (1794–1847) and Lambert Adolphe Jacques Quetelet (1796–1874) were investigating very similar problems, presenting their results to the L'Académie Royale des Sciences et Belles-Lettres de Bruxelles [54]. At the end of a paper by Dandelin, presented on June 4 of 1825, there is the main formula of inversion [55] (see Eq. (39) below) and at the end of a longer paper by Quetelet [56], presented on November 5 of the same year, a 3-page note is appended which contains – probably for the first time in "analytical" form – the transformation formula

$$\hat{x} = \frac{r_0^2\, x}{x^2 + y^2}, \quad \hat{y} = \frac{r_0^2\, y}{x^2 + y^2}, \tag{38}$$

for an inversion on a circle with radius r_0.

The transformation (38) is also mentioned by Julius Plücker (1801–1868) in the first volume of his textbook from 1828 [57].

Other important early contributions to the subject (mostly ignored in the literature) are those of the Italian mathematician Giusto Bellavitis (1803–1880) in 1836 and 1838 [58].

Now back to the mathematics [59, 60]! The basic geometrical idea is the following (see Fig. 3): Given a circle with radius r_0 and origin O in the plane, draw a line from the origin to a point P *outside* the circle with a distance r from the origin O away. If a point $\hat{P}$ on the same line *inside* the circle and with the distance $\hat{r}$ form the origin obeys the relation

$$r\,\hat{r} = r_0^2, \tag{39}$$

then the point $\hat{P}$ is called "inverse" to P and vice versa.

The last equation obviously is a consequence of Eqs. (38). If now P traces out a curve then $\hat{P}$ describes an "inverse" curve, e.g. circles are mapped onto circles (see below). P and $\hat{P}$ were also called "conjugate". Such points have many interesting geometrical properties, e.g. if one draws a circle through the points P and $\hat{P}$, with its origin on the line connecting the two inverse points, then the new circle is orthogonal to the old one! For many more interesting properties of such systems see the textbooks mentioned in [59, 60].

If

$$B \equiv (x-\alpha)^2 + (y-\beta)^2 - \rho^2 = x^2 + y^2 - 2\alpha\, x - 2\beta\, y + C = 0, \;\; C = \alpha^2 + \beta^2 - \rho^2, \tag{40}$$

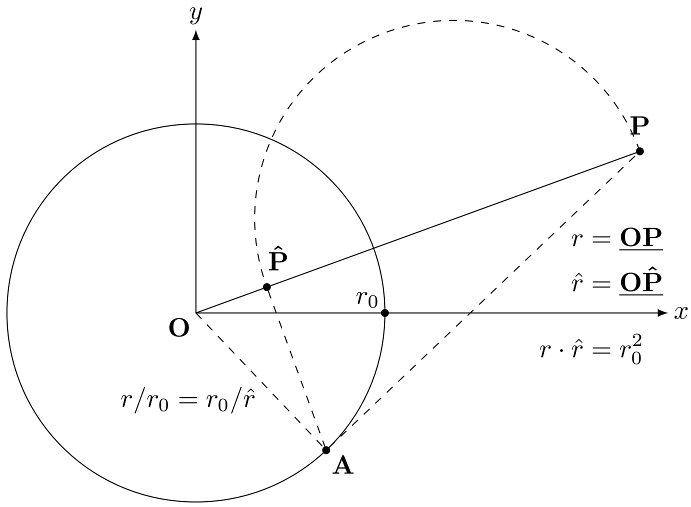

Fig. 3 Inversion on a circle with radius $\mathbf{r_0}$: A point P outside the circle with distance r from the center O is mapped onto a point $\hat{P}$ on the line $\underline{OP}$ with distance $\hat{r} = r_0^2/r$. A line from P tangent to the circle at A generates several similar rectangular triangles corresponding sides of which obey $r/r_0 = r_0/\hat{r}$. A circle through P and $\hat{P}$ with its origin on $\underline{\hat{P}P}$ is orthogonal to the original one.

is any circle in the plane with radius ρ, then the "inverse" circle $\hat{B}$ generated by the transformation (38) has the constants

$$\hat{\alpha} = \frac{r_0^2\,\alpha}{C}, \quad \hat{\beta} = \frac{r_0^2\,\beta}{C}, \quad \hat{C} = \frac{r_0^4}{C}, \quad \hat{\rho}^2 = \frac{r_0^4}{C^2}\,\rho^2. \tag{41}$$

Eqs. (38) show that the "inverse" of the origin $(x = 0, y = 0)$ is infinity! If a circle (40) passes through the origin, then $C = 0$ and it follows from the last of the Eqs. (41) that $\hat{\rho} = \infty$, i.e. the image of a circle passing through the origin is one with an infinite radius, that is a straight line! It follows from the Eqs. (38), (40) and $C = 0$ that this image straight line obeys the equation

$$\alpha\,\hat{x} + \beta\,\hat{y} - r_0^2/2 = 0. \tag{42}$$

On the other hand, a straight line given by

$$b_1\,x + b_2\,y + g = 0, \quad g \neq 0, \tag{43}$$

is mapped onto the circle

$$(\hat{x} + r_0^2\,b_1/2g)^2 + (\hat{y} + r_0^2\,b_2/2g)^2 = (r_0^2\,b_1/2g)^2 + (r_0^2\,b_2/2g)^2. \tag{44}$$

In order to have the mapping (38) one-to-one one has to add a *point* at infinity (not a straight line as in projective geometry!). The situation is completely the same as in the case of stereographic projections discussed in example II of Sect. 1.2.1 above. Thus, the set (totality) of circles and straight lines in the plane is mapped onto itself. In this framework points of the plane are interpreted as being given by circles with radius 0.

Later the mathematician August Ferdinand Möbius (1790–1868) called the joint sets of circles, straight lines and their mappings by reciprocal radii "Kreisverwandtschaften" (circle relations) [61]. Behind this notion is an implicit characterization of group theoretical properties which were only identified explicitly later when group theory for continuous transformation groups became established.

Analytically the "Kreisverwandtschaften" are characterized by the transformation formulae (35) (nowadays called "Möbius transformations"):

$$z \to \hat{z} = \frac{a\,z + b}{c\,z + d}, \quad z = \frac{d\,\hat{z} - b}{-c\,\hat{z} + a}, \quad a\,d - b\,c \neq 0, \tag{45}$$

implying

$$(d\,\hat{x})^2 + (d\,\hat{y})^2 = \frac{|a\,d - b\,c|^2}{|c\,z + d|^4}\,[(dx)^2 + (dy)^2]. \tag{46}$$

Multiplying numerators and denominators in Eq. (45) by an appropriate complex number one can normalize the coefficients such that

$$a\,d - b\,c = 1. \tag{47}$$

The last equation implies that 6 real parameters of the 4 complex numbers $a, \ldots, d$ are independent. In group theoretical language: the transformations (45) form a 6-dimensional group.

Important special cases are the linear transformations

$$z \to \hat{z} = a\,z + b, \tag{48}$$

consisting of (2-dimensional) translations $T_2[b]$, a (1-dimensional) rotation $D_1[\phi]$ and a (1-dimensional) scale transformation (dilatation) $S_1[\gamma]$:

$$T_2[b] :\ z \to z + b,\ b = b_1 + i\,b_2;\ \ D_1[\phi] : z \to e^{i\phi}z,\ \phi = \arg a;\ \ S_1[\gamma] : z \to e^{\gamma}z,\ e^{\gamma} = |a|. \tag{49}$$

Of special interest here is the discrete transformation

$$\bar{R} : \quad z \to \hat{z} = \frac{r_0^2}{z} = \frac{r_0^2}{x^2 + y^2}\,(x - i\,y). \tag{50}$$

This is the inversion (38) followed by a reflection with respect to the x-axis. Notice that the r.h. side $1/z$ is a *meromorphic* function on the complex plane with a pole at the point $z = 0$ that is mapped onto the "point" ∞ which has to be "joined" to the complex plane.

Another analytical implementation of the "Kreisverwandtschaften" is

$$z \to \hat{z} = \frac{a\,z^* + b}{c\,z^* + d}, \quad a\,d - b\,c = 1, \quad z^* = x - i\,y, \tag{51}$$

with an obvious corresponding expression for the relation (46).

The inversion (38) itself is given by

$$R : \quad \hat{z} = \frac{r_0^2}{z^*}, \quad z^* = x - i\,y. \tag{52}$$

Here the orientation of angles is inverted, contrary to the transformation (50).

The combination $C_2[\beta] = R \cdot T_2[\beta] \cdot R$, where $T_2[\beta]$ denotes the translations $z \to z + \beta$, $\beta = \beta_1 + i\,\beta_2$, yields

$$R \cdot T_2[\beta] \cdot R = C_2[\beta] : \quad z \to \hat{z} = \frac{z + \beta|z|^2}{1 + 2(\beta_1\,x + \beta_2\,y) + |\beta|^2\,|z|^2}, \tag{53}$$

which constitutes another 2-dimensional abelian subgroup, because $R^2 = 1$ and $T_2[\beta]$ is abelian.

It is instructive to see which transformation is induced on the sphere of radius a by the inverse stereographical projection (21) when applied to the inversion (52). We can write the Eqs. (21) as

$$\sigma = \xi + i\,\eta = \frac{4\,a^2\,z}{4\,a^2 + |z|^2}, \quad \zeta = \frac{2a\,|z|^2}{4a^2 + |z|^2}. \tag{54}$$

Taking for convenience $a = 1/2$ and $r_0 = 1$, we get for the inversion (52):

$$\sigma \to \hat{\sigma} = \frac{\hat{z}}{1+|\hat{z}|^2} = \frac{z}{1+|z|^2} = \sigma, \quad \zeta \to \hat{\zeta} = \frac{|\hat{z}|^2}{1+|\hat{z}|^2} = \frac{1}{1+|z|^2} = 1-\zeta, \tag{55}$$

i.e. the points (ξ, η, ζ) on the sphere are reflected on the plane $\zeta = 1/2$. The south pole of the sphere which corresponds to the origin $(x = 0, y = 0)$ of the plane is mapped onto the north pole which corresponds to the point ∞ of the plane. And vice versa.

Contrary to the non-linear transformation (52) the transformation (55) is a (inhomogeneous) linear and continuous one for the coordinates of the sphere. We shall see below that such a linearization is possible for all the transformation (45) or (51) by introducing appropriate homogeneous coordinates!

A few remarks on a more modern aspect: If $m(\tau),\ m(\tau = 0) = 1$ (group identity), denotes the elements of any of the above 6 real one-parameter transformation subgroups and $f(z = x + i\,y)$ a smooth function on the complex plane, then

$$\tilde{V} f(z) = \lim_{\tau\to 0} \frac{f[m(\tau)\,z] - f(z)}{\tau} \tag{56}$$

defines a vector field on the plane. In terms of the coordinates x and y these are for the individual groups

$$T_2[b]: \quad \tilde{P}_x = \partial_x, \ \ \tilde{P}_y = \partial_y; \tag{57}$$

$$D_1[\phi]: \quad \tilde{L} = x\,\partial_y - y\,\partial_x; \tag{58}$$

$$S_1[\gamma]: \quad \tilde{S} = x\,\partial_x + y\,\partial_y; \tag{59}$$

$$C_2[\beta]: \quad \tilde{K}_x = (y^2 - x^2)\,\partial_x - 2x\,y\,\partial_y, \ \ \tilde{K}_y = (x^2 - y^2)\,\partial_y - 2x\,y\,\partial_x. \tag{60}$$

These vector fields form a Lie algebra which is isomorphic to the real Lie algebra of the Möbius group:

$$[\tilde{P}_x, \tilde{P}_y] = 0, \tag{61}$$

$$[\tilde{L}, \tilde{P}_x] = -\tilde{P}_y, \quad [\tilde{L}, \tilde{P}_y] = \tilde{P}_x; \tag{62}$$

$$[\tilde{S}, \tilde{P}_x] = -\tilde{P}_x, \quad [\tilde{S}, \tilde{P}_y] = -\tilde{P}_y; \tag{63}$$

$$[\tilde{S}, \tilde{L}] = 0\,; \tag{64}$$

$$[\tilde{L}, \tilde{K}_x] = -\tilde{K}_y, \quad [\tilde{L}, \tilde{K}_y] = \tilde{K}_x; \tag{65}$$

$$[\tilde{S}, \tilde{K}_x] = \tilde{K}_x, \quad [\tilde{S}, \tilde{K}_y] = \tilde{K}_y; \tag{66}$$

$$[\tilde{P}_x, \tilde{K}_x] = [\tilde{P}_y, \tilde{K}_y] = -2\,\tilde{S}; \tag{67}$$

$$[\tilde{P}_x, \tilde{K}_y] = -[\tilde{P}_y, \tilde{K}_x] = 2\,\tilde{L}. \tag{68}$$

Here we have already several of the essential structural elements of conformal groups we shall encounter later:

1. The dilatation operator $\tilde{S}$ determines the dimensions of length of the operators $\tilde{P}_i$, $\tilde{L}$ and $\tilde{K}_i, i = x, y$, namely $[L^{-1}]$, $[L^0]$ and $[L^1]$ as expressed by the Eqs. (63), (64) and (66).
2. The Eqs. (67) and (68) show that the Lie algebra generators $\tilde{L}$ and $\tilde{S}$ can be obtained from the commutators of $\tilde{P}_i$ and $\tilde{K}_i,\ i = x, y$. But we know from Eq. (53) that

   $$\tilde{K}_i = R \cdot \tilde{P}_i \cdot R, \tag{69}$$

   which means that the whole Lie algebra of the Möbius group can be generated from the translation generators $\tilde{P}_i$ and the inversion R alone!

3. We further have the relations

$$R \cdot \tilde{S} \cdot R = -\tilde{S}, \quad R \cdot \tilde{L} \cdot R = \tilde{L}. \tag{70}$$

These properties indicate the powerful role of the discrete transformation R, mathematically and physically! An appropriate name for R would be "length inversion (operator)"!

Many properties of the inversion (38) for the plane were investigated for the 3-dimensional space, too, by the authors mentioned above (Durrande, Steiner, Plücker, ..., Möbius etc.), without realizing, however, that in 3 dimensions the transformation by reciprocal radii was essentially the only non-linear conformal mapping, contrary to the complex plane and its extensions to Riemann surfaces with their wealth of holomorphic and meromorphic functions. This brings us to the next Sect. :

2.3 William Thomson, Joseph Liouville, Sophus Lie, other mathematicians and James Clerk Maxwell

Prodded by his ambitious father, the mathematics professor James Thomson (1786–1849), in January of 1845 the young William Thomson (1824–1907) – later Baron Kelvin of Largs – spent four and a half months in Paris in order to get acquainted, study and work with the well-known mathematicians there [62]. His best Paris contacts Thomson had with Joseph Liouville (1809–1882) whose protégé he became [63]. Back in Cambridge, in October 1845 Thomson wrote Liouville a letter in which he proposed to use the relation (39) for a sphere of radius r_0 in oder to solve certain (boundary) problems in electrostatics, refering to discussions the two had in Paris. Excerpts from that letter were published immediately by Liouville in the journal he edited [64]. In June and September 1846 Thomson sent two more letters excerpts of which Liouville published in 1847 [65], directly followed by a long commentary by himself [66]. In the first of these letters Thomson introduced the mapping

$$R: \; x \to \xi = \frac{x}{x^2+y^2+z^2}, \; y \to \eta = \frac{y}{x^2+y^2+z^2}, \; z \to \zeta = \frac{z}{x^2+y^2+z^2}, \tag{71}$$

and pointed out that the function $\hat{h}(\vec{x}) = h(\vec{x}/r)/r$, $r = (x^2+y^2+z^2)^{1/2}$ is a solution of the Laplace equation (5), if $h(\vec{x})$ is a solution. In his commentary Liouville discussed in detail several properties of the mapping (71) and gave it the name "transformation *par rayons vecteurs réciproques*, relativement à l'origine O" (italics by Liouville), from which the usual expression "transformation by reciprocal radii" derives.

Afterwards Liouville made the important discovery that the transformation (71), combined with translations, is actually the only generic conformal transformation in $\mathbb{R}^3$, contrary to the situation in the plane [67]!

Liouville's result kindled a lot of fascination among mathematicians and brought quite a number of generalizations and new proofs:

At the end of a paper by Sophus Lie (1842–1899), presented by A. Clebsch in April 1871 to the Royal Society of Sciences at Göttingen, Lie concluded that the orthogonal transformations and those by reciprocal radii belong to the most general ones which leave the quadratic form

$$\sum_{\nu=1}^{n} (dx_\nu)^2 = 0 \tag{72}$$

invariant [68]. He does not mention the condition $n > 2$ nor does he quote Liouville's proof for $n = 3$. In a long paper from October and November 1871 Lie gave a different proof for Liouville's theorem (which he quotes now) and points out in a footnote that the results of his paper [68] imply a corresponding generalization for arbitrary $n > 2$ [69]. He provided the details of the proof for $n > 2$ in an article from 1886 [70] and in volume III of his "Theorie der Transformationsgruppen" [71] from 1893. In the meantime other proofs for the general case $n > 2$ had appeared: one by a German secondary school teacher, R. Beez [72], and another one by Gaston Darboux (1842–1917) [73]. For a more modern one see [74].

For the case $n = 3$ there are about a dozen new proofs of Liouville's theorem till around 1900 [75–86].

Most important, however, for the influence of Thomson's work on the physics community was that James Clerk Maxwell (1831–1879) devoted a whole chapter in his "Treatise" to applications of the inversion – combined with the notion of virtual electric images – in electrostatics [87]. Maxwell's high opinion of Thomson's work is also evident from his review (in Nature) [88] of the reprint volume of Thomson's papers [65]. Maxwell says there:
" ... Thus Thomson obtained the rigorous solution of electrical problems relating to spheres by the introduction of an imaginary electrified system within the sphere. But this imaginary system itself next became the subject of examination, as the result of the transformation of the external electrified system by reciprocal *radii vectores*. By this method, called that of electrical inversion, the solution of many new problems was obtained by the transformation of problems already solved. ... If, however, the mathematicians were slow in making use of the physical method of electric inversion, they were more ready to appropriate the geometric idea of inversion by reciprocal *radii vectores*, which is now well known to all geometers, having been, we suppose, discovered and re-discovered repeatedly, though, unless we are mistaken, most of these discoveries are later than 1845, the date of Thomson's paper. ..." [89].

2.4 Gaston Darboux and the linear action of the conformal group on "polyspherical" coordinates

We now come to a *global* aspect of the action of the conformal group which plays a major role in the modern development of its applications (see Subsects. 7.2 and 7.4 below): We have already seen that the mapping (38) sends the origin of the plane to infinity and vice versa. Similar to what is being done in projective geometry where one adds an "imaginary" straight line at infinity one now adds a point at infinity in order to have the mapping (38) one-to-one. Topologically this means that one makes the non-compact plane to a compact 2-dimensional surface S^2 of the sphere. This is implemented by the stereographic projection (21). As the projection is conformal it preserves an essential part of the Euclidean metric structure of the plane, e.g. orthogonal curves on the sphere are mapped onto orthogonal curves in the plane. In addition the non-linear transformations (52) become linear on S^2 if one introduces homogeneous coordinates in the plane and in the associated space $\mathbb{R}^3$ in which the sphere S^2 is embedded, i.e. the conformal transformations act continuously on S^2. This does not seem to be very exciting for the plane and the sphere S^2, but it becomes important for the Minkowski space (1) where the inversion (3) is singular on the 3-dimensional light cone $(x, x) = 0$. But the essential ingredients of the idea can already be seen in the case of the plane and the sphere S^2 which also shows the close relationship between stereographic projections and mappings be reciprocal radii!

2.4.1 Tetracyclic coordinates for the compactified plane

In a short note from 1869 Darboux pointed out [90] that one could generate a system of orthogonal curvilinear coordinates in the plane by projecting them stereographically from a given system on the surface of a sphere in space. More generally, he observed that properties of an $\mathbb{R}^{n-1}$ could be dealt with by considering the corresponding properties on the $(n-1)$-dimensional surface S^{n-1} of a sphere in an $\mathbb{R}^n$. He discussed the details for $n = 3, 4$ in later publications, especially in his monograph of 1873 [91]. The following is a brief summary of the main ideas, using also later textbooks on the subject [59, 60, 92, 93]:

First one introduces homogeneous coordinates on $\mathbb{R}^2$ and $\mathbb{R}^3$:

$$x = y^1/k,\ y = y^2/k, \quad (y^1, y^2, k) \neq (0, 0, 0); \tag{73}$$
$$\xi = \eta^1/\kappa,\ \eta = \eta^2/\kappa,\ \zeta = \eta^3/\kappa, \quad (\eta^1, \eta^2, \eta^3, \kappa) \neq (0, 0, 0, 0).$$

Mathematically the new homogeneous coordinate k is just a real number which – in addition – can be given an obvious physical interpretation [94, 95]: As the coordinates x and y have the dimension of length, one can interpret k as providing the length scale by giving it the dimension $[L^{-1}]$ so that the coordinates y^1 and y^2 are dimensionless. For the sphere from Eq. (20) we get now

$$\begin{aligned} Q(\vec{\eta},\,\kappa) &\equiv (\eta^1)^2+(\eta^2)^2+(\eta^3)^2-2(a\,\kappa)\,\eta^3 = \\ &= (\eta^1)^2+(\eta^2)^2-2\eta^3\,\chi \equiv Q(\vec{\eta},\,\chi)=0, \\ &\quad \chi = a\,\kappa-\eta^3/2, \quad \vec{\eta}=(\eta^1,\eta^2,\eta^3). \end{aligned} \tag{74}$$

Here a corresponding physical dimensional interpretation of κ is slightly more complicated as the system has already the intrinsic fixed length a : Now – like k in the plane – the carrier of the dimension of an inverse length is the coordinate χ from Eqs. (74). This follows immediately from the transformation formulae (20) which may be written as

$$\sigma\,y^1=\eta^1, \quad \sigma\,y^2=\eta^2, \quad \sigma\,(a\,k)=\chi, \quad \sigma\neq 0. \tag{75}$$

Here σ is an arbitrary non-vanishing real number which drops out when the ratios in Eqs. (20) or (73) are formed. It follows that

$$\eta^3=\sigma\,\frac{(y^1)^2+(y^2)^2}{2a\,k}, \quad a\,\kappa=\sigma\left(a\,k+\frac{(y^1)^2+(y^2)^2}{4a\,k}\right). \tag{76}$$

We could also start from the Eqs. (21) and get

$$\begin{aligned} \rho\,\xi=4(a\,k)\,y^1, \quad \rho\,\eta=4(a\,k)\,y^2, \quad \rho\,\zeta=2[(y^1)^2+(y^2)^2], \\ \rho\,(a\,\kappa)=4(a\,k)^2+(y^1)^2+(y^2)^2, \quad \rho\neq 0. \end{aligned} \tag{77}$$

The two formulations coincide for $\rho\,\sigma=4(a\,k)$.

We see that we can characterize the points in the plane – including the "point" ∞ – by 3 ratios of 4 homogeneous coordinates which in addition obey the bilinear relation

$$Q(\vec{\eta},\,\chi)=0. \tag{78}$$

It follows from Eqs. (73) that the point ∞ lies on the projective straight line $k=0$. According to Eqs. (75) and (74) this implies $\eta^3/\kappa=2a$ and $(\eta^1,\eta^2)=(0,0)$, i.e. the coordintes of the north pole.

The action of the different subgroups of the Möbius group as discussed in Sect. 2.2 on the homogeneous coordinates (73) may be described as follows:

The *scale transformation*

$$S_1[\gamma]: \quad z\to z'=e^{\gamma}\,z \tag{79}$$

can be implemented by

$$\tau\,y^{1\,\prime}=y^1, \quad \tau\,y^{2\,\prime}=y^2, \quad \tau\,k'=e^{-\gamma}\,k, \tag{80}$$

where τ again is an arbitrary real number $\neq 0$. As y^1 and y^2 are dimensionless and k has the dimension of an inverse length, properties which should not be changed by the transformation, we put $\tau=1$. As to such a choice of the, in principle, arbitrary real number $\tau\neq 0$ see below. It then follows from Eqs. (75) that

$$\chi\equiv a\,\kappa-\eta^3/2\;\to\;\chi'=e^{-\gamma}\,\chi, \tag{81}$$

i.e. the combination χ has the dimension of an inverse length. We have $Q(\vec{\eta}\,',\,\chi')=Q(\vec{\eta},\,\chi)$.

For the *translation*

$$T_2[b_1]: \quad x\to x'=x+b_1, \quad y\to y'=y, \tag{82}$$

we obtain accordingly

$$\tau\, y^{1\,\prime} = y^1 + b_1\, k, \quad \tau\, y^{2\,\prime} = y^2, \quad \tau\, k' = k. \tag{83}$$

Again taking $\tau = 1$ we get

$$\eta^1 \to \eta^{1\,\prime} = \eta^1 + (b_1/a)\,\chi, \quad \eta^2 \to \eta^{2\,\prime} = \eta^2, \tag{84}$$

$$\eta^3 \to \eta^{3\,\prime} = (b_1/a)\,\eta^1 + \eta^3 + (b_1/a)^2/2\,\chi, \quad \chi \to \chi' = \chi. \tag{85}$$

This transformation also leaves the quadratic form $Q(\vec{\eta},\,\chi)$ invariant: $Q(\vec{\eta}\,',\,\chi') = Q(\vec{\eta},\,\chi)$.
The translation $x \to x,\ y \to y + b_2$ can be treated in the same way.

For the *inversion*

$$R: \quad x \to x' = \frac{r_0^2\, x}{x^2 + y^2}, \quad y \to y' = \frac{r_0^2\, y}{x^2 + y^2}, \tag{86}$$

one obtains

$$y^{1\,\prime} = y^1, \quad y^{2\,\prime} = y^2, \quad k' = \frac{(y^1)^2 + (y^2)^2}{r_0^2\, k}. \tag{87}$$

This yields

$$R: \quad \eta^{1\,\prime} = \eta^1, \quad \eta^{2\,\prime} = \eta^2, \quad y^{3\,\prime} = \frac{r_0^2}{2\,a^2}\,\chi, \quad \chi' = \frac{2\,a^2}{r_0^2}\,\eta^3. \tag{88}$$

We again have $Q(\vec{\eta}\,',\,\chi') = Q(\vec{\eta},\,\chi)$. Invariance of Q under rotations in the (x, y)–plane and the corresponding (ξ, η)–plane, with $k,\ \kappa$ and ζ fixed, is obvious.

Introducing the coordinates

$$\xi^1 = \eta^1, \quad \xi^2 = \eta^2, \quad \xi^3 = \frac{1}{\sqrt{2}}(\chi + \eta^3), \quad \xi^0 = \frac{1}{\sqrt{2}}(\chi - \eta^3), \tag{89}$$

implies

$$Q(\vec{\eta},\,\chi) = Q(\xi, \xi) = (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 - (\xi^0)^2. \tag{90}$$

Thus, we see that the 6-dimensional conformal group of the plane – Möbius group (45) with the normalization (47) – is isomorphic to the 6-dimensional pseudo-orthogonal (“Lorentz”) group $O(1, 3)/\mathbb{Z}_2$ (division by $\mathbb{Z}_2\ : \xi \to \xi$ or $-\xi$, because the coordinates ξ are homogeneous ones). The inversion R, Eq. (86), e.g. is implemented by the “time reversal” $\xi^0 \to -\xi^0$! As ξ is equivalent to $-\xi$, “time reversal” here is equivalent to “space reflection”: $\xi^0 \to \xi^0,\ \xi^j \to -\xi^j,\ j = 1,\, 2,\, 3$.

The homogeneous coordinates $\vec{\eta},\ \kappa$ of Eq. (73) or any linear combination of them, e.g. (89), were called “*tetracyclic*” coordinates of the points in the plane (the point ∞ included), i.e. “four-circle” coordinates (from the Greek words “tetra” for four and “kyklos” for circle) [59, 60, 92, 93]. The geometrical background for this name is the following: We have seen above, Eq. (25), that a circle on the sphere may be characterized by the plane passing through the circle. In homogeneous coordinates Eq. (25) becomes

$$c_1\,\eta^1 + c_2\,\eta^2 + c_3\,\eta^3 + c_0\,\kappa = 0. \tag{91}$$

The four different planes $\eta^1 = 0$; $\eta^2 = 0$; $\eta^3 = 0$ or $\kappa = 0$ correspond to four circles in the plane (which may have radius ∞ , i.e. they are straight lines). On the other hand, let, in the notation of Eq. (40),

$$B_j \equiv (x - \alpha_j)^2 + (y - \beta_j)^2 - \rho_j^2 = x^2 + y^2 - 2\alpha_j\, x - 2\beta_j\, y + C_j = 0, \;\; j = 1, 2, 3, 4, \tag{92}$$

be four different and arbitrary circles in the plane, each of which is determined by three parameters $\alpha_j,\ \beta_j$ and C_j or the radius ρ_j. Any other circle $B = 0$ in the plane can be characterized by the relation

$$B = \sum_{j=1}^{4} \eta^j \, B_j = 0, \quad (\eta^1, \eta^2, \eta^3, \eta^4) \neq (0,0,0,0), \tag{93}$$

which constitute 4 homogeneous equations for the 3 inhomogeneous parameters of the fifth circle. Its radius squared ρ^2 becomes proportional to a bilinear form of the homogeneous coordinates η^j. If the new circle (93) is a point, i.e. $\rho = 0$, then the η^j obey a quadratic relation like (74) or (90) (with $Q(\xi, \xi) = 0$). This is the geometrical background for the term "tetracyclic" coordinates for points in the plane. It is a variant of the term "pentaspherical" coordinates originally introduced by Darboux in the corresponding case of characterizing points in 3-dimensional space in terms of five (Greek: "penta") homogeneous coordinates which obey a bilinear relation [96].

2.4.2 Polyspherical coordinates for the extended $\mathbb{R}^n$, $n \geq 3$

Let x^μ, $\mu = 1, \ldots, n$ be the cartesian coordinates of an $\mathbb{R}^n$ with the bilinear form (15). Then, without refering to an explicit $(n+1)$-dimenional geometrical background, so-called "polyspherical" coordinates $y^\mu,\ \mu = 1, \ldots, n,\ k$ and q, can be introduced by

$$x^\mu = y^\mu / k, \quad (y, y) - k\, q = 0. \tag{94}$$

Here k has the dimension of an inverse length, and q that of a length.

The conformal transformations in such a $\mathbb{R}^n$ consists of n translations $T_n[b]$, $n\,(n-1)/2$ pseudo-rotations $D_{n(n-1)/2}[\phi_\nu]$, one scale transformation $S_1[\gamma]$, n "special conformal" transformations of the type (53): $C_n[\beta] = R \cdot T_n[\beta] \cdot R$ and discrete transformations like R etc. Combined these make a transformation group of dimension $(n+1)(n+2)/2$.

If the bilinear form (15) is a "lorentzian" one,

$$(x, x) = (x^0)^2 - (x^1)^2 - \cdots - (x^{n-1})^2, \tag{95}$$

then the associated bilinear form (94) is

$$Q(y, y) = (y^0)^2 + (y^{n+1})^2 - (y^1)^2 - \cdots - (y^n)^2, \quad k = y^n + y^{n+1},\ q = y^n - y^{n+1}. \tag{96}$$

In this case one would properly speak of "poly-hyperboloidical", and for $n = 4$ of "hexa-hyperboloidical" coordinates ("hexa": Greek for six)!

The conformal group of the n-dimensional Minkowski space now coresponds to the group $O(2, n)/\mathbb{Z}_2$. Its global structure and that of the manifold $Q(y, y) = 0$ will be discussed in Sect. 7.2 below. As already discussed for the plane, the division by $\mathbb{Z}_2$ comes from the fact that one can multiply the homogeneous coordinates y^ν in Eq. (96) by an arbitrary real number $\rho \neq 0$ without affecting the coordinates x^μ in Eq. (94).

If one now wants to discuss conformally invariant or covariant differential equations ("field equations") of functions $F(y)$ on the manifold $Q(y, y) = 0$ one has to take into account the homogeneity of the coordinates y in Eq. (94) and the condition $(y, y) - k\, q = 0$. The work on this task was started by Darboux in the case of potential theory [97], extended by Pockels and Bôcher [92,93], later discussed by Paul Adrien Maurice Dirac (1902–1984) [98] and more recently by other authors, e.g. [95, 99–101]. As the subject is more technical I refer to those papers and reviews for details.

3 Einstein, Weyl and the origin of gauge theories

3.1 Mathematical beauty versus physical reality and the far-reaching consequences

In November 1915 Einstein had presented the final version of his relativistic theory of gravitation in the mathematical framework of Riemannian geometry. Here the basic geometrical field quantities are the coefficients $g_{\mu\nu}(x),\ \mu,\ \nu = 0, 1, 2, 3$ of the metric form (summation convention)

$$(ds)^2 = g_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu. \tag{97}$$

The local lenghts ds are assumed to be determined by physical measuring rods and clocks (made of atoms, molecules etc.). A basic assumption of Riemannian geometry applied to gravity is that the physical units defined by those instruments are locally the same everywhere and at all time, independent of the gravitational fields present: we assume that hydrogen atoms etc. and their energy levels locally do not differ from each other everywhere in our cosmos and have not changed during its history.

In 1918 Hermann Weyl proposed to go beyond this assumption in order to incorporate electromagnetism and its charge conservation (for a more elaborate account of the following see the recent reviews [102–105]): in Riemannian geometry *parallel transport* of a vector ("yardstick") $a = a^\mu \partial_\mu$ does *not change its length* when brought from the point $P(x)$ to a neighbouring point $P(x + \delta x)$. This means that $\delta[g_{\mu\nu}(x)\, a^\mu a^\nu] = 0$ (the covariant derivative of $g_{\mu\nu}$ vanishes, here formally characterized by $\delta g_{\mu\nu} = 0$).

Weyl now allows for a geometrical structure in which infinitesimal *parallel transport* of a vector can result in a change of length, too, which is characterized by the postulate that this change is given by

$$\delta g_{\mu\nu}(x) = A(x)\, g_{\mu\nu}(x),\ \ A(x) = A_\mu(x)\, \delta x^\mu. \tag{98}$$

For the Christoffel symbols of the first kind (which determine the parallel transport) this leads to the modification

$$\Gamma_{\lambda,\mu\nu} + \Gamma_{\mu,\lambda\nu} = \partial_\nu g_{\lambda\mu} + g_{\lambda\mu}\, A_\nu; \tag{99}$$

$$\Gamma_{\lambda,\mu\nu} = \frac{1}{2}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) + \frac{1}{2}(g_{\lambda\mu}\, A_\nu + g_{\lambda\nu}\, A_\mu - g_{\mu\nu}\, A_\lambda).$$

The relation (98) may be rephrased as follows: Let l be the "physical" length $l = ds$ (Eq. (97)) of a vector $a = a^\mu \partial_\mu$ at $P(x)$. If a is parallel transported to a neighbour point $P(x+\delta x)$ then the change of its length l is given by

$$\delta l = l\, A, \tag{100}$$

which vanishes in Riemannian geometry. If one parallel transports a vector of length l_{P_1} from P_1 along a curve to P_2, then integration of Eq. (100) gives the associated change

$$l_{P_2} = l_{P_1}\, e^{\int_{P_1}^{P_2} A}. \tag{101}$$

Here the integral is path-dependent if not all $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ vanish (Stokes' theorem).

Multiplying the metrical coefficients $g_{\mu\nu}(x)$ by a scale factor $\omega(x) > 0$ leads to the joint transformations:

$$g_{\mu\nu}(x) \to \omega(x)\, g_{\mu\nu}(x), \quad A \to A - \delta\omega/\omega = A - \delta(\ln\omega),\ \ \delta\omega = \partial_\mu \omega(x)\, \delta x^\mu. \tag{102}$$

All this strongly suggests to identify (up to a constant) the four $A_\mu(x)$ with the electromagnetic potentials and the transformation (102) with a gauge transformation affecting simultaneously both, gravity and electromagnetism. Making a special choice for ω Weyl called "Eichung" (= "gauge"). In this way the term entered the realm of physics [106].

On March 1, 1918, Weyl wrote a letter to Einstein announcing a forthcoming paper on the unification of gravity and electromagnetism and asking whether the paper could be presented by Einstein to the Berlin Academy of Sciences [107]. Einstein reacted enthusiastically on March 8 and promised to present Weyl's paper [108]. After receiving it he called it (on April 6) "a first rank stroke of a genius", but that he could not get rid of his "Massstab-Einwand" (measuring rod objection) [109], probably alluding to discussions the two had at the end of March in Berlin during Weyl's visit.

This was the beginning of a classical controversy over mathematical beauty versus physical reality! On April 8 Einstein wrote "apart from its agreement with reality it is in any case a superb achievement of thought" [110], and again on April 15: "As beautiful as your idea is, I have to admit openly that according to my view it is impossible that the theory corresponds to nature" [111]. Einstein's first main objection concerned the relation (101): In the presence of electromagnetic fields $F_{\mu\nu}$ two "identical" clocks could run differently after one of them was moved around on a closed path! Einstein communicated his physical objections in a brief appendix to Weyl's initial paper he presented to the Academy in May 1918 [112]. The lively exchange between Einstein and Weyl continued till the end of the year, with Weyl trying hard to persuade Einstein. To no avail: on Sept. 27 Einstein wrote: "How I think with regards to reality you know already; nothing has changed that. I know how much easier it is to persuade people, than to find the truth, especially for someone, who is such an unbelievable master of depiction like you." [113]. In a letter from Dec. 10 Weyl said, disappointed: " So I am hemmed in between the belief in your authority and my insight. ... I simply cannot otherwise, if I am not to walk all over my mathematical conscience" [114]. Einstein's answer from Dec. 16 is quite conciliatory: "I can only tell you that all I talked to, from a mathematical point of view spoke with the highest admiration about your theory and that I, too, admire it as an edifice of thoughts. You don't have to fight, the least against me. There can be no question of anger on my side: Genuine admiration but unbelief, that is my feeling towards the matter." [115]. Einstein's other main objection concerned the relation (99) which determines geodetic motions: it implies that an uncharged particle would nevertheless be influenced by an electromagnetic field!

Einstein was right as far as gravity and classical electrodynamics is concerned. But Weyl's idea found an unexpected rebirth and modification in the quantum theory of matter [102–105]: In 1922 Erwin Schrödinger (1887–1961) observed – by discussing several examples – that the Bohr-Sommerfeld quantization conditions are compatible with Weyl's gauge factor (101) if one replaces the real exponent by the imaginary one

$$\frac{i\,e}{\hbar}\int A, \qquad A = A_\mu dx^\mu, \tag{103}$$

where the $A_\mu(x)$ are now the usual electromagnetic potentials [116]. In 1927 Fritz London (1900–1954) reinterpreted Weyl's theory in the framework of the new wave mechanics [117]: like Schrödinger, whom he quotes, London replaces the real exponent in Weyl's gauge factor (101) by the expression (103) and assumes that a length l_0 when transported along a closed curve in a nonvanishing electromagnetic field acquires a phase change

$$l_0 \to l = l_0\, e^{(ie/\hbar)\int A}, \tag{104}$$

without saying why a length could become complex now. He then argues – in a way which is difficult to follow – that

$$\psi(x)/l(x) = |\psi|/l_0 = \text{const.}, \tag{105}$$

where $\psi(x)$ is a wave function which now posesses the phase factor from Eq. (104).

In two impressive papers from 1929 [118] and 1931 [119] Weyl himself revoked his approach from 1918 and reinterpreted his gauge transformations in the new quantum mechanical framework as implemented by

$$\psi \to e^{i\,e\,f/\hbar}\,\psi, \quad A_\mu \to A_\mu + \partial_\mu f, \quad \partial_\mu \to \partial_\mu - \frac{i\,e}{\hbar}\,A_\mu, \quad f(x) = -\ln\omega(x), \tag{106}$$

giving credit to Schrödinger and London in the second paper [119]. Especially this second paper with its conceptually brilliant and broad analysis as to the importance of geometrical ideas in physics and mathematics provided the basis for the great future of gauge theories in physics and that of fiber bundles in mathematics.

Also stimulated by Weyl's idea, another interesting attempt to unify gravitation and electromagnetism was that of Theodor Kaluza (1885–1954) who started from a 5–dimensional Einsteinian gravity theory with a compactified 5th dimension [120]. As to further developments (O. Klein and others) of this approach see [102].

3.2 Conformal geometries

Despite its (preliminary) dead end in physics, Weyl's ideas were of considerable interest in mathematical differential geometry. Weyl discussed them in several articles [121] and especially, of course, in his textbooks [122]. An important new notion in these geometries was that of the (conformal) *weight* e of geometrical quantities $Q(x,\, g_{\mu\nu})$ like tensors or tensor densities which depend on the $g_{\mu\nu}$ and their derivatives [123]: The covariant metric tensor has the weight 1, thus

$$\text{if}\;\; g_{\mu\nu}(x) \to \omega(x)\, g_{\mu\nu}(x), \;\; \text{then} \quad Q(x,\, \omega\, g_{\mu\nu}) = \omega^{e}(x)\, Q(x,\, g_{\mu\nu}). \tag{107}$$

Only quantities of weight $e = 0$ are conformal invariants.
Even Einstein contributed to this kind of geometry [124], defined by the invariance of the bilinear form

$$g_{\mu\nu}(x)\, dx^{\mu} \otimes dx^{\nu} = 0, \tag{108}$$

which also characterizes light rays and their associated causal cones. Einstein was interested in the relationship between "Riemann-tensors" and "Weyl-tensors". Most of the mathematical developments of these conformal geometries are summarized in the second edition of a textbook by Jan Arnoldus Schouten (1883–1971) who himself made substantial contributions to the subject [125].

For a concise summary of conformal transformations in the sense of Weyl and their role in a modern geometrical framework of General Relativity and associated field equations see, e.g. [126].

3.3 Conformal infinities

We have seen in Sect. 2.2 that it can have advantages to map the "point" ∞ and its neighbourhoods into a finite one – either by a reciprocal radii transformation in the plane or by a stereographic projection onto the sphere S^2 –, if one wants to investigate properties of geometrical quantities near ∞. Similarly, Weyl's conformal transformations (107) have been used to develop a sophisticated analysis of the asymptotic behaviour of space-time manifolds, especially for those which are asymptotically flat [127], also possibly with a change of topological properties. They even have become an important tool for the *numerical* analysis of black holes physics etc. [128].

Furthermore, as the rays of electromagnetic and gravitational radiation obey the relation (108) and as these rays form the boundaries ("light cones") between causally connected and causally disconnected regions, Weyl's conformal transformations play also an important role in the causal analysis of space-time structures [129].

4 Emmy Noether, Erich Bessel-Hagen and the (partial) conservation of conformal currents

4.1 Bessel-Hagen's paper from 1921 on the conformal currents in electrodynamics

I indicated already in Sect. 1.1 that the form invariance of Maxwell's equations with respect to conformal space-time transformations as discovered by Bateman and Cunningham does not necessarily imply new conservation laws. This point was clarified in 1921 in an important paper by Erich Bessel-Hagen (1898–1946):

In July 1918 Felix Klein (1849–1925) had presented Emmy Noether's (1882–1935) seminal paper with her now two famous theorems on the consequences of the invariance of an action integral either under an r-dimensional continuous (Lie) group or under an "infinite"-dimensional (gauge) group the elements of which depend on r arbitrary functions [130, 131]. The former leads to r conservation laws (first theorem), whereas the latter entails r identities among the Euler-Lagrange expressions for the field equations (second theorem, e.g. the 4 Bianchi identities as a consequence of the 4 coordinate diffeomorphisms).

In the winter of 1920 Klein encouraged Bessel-Hagen to apply the first theorem to the conformal invariance of Maxwell's equations as discovered by Bateman and Cunningham.

Bessel-Hagen's paper [132] contains a number of results which are still generic examples for modern applications of the theorems: He first generalized Noether's results by not requiring the invariance of $L\,dx^1\cdots dx^m$ inside the action integral, but by allowing for an additional total divergence $\partial_\mu b^\mu$ which is also linear in the infinitesimal group parameters or (gauge) functions. Because of the importance of Noether's first theorem let me briefly summarize its content:
Suppose the differential equations for n fields $\varphi^i(x),\ i = 1,\ldots,n,\ x = (x^1,\ldots,x^m)$, are obtained from an action integral

$$A = \int_G dx^1\cdots dx^m\, L(x;\varphi^i,\,\partial_\mu\varphi^i). \tag{109}$$

Let

$$x^\mu \to \hat{x}^\mu = x^\mu + \delta x^\mu\,;\ \varphi^i(x) \to \hat{\varphi}^i(\hat{x}) = \varphi^i(x) + \delta\varphi^i = \varphi^i(x) + \tilde{\delta}\varphi^i + \partial_\mu\varphi^i\,\delta x^\mu, \tag{110}$$

be infinitesimal transformations which imply

$$\begin{aligned}\delta A &= \int_{\hat{G}} d\hat{x}^1\cdots d\hat{x}^m\, L[\hat{x};\hat{\varphi}^i(\hat{x}),\hat{\partial}\hat{\varphi}^i(\hat{x})] - \int_G dx^1\cdots dx^m\, L[x;\varphi^i(x),\partial\varphi^i(x)]\\ &= \int_G dx^1\cdots dx^m\,[E_i(\varphi)\,\tilde{\delta}\varphi^i - \partial_\mu j^\mu(x;\,\varphi,\,\partial\varphi;\delta x,\,\delta\varphi)],\end{aligned} \tag{111}$$

$$E_i(\varphi) \equiv \frac{\partial L}{\partial\varphi^i} - \partial_\mu\frac{\partial L}{\partial(\partial_\mu\varphi^i)}, \tag{112}$$

$$j^\mu = T^\mu{}_\nu\,\delta x^\nu - \frac{\partial L}{\partial\varphi^i}\,\delta\varphi^i + b^\mu(x;\varphi,\partial\varphi;\delta x,\delta\varphi),\ \ T^\mu{}_\nu = \frac{\partial L}{\partial(\partial_\mu\varphi^i)}\,\partial_\nu\varphi^i - \delta^\mu_\nu\,L. \tag{113}$$

From $\delta A = 0$ and since the region G is arbitrary we get the general variational identity

$$\partial_\mu j^\mu = -E_i(\varphi)\,\tilde{\delta}\varphi^i. \tag{114}$$

If

$$\delta x^\mu = X^\mu_\rho(x)\,a^\rho,\quad \delta\varphi^i = \Phi^i{}_\rho(x,\varphi)\,a^\rho,\quad b^\mu = B^\mu_\rho(x,\varphi,\partial\varphi)\,a^\rho,\ |a^\rho| \ll 1,\ \rho = 1,\ldots,r, \tag{115}$$

where the a^ρ are r independent infinitesimal group parameters, then one has r conserved currents

$$j^\mu_\rho(x) = T^\mu{}_\nu\,X^\nu_\rho - \frac{\partial L}{\partial(\partial_\mu\varphi^i)}\,\Phi^i{}_\rho + B^\mu_\rho,\ \rho = 1,\ldots,r, \tag{116}$$

i.e. we have

$$\partial_\mu j^\mu_\rho(x) = 0,\ \rho = 1,\ldots,r, \tag{117}$$

for solutions $\varphi^i(x)$ of the field equations $E_i(\varphi) = 0,\ i = 1, \ldots, n$.

As a first application Bessel-Hagen shows that such an additional term b occurs for the n-body system in classical mechanics if one wants to derive the uniform center of mass motion from the 3-dimensional special Galilean group $\vec{x}_j \to \vec{x}_j + \vec{u}\,t,\ j = 1, \ldots, n$.

In electrodynamics (4 space-time dimensions) he starts from the free Lagrangean density

$$L = -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \frac{\partial L}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}, \quad E^\nu(A) = \partial_\mu F^{\mu\nu}. \tag{118}$$

Requiring the 1-form $A_\mu(x)dx^\mu$ to be invariant implies

$$\delta A_\mu = -\partial_\mu(\delta x^\nu)\,A_\nu(x). \tag{119}$$

Notice that only those δx^μ and δA_μ are of interest here which leave the quantity $L\,dx^0dx^1dx^2dx^3$ invariant (up to a total divergence) with L from Eq. (118). For the (infinitesimal) gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu f(x), \quad \delta A_\mu = \partial_\mu f(x), \quad |f(x)| \ll 1, \quad f(x) = \ln\omega(x), \tag{120}$$

Noether's second theorem gives the identity $\partial_\nu E^\nu(A) = \partial_\nu\partial_\mu F^{\mu\nu} = 0$ (which implies charge conservation if $E^\nu(A) = j^\nu(x)$). The "canonical" energy-momentum tensor

$$T^\mu{}_\nu = -F^{\mu\kappa}\partial_\nu A_\kappa + \frac{1}{4}\,\delta^\mu_\nu F^{\kappa\lambda}F_{\kappa\lambda} \tag{121}$$

is not symmetric and not gauge invariant. Its relation to the symmetrical and gauge invariant energy-momentum tensor $\Theta^{\mu\nu}$ is given by

$$T^{\mu\nu} = \Theta^{\mu\nu} - F^{\mu\kappa}\partial_\kappa A^\nu, \quad \Theta^{\mu\nu} = F^{\mu\kappa}F_\kappa{}^\nu + \frac{1}{4}\,\eta^{\mu\nu}F^{\kappa\lambda}F_{\kappa\lambda}, \tag{122}$$

where $\eta_{\mu\nu}$ represents the Lorentz metric from Eq. (1).
Combining the variations (119) and (120) with the relation (122) here gives for the current (113) ($b^\mu = 0$)

$$j^\mu = \Theta^\mu{}_\nu\delta x^\nu + F^{\mu\nu}\partial_\nu(f - A_\lambda\delta x^\lambda). \tag{123}$$

As $f(x)$ is an arbitrary function, Bessel-Hagen argues, we can choose the gauge

$$f(x) = A_\nu\,\delta x^\nu \tag{124}$$

for any given δx^ν, so that now

$$j^\mu = \Theta^\mu{}_\nu\delta x^\nu, \tag{125}$$

which is invariant under another transformation (120). For a gauge (124) the variation $\tilde{\delta}A_\mu$ (see Eq. (110)) takes the form

$$\tilde{\delta}A_\mu = F_{\mu\nu}\delta x^\nu, \tag{126}$$

so that finally

$$\partial_\mu(\Theta^\mu{}_\nu\delta x^\nu) = E^\mu(A)F_{\mu\nu}\delta x^\nu. \tag{127}$$

For non-vanishing charged currents j^μ one has

$$E^\nu(A) = \partial_\mu F^{\mu\nu} = j^\nu, \quad j^\mu F_{\mu\nu} = -F_{\nu\mu}j^\mu = -f_\nu, \tag{128}$$

where f_ν is the (covariant) relativistic force density. Eq. (127) can therefore also be written as

$$\partial_\mu(\Theta^\mu{}_\nu \delta x^\nu) = -f_\nu \delta x^\nu. \tag{129}$$

Now let $S^{\mu\nu}$ be a symmetrical *mechanical* energy-momentum tensor such that

$$\partial_\mu S^{\mu\nu} = f^\nu, \quad S^{\mu\nu} = S^{\nu\mu}, \tag{130}$$

then Eq. (129) may be rewritten as

$$\partial_\mu[(\Theta^\mu{}_\nu + S^\mu{}_\nu)\delta x^\nu] = S^\mu{}_\nu \partial_\mu(\delta x^\nu). \tag{131}$$

This is an important equation from Bessel-Hagen's paper.

As $\delta x^\nu =$ const. for tranlations and $\delta x^\mu = \omega^\mu{}_\nu x^\nu$, $\omega_{\nu\mu} = -\omega_{\mu\nu}$, for homogeneous Lorentz transformations, one sees immediately that the associated currents are conserved for the full system. The situation is more complicated for the currents associated with the scale transformations and the 4 special conformal transformations corresponding to those in Eq. (53):

$$S_1[\gamma] : x^\mu \to \hat{x}^\mu = e^\gamma\, x^\mu, \quad \mu = 0,1,2,3, \tag{132}$$

$$\delta x^\mu = \gamma\, x^\mu, \quad |\gamma| \ll 1. \tag{133}$$

$$C_4[\beta] : x^\mu \to \hat{x}^\mu = [\, x^\mu + (x,x)\,\beta^\mu]/\sigma(x;\beta), \quad \mu = 0,1,2,3, \tag{134}$$

$$\sigma(x;\beta) = 1 + 2(\beta,x) + (\beta,\beta)\,(x,x)\,;$$

$$\delta x^\mu = (x,x)\,\beta^\mu - 2\,(\beta,x)\,x^\mu, \; |\beta^\mu| \ll 1. \tag{135}$$

From the infinitesimal scale transformation (133) one obtains

$$\partial_\mu s^\mu(x) = S^\mu{}_\mu, \quad s^\mu(x) = (\Theta^\mu{}_\nu + S^\mu{}_\nu)x^\nu. \tag{136}$$

And the four currents associated with the transformations (135) obey the relations

$$\partial_\mu k^\mu_\rho(x) = 2\,x_\rho\, S^\mu{}_\mu, \quad k^\mu_\rho(x) = (\Theta^\mu{}_\nu + S^\mu{}_\nu)[2\,x^\nu x_\rho - (x,x)\delta^\nu{}_\rho], \; \rho = 0,1,2,3. \tag{137}$$

Thus, for a vanishing electromagnetic current density $j^\mu = 0\,(S_{\mu\nu} = 0)$ the five currents (136) and (137) are conserved, but for $j^\mu \neq 0$ this is only the case if the trace $S^\mu{}_\mu$ vanishes. In general this does not happen! This was observed by Bessel-Hagen, too. Let me give two simple examples (with $c = 1$): For a charged relativistic point particle with (rest) mass m one may take

$$S^{\mu\nu}(x) = m \int_{-\infty}^{+\infty} d\tau\, \dot{z}^\mu \dot{z}^\nu\, \delta^4[x - z(\tau)], \tag{138}$$

where $z(\tau)$ describes the orbit of the particle in Minkowski space. $S^{\mu\nu}(x)$ has the properties

$$\partial_\mu S^{\mu\nu} = m \int_{-\infty}^{+\infty} d\tau\, \ddot{z}^\nu\, \delta^4[x - z(\tau)] = f^\nu(x), \quad S^\mu{}_\mu = m \int_{-\infty}^{+\infty} d\tau\, \delta^4[x - z(\tau)]. \tag{139}$$

Thus, for a non-vanishing mass the trace does not vanish and the five currents (136) and (137) are not conserved.

For a relativistic ideal fluid with invariant energy density $\epsilon(x)$ and invariant pressure $p(x)$ one has

$$S^{\mu\nu}(x) = [\epsilon(x) + p(x)]u^\mu(x)u^\nu(x) - p(x)\eta^{\mu\nu}, \tag{140}$$

which has the trace

$$S^{\mu}{}_{\mu} = \epsilon(x) - 3\,p(x), \tag{141}$$

which in general does not vanish either. It does so for a gas of massless particles and approximately so for massive particles at extremely high energies.

If the vector field $A_\mu(x)$ is coupled to a (conserved) current $j^\mu(x)$ then the Lagrange density

$$L = -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu \tag{142}$$

yields the field equations $\partial_\mu F^{\mu\nu} = j^\nu$ and the canonical energy-momentum tensor is now

$$T^{\mu}{}_{\nu} = -F^{\mu\kappa}\partial_\nu A_\kappa + \delta^\mu_\nu\left(\frac{1}{4}\,F^{\kappa\lambda}F_{\kappa\lambda} + j^\mu A_\mu\right). \tag{143}$$

Its divergence is

$$\partial_\mu T^{\mu}{}_{\nu} = (\partial_\nu j^\lambda)A_\lambda, \tag{144}$$

i.e. an *external* current in general leads to "violation" of energy and momentum conservation for the *electromagnetic* subsystem. This is just a different version of Bessel-Hagen's analysis from above.

Nevertheless, the field equations $\partial_\mu F^{\mu\nu} = j^\nu$ and the expression $j^\mu A_\mu\, dx^0 dx^1 dx^2 dx^3$ are invariant under the transformations (133) and (135) if (see Eq. (119))

$$\delta A_\mu = -\gamma\,A_\mu, \quad \delta j^\mu = -3\,\gamma\,j^\mu; \tag{145}$$

$$\begin{aligned} \delta A_\mu &= 2\,(\beta,x)\,A_\mu + 2[\beta_\mu(x,A) - x_\mu(\beta,A)], \\ \delta j^\mu &= 6\,(\beta,x)\,j^\mu + 2[\beta^\mu(x,j) - x^\mu(\beta,j)], \end{aligned} \tag{146}$$

but this does not lead to new conservation laws if the current is an external one. Only if j^μ is composed of dynamical fields, e.g. like the Dirac current $j^\mu = \bar\psi\gamma^\mu\psi$, conservation laws for the total system may exist! (See also Sect. 6.3).

4.2 Invariances of an action integral versus invariances of associated differential equations of motion

A simple but illustrative example for the form invariance of an equation of motion without an additional conservation law is the following:

A year before he presented E. Noether's paper to the Göttingen Academy Felix Klein raised another interesting question concerning "dynamical" differential equations, their symmetries and conservation laws: In 1916 Klein had asked Friedrich Engel (1861–1941), a long-time collaborator of S. Lie, to derive the 10 known classical conservation laws of the gravitational n-body problem by means of the 10-parameter Galilei Group, using group theoretical methods applied to differential equations as developed by Lie. Engel did this by using Hamilton's equations and the invariance properties of the canonical 1-form $p_j dq^j - H dt$ [133]. Thus, he essentially already used the invariance of Ldt !
Then Klein noticed that the associated *equations of motion*, e.g.

$$m\,\frac{d^2\vec{x}}{d\,t^2} = -G\,\frac{\vec{x}}{r^3}, \quad r = |\vec{x}|, \tag{147}$$

are also invariant under the transformation

$$\vec{x} \to \vec{x}\,' = \lambda^2\,\vec{x}, \quad t \to t' = \lambda^3\,t, \quad \lambda = \text{const.}\,, \tag{148}$$

and he, therefore, asked Engel in 1917 whether this could yield a new conservation law. Engel's answer was negative [134]. This had already been noticed in 1890 by Poincaré in his famous work on the 3-body problem [135].

Nevertheless, such joint scale transformations of space and time coordinates like (148) may be quite useful, as discussed by Landau and Lifshitz in their textbook on mechanics [136]. In this context they also mention the virial theorem: If the potential $V(\vec{x})$ is homogeneous in $\vec{x}$ of degree k and allows for bounded motions such that $|\vec{x}\cdot\vec{p}| < M < \infty$, then one gets for the *time averages* of the kinetic energy $T = \vec{v}\cdot\vec{p}/2 = [d(\vec{x}\cdot\vec{p})/dt - \vec{x}\cdot \text{grad}V(\vec{x})]/2$ and the potential energy V:

$$\langle T\rangle = \frac{k\,E}{k+2}, \quad \langle V\rangle = \frac{2\,E}{k+2}. \tag{149}$$

These relations break down for $k = -2$, i.e. for potentials $V(\vec{x})$ homogeneous of degree -2. Here genuine scale invariance comes in (not mentioned by Landau and Lifshitz!): For such a potential the expression $L\,dt = (T - V)\,dt$ is invariant with respect to the transformation

$$\vec{x} \to \vec{x}\,' = \lambda\,\vec{x}, \quad t \to t' = \lambda^2\,t. \tag{150}$$

This implies the conservation law

$$S = 2\,E\,t - \vec{x}\cdot\vec{p} = \text{const.}\,. \tag{151}$$

Thus, if $E \neq 0$, the term $\vec{x}\cdot\vec{p} = 2Et - S$ cannot be bounded in the course of time.

For such a potential there is another conservation law, namely

$$K = 2\,E\,t^2 - 2\,\vec{x}\cdot\vec{p}\,t + m\,\vec{x}^{\,2} = \text{const.}\,, \tag{152}$$

which, together with the constant of motion (151), determines $r(t)$ without further integration. The quantity K can be derived, according to Noether's method, from the infinitesimal transformations

$$\delta\vec{x} = 2\,\alpha\,\vec{x}\,t, \quad \delta t = 2\,\alpha\,t^2,\ |\alpha| \ll 1, \tag{153}$$

which leave $L\,dt$ invariant up to a total derivative term $b\,dt,\ b = m\,d(\vec{x}^{\,2})/dt$. Using Poisson brackets the three constants of motion $H = T + V$, S and K form the Lie algebra of the group $SO(1,2)$ [137].

That n-body potentials V which are homogeneous of degree -2 with repect to their spatial coordinates have two additional conservation laws of the type (151) and (152) was already discussed by Aurel Wintner (1903–1958) in his impressive textbook [138], without, however, recognizing the group theoretical background.

Joint scale transformations of quantities with different physical dimensions and appropriate functions of them was in particular advocated by Lord Rayleigh (John William Strutt, 1842–1919) [139].

5 An arid period for conformal transformations from about 1921 to about 1960

From 1921 on conformal transformations and symmetries did not play a noticeable role in physics or mathematics. The physics community was almost completely occupied with the new quantum mechanics and its consequences for atomic, molecular, solid state, nuclear physics etc.. In mathematics there were several papers on the properties on conformal geometries as a consequence of Weyl's work, mostly technically ones (see [125] for a long list of references), but also some as to classical field equations of physics.

With regard to the finite dimensional scale and special conformal transformations there was Dirac's important paper from 1936 [98] and – independently – about the same time the beginning of interpreting the special conformal ones as transformations from an inertial system to a system which moves with a constant acceleration with respect to the inertial one, an interpretation which ended in a dead end about 25 years later and brought the group into discredit.

5.1 Conformal invariance of classical field equations in physics

In 1934 Schouten and Haantjes discussed conformal invariance of Maxwell's equations and of the associated continuity equations for energy and momentum in the framework of Weyl's conformal geometry [140].

In 1935 Dirac proved invariance of Maxwell's equations with currents under the 15-parameter conformal group by rewriting them in terms of the hexaspherical coordinates $y^0, \ldots, y^5$, $(y^0)^2 + (y^5)^2 - (y^1)^2 - \cdots - (y^4)^2 = 0$, discussed in Sect. 2.4 above. In addition he wrote down a spinor equation

$$\beta^\mu \gamma^\nu M_{\mu\nu}\psi = \tilde{m}\,\psi, \quad M_{\mu\nu} = y_\mu \partial_\nu - y_\nu \partial_\mu,\ \mu,\nu = 0,\ldots,5, \tag{154}$$

where the β^μ and the γ^ν each are anticommuting 4×4 matrices, ψ is a 4-component spinor and the 15 operators $M_{\mu\nu}$ correspond to the Lie algebra generators of the group $SO(2,4)$. The (dimensionless) number $\tilde{m}$ cannot be interpreted as a mass (it is related to a Casimir invariant of $SO(2,4)$). Dirac tried to deduce from the spinor Eq. (154) his original one, but did not succeed. This is no surprise: Eq. (154) is invariant under the 2-fold covering group $SU(2,2)$ of the identity component of $SO(2,4)$ the 15 generators of which can be expressed by the four Dirac matrices $\gamma^\mu, \mu = 0,1,2,3$ and their eleven independent products $\gamma^\mu\gamma^\nu$, $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$, $\gamma^\mu\gamma^5$. But in order to incorporate space reflections, one has to pass to 8×8 matrices. The connection was later clarified by Hepner [141].

Also in 1935 Brauer and Weyl analysed spinor representations of pseudo-orthogonal groups in n dimensions using Clifford algebras and clarified the topological structure of these groups for real indefinite quadratic forms [142].

Dirac's paper prompted only a few others at that time [143–145].

In 1936 Schouten and Haantjes showed [146], in the framework of Weyl's conformal geometry, that in addition to Maxwell's equations also the equations of motions (geodetic equations in the presence of Lorentz forces) for a point particle are conformally invariant if one transforms the mass m as an inverse length:

$$m\,c/\hbar \to \omega^{-1/2}\,m\,c/\hbar, \tag{155}$$

where ω is defined in Eq. (102). They also showed that the Dirac equation with non-vanishing mass term is invariant in the same framework, using space-time dependent Dirac γ-matrices as introduced by Schrödinger and Valentin Bargmann (1908–1989) in 1932 [147] without mentioning the two.

This was the first time that conformal invariance was enforced by transforming the mass parameter, too. It was used and rediscovered frequently later on. Schouten and Haantjes did not discuss whether this *formal invariance* would also imply a new conservation law as discussed by Bessel-Hagen!

In 1940 Haantjes discussed this mass transformation for the special conformal transformations (134) applied to the relativistic Lorentz force [148]. He did the same about a year later for the usual Dirac equation with mass term [149].

In 1940 Pauli argued that the Dirac equation as formulated for General Relativity by Schrödinger and Bargmann could only be conformally invariant in the sense of Weyl if the mass term vanishes and added in a footnote at the end that in any conformally invariant theory the trace of the energy momentum tensor vanishes and that this could never happen for systems with non-vanishing masses [150].

In 1956 Mclennan discussed the conformal invariance and the associated conserved currents for free massless wave equations with arbitrary spins [151].

5.2 The acceleration "aberration"

Immediately after the papers by Bateman and Cunningham the conformal transformations were discussed as a coordinate change between relatively accelerated systems by Hassé [152].

Prompted by Arthur Milne's (1896–1950) controversial "Kinematical Relativity" [153], Leigh Page (1884–1952) in 1935 proposed a "New Relativity" [154] in which the registration of light signals should replace Einstein's rigid measuring rods and periodical clocks. He came to the conclusion that in such a framework not only reference frames which move with a constant relative velocity are equivalent, but also those which move with a relative constant acceleration! There was an immediate reaction to Page's papers by Howard Percy Robertson (1903–1961) [155] who had already written critically before on Milne's work [156]. Robertson argued that Page's framework should be accomodated within the kinematical one of General Relativity: He showed that the line element (Robertson-Walker)

$$ds^2 = d\tau^2 - f^2(\tau)(dy_1^2 + dy_2^2 + dy_3^2), \tag{156}$$

where $f(\tau)$ depends on a constant acceleration between observers Page had in mind, can be transformed into

$$ds^2 = [(t^2 - r^2/c^2)/t^2]^2 \, [dt^2 - (dx_1^2 + dx_2^2 + dx_3^2)/c^2]. \tag{157}$$

In addition he argued that the change of coordinate systems between observers with relative constant accelerations as described by Eq. (156) can be implemented by certain conformal transformations of the Minkowski-type line element (157) which Page had essentially used.

Robertson further pointed out that the line element (156) should better be related to a de Sitter universe. He emphasized that the line element (157), too, should be interpreted in the kinematical framework of general relativity and not in the one of special relativity, as Page had done. At the end he also warned that one should not identify the conformal transformations he had used in connection with Eq. (157) with the ones Bateman and Cunningham had discussed in the framework of special relativistic electrodynamics.

Page hadn't mentioned conformal transformations at all and Robertson didn't mean to suggest that the transformations his Ph.D. adviser Bateman had used in 1908/9 were to be interpreted as connecting systems of constant relative acceleration, but the allusion got stuck and dominated the physical interpretation of the special conformal transformations for almost thirty years.

Page's attempt was also immediately interpreted in terms of the conformal group by Engstrom and Zorn, but without refering to accelerations [157].

In 1940 Haantjes also proposed [148] to interpret the transformations (134) as coordinate changes between systems of constant relative acceleration, using hyperbolic motions [158] without mentioning them explicitly: Take $\beta = (0, b, 0, 0)$ and $x^0 = t$, $x^1 \equiv x$, $x^2 = x^3 = 0$. Then the spatial origin $(\hat{x} = 0, \hat{x}^2 = 0, \hat{x}^3 = 0)$ of the new system moves in the original one according to

$$0 = x(t) + b\,[t^2 - x^2(t)], \quad x^2 = 0, \quad x^3 = 0, \tag{158}$$

where

$$\sigma(x; b) = 1 - 2b\,x(t) - b^2[t^2 - x^2(t)] = 1 - b\,x(t) \tag{159}$$

does not vanish because $t^2 + 1/(4b^2) > 0$. Haantjes does not mention Page nor Robertson. He summarizes his interpretation again in [149].
On the other hand, the time $\hat{t}$ of the moving system runs as

$$\hat{t} = \frac{t}{1 - b^2\,t^2} \tag{160}$$

at the point $(x = 0,\ x^2 = 0,\ x^3 = 0)$, i.e. it becomes singular for $(b\,t)^2 = 1$.
The situation is similarly bewildering for the corresponding motion of a point $(\hat{x} = a \neq 0, \hat{x}^2 = 0, \hat{x}^3 = 0)$ in the original system: Instead of Eq. (158) we have now

$$b\,(ab + 1)[t^2 - x^2(t)] + (2ab + 1)\,x(t) - a = 0. \tag{161}$$

As b is arbitray, we may choose $a\,b = -1$ and get from (161) that $x = -a = 1/b$ and $\sigma(x; b) = -b^2\,t^2$. If $a\,b = -1/2$, then (161) takes the special form $t^2 - x^2(t) + 1/b^2 = 0$ and $\sigma(x; b) = 2(1 - b\,x)$.

Many more such strange features may be added if one wants to keep the acceleration interpretation! The crucial point is that the transformations (134) are those of the Minkowski space and its associated inertial frames and that one should find an interpretation within that framework. Accelerations are – as Robertson asserted – an element of General Relativity! We come back to this important point below.

From 1945 till 1951 there were several papers by Hill on the acceleration interpretation of the conformal group [159] and about the same time work by Infeld and Schild on kinematical cosmological models along the line of Robertson involving conformal transformations [160].

Then came a series of papers by Ingraham on conformal invariance of field equations, also adopting the acceleration interpretation [161].

Bludman, in the wake of the newly discovered parity violation, discussed conformal invariance of the 2-component neutrino equation and the associated γ^5-invariance of the massless Dirac equation [162], also mentioning the acceleration interpretation.

The elaborate final attempt to establish the conformal group as connecting reference systems with constant relative acceleration came from Rohrlich and collaborators [163], till Rohrlich in 1963 conceded that the interpretation was untenable [164]!

6 The advance of conformal symmetries into relativistic quantum field theories

6.1 Heisenberg's (unsuccesful) non-linear spinor theory and a few unexpected consequences

Attempting to understand the mesonic air showers in cosmic rays, to find a theoretical framework for the ongoing discoveries of new "elementary" particles, and to cure the infinities of relativistic non-linear quantum field theories, Werner Heisenberg (1901–1976) in the 1950s proposed a non-linear spinor theory as a possible ansatz [165]. Spinors, because one would like to generate half-integer and integer spin particle states, non-linear, because interactions among the basic dynamical quantum fields should be taken into account on a more fundamental level, without starting from free particle field equations and inventing a quantum field theory for each newly discovered particle. The theory constituted a 4-fermion coupling on the Lagrangean level which was not renormalizable according to the general wisdom. This should be taken care of by introducing a Hilbert space with an indefinite metric (such introducing a plethora of new problems which were among the reasons why the theoretical physics community after a while rejected the theory). Heisenberg and collaborators associated the final version

$$\gamma^\mu \partial_\mu \psi \pm l^2 \gamma^\mu \gamma^5 \psi \, (\overline{\psi} \gamma_\mu \gamma^5 \psi) = 0 \tag{162}$$

of their basic field equation with several symmetries [166], from which I mention two which – after detours – had a lasting influence on future quantum field theories:

In order to describe the isospin it was assumed that – in analogy to a ferromagnet – the ground state carries an infinite isospin from which isospins of elementary particles emerge like spin waves. This appears to be the first time that a degenerate ground state was introduced into a relativistic quantum field theory. It was soon recognized by Nambu [167] that the analogy to superconductivity was more fruitful for particle physics.

As spinors ψ in the free Dirac equation have the dimension of length $[L^{-3/2}]$ ($\psi^+\psi$ is a spatial probability density) the Eq. (162) is invariant under the scale transformation [166]

$$\psi(x, l) \to \psi'(\rho\, x; \rho\, l) = \rho^{-3/2}\, \psi(x, l), \quad l \to l' = \rho\, l, \quad \rho = e^\gamma . \tag{163}$$

Here the length l serves as a coupling constant which is not dimensionless. So Heisenberg et al. do the same what Schouten and Haantjes had done previously [146] with the mass parameter, namely to rescale it, too. As this does not lead to a new conservation law the authors had to argue their way around that problem and they related possible associated discrete quantum numbers to the conservation of lepton numbers!

Though this attempt was not successful with respect to Eq. (163), it raised interest as to the possible role of scale and conformal transformations in field theories and particle physics: Already early my later teacher Fritz Bopp (1909–1987) had shown interest in them [168]. Feza Gürsey (1921–1992) discussed the non-linear equation

$$\gamma^\mu \partial_\mu \psi + \lambda\, (\overline{\psi}\, \psi)^{1/3}\, \psi = 0 \tag{164}$$

as an alternative which is genuine scale and conformal invariant [169], though the cubic root is, of course, a nuisance. McLennan added the more interesting example [170]

$$\Box \varphi + \lambda(\varphi^* \varphi)\, \varphi = 0, \tag{165}$$

where $\varphi(x)$ is a complex scalar field in 4 dimensions with a dimension of lengh $[L^{-1}]$ and the coupling constant λ is dimensionless.

Immediately after the paper by Heisenberg et al. with the scale transformation (163) appeared, Julius Wess (1934–2007) analysed their interpretation of that transformation for the example of a free massive scalar quantized field, rescaling the mass parameter, too. He found no conservation law in the massive case, but a time dependent generator for the scale transformation of the field [171].

A year later Wess published a paper [172] in which he investigated the possible role of the conformal transformations (134) in quantum field theory, too, by discussing their role for free massless scalar, spin-one-half and electromagnetic vector fields, including the associated conserved charges and the symmetrization of the canonical energy-momentum tensor. He also analysed the conformal invariance of the 2-point functions. He further pointed out that the generator of scale transformations has a continuous spectrum and cannot provide discrete lepton quantum numbers. He finally mentioned that in the case of a non-vanishing mass of the scalar field one can ensure invariance if one transforms the mass accordingly. But no conservation law holds in that case.

Wess also observed that the transformations (134) can map time-like Minkowski distances into space-like ones and vice versa, because

$$(\hat{x} - \hat{y}, \hat{x} - \hat{y}) = \frac{1}{\sigma(x;\beta)\, \sigma(y;\beta)}\, (x - y, x - y), \tag{166}$$

where

$$\sigma(x;\beta) = 1 + 2(\beta, x) + (\beta,\beta)\, (x,x) = (\beta,\beta) \left(x + \frac{\beta}{(\beta,\beta)}, x + \frac{\beta}{(\beta,\beta)} \right). \tag{167}$$

The sign of the product $\sigma(x;\beta)\, \sigma(y;\beta)$ may be negative! This mix-up of the causal structure for the Minkowski space was, of course, a severe problem [173]. Similarly the possibility that $\sigma(x;\beta)$ from Eq. (167) can vanish. At least locally causality is conserved because

$$\eta_{\mu\nu} d\hat{x}^\mu \otimes d\hat{x}^\nu = \frac{1}{[\sigma(x;\beta)]^2}\, \eta_{\mu\nu} dx^\mu \otimes dx^\nu. \tag{168}$$

Wess does not say anything about possible physical or geometrical interpretations of the conformal group!

6.2 A personal interjection

In 1959 I was a graduate student in theoretical physics and had to look for a topic of my diploma thesis. Being at the University of Munich it was natural to go to Bopp, Sommerfeld's successor, who had been working on problems in quantum mechanics and quantum field theory. He suggested a topic he was presently working on and which had to do with the fusion of massless spin-one-half particles in an unconventional mathematical framework I did not like. Having learned from talks in the nearby Max-Planck-Institute for Physics about scale transformations as treated in [166] (Heisenberg and his Institute had moved from Göttingen to Munich the year before), and knowing about Bopp's interest in the conformal group, I asked him whether I could take that subject. Bopp was disappointed that I did not like his original suggestion, but, being kind and conciliatory as usual, he agreed that I work on the conformal group!

When studying the associated literature, I got confused: whereas the interpretation of the scale transformations (dilatations) (132) wasn't so controversial I couldn't make sense of the acceleration interpretation for the conformal transformations (134) (see Sect. 5.2 above)! I knew I had to find a consistent interpretation in order to think about possible physical applications.

I was brought on the right track by the observation that there was a very close relationship between invariance or non-invariance of a system with respect to scale and conformal transformations: If the dilatation current was conserved, so were the 4 conformal currents, if the dilatation current was not conserved, then neither were the conformal ones, the divergences of the latter being proportional to the divergence of the former one (see, e.g. Eqs. (137)), at least in the cases I knew then. A simple example is given by a free relativistic particle [174]: It follows from the infinitesimal transformations (133) and (135) that the – possibly – conserved associated "momenta" are given by ($c = 1$)

$$s = (x, p) = E\,t - \vec{x}\cdot\vec{p}, \quad E = (\vec{p}^{\,2} + m^2)^{1/2}. \tag{169}$$

$$h^\mu = (x, x)\,p^\mu - 2\,(x, p)\,x^\mu, \quad \mu = 0, 1, 2, 3; \tag{170}$$

$$h^0 = (t^2 - r^2)\,E - 2\,s\,t, \quad r = |\vec{x}|; \quad \vec{h} = (t^2 - r^2)\,\vec{p} - 2\,s\,\vec{x}. \tag{171}$$

Inserting

$$\vec{x}(t) = (\vec{p}/E)\,t + \vec{a} \tag{172}$$

into those momenta gives

$$s = -\vec{a}\cdot\vec{p} + (m^2/E)\,t, \tag{173}$$

$$h^0 = -\vec{a}^{\,2}\,E - (m^2/E)\,t^2, \quad \vec{h} = 2\,(\vec{a}\cdot\vec{p})\,\vec{a} - \vec{a}^{\,2}\,\vec{p} - (m^2/E)[(\vec{p}/E)\,t^2 + 2\,\vec{a}\,t], \tag{174}$$

which shows that the quantities s and h^μ are constants for a free relativistic particles only in the limits $m \to 0$ or $E \to \infty$! Both types are either conserved, or not conserved, simultaneously.

More arguments for the conceptual affinities of scale and conformal transformations came from their group structures, especially as subgroups of the 15-dimensional full conformal group. These features may be infered from the Lie algebra (now with hermitean generators, the Poincaré Lie algebra left out; compare also the algebra from Eqs. (61) - (68), including the related group definitions of the operators):

$$[S,\, M_{\mu\nu}] = 0, \quad i\,[S,\, P_\mu] = -P_\mu, \quad i\,[S,\, K_\mu] = K_\mu, \;\; \mu, \nu = 0, 1, 2, 3, \tag{175}$$

$$[K_\mu,\, K_\nu] = 0, \tag{176}$$

$$i\,[K_\mu,\, P_\nu] = 2\,(\eta_{\mu\nu}\,S - M_{\mu\nu}), \tag{177}$$

$$i\,[M_{\lambda\mu},\, K_\nu] = (\eta_{\lambda\nu}\,K_\mu - \eta_{\mu\nu}\,K_\lambda). \tag{178}$$

These relations show that the transformations (132) and (134) combined form a subgroup (generated by S and K_μ), that K_μ and P_ν combined do not form a subgroup, but generate scale transformations and homogeneous Lorentz transformations. As

$$K_\mu = R\cdot P_\mu \cdot R, \tag{179}$$

where R is the inversion (3), one can generate the 15-dimensional conformal group by translations and the discrete operation R alone [94]!

Now all different pieces of the interpretation puzzle presented by the transformations (134) fell into the right places if one – inspired by Weyl's conformal transformations (102) and (107) – interpreted them as space-time dependent scale transformations. However, whereas Weyl's factor $\omega(x)$ is arbitrary, the corresponding factor $1/\sigma^2(x;\beta)$ in Eq. (168) has a special form induced by the coordinate transformations (134). For that reason I called them "special conformal transformations" in [94], a name that has remained.

So the proposal was to interpret scale and special conformal transformations as (length) "gauge transformations" of the Minkowski space [95], an interpretation which has been adopted generally by now.

Having a consistent interpretation did not immediately settle the question where those transformations could be physically useful! A first indication came from the relations (173)–(174) which show that the very high energy limit may be a possible realm for applications. It was helpful that – at that time – interesting interaction terms with dimensionless coupling constants like $\overline{\psi}\gamma^\mu\psi\, A_\mu$, $\overline{\psi}\gamma^5\psi\, A$, φ^4 were also scale *and* conformal invariant [174]. This led to Born approximations at very high energies and very large momentum transfers which were compatible with scale invariance [175]. However, the experimental hadronic elastic and other "exclusive" cross sections behaved quite differently. The way out was the deliberation that in these reactions the scale invariant short distance properties were hidden behind the strong rearrangment effects of the long range mesonic clouds which were accompanied by the emission of a large number of secondary particle into the final states, like the emission oft "soft" photons in the scattering of charged particles.

A somewhat crude Bremsstrahlung model showed successfully how this mechanism could work and how to relate scale invariance to the "inclusive" cross section (i.e. after summation over all final state channels) in inelastic electron-nucleon scattering [176]. A very similar result was obtained by Bjorken about the same time by impressively exploiting current algebra relations [177]. His scaling predictions for "deep-inelastic" electron-nucleon scattering generated considerable general interest in the field [178]. Soon scale invariance at short distances found its proper place in applications of quantum field theories to high energy problems in elementary particle physics (see below).

The situation for special conformal transformations was more difficult at that time: First, there was their long bad reputation of being related to a somewhat obscure coordinate change with respect to accelerated systems! I always felt the associated resistance any time I gave a talk on my early work [179]. Also, it appeared that scale invariance was the dominating symmetry because special conformal invariance seemed to occur in the footsteps of scale invariance. This changed drastically later, too.

6.3 Partially conserved dilatation and conformal currents, equal-time commutators and short-distance operator-expansions

While I was in Princeton (University) in 1965/66, I was joined by the excellent student Gerhard Mack whom I had "acquired" in 1963 as my very first diploma student in Munich. In Princeton it was not easy to persuade Robert Dicke (1916–1997) who was in charge of admissions that Mack would be an adequate graduate student of the physics department, but I succeeded! Around that time the work on physical consequences from Murray Gell-Mann's algebra of currents was at the forefront of activities in theoretical particle physics [180, 181]. In discussions with John Cornwall who had invited me for a fortnight to UC Los Angeles to talk about my work, the idea came up to incorporate broken scale and conformal invariance into the current algebra framework. Back in Princeton I suggested to Gerhard Mack to look into the problem. This he did with highly impressive success: When we both were in Bern in 1966/67 as guests of the Institute for Theoretical Physics (the invitation arranged by Heinrich Leutwyler), he completed his Ph.D. thesis on the subject [182] and got the degree in Februar 1967 from the University of Bern. An extract of the thesis was published in 1968 [183].

In the next few years the subject almost "exploded". It is impossible to cover the different lines of development in these brief notes and I refer to several of the numerous reviews [99, 184–202] on the field. I here shall briefly indicate only the most salient steps till today:

The crucial new parameter which is associated with scale (and special conformal) transformations is the length dimension l of a physical quantity A: It is said to have the length dimension l_A if it transforms under the group (132) as [94]

$$S_1[\gamma]: \quad A \to \hat{A} = \rho^{l_A} A, \quad \rho = e^{\gamma}; \tag{180}$$

or, if $F(x)$ is a field variable,

$$F(x) \to \hat{F}(\hat{x}) = \rho^{l_F} F(x), \quad \hat{x} = \rho\, x. \tag{181}$$

If A or $F(x)$ are corresponding operators and scale invariance holds, then

$$e^{i\,\gamma\,S} A\, e^{-i\,\gamma\,S} = \rho^{l_A} A, \quad e^{i\,\gamma\,S} F(x)\, e^{-i\,\gamma\,S} = \rho^{-l_F} F(\rho\, x), \tag{182}$$

where S is the hermitean generator of the scale transformation (= dilatation). The “infinitesimal” versions of these relations are

$$i\,[S,\, A] = l_A\, A, \quad i\,[S,\, F(x)] = (-l_F + x^{\mu}\partial_{\mu})\, F(x). \tag{183}$$

The corresponding relations for the generators K_μ of the special conformal transformations are

$$i\,[K_\mu,\, F(x)] = [-2\, x_\mu(-l_F + x^{\nu}\partial_\nu) + (x,x)\partial_\mu + 2\, x^{\nu}\Sigma_{\mu\nu}]\, F(x), \tag{184}$$

where the $\Sigma_{\mu\nu}$ are the spin representation matrices of F with respect to the Lorentz group. The “classical” or “canonical” dimension of scalar and vector fields $\varphi(x)$ and $A_\mu(x)$ in 4 space-time dimensions are $l_\varphi = l_A = -1$ (the classical action integral $\int d^4x\, L$ has vanishing length dimension), a Dirac spinor $\psi(x)$ has $l_\psi = -3/2$.
It follows from the commutation relations (175) that $M_{\mu\nu}$, P_μ and K_μ have the dimensions 0, -1 and $+1$, respectively. As a mass parameter m has – in natural units (see Eq. (155)) – length dimension -1, one defines the “mass dimension” $d_F = -l_F$, in order to avoid the minus signs in case of the usual fields.

For a given scale invariant system one expects the generator S to be the space integral

$$S = \int dx^1 dx^2 dx^3\, s^0(x) \tag{185}$$

of the component s^0 of the dilatation current $s^{\mu}(x)$, where – according to Eq. (116) –

$$s^{\mu}(x) = T^{\mu}{}_{\nu}\, x^{\nu} + \sum_i d_i \frac{\partial L}{\partial(\partial_\mu \varphi^i)} \varphi^i. \tag{186}$$

Using the equations of motions $E(\varphi^i) = 0$ (Eq. (112)) and the expression (113) for the canonical energy-momentum tensor, we get for the divergence

$$\begin{aligned}\partial_\mu s^{\mu} &= T^{\mu}{}_{\mu} + \sum_i d_i \frac{\partial L}{\partial \varphi^i}\, \varphi^i + d_i\, \frac{\partial L}{\partial(\partial_\mu \varphi^i)}\, \partial_\mu \varphi^i \\ &= -4\, L + \sum_i d_i\, \varphi^i\, \frac{\partial L}{\partial \varphi^i} + (d_i + 1)\, \partial_\mu \varphi^i\, \frac{\partial L}{\partial(\partial_\mu \varphi^i)}.\end{aligned} \tag{187}$$

For a large class of models the divergence of the special conformal currents is proportional to the divergence (187), namely

$$\partial_\mu k^{\mu}{}_{\lambda}(x) = 2\, x_\lambda\, \partial_\mu s^{\mu}(x), \quad \lambda = 0, 1, 2, 3. \tag{188}$$

For this and possible exceptions see [99, 199, 203]. Compare also Eq. (137) above.

If the divergence $\partial_\mu s^\mu$ vanishes then the generator S from Eq. (185) is – formally at least – independent of time and it follows from the commutation relations (175) that the mass squared operator obeys

$$M^2 = P_\mu P^\mu \to \hat{M}^2 = e^{i\gamma S}\, M^2\, e^{-i\gamma S} = \rho^{-2}\, M^2, \tag{189}$$

i.e. either $M^2 = 0$ or M^2 has a continuous spectrum. In general, however, the physical spectrum of M^2 is more complicated and the divergence (187) will not vanish. A very simple example is provided by a free massive scalar field which has (ignoring problems as to the product of field operators at the same point in the quantum version!)

$$L = \frac{1}{2}(\partial_\mu \varphi\, \partial^\mu \varphi - m^2\, \varphi^2), \quad \partial_\mu s^\mu = m^2\, \varphi^2. \tag{190}$$

(As to problems with the energy-momentum tensor for quantized scalar fields see [181, 189], the references given there and the literature quoted in Sect. 6.4 below [213].)

In such a case as (190) the generator (185) becomes time dependent: $S = S(x^0)$. Nevertheless, using canonical equal-time commutation relations for the basic field variable and their conjugate momenta the second of the relations (183) may still hold (formally) and can be exploited, mostly combined with the fact that the divergence is a local field with spin 0, carrying certain internal quantum numbers [183–185, 204]. In this approach the dimensions d_i of the fields φ^i are considered to be the prescribed canonical ones.

Postulating the existence of equal-time commutators is seriously not tenable in the case of interacting fields which may also acquire non-canonical dimensions. This was first clearly analyzed by Kenneth Wilson in the framework of his operator product expansion [205]: In quantum field theories the product of two "operators" $A(x)$ and $B(y)$ is either badly defined or highly singular in the limit $y \to x$. Refering to experience with free fields and perturbation theory Wilson first postulated that

$$\text{for} \quad y \to x: \quad A(x)\, B(y) \simeq \sum_n C_n(x-y)\, O_n(x), \tag{191}$$

where the generalized functions $C_n(x-y)$ beome singular on the light cone $(x-y)^2 = 0$ and the operators $O_n(x)$ are local fields.

Second Wilson implemented asymptotic scale invariance by attributing (mass) dimensions $d_A,\ d_B$ and d_n to the fields in Eq. (191). Postulating the property (182) for both sides of that (asymptotic) relation implies

$$C_n[\rho\,(x-y)] = \rho^{d_n - d_A - d_B}\, C_n(x-y), \tag{192}$$

i.e. the $C_n(x)$ are homogeneous "functions" of degree $d_n - d_A - d_B$.
This approach provided an important operational framework for the concept "asymptotic scale invariance". It was soon applied successfully to deep inelastic lepton-hadron scattering processes, with A and B electromagnetic current operators $j_\mu(x)$ of hadrons [185, 188, 206–208].

6.4 Anomalous dimensions, Callan-Symanzik equations, conformal anomalies and conformally invariant n-point functions

Wilson had already stated that for interacting fields the dimensions $d_A,\ d_B$ and d_n need not be canonical. The intuitive reason for this is the following: even if one starts with a (classical) massless scale invariant theory in lowest order, one (always) has to break the symmetry in higher order perturbation theory for renormalizable field theories through the introduction of regularization schemes which involve length (mass) parameters like a cutoff, e.g. related to a mass (re)normalization point μ. This leads to anomalous positive corrections to the canonical dimensions

$$d \to d + \gamma(g),\ \ \gamma(g) \geq 0, \tag{193}$$

where g is the physical coupling constant.

Curtis Callan and Kurt Symanzik (1923–1983) in 1970 showed (independently) how a change of the scale parameter μ and an associated one in the coupling g affects the asymptotic behaviour of the nth proper (1-particle irreducible) renormalized Green function in momentum space [209]. For a single scalar field its asymptotic behaviour is governed by the partial differential (Callan-Symanzik) equation

$$\left(\mu\,\frac{\partial}{\partial\mu}+\beta(g)\,\frac{\partial}{\partial g}-n\,\gamma(g)\right)\Gamma^{(n)}_{as}(p;g,\mu)=0. \tag{194}$$

The function $\beta(g)$ and the anomalous part $\gamma(g)$ of the dimension may be calculated in perturbation theory. This material is discussed in standard textbooks [210–212] and therefore I shall not discuss it further here.

As there is no regularization scheme which preserves scale and special conformal invariance, the quantum version of the trace of the energy momentum tensors and the divergences of the dilatation and special conformal currents in general contain an anomaly which for a number of important models is proportional to $\beta(g)$, where $\beta(g)$ is the same function as in Eq. (194) [213]. Thus, only if $\beta(g)$ vanishes identically or has a zero (fix point) at some $g = g_1 \neq 0$ can the quantum version of that field theory be dilatation and conformally invariant. The vanishing of $\beta(g)$ in all orders of perturbation theory, and perhaps beyond, appears to happen for the $\mathcal{N} = 4$ superconformal pure Yang-Mills theory (see Sect. 7.3 below).

Early it seemed, at least in Lagrangean quantum field theory [99,203], that in most cases scale invariance entails special conformal invariance, e.g. the 2-point function

$$\langle 0|\varphi(x)\,\varphi(0)|0\rangle = \text{const.}\,(x^2-i0)^{-d} \tag{195}$$

of a scale invariant system with scalar field $\varphi(x)$ is also conformally invariant. Here the dimension d of the scalar field in general remains to be determined by the interactions.

But then in 1970 Schreier [214], Polyakov [215] and Migdal [216] realized that conformal invariance imposes *additional* restrictions on the 3-point functions, leaving only a few parameters to be determined by the dynamics.

Migdal also proposed a self-consistent scheme ("bootstrap") in terms of Dyson-Schwinger equations with Bethe-Salpeter-like kernels which, in principle, would allow to find solutions for the theory. This soon led to an impressive development in the analysis of conformally invariant 3- and 4-dimensional quantum field theories in terms of (euclidean) n-point functions, reviews of which can be found in [192–194, 196, 197].

7 Conformal quantum field theories in 2 dimensions, global properties of conformal transformations, supersymmetric conformally invariant systems, and "postmodern" developments

7.1 2-dimensional conformal field theories

Polyakov's paper [215] was a first important step for conformal invariance into the realm of statistical physics, but progress was somewhat slow due to the fact that the additional symmetry group was just 3-dimensional (scale transformations were already established within the theory of critical points in 2nd order phase transitions). However applications of conformal invariance again "exploded" after the seminal paper [217] by Belavin, Polyakov and Zamolodchikov on *2-dimensional* conformal invariant quantum field theories with now "infinite-dimensional" conformal Lie algebras generated by the (Witt) operators

$$l_n = z^{n+1}\,\frac{d}{dz}, \quad n=0,\pm1,\pm2,\ldots, \quad [l_m, l_n]=(m-n)\,l_{m+n}, \tag{196}$$

if one uses complex variables, and the corresponding ones with complex conjugate variables $\bar{z}$ where $\bar{l}_n = \bar{z}^{n+1}d/d\bar{z}$. The generators (57) - (60) form a (real) 6-dimensionsional subalgebra which generates the group (45). Its complex basis here is $l_{-1},\, l_0,\, l_1$ and $\bar{l}_{-1},\, \bar{l}_0,\, \bar{l}_1$.

The "quantized" version of the algebra (196) generated by corresponding operators L_n, the Virasoro algebra, contains anomalies which can be interpreted as those of the 2-dimensional energy-momentum tensor which, being conserved and having a vanishing trace, has only 2 independent components one of which can be (Laurent) expanded in terms of the L_n, the other one in terms of the $\bar{L}_n$ [218].
The Virasoro algebra first played an important role for the euclidean version of the 2-dimensional bosonic string world sheet [202,219,220]. However, with the work of Belavin et al. [217] 2-dimensional conformal quantum field theory became a subject by its own, first, because it allowed rich new insights into the intricate structures of such theories [198,200,202,221], and second, because it allowed for applications in statistical physics for, e.g. phase transitions in 2-dimensional surfaces [200,222–225].

7.2 Global properties of conformal transformations

The transformations (3) and (134) become singular on light cones. As was already known in the 19th century, this can be taken care of geometrically in terms of the polyspherical coordinates discussed in Sect. 2.4 above which allow to extend the usual Minkowski space and have all conformal transformations act linearly on the extension. One can even give a physical interpretation of the procedure [94,95]: Introducing homogeneous coordinates

$$x^\mu = y^\mu/k, \quad \mu = 0,\, 1,\, 2,\, 3, \tag{197}$$

one can interpret k as an initially Poincaré invariant length scale which transforms as

$$k \to \hat{k} = e^{-\gamma}\, k, \tag{198}$$

$$k \to \hat{k} = \sigma(x;\beta)\, k, \tag{199}$$

with respect to the groups (132) and (134). The limit $\sigma(x;\beta) \to 0$ then means that at the points with the associated coordinates x the new scale $\hat{k}$ becomes infinitesimally small so that the new coordinates $\hat{x}^\mu$ become arbitrarily large whereas the dimensionless coordinates y^μ stay the same [94,95]. The scale coordinate $\hat{k}$ can also become negative now.

In order to complete the picture, one introduces the dependent coordinate

$$q = (x,x)\, k, \quad \text{or} \quad Q(y;q,k) \equiv (y,y) - q\, k = 0, \tag{200}$$

where q has the dimension of length, transforms as

$$q \to \hat{q} = e^{\gamma}\, q \tag{201}$$

under dilatations and remains invariant under special conformal transformations. So we have now

$$\begin{aligned} C_4[\beta]\;: \quad & \hat{y}^\mu = y^\mu + \beta^\mu\, q, \\ & \hat{k} = 2\,(\beta,y) + k + (\beta,\beta)\, q, \\ & \hat{q} = q. \end{aligned} \tag{202}$$

These transformations leave the quadratic form $Q(y;q,k)$ itself invariant, not only the equation $Q = 0$. The same holds for the translations $T_4[a] : x^\mu \to x^\mu + a^\mu$ which act on the new coordinates as

$$\begin{aligned} T_4[a]\;: \quad & \hat{y}^\mu = y^\mu + a^\mu\, k, \\ & \hat{k} = k, \\ & \hat{q} = 2\,(\beta,y) + (\beta,\beta)\, k + q. \end{aligned} \tag{203}$$

It is important to keep in mind that a given 6-tuple $(y^0,\, y^1,\, y^2, y^3,\, k,\, q)$, with $Q(y;k,q)=0$, is only one representative of an equivalence class of such 6-tuples which can differ by an arbitrary real multiplicative number $\tau \neq 0$ and still describe the same point in the 4-dimensional physical space:

$$(y^0,\, y^1,\, y^2,\, y^3,\, k,\, q) \cong \tau\,(y^0,\, y^1,\, y^2,\, y^3,\, k,\, q), \quad \tau \neq 0. \tag{204}$$

As k is a Lorentz scalar, the quadratic form (y,y) is – like (x,x) – invariant under the homogenous Lorentz group $O(1,3)$. So the 15-parameter conformal group C_{15} of the Minkowski space M^4 leaves the quadratic form $Q(y;k,q)$ invariant. Writing

$$k = y^4 + y^5, \quad q = y^4 - y^5, \tag{205}$$

$$Q(y,y) \equiv (y^0)^2 + (y^5)^2 - (y^1)^2 - (y^2)^2 - (y^3)^2 - (y^4)^2 \tag{206}$$
$$= y^T \cdot \eta \cdot y = \eta_{ij} y^i\, y^j,$$

one sees how the group C_{15} has to be related to the group $O(2,4)$. Because of the equivalence relation (204) one has

$$C_{15} \cong O(2,4)/Z_2(6), \quad Z_2(6): y \to y, \text{ and } y \to -y. \tag{207}$$

$Z_2(6)$ is the center of the identity component $SO^{\uparrow}(2,4)$ of $O(2,4)$ to which $C_4[\beta]$ and $S_1[\gamma]$ belong, too.

Like $O(1,3)$ the group $O(2,4)$ – and therefore C_{15}, too – consists of 4 disjoint pieces: If $w \in O(2,4)$ is the transformation matrix defined by

$$y^i \to \hat{y}^i = w^i{}_j\, y^j, \quad i = 0, \ldots, 5, \quad w^T \cdot \eta \cdot w = \eta, \tag{208}$$

then the pieces are characterized by [142]

$$\det w = \pm 1, \quad \epsilon_{05}(w) \equiv \text{sign}(w^0{}_0 w^5{}_5 - w^0{}_5 w^5{}_0) = \pm 1, \tag{209}$$

the identity component $SO^{\uparrow}(2,4)$ being given by $\det w = 1,\ \epsilon_{05}(w) = 1$. For, e.g. the inversion (3) we have

$$R: \quad k \to q,\, q \to k, \text{ or } y^0 \to y^0, \cdots, y^4 \to y^4,\, y^5 \to -y^5;\ \det w = -1,\ \epsilon_{05}(w) = -1. \tag{210}$$

For these and other details I refer to the literature [198, 226–229].

Taking the equivalence relation (204) into account, one can express $Q(y,y) = 0$ as

$$(y^0)^2 + (y^5)^2 = (y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2 = 1, \tag{211}$$

and one sees that the extended Minkowski space on which the group C_{15} acts continuously is compact and topologically given by

$$M_c^4 \simeq (S^1 \times S^3)/Z_2(6), \tag{212}$$

which essentially says that time is compactified to S^1 and space to S^3. So time becomes periodic! Now one needs at least four coordinate neighbourhoods ("charts") in order to cover the manifold (212) [226, 227]. This can be related to the fact that the maximally compact subgroup of the identity component of $O(2,4)$ is $SO(2) \times SO(4)$.

In order to get rid of the periodical time one has to pass to the universal covering space

$$M_c^4 \to \tilde{M}^4 \simeq \mathbb{R} \times S^3. \tag{213}$$

If $\tilde{M}^4$ is parametrized as

$$\tilde{M}^4 = \{(\tau, \vec{e});\, \tau \in \mathbb{R},\, \vec{e} = (e^1,\, e^2,\, e^3, e^4) \in S^3,\, \vec{e}^{\,2} = 1\}, \tag{214}$$

then that coordinate chart for M_c^4 which describes the relations (211) for $k = y^0 + y^5 > 0$, namely M^4, may be given by

$$y^0 = \sin\tau, \quad y^5 = \cos\tau, \quad \tau \in (-\pi,\, +\pi), \quad \vec{y} = \vec{e}, \quad e^4 + \cos\tau > 0. \tag{215}$$

leading to the Minkowski space coordinates

$$x^0 = \frac{\sin\tau}{e^4 + \cos\tau}, \quad x^j = \frac{e^j}{e^4 + \cos\tau}, \quad j = 1,\, 2,\, 3, \tag{216}$$

from which it follows that

$$\eta_{\mu\nu}\, dx^\mu \otimes dx^\nu = \frac{1}{(e^4 + \cos\tau)^2}\,(d\tau \otimes d\tau - \sum_{i=1}^{4} de^i \otimes de^i), \quad \vec{e} \cdot d\vec{e} = 0, \tag{217}$$

or more generally – according to Eqs. (197), (205) and (206) –

$$\eta_{\mu\nu}\, dx^\mu \otimes dx^\nu = \frac{1}{k^2}\,(\eta_{ij}\, dy^i \otimes dy^j), \quad y_j dy^j = 0. \tag{218}$$

Global structures associated with conformal point transformations were first analyzed by Kuiper [230].

Very important in this context is the question of possible causal structures on those manifolds, i.e. whether one can give a conformally invariant meaning to time-like, space-like and light-like. We know already that such a global causal structure cannot exist on M^4 (Eq. (166)), but that a local one is possible (Eqs. (168) and (218)). This generalizes to the compact manifold M_c^4 [226].

However, the universal covering $\tilde{M}^4$ does allow for a *global conformally invariant causal structure* with respect to the universal covering group $\widetilde{SO^{\uparrow}(2,4)}$, namely a point $(\tau_2, \vec{e}_2) \in \tilde{M}^4$ is *time-like later* than $(\tau_1, \vec{e}_1)$ if

$$\tau_2 - \tau_1 > \mathrm{Arccos}(\vec{e}_1 \cdot \vec{e}_2), \tag{219}$$

where $y = \mathrm{Arccos}(x)$ means the principal value $y \in [0,\, \pi]$. More in the literature [198, 226–229, 231]. That the universal covering space (214) allows for a conformally invariant structure was first observed by Segal [231].

Now the group $O(2,4)$ is also the invariance group ("group of motions") of the anti-de Sitter space

$$AdS_5: \quad Q(u,u) = (u^0)^2 + (u^5)^2 - (u^1)^2 - (u^2)^2 - (u^3)^2 - (u^4)^2 = a^2, \tag{220}$$

which has the topological structure $S^1 \times \mathbb{R}^4$ and is, therefore, multiply connected, too. It has the universal covering $\mathbb{R} \times \mathbb{R}^4$ or $\mathbb{R} \times S^4$ if one compactifies the "spatial" part [232].

In the following sense the Minkowski space (216) may be interpreted as part of the boundary of the space (220) [241, 249]: Introducing the coordinates

$$u^0 = (a^2 + \lambda^2)^{1/2}\, \sin\tau, \quad u^5 = (a^2 + \lambda^2)^{1/2}\, \cos\tau, \quad \vec{u} = \lambda\, \vec{e}, \quad \lambda > 0, \quad \vec{e} \in S^3, \tag{221}$$

the ratios

$$\xi^0 = \frac{u^0}{u^4 + u^5}, \quad \xi^j = \frac{u^j}{u^4 + u^5}, \quad j = 1,\, 2,\, 3, \tag{222}$$

approach the limits (216) if $\lambda \to \infty$. Such limits may be taken for other charts of the compact space (212), too.

This property is at the heart of tremendous activities in a part of the mathematical physics community during the last decade (see the last Sect. below).

As the group $O(2,4)$ is infinitely connected (because it contains the compact subgroup $SO(2) \cong S^1$) it has also infinitely many covering groups, the double covering $SU(2,2)$ being one of the more important ones. The theory of irreducible representations of $O(2,4)$ and its covering groups also belongs to its global aspects. I here mention only a few of the relevant references on the discussions of those irreducible representations [233].

7.3 Supersymmetry and conformal invariance

An essential key to the compatibility of conformal invariance and supersymmetries is the generalization of the Coleman-Mandula theorem [234] by Haag, Łopuszánski and Sohnius [235]: Coleman and Mandula had settled a long dispute about the possible non-trivial "fusion" of space-time (Poincaré) and internal symmetries. From reasonable assumptions they deduced that one can only have a non-trivial S-matrix, if Poincaré and internal symmetry group "decouple", i.e. form merely a direct product. One of their postulates was the existence of a finite mass gap, thus excluding conformal invariance. Soon afterward supersymmetries were discovered [236–241] the fermionic generators of which had non-trivial commutators with those of the Poincaré group. Haag et al. not only generalized Coleman's and Mandula's results in the massive case by including supersymmetric charges, but they also discussed the case of massless particles and found a unique structure: now the fermionic supercharges can generate the 15-dimensional Lie algebra of the conformal group *and* internal unitary symmetries $U(\mathcal{N})$, $\mathcal{N} = 1, 2, \dots, 8$. For their result the inclusion of the generators K_μ of the special conformal transformations was essential.

One very important feature of supersymmetries is that they reduce the number of divergences for the conventional quantum field theories, making the usual associated renormalization procedures at least partially superfluous (so-called "non-renormalization theorems") [236–240]. Of special interest here are the $\mathcal{N} = 4$ superconformal quantum Yang-Mills theories in 4 space-time dimensions : they are finite, their β-function (see Sect. 6.4 above) vanishes [241–244] and, therefore, the trace of their energy-momentum tensor, too, thus implying scale and conformal invariance on the quantum level! The Lorentz spin content of this system of massless fields is: 1 (Yang-Mills) vector field with two helicities and an $SU(N)$ gauge group, 4 spin $1/2$ fields with two helicities each and 6 scalar fields.

Though this model is as similar to physical realities as an ideal mathematical sphere is similar to the real earth with its mountains, oceans, forests, towns etc., it is nevertheless a striking and interesting idealization!

7.4 "Postmodern" developments

7.4.1 AdS/CFT correspondence

The last 10 years have seen thousands (!) of papers which are centered around a conjecture by Maldacena [245] related to the limit $\lambda \to \infty$ of Eq. (222) above, namely that the 4-dimensional conformally compactified Minkowski space (or its universal covering) is the boundary of the 5-dimensional anti-de Sitter space AdS_5 (or its universal covering correspondingly). The conjecture is that the superconformal $\mathcal{N} = 4$, $SU(N)$ Yang-Mills theory on Minkowski space "corresponds" (at least in the limit $N \to \infty$) to a (weakly coupled) supergravity theory on AdS_5 accompanied by a Kaluza-Klein factor S^5 , both factors being related to a superstring in 10 dimensions of type $II\,B$ (closed strings with massless right and left moving spinors having the same chirality) by some kind of low energy limit.

The conjecture was rephrased by Witten [246] in proposing that the correlation (Schwinger) functions of the superconformal Yang-Mills theory may be obtained as asymptotic (boundary) limits of 5-dimenional supergravity on AdS_5 plus Kaluza-Klein modes related to the compact S^5, by means of the associated generating partition functions for the supergravity and the gauge theory, respectively. The dimensions of operators in the superconformal gauge theory could be determined from masses in the AdS_5 supergravity theory (for vanishing masses those dimensions become "canonical"). An important point is that strongly coupled Yang-Mills theories correspond to weakly coupled supergravity, thus allowing – in principle – to calculate strong coupling effects in the gauge sector by perturbation theory in the corresponding supergravity sector.

The hypothesis has caused a lot of excitement, with (partial) confirmations for special cases or models and also by using the conjecture as a working hypothesis for the analysis of certain problems, e.g. strong-coupling problems in gauge theories.

I am unable to do justice to the many works and people working in the field and I refer to reviews for further insights [247].

The conjecture as phrased by Witten suggests the question whether such or a similar correspondence can perhaps be achieved without refering to superstrings. This question was investigated with some success by Rehren in the framework of algebraic quantum field theory [248]. A recent summary of results can be found in the Thesis [249].

7.4.2 "Unparticles"

Physical systems with a discrete mass spectrum like the standard model of elementary particles cannot be scale and conformally invariant. Last year Georgi suggested that nevertheless conformally covariant operators with definite (dynamical) scale dimension d_U from an independent conformally invariant field theory might couple to standard model operators: At suffiently high energies this could lead to a "non-standard" loss of energy and momentum in the form of "unparticles" in very high energy reactions of standard model particles [250]. Though still extremely speculative this might lead to another interesting application of conformal symmetries, at least theoretically!

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as to Ebenezer Cunnigham see W. H. McCrea, Bull. London Math. Soc. **10**, 116–126 (1978).

[7] See, e.g. J. D. North, The Astrolabe; Scient. American **230**, 96–106 (Jan. 1974); see also the next Refs. [8–12].

[8] O. Neugebauer, The early history of the astrolabe; Isis **40**, 240–256 (1949);
O. Neugebauer, A History of Ancient Mathematical Astronomy, in 3 Parts (Studies in the History of Mathematics and Physical Sciences **1**, Springer, Berlin etc., 1975); on Hipparch: Part I, pp. 274–343; on stereographic projection with respect to Hipparch and Ptolemaeus: Part II, pp. 857–879. On Hipparch see also
Dict. Scient. Biography **15**, Suppl. I (Charles Scribner's Sons, New York, 1981), pp. 207–224 (by G. J. Toomer).

[9] C. Ptolemaeus, Planisphaerium, in: Claudii Ptolemaei Opera Quae Exstant Omnia, vol. II (Opera Astronomica Minora), ed. by J. L. Heiberg (Teubner, Leipzig, 1907) pp. 222–259; German translation by J. Drecker, Isis **9**, 255–278 (1927);
On Ptolemaeus and his work see Neugebauer [8] here Part II, pp. 834–941.
On the later history of that work by Ptolemaeus and the commenting notes by the Spanish-Arabic astronomer and mathematician Maslama (around 1000) see
P. Kunitzsch and R. Lorch, Maslama's Notes on Ptolemy's *Planisphaerium* and Related Texts; Sitzungsber. Bayer. Akad. Wiss., Philos.-Histor. Klasse, **1994**, Heft 2 (Verlag Bayer. Akad. Wiss., Munich, 1994).
As to the recent discussions on Ptolemaeus's possible mathematical insights regarding the stereographic projection he used, see
J. L. Berggren, Ptolemy's Maps of Earth and the Heavens: A New Interpretation; Arch. Hist. Exact Sci. **43**, 133–144 (1991);
R. P. Lorch, Ptolemy and Maslama on the Transformation of Circles into Circles in Stereographic Projection; Arch. Hist. Exact Sci. **49**, 271–284 (1995).

[10] See Appendix 3 of the next Ref. [11]: "Al-Farghānī's Proof of the Basic Theorem of Stereographic Projection" by N. D. Sergeyeva and L. M. Karpova, pp. 210–217;
Al-Farghānī, On the Astrolabe, Arabic Text Edition with Translation and Commentary by R. Lorch (Boethius Texte u. Abhandl. Geschichte Mathem. u. Naturw. **52**, Franz Steiner Verlag, Stuttgart, 2005).

[11] Jordanus de Nemore and the Mathematics of Astrolabes: *De Plana Spera*; an Edition with Introduction, Translation and Commentary by R.B. Thomson (Studies and Texts **39**, Pontifical Institute of Mediaeval Studies, Toronto, 1978).

[12] Christopheri Clavii Bambergensis e Societate Iesu Astrolabium (Romae, Impensis Bartholomaei Grassi. Ex Typographia Gambiana. 1593) here Liber [Book] II, Probl. 12 Propos. 15 and Probl. 13 Propos. 16 on pp. 518–524.
At the beginning of the book Clavius has a page on which he announces ten not yet known discoveries with proofs ("Qvae in aliorum astrolabiis non traduntur, sed in hoc nunc primum inuenta sunt, ac demonstrata"). Under point X. he lists: Various determinations of the magnitudes of angles in spherical triangles, noticed by nobody up to now. ("Variae determinationes magnitudinis angulorum in triangulis sphaericis, a nemine hactenus animaduersae.") The text is available from Fondos Digitalizados de la Universidad de Sevilla: http://fondosdigitales.us.es/books/ ; the second edition of the Astrolabium from 1611 (published in Mainz, Germany) as part II of the third volume of Clavius' Collected Papers is available from the Mathematics Library of the University of Notre Dame: http://mathematics.library.nd.edu/clavius/. The above mentioned problems and propositions are on pages 241–244 of the 2. Edition.
On Christopher Clavius see
Dict. Scient. Biography **3** (Charles Scribner's Sons, New York, 1971) pp. 311–312; by H.L.L. Busard;
J. M. Lattis, Between Copernicus and Galilei, Christoph Clavius and the Collapse of Ptolemaic Cosmology (The University of Chicago Press, Chicago and London, 1994); this otherwise informative book contains nothing on Clavius' work on the astrolabe.

[13] S. Haller, Beitrag zur Geschichte der konstruktiven Auflösung sphärischer Dreiecke durch stereographische Projektion; Bibliotheca Mathematica (Neue Folge) **13**, 71–80 (1899); the investigation was proposed by the mathematician and historian of mathematics Anton von Braunmühl; see his Vorlesungen über Geschichte der Trigonometrie, 1. Teil. Von den ältesten Zeiten bis zur Erfindung der Logarithmen (Teubner, Leipzig, 1900) here footnote 2) on page 190, where he refers to Haller's work.
Haller's paper is somewhat bewildering because his order and numbering of the problems he selects and discusses is different from those of Clavius without him saying so. But his main conclusion that Clavius showed the stereographic projection to be conformal appears to be correct. See also von Braunmühl's brief remark in Archiv der Mathematik und Physik (3. Reihe) **8**, 93 (1905).

[14] Results of those rediscoveries of Harriot's work are summarized in
Thomas Harriot, Renaissance Scientist, ed. by J. Shirley (Clarendon Press, Oxford, 1974) where J. V. Pepper's contribution (Harriot's earlier work on mathematical navigation: theory and praxis) on pp. 54-90 is of special interest.
J. W. Shirley, Thomas Harriot: A Biography (Clarendon Press, Oxford, 1983).

[15] Articles relevant to our priority problem are
E. G. R. Taylor, The Doctrine of nauticall triangles compendious, I. Thomas Hariot's manuscript; The Journ. of the Inst. of Navigation (London) **6**, 131–140 (1953);
D. H. Sadler, The Doctrine of nauticall triangles compendious, II. Calculating the meridonal parts; The Journ. of the Inst. of Navigation (London) **6**, 141–147 (1953);
J. A. Lohne, Thomas Harriot als Mathematiker; Centaurus **11**, 19–45 (1965); Harriot's proof that the stereographical projection is conformal is discussed on pp. 25–27;
R. C. H. Tanner, Thomas Harriot as Mathematician; Physis **9**, 235–247 and 257–292 (1967);
J. V. Pepper, Harriot's calculation of the meridional parts as logarithmic tangents; Arch. Hist. Exact Sciences **4**, 359–413 (1967/68); Harriot's proof of conformality (in Latin) is transcribed and translated into English on pp. 411–412;
J. V. Pepper, The Study of Thomas Harriot's Manuscripts, II. Harriot's unpublished papers; Hist. of Science **6**, 17–40 (1967)
Harriot, Thomas; Dict. Scient. Biography **4** (Charles Scribner's Sons, New York, 1972) pp. 124–129; by J.A. Lohne;
J. A. Lohne, Essays on Thomas Harriot; Arch. Hist. Exact Sciences **20**, 189–312 (1979);
Harriot was also the first one (on July 26, 1609) to use the just invented telescope (1608 in the Netherlands) to observe the moon, even before Galilei did that at the end of 1609. He will be celebrated, 400 years after his pioneering achievement, in 2009 as "England's Galilei" (http:www.telescope400.org.uk/harriot.htm).

[16] See Pepper, [14] and [15].

[17] According to Taylor, Saddler and especially Pepper, [15], Harriot in 1594 used numerical tables from an earlier book by Clavius: Theodosii Tripolitae sphaericorum libri III, Romae, Basaus, 1586; whether and when

Clavius' Astrolabium from 1593 (or 1611) arrived in England and could have been seen by Harriot needs further investigation. Perhaps a book written by a Jesuit was banned at that time in England!

[18] E. Halley, An easy demonstration of the analogy of the logarithmic tangents, to the meridian line, or sum of the secants: with various methods for computing the same to the utmost exactness; The Philos. Transact. Royal Soc. London **19**, 202–214 (1698); reprinted in the abridged version from 1809 of the Transactions for the period 1665–1800: Vol. IV (1694–1702), pp. 68–77; this abridged version (which has still the complete text of the articles) is available from the Internet Archive: http://www.archive.org/
Halley: " Lemma II: In the stereographic projection, the angles, under which the circles intersect each other, are in all cases equal to the spherical angles they represent: which is perhaps as valuable a property of this projection as that of all circles of the sphere on its appearing circles; but this, not being commonly known, must not be assumed without a demonstration."
After his proof Halley says: " This lemma I lately received from Mr. Ab. de Moivre, though I since understand from Dr. Hook that he long ago produced the same thing before the society. However the demonstration and the rest of the discourse is my own."
As to Halley see the recent comprehensive biography
A. Cook, Edmond Halley, Charting the Heavens and the Seas (Clarendon Press, Oxford, 1998).

[19] A. S. Wightman, Relativistic invariance and quantum mechanics (notes by A. Barut); Suppl. Nuovo Cim. (Ser. 10) **14**, 81–94 (1959).

[20] As to the early history of map projections see, e. g.
J. Keuning, The history of geographical map projections until 1600; Imago Mundi **12**, 1–24 (1955);
J.P. Snyder, Flattening the Earth: Two Thousand Years of Map Projections (University of Chicago Press, Chicago and London, 1993).

[21] On Mercator's projection see, e. g.
J.B. Calvert, http://mysite.du.edu/∼jcalvert/math/mercator.htm .
C.A. Furuti, http://www.progonos.com/furuti/MapProj/ .
E.W. Weisstein,
http://mathworld.wolfram.com/MercatorProjection.html .

[22] As to Mercator see
A. Taylor, The World of Gerhard Mercator, The Mapmaker Who Revolutionised Geography (Harper Perennial, London etc., 2004).
M. Monmonier, Rhumb Lines and Map Wars, A Social History of the Mercator Projection (The University of Chicago Press, Chicago and London, 2004).

[23] J. H. Lambert, Anmerkungen und Zusätze zur Entwerfung der Land- und Himmelscharten, in: Beyträge zum Gebrauche der Mathematik und deren Anwendung durch J. H. Lambert, Mit Kupfern und Tafeln, III. Theil (Verlag der Buchhandlung der Realschule, Berlin, 1772) pp. 105–199; that special chapter was later edited seperately by A. Wangerin (Ostwald's Klassiker der Exakten Wissenschaften **54**; Verlag W. Engelmann, Leipzig, 1894); available from University of Michigan Historical Math Collection, UMDL; http://quod.lib.umich.edu/u/umhistmath/; English translation: J. H. Lambert, Notes and comments on the composition of terrestrial and celestial maps (1772), translated and introduced by W. R. Tobler (Michigan Geographical Publications **8**, Dept. of Geography, Univ. Michigan, Ann Arbor, 1972).

[24] J. B. Calvert, http://mysite.du.edu/∼jcalvert/math/lambert.htm .
C.A. Furuti, http://www.progonos.com/furuti/MapProj/ .
E.W. Weisstein, http://mathworld.wolfram.com/LambertConformalConicProjection.html .

[25] Francisci Aguilonii, e Societate Iesu, Opticorum Libri Sex, Philosophis iuxta ac Mathematicis utiles (Ex Officina Plantiniana, Apud Viduam et Filios Io. Moreti, Antwerpiae, 1613) [From Franciscus Aguilonius, Six Books on Optics, equally useful for Philosophers and Mathematicians]; as to François d'Aiguillon (or de Aguilón) see A. Ziggelaar S. J., François de Aguilón, S. J. (1567–1617), Scientist and Architect (Bibliotheca Instituti Historici S. I. **44**; Institutum Historicum S.I., Roma, 1983); also:
The Scientific Revolution – 700 Biographies – Catalogue (Richard S. Westfall; Robert A. Hatch, Univ. Florida): http://web.clas.ufl.edu/users/rhatch/pages/03-Sci-Rev/SCI-REV-Home/ .

[26] The engravings can be seen under http://www.faculty.fairfield.edu/jmac/sj/scientists/aguilon.htm ; the last one for the 6th part shows three Putti performing and discussing the stereographic projection of a globe – held by the giant Atlas – onto the floor below! The original (on p. 452) has a size of about 14×10 cm^2 . (The year of birth – 1546 – for F. d'Aiguillon given on that Internet page is wrong!).

[27] On page 498: ... Secundum, ex contactu, quod & Stereographice non incongruè potest appelari: quare ut ea vox in usum venire liberè possit, dum alia melior non occurit, Lector, veniam dabis. ... In his introduction to the special chapter on stereographic projections ("De Stereographice Altero Proiectionis Genere Ex Oculi Contactu") on the pages 572/73 he justifies the term in more detail and compares it, e.g. to "Stereometria" which measures the capacity and extensions of bodies ("corporum dimensiones capacitatesque metitur") [the Greek word "stereos" means "rigid, solid, massive"].

[28] L. Euler, De representatione superficiei sphericae super plano [1]. De proiectione geographica superficiei [2]. De proiectione geographica de Lisliana in mappa generali imperii russici usitata [3]. The papers are published in: L. Euler, Opera Omnia, Ser. I, **28**, edited by A. Speiser (Orell Füssli, Zurich, 1955) pp. 248–275; 276–287; 288–297. German translation and edition by A. Wangerin: Drei Abhandlungen über Kartenprojection, von Leonhard Euler (1777.) (Ostwald's Klassiker der Exakten Wissenschaften **93**; Verlag W. Engelmann, Leipzig, 1898). See also Speiser's editorial remarks on Euler's work and its relationships to that of Lambert and Lagrange (pp. XXX-XXXVII of Opera Omnia **28**).

[29] L. Euler, Considerationes de traiectoriis orthogonalibus, Opera Omnia, [28], pp. 99–119. German translation in: Zur Theorie Komplexer Funktionen, Arbeiten von Leonhard Euler, eingeleitet und mit Anmerkungen versehen von A. P. Juschkewitsch (Ostwalds Klassiker der Exakten Wissenschaften **261**, Akad. Verlagsges. Geest & Portig K.-G., Leipzig, 1983) here article III; article IV gives again the German translation of the first of the three aricles of [28].

[30] V. Kommerell, Analytische Geometrie der Ebene und des Raumes, Chap. XXIV in: Vorlesungen über Geschichte der Mathematik, herausgeg. von M. Cantor, Bd. **IV**, von 1759 bis 1799 (Teubner, Leipzig, 1908) pp. 451–576; here pp. 572–576.

[31] J. L. de Lagrange, Sur la Construction des Cartes Géographiques; Œuvres de Lagrange **4** (Gauthier-Villars, Paris, 1869) here pp. 637–692; available from Gallica; http://gallica.bnf.fr/ . German translation and edition by A. Wangerin: Über Kartenprojection. Abhandlungen von Lagrange (1779) and Gauss (1822). (Ostwald's Klassiker der Exakten Wissenschaften **55**; Verlag W. Engelmann, Leipzig, 1894).

[32] C. F. Gauss, Allgemeine Auflösung der Aufgabe: Die Theile einer gegebenen Fläche auf einer anderen gegebenen Fläche so abzubilden dass die Abbildung dem Abgebildeten in den kleinsten Theilen ähnlich wird; Carl Friedrich Gauss, Werke **4** (Königl. Gesellsch. Wiss. Göttingen, Göttingen, 1873) pp. 189–216; available from Göttinger Digitalisierungszentrum (GDZ): http://gdz.sub.uni-goettingen.de/; from Gallica: gallica.bnf.fr/; from Internet Archive: http://www.archive.org/.
English translation: C. F. Gauss, General solution of the Problem: to represent the Parts of a given Surface on another given Surface, so that the smallest Parts of the Representation shall be similar to the corresponding Parts of the Surface represented; The Philosophical Magazine and Annals of Philosophy, New Ser., **IV**, 104–113 and 206–215 (1828); available from Internet Archive: http://www.archive.org/. French translation by L. Laugel: Solution Générale de ce Problème: Représenter les... [Représentation Conforme] par C.-F. Gauss (Hermann & Fils, Paris, 1915); available from Gallica (under Gauss): http://gallica.bnf.fr/.

[33] C. F. Gauss, Untersuchungen über Gegenstände der Höheren Geodaesie, Gauss, Werke **4** (see [32]) pp. 261–300; here p. 262: "... und ich werde daher dieselben [Abbildungen] *conforme* Abbildungen oder Übertragungen nennen, indem ich diesem sonst vagen Beiworte eine mathematisch scharf bestimmte Bedeutung beilege." (*italics* by Gauss himself). (... I, therefore, shall call these [maps] *conformal* maps or assignments by giving that otherwise vague attribute a sharp mathematical meaning.)

[34] F. T. Schubert, De proiectione sphaeroidis ellipticae geographica. Dissertatio prima. (presented on May 22, 1788), Nova Acta Academiae Scientiarum Imperialis Petropolitanae **V** (Petropoli Typis Academiae Scientiarum, 1789) pp. 130–146. The paper deals with possible corrections to the known stereographic projections from a sphere to a plane if one replaces the sphere by a rotationally symmetric spheroid like in the case of the earth. The meridians now become ellipses and the question is whether they, too, can be mapped onto a plane, preserving angles. In this context Schubert speaks on p. 131 of "proiectio figurae ellipticae conformis". I found the reference to Schubert in the article by Kommerell mentioned before [30], pp. 575–576. As to Schubert see the article by W. R. Dick in: Neue Deutsche Biographie **23** (Duncker & Humblot, Berlin, 2007) pp. 604–605.

[35] B. Riemann, Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse, in: Bernhard Riemann, Collected Papers, newly ed. by R. Narasimhan (Springer, Berlin etc.; Teubner Verlagsgesellschaft, Leipzig, 1990) pp. 35–80 (pp. 3–48 in the edition from 1892); older edition available from UMDL: http://quod.lib.umich.edu/u/umhistmath/.

[36] See, e.g. L. V. Ahlfors, Complex Analysis, An Introduction to the Theory of Analytic Functions of One Complex Variable, 3rd Ed. (McGraw-Hill, New York etc., 1979) here Chap. 6 with the details of assumptions and proof.

[37] As to the prehistory (d'Alembert, Euler, Laplace) and Cauchy's part in formulating the conditions see
B. Belhoste, Augustin-Louis Cauchy, A Biography (Springer, New York etc., 1991);
F. Smithies, Cauchy and the Creation of Complex Function Theory (Cambridge University Press, Cambridge UK, 1997).
As to the relations between Cauchy and Gauss see
K. Reich, Cauchy and Gauß. Cauchys Rezeption im Umfeld von Gauß; Arch. Hist. Exact Sci. **57**, 433–463 (2003).

[38] See , e.g. as one of many possible examples:
R. Schinzinger and P. A. A. Laura, Conformal Mapping: Methods and Applications (Elsevier, Amsterdam etc., 1991).

[39] C. F. Gauss, Disquisitiones generales circa superficies curvas, Werke **4** (see [32]) pp. 217–258; German translation and edition by A. Wangerin: Allgemeine Flächentheorie (Disquisitiones...) von Carl Friedrich Gauss (1827) (Ostwald's Klassiker der Exakten Wissenschaften **5**, Verlag Wilhelm Engelmann, Leipzig, 1889); English translation with notes and a bibliography by J.C. Morehead and A.M. Hiltebeitel: Karl Friedrich Gauss, General Investigations of Curved Surfaces of 1827 and 1825 (The Princeton University Library, Princeton, 1902); available from Internet Archive: http://www.archive.org/. A French translation of 1852 is available from Gallica: http://gallica.bnf.fr/.

[40] G. Monge, Application de l'analyse à la géométrie: à l'usage de l'École Impériale Polytechnique, 4. éd. (Bernard, Paris, 1809); previously (1795, 1800/1801, 1807) published as: Feuilles d'Analyse appliquée à la Géométrie, à l'usage de l'École Polytechnique, Paris. We shall encounter the 5th edition from 1850, newly edited by Liouville, below [67].

[41] An excellent survey on the history of synthetic geometry is
E. Kötter, Die Entwicklung der synthetischen Geometrie von Monge bis auf Staudt (1847); Jahresber. d. Deutschen Mathematiker-Vereinig. **5** (1901), 2. Heft (about 500 pages); available (under "Jahresberichte...") from GDZ: http://gdz.sub.uni-goettingen.de/.

[42] A typical example for "synthetic" reasoning in our context is an article by Jean Nicolas Pierre Hachette (1769–1834), published in 1808, in which he proves that stereographic projections (which he calls "perspective d'une sphère") have the properties: 1. circles are mapped onto circles and 2. angles are preserved. There is not a single formula, only a reference to a figure and to certain of its points identified by capital letters:
J. N. P. Hachette, De la perspective d'une sphère, dans laquelle les cercles tracés sur cette sphère sont représentés par d'autres cercles; Correspondence sur l'École Impériale Polytechnique, Avril 1804–Mai 1808, t. **1**, 362–364 (1808); The "Correspondance" was edited by Hachette, mainly with articles of himself.

[43] B. C. Patterson, The origins of the geometric principle of inversion, Isis **19**, 154–180 (1933).

[44] Jacob Steiners Gesammelte Werke, Bde. 1 and 2, hrsg. von K. Weierstrass (G. Reimer, Berlin, 1881 and 1882).

[45] F. Bützberger, Über Bizentrische Polynome, Steinersche Kreis- und Kugelreihen und die Erfindung der Inversion (Teubner, Leipzig, 1913); I was not able to see this rare book, but learnt about its content from the summary by A. Emch, The discovery of inversion; Bull. Amer. Math. Soc. **20**, 412–415 (1914); in 1928 Emch examined Steiner's unpublished notes and manuscripts himself in Bern: A. Emch, Unpublished Steiner manuscripts; Amer. Math. Monthly **36**, 273–275 (1929).

[46] Jakob Steiner, Allgemeine Theorie über das Berühren und Schneiden der Kreise und der Kugeln, hrsg. von R. Fueter and F. Gonseth (Veröff. Schweiz. Mathem. Gesellschaft **5**, Orell Füssli Verlag, Zürich und Leipzig, 1931).

[47] J. B. Durrande, Géométrie élémentaire. Théorie élémentaire des contacts des cercles, des sphères, des cylindres et des cônes; Annales de Mathématiques pures et appliquées **11**, 1–67 (1820–21); the paper was published on July 1, 1820. In the following the journal will briefly be called "Annales de Gergonne" (see [49] below); all volumes of that journal are available from NUMDAM: http://www.numdam.org/.

[48] After the introduction Durrande starts his main text (§. 1. Des pôles et polaires, [on p. 5]) with:
"1. Nous appellerons, à l'avenire, *pôles conjugués* d'un cercle, deux points en ligne droite avec son centre, et du même côté de ce centre, tels que le rayon du cercle sera moyen proportionnel entre leurs distance à son centre." Durrande does not give a formula, but that he means the relation (39) in my text above is evident from what he says about immediate consequences: "... 2.$^\circ$ que de ce deux points l'un est toujours intérieur au cercle

et l'autre extérieur au cercle, de telle sort que, plus l'un s'eloigne du centre, plus l'autre s'en approche; 3.° que le sommet d'un angle circonscrit au cercle et le milieu de sa corde de contact sont deux pôles conjugués l'un à l'autre. " In the notation used in Eq. (39) the last means that $\hat{r}/r_0 = r_0/r$ (see Fig. 3). In Sect. II of the paper the concept is generalized to 3 dimensions, spheres etc. Thus, there can be no doubts that Durrande deserves the credit for priority. Patterson mentions only Durrande's last paper from 1825 (see [50]) on page 179 of his article and appears to have overlooked the one from July 1820. There is a lot of overlap between Durrande's article of 1820 and Steiner's manuscript [46] which does not mention Durrande at all. Perhaps it was his knowledge of Durrande's work which let Steiner hesitate to publish that manuscript of his own!

[49] The Annales de Mathématiques pures et appliquées appeared from 1810 to 1832 (in Nîmes) and were personally founded, lively and critically edited by the French mathematician Joseph Diaz Gergonne (1771–1859), a student of Gaspard Monge (see [40] and [67]), and were therefore called "Annales de Gergonne".

[50] J. B. Durrande [called "feu", i.e. deceased], Solution de deux des quatre problèmes de géometrie proposés à la page 68 du XI.^{e} volume des Annales, et deux autres problèmes analogues; Annales de Gergonne **16**, 112–117 (1825/26) the issue IV of the journal was published in Oct. 1925; at the end of the article is a footnote by Gergonne: "M. Durrande, déja très-gravement malade lorsqu'il nous adressa ce qu'on vient de lire, nous avait annoncé la solution de deux autres problèmes de l'endroit cité. Il a terminé sans carrière sans l'avoir pu mettre par écrit."

[51] J. B. Durrande, Solution du premier des deux problèmes de géométrie proposés à la page 92 de ce volume; Annales de Gergonne **5**, 295–298 (1814/15) publ. March 1815; Gergonne's footnote on page 295: " M. Durrande est un géomètre de 17 ans, qui a appris les mathématiques sans autre secours que celui des livres."

[52] J. B. Durrande, Géométrie élémentaire. Démonstration des propriétés des quadrilatères à la fois inscriptibles et circonscribtibles au cercle; Annales de Gergonne **15**, 133–145 (1824/25).

[53] I haven't found any obituary or similar notice concerning Durrande. Michel Chasles (1793–1880) in his Aperçu mentions just one paper by Durrande in a footnote: M. Chasles, Aperçu Historique sur l'Origine et le Développement des Méthodes en Géométrie particulièrement de Celles qui se Rapportent à la Géométrie Moderne..., 3. Édition, conforme à la première (Gauthier-Villars, Paris, 1889) p. 238. The Grande Encyclopédie from 1892 has an article on the mathematician A. Durrande, born Nov. 1831, which ends with the remarks "Un de ses parents [relatives], *Jean-Baptiste* Durrande, géomètre précoce, mort en 1825 à vingt-sept ans, fut élève et le collaborateur de Gergonne et publia, dans les Annales de ce dernier, huit mémoires remarquables de géométrie: le premier et de 1816.": La Grande Encyclopédie Inventaire Raisonné des Sciences, des Lettres et des Arts par une Société de Savants et de Gens de Lettres **15**, Duel – Eœtvœs, (Société Anonyme de la Grande Encyclopédie, Paris, 1892) p. 131.

[54] For more details on the work of Dandelin and Quetelet see Patterson [43]. The Nouveaux Mémoires de L'Académie Royal des Sciences et Belles-Lettres de Bruxelles are available from GDZ: http://gdz.sub.uni-goettingen.de/.

[55] P. G. Dandelin, Sur les intersections de la sphère et d'un cône de second degré; Nouveaux Mémoires de L'Académie Royal des Sciences et Belles-Lettres de Bruxelles **4**, 1–10 (1827).

[56] A. Quetelet, Résumé d'une nouvelle théorie des caustiques, suivi de différentes applications à la théorie des projections stéréographiques; Nouveaux Mémoires de L'Académie Royal des Sciences et Belles-Lettres de Bruxelles **4**, 79–109 (1827); the appended Note is on pp. 111–113.

[57] J. Plücker, Analytisch-Geometrische Entwicklungen, 1. Band (G.D. Baedeker, Essen, 1828), here pp. 93 .... Plücker's preface is from Sept. 1827; book available from UMDL: http://quod.lib.umich.edu/u/umhistmath/. Plücker later discussed the mapping in more detail: J. Plücker, Analytisch-geometrische Aphorismen, V. Über ein neues Übertragungs-Princip; Journ. reine u. angew. Mathem. **11**, 219–225 (1834); available from GDZ: http://gdz.sub.uni-goettingen.de/.

[58] G. Bellavitis, Teoria delle figure inverse, e loco uso nella Geometria elementare; Annali delle Scienze del Regno Lombardo-Veneto, Opera Periodica, **6**, 126–141 (1836); available from Google Book Search (copied from the Harvard Library). The associated figures at the end of the volume are very badly reproduced!
G. Bellavitis, Saggio di geometria derivata; Nuovo saggi della Reale Academia di Scienze, Lettere ed Arti di Padova **4**, 243–288 (1838).

[59] An impressive but quite subjective summary of the historical developments is given in the classical textbook by Felix Klein:
F. Klein, Vorlesungen über höhere Geometrie, 3. Aufl., bearb. und hrsg. von W. Blaschke (Die Grundlehren der mathematischen Wissenschaften **22**, Verlag Julius Springer, Berlin, 1926).

Good textbooks on the subject from the beginning of the 20th century are
K. Doehlemann, Geometrische Transformationen, II. Theil: Die quadratischen und höheren, birationalen Punkttransformationen (Sammlung Schubert **28**, G. J. Göschen'sche Verlagshandlung, Leipzig, 1908); available from UMDL: http://quod.lib.umich.edu/u/umhistmath/.
J. L. Coolidge, A Treatise on the Circle and the Sphere (At the Clarendon Press, Oxford, 1916); also available from UMDL: http://quod.lib.umich.edu/u/umhistmath/.
A useful historical overview is also contained in
J. L. Coolidge, A History of Geometrical Methods (At the Clarendon Press, Oxford, 1940).
Julian Lowell Coolidge (1873–1954) was a younger colleague at Harvard of Maxime Bôcher who worked with Klein in Göttingen around 1890, won a prize there for an essay on the subject which concerns us here. That essay was acknowledged afterwards as a Ph.D. Thesis by the "Philosophische Fakultät der Universität Göttingen", with Felix Klein as the referee. Bôcher extended that work to a beautiful book which we will encounter below [93]. As to Bôcher see [93] below, too.

[60] Quite helpful and informative for the subject of the present paper are also several articles in the Encyklopädie der Mathematischen Wissenschaften, mit Einschluss ihrer Anwendungen, III. Band in 3 Teilen: Geometrie, redig. von W. Fr. Meyer u. H. Mohrmann (Teubner, Leipzig, 1907–1934); Bd. III, 1. Teil, 1. Hälfte: article 4b. by G. Fano, article 7. by E. Müller; Bd. III, 1. Teil, 2. Hälfte: article 9. by M. Zacharias; Bd. III, 2. Teil, 2. Hälfte. B.: article 11. by L. Berzolari; Bd. III, 3. Teil: article 6. by A. Voss, article 7. by H. Liebmann, article 9. by E. Salkowski.
All volumes are available from GDZ: http://gdz.sub.uni-goettingen.de/.

[61] A. F. Möbius, Ueber eine neue Verwandtschaft zwischen ebenen Figuren; originally published in 1853, reprinted in: August Ferdinand Möbius, Gesammelte Werke **2**, hrsg. von F. Klein (Verlag von S. Hirzel, Leipzig, 1886) pp. 205–218; available from Gallica-Mathdoc: http://portail.mathdoc.fr/OEUVRES;
Die Theorie der Kreisverwandtschaft in rein geometrischer Darstellung; originally publ. in 1855, reprinted in Ges. Werke **2**, pp. 243–314;
as to the later analytical developments see [59, 60] and, e.g. C. Carathéodory, Conformal Representation, 2nd ed. (Cambridge Tracts in Mathematics and Mathematical Physics **28**, Cambridge University Press, Cambridge, 1952).

[62] As to the life and work of William Thomson see:
S. P. Thompson, The Life of William Thomson, Baron Kelvin of Largs, in two volumes (Macmillan and Co., London, 1910) here vol. I, Chap. III. Biography available under http://www.questia.com/PM.qst.

[63] As to Liouville see the impressive biography by Lützen: J. Lützen, Joseph Liouville 1809–1882: Master of Pure and Applied Mathematics (Studies in the History of Mathematics and Physical Sciences **15**, Springer, New York etc., 1990); Liouville's relations to Thomson are discussed on pp. 135–146 and pp. 727–733.
That period of differential geometry is also covered in
K. Reich, Die Geschichte der Differentialgeometrie von Gauß bis Riemann (1828–1868); Arch. Hist. Exact Sciences **11**, 273–382 (1973).

[64] Extrait d'une lettre de M. William Thomson à M. Liouville; Journ. Mathém. Pure et Appliquées **10**, 364–367 (1845); the journal is available from Gallica: http://gallica.bnf.fr/.

[65] Extraits de deux lettres adressées à M. Liouville, par M. William Thomson; Journ. Mathém. Pure et Appliquées **12**, 256–264 (1847);
Thomson's three letters and Liouville's comments are reprinted in: W. Thomson, Reprints of Papers on Electrostatics and Magnetism (Macmillan & Co., London, 1872) article XIV: Electric Images; pp. 144–177; the text is available from Gallica: http://gallica.bnf.fr/.
Thomson might have been inspired to use the inversion by an article of Stubbs in the Philosophical Magazine: J. W. Stubbs, On the application of a New Method to the Geometry of Curves and Curve Surfaces; The London, Edinburgh, and Dublin Philos. Magaz. and Journ. of Science **23**, 338–347 (1843); the journal is available from Internet Archive: http://www.archive.org/. That article and two related others by J. K. Ingram were presented in 1842 to the Dublin Philosophical Society, see Patterson, [43], pp. 175/76.
That reprint volume of Thomson's papers contains several early publications (in the Cambridge and Dublin Mathematical Journal) by him to problems in electrostatics in which he applies the inversion: reprints V: On the mathematical theory of electricity in equilibrium (pp. 52–85).

[66] Note au sujet de l'article précédent, par J. Liouville; Journ. Mathém. Pure et Appliquées **12**, 265–290 (1847).

[67] Liouville published his proof in an appendix to a new edition of Monge's book ([40]), the publication of which he had organized: G. Monge, Application de l'Analyse à la Géométrie, 5. édition, revue, corrigée et annotée par M. Liouville (Bachelier, Paris, 1850), Note VI (pp. 609–616): Extension au cas des trois dimensions de la question du tracé géographique. The book is available from Google Book Search: http://books.google.de/ (copied from the Harvard Library).
In his proof Liouville makes use of an earlier paper by Gabriel Lamé (1795–1870): G. Lamé, Mémoire sur les coordinées curvilignes; Journ. Mathém. Pure et Appliquées **5**, 313–347 (1840). In 1850 Liouville published a 1-page summary of his result: J. Liouville, Théorème sur l'équation $dx^2 + dy^2 + dz^2 = \lambda\,(d\alpha^2 + d\beta^2 + d\gamma^2)$; Journ. Mathém. Pure et Appliquées **15**, 103 (1850); here he refers to another paper by Lamé: G. Lamé, Mémoire sur les surfaces orthogonales et isothermes; Journ. Mathém. Pure et Appliquées **8**, 397–434 (1843); and to a more recent paper by Pierre Ossian Bonnet (1819–1892): O. Bonnet, Sur les surfaces isothermes et orthogonales; Journ. Mathém. Pure et Appliquées **14**, 401–416 (1849);
This important part of Liouville's work is described by Lützen, [63], on pp. 727–733. In his version of the early history of the transformation by reciprocal radii Lützen relies essentially on Patterson, [43]. with its partial deficiency of overlooking the work by Durrande, see Sect. 2.2 above.
[68] S. Lie, Ueber diejenige Theorie eines Raumes mit beliebig vielen Dimensionen, die der Krümmungs-Theorie des gewöhnlichen Raumes entspricht; Nachr. Königl. Gesellsch. Wiss. u. d. G. - A. - Univ. Göttingen **1871**, 191–209; available from GDZ: http://gdz.sub.uni-goettingen.de ; reprinted in: S. Lie, Gesammelte Abhandlungen I, hrsg. von F. Engel u. P. Heegaard (Teubner, Leipzig, 1934 and H. Aschehoug & Co., Oslo, 1934) pp. 215–228; editorial comments on this paper in: Sophus Lie, Gesam. Abh., Anmerk. z. ersten Bde., hrsg. von F. Engel u. P. Heegaard (Teubner, Leipzig, 1934 and H. Aschehoug & Co., Oslo, 1934) pp. 734–743. The editors question here that Lie has actually shown what he later claimed to be a generalization of Liouville's result.
[69] S. Lie, Ueber Complexe, insbesondere Linien- und Kugel-Complexe, mit Anwendung auf die Theorie partieller Differentialgleichungen; Mathem. Annalen **5**, 145–256 (1872) here pp. 183–186; available from GDZ: http://gdz.sub.uni-goettingen.de; reprinted in: S. Lie, Gesamm. Abhandl. II,1, hrsg. von F. Engel u. P. Heegaard (Teubner, Leipzig, 1935 and H. Aschehoug & Co., Oslo, 1935) pp. 1–121; editorial comments on this paper in: Sophus Lie, Gesam. Abh., Anmerk. z. zweiten Bde., hrsg. von F. Engel u. P. Heegaard (Teubner, Leipzig, 1937 and H. Aschehoug & Co., Oslo, 1937) pp. 854-901.
[70] S. Lie, Untersuchungen über Transformationsgruppen. II; Archiv f. Mathematik og Naturvidenskab **10**, 353–413 (Kristiania 1886); reprinted in: S. Lie, Gesamm. Abhandl. V,1, hrsg. von F. Engel u. P. Heegaard (Teubner, Leipzig, 1924 and H. Aschehoug & Co., Oslo, 1924) pp. 507–552; editorial comments on that paper on pp. 747–753.
[71] S. Lie, unter Mitwirkung von F. Engel, Theorie der Transformationsgruppen III (Teubner, Leipzig, 1893; reprinted by Chelsea Publ. Co. New York, 1970), Kap. 17 u. 18 (pp. 314–360); here Lie also presents the algebraic structure of the generators for the infinitesimal transformations ("Lie algebra") and shows its dimensions to be $(n+1)(n+2)/2$.
[72] R. Beez, Ueber conforme Abbildung von Mannigfaltigkeiten höherer Ordnung; Zeitschr. Mathem. u. Physik: Organ f. Angew. Mathem. **20**, 253–270 (1875); Beez was not aware of Lie's papers [68] and [69].
[73] G. Darboux, Mémoire sur la théorie des coordonnées et des systèmes orthogonaux, III; Annales Scientif. de l'É.N.S. (Sér. 2) **7**, 275–348 (1878), here § XIII (pp. 282–286);
available from NUMDAM: http://www.numdam.org/.
[74] J. Levine, Groups of motions in conformally flat spaces; Bull. Amer. Math. Soc. **42**, 418–422 (1936) and **45**, 766–773 (1939).
[75] J.-N. Haton de la Goupillière, Méthodes de transformation en géométrie et en physique mathématique; Journ. l'École Impériale Polytechnique 25, 153–204 (1867).
[76] J. C. Maxwell, On the condition that, in the transformation of any figure by curvilinear coordinates in three dimensions, every angle in the new figure shall be equal to the corresponding angle in the original figure; Proc. London Mathem. Soc. (Ser. I) **4**, 117–119 (1872); Maxwell quotes only Haton de la Goupillière.
The paper is reprinted in: The Scientific Papers of James Clerk Maxwell, ed. by W.D. Niven, vol. II (At the University Press, Cambridge, 1890), paper L (pp. 297–300); text is available from; Internet Archives: http://www.archive.org/ or Gallica: http://gallica.bnf.fr/.
[77] L. Bianchi, Sulla trasformazione per raggi vettori reciproci nel piano e nelle spazio; Giorn. di Matematiche ad Uso degli Studenti delle Università Italiane **17**, 40–42 (1879); another proof is contained in:
L. Bianchi, Lezioni di Geometria Differenziale (Enrico Spoerri, Pisa, 1894) Cap. XVIII (pp. 450–468); the

proof of Liouville's theorem is in § 273 (pp. 460–462); text available from UMDL: http://quod.lib.umich.edu/u/umhistmath/. German translation: Vorlesungen über Differentialgeometrie (Teubner, Leipzig, 1899) available from GDZ: http://gdz.sub.uni-goettingen.de/.

[78] A. Capelli, Sulla limitata possibilità di trasformazioni conformi nello spazio; Annali di Matematica Pura ed Applicata (Ser. 2) **14**, 227–237 (1886/87). Capelli quotes Liouville [67], but the wrong note number (IV) and the wrong year (1860).

[79] E. Goursat, Sur les substitutions orthogonales et les divisions régulières de l'espace; Ann. Scient. l'École Norm. Sup. (Ser. 3) **6**, 9–102 (1889); available from NUMDAM: http://www.numdam.org/.

[80] Tait presented 3 papers on related problems to the Royal Society of Edinburgh, all in his beloved language of quaternions, the first one in 1872, the second one in 1877 and the third one, in which he recognizes the work of Liouville, in 1892. They are reprinted in his scientific papers: P. G. Tait, On orthogonal isothermal surfaces I; in Scientific Papers I (At the University Press, Cambridge, 1898) paper XXV (pp. 176–193). Note on vector conditions of integrability; in Scientific Papers I, paper XLIV (pp. 352–356). Note on the division of space into infinitesimal cubes; Scientific Papers II (appeared 1900), paper CV (pp. 329–332); texts available from Gallica: http://gallica.bnf.fr/.

[81] S. Lie and G. Scheffers, Geometrie der Berührungstransformationen I (Teubner, Leipzig, 1896) here Kap. 10.

[82] A. Giacomini, Sulla inversione per raggi vettori reciproci; Giorn. di Matematiche ad Uso degli Studenti delle Università Italiane (Ser. 2) **35**, 125–131 (1897).

[83] A. R. Forsyth, On those transformations of the coordinates which lead to new solutions of Laplace's equation; Proc. London Math. Soc. (Ser. I) **29**, 165–206 (1897/98).

[84] J. E. Campbell, On the transformations which do not alter the equation $\partial_x^2 U + \partial_y^2 U + \partial_z^2 U = 0$; Messenger Mathem. **28**, 97–102 (1898).

[85] T. J. I'A. Bromwich, Conformal space transformations; Proc. London Math. Soc. (Ser. I) **33**, 185–192 (1900).

[86] G. Darboux, Sur les transformations conformes de l'espace à trois dimensions; Archiv Mathematik u. Physik (3. Reihe) **1**, 34–37 (1901).

[87] J. C. Maxwell, A Treatise on Electricity and Magnetism, vol. I (At the Clarendon Press, Oxford, 1873) here Chap. XI: Theory of electric images and electric inversion. (pp. 191–225). Maxwell's paper from 1872 [76] is obviously related to this chapter. Maxwell's Treatise is available from Gallica: http://gallica.bnf.fr/.

[88] Review of: Electrostatics and Magnetism, *Reprint of Papers on Electrostatics and Magnetism*. By Sir W. Thomson,.... (London: Macmillan and Co., 1872); Nature **7**, 218–221 (1872/73), issue no. 169 from Jan. 1873. No author of that review is mentioned, but it is reprinted as article LI (pp. 301–307) in vol. II of Maxwell's Scientific Papers [76].

[89] Indeed, in 1860 Carl Gottfried Neumann (1832–1925) (the one from the boundary conditions) rediscovered the usefulness of the inversion for solving problems in potential theory: C. Neumann, Geometrische Methode, um das Potential der, von einer Kugel auf innere oder äussere Punkte ausgeübten, Wirkung zu bestimmen; Annalen der Physik und Chemie (2. Folge) **109**, 629–632 (1860); available from Gallica: http://gallica.bnf.fr/. A year later he published a brochure of 10 pages on the subject: C. Neumann, Lösung des allgemeinen Problems über den stationären Temperaturzustand einer homogenen Kugel ohne Hülfe von Reihen-Entwicklungen: nebst einigen Sätzen zur Theorie der Anziehung (Verlag Schmidt, Halle/Saale, 1861). In a book from 1877 Neumann acknowledges Thomson's priority: C. Neumann, Untersuchungen über das logarithmische und Newton'sche Potential (Teubner, Leipzig, 1877) The inversion is discussed on pp. 54–68 and 355–359; the reference to Thomson is in a footnote on page 55; available from UMDL: http://quod.lib.umich.edu/u/umhistmath/.

[90] G. Darboux, Sur une nouvelle série de système orthogonaux algébrique; Comptes Rendus l'Académie des Sciences **69**, 392–394 (1869); available from Gallica: http://gallica.bnf.fr/.

[91] G. Darboux, Sur une classe remarquable de courbes et de surfaces algébriques et sur la théorie des imaginaires (Gauthier-Villars, Paris, 1873); available from Gallica: http://gallica.bnf.fr/. Second printing in 1896 by A. Hermann, Paris; available from Cornell Univ. Library Historical Math Monographs: http://dlxs2.library.cornell.edu/m/math/.

[92] F. Pockels, Über die Partielle Differentialgleichung $\Delta u + k^2 u = 0$ und deren Auftreten in der Mathematischen Physik, mit einem Vorwort von Felix Klein (Teubner, Leipzig, 1891) here pp. 195–206; available from Cornell Univ. Library Historical Math Monographs: http://dlxs2.library.cornell.edu/m/math/.

[93] M. Bôcher, Ueber die Reihenentwicklungen der Potentialtheorie, mit einem Vorwort von Felix Klein (Teubner, Leipzig, 1894).

As to M. Bôcher see G. D. Birkhoff, The scientific work of Maxime Bôcher; Bull. Amer. Math. Soc. **25**, 197–215 (1919); available under http://www.math.harvard.edu/history/bocher/ ;
W. F. Osgood, The life and services of Maxime Bôcher; Bull. Amer. Math. Soc. **25**, 337–350 (1919).

[94] H. A. Kastrup, Zur physikalischen Deutung und darstellungstheoretischen Analyse der konformen Transformationen von Raum und Zeit; Ann. Physik (Leipzig), 7. Folge, **9**, 388–428 (1962).

[95] H. A. Kastrup, Gauge Properties of the Minkowski Space; Phys. Rev. **150**, 1183–1193 (1966).

[96] G. Darboux, Leçons sur la Théorie Générales des Surfaces et les Applications Géométriques du Calcul Infinitésimal, Partie I (Gauthier-Villars, Paris, 1887) here p. 213.
Available from UMDL: http://quod.lib.umich.edu/u/umhistmath/.
Equivalent coordinates were also introduced in 1868 in a short (unpublished) note by William Kingdon Clifford (1845–1879), On the powers of spheres; in: W. K. Clifford, Mathematical papers, ed. by R. Tucker (Macmillan and Co., London, 1882) paper 34, pp. 332–336.

[97] G. Darboux, Sur l'application des méthodes de la physique mathématique à l'étude des corps terminés par des cyclides; Comptes Rendus de l'Académie des Sciences **83**, 1037–1040, 1099–1102 (1876); available from Gallica: http://gallica.bnf.fr/.

[98] P. A. M. Dirac, Wave equation in conformal space; Ann. Math. **37**, 429–442 (1936); Dirac does not mention that he uses the polyspherical coordinates, invented in the 19th century. But at the end of his paper he thanks the mathematician Oswald Veblen, "whose general theory of conformal space suggested the present, more special, methods and...". Veblen had used those coordinates about the same time:
O. Veblen, Geometry of four-component spinors; Proc. Nat. Acad. Sci. USA **19**, 503–517 (1933);
Formalism for conformal geometry; Proc. Nat. Acad. Sci. USA **21**, 168–173 (1935);
see also [143].

[99] G. Mack and A. Salam, Finite-component field representations of the conformal group; Ann. Phys. (N.Y.) **53**, 174–202 (1969).

[100] T. H. Go and D. H. Mayer, The Klein-Gordon operator in conformal space; Rep. Math. Phys. **5**, 187–202 (1974);
D. H. Mayer, Vector and tensor fields on conformal space; Journ. Math. Phys. **16**, 884–893 (1975).

[101] See also the reviews [195, 197] and the literature quoted in Sect. 7.2.

[102] L. O'Raifeartaigh, The Dawning of Gauge Theory (Princeton Series in Physics, Princeton University Press, Princeton NJ, 1997); contains edited reprints and translations of important papers in the development of gauge theories.

[103] L. O'Raifeartaigh and N. Straumann, Gauge theory: Historical origins and some modern developments; Rev. Mod. Phys. **72**, 1–23 (2000).

[104] Hermann Weyl's *Raum – Zeit – Materie* and a General Introduction to His Scientific Work, ed. by E. Scholz (DMV Seminar **30**, Birkhäuser Verlag, Basel etc., 2001) with contributions by R. Coleman and H. Korté, H. Goenner, E. Scholz, S. Sigurdsson, N. Straumann.

[105] H. F. M. Goenner, On the History of Unified Field Theories; Living Reviews,
http://www.livingreviews.org/lrr-2004-2.

[106] The term first appears in the third of a series of Weyl's papers on the subject:
H. Weyl, Eine neue Erweiterung der Relativitätstheorie; Ann. Physik (Leipzig), IV. Folge, **59**, 101–133 (1919); reprinted in Hermann Weyl, Gesamm. Abhandl. (abbr. as HWGA in the following), hrsg. von K. Chandrasekharan, vol. II (Springer, Berlin etc., 1968) here paper 34, pp. 55–87.

[107] Letter of Weyl to Einstein, The Collected Papers of Albert Einstein (abbr. CPAE in the following), ed. R. Schulmann et al., vol. **8 B** (Princeton Univ. Press, Princeton NJ, 1998) here Doc. 472.

[108] CPAE **8 B**, Doc. 476.

[109] CPAE **8 B**, Doc. 498.

[110] CPAE **8 B**, Doc. 499.

[111] CPAE **8 B**, Doc. 507.

[112] H. Weyl, Gravitation und Elektrizität; Sitzungsber. Königl. Preuß. Akad. Wiss. Berlin **1918**, 465–480; reprinted in HWGA II, paper 31, pp. 28–42 (with Einstein's comment, Weyl's reply to that and with a comment by Weyl from 1955). English translation in [102], pp. 24–37. Einstein's comment is also reprinted in CPAE **7** (The Berlin Years: Writings 1918–1921), Doc. 8.

[113] CPAE **8 B**, Doc. 626.

[114] CPAE **8 B**, Doc. 669.

[115] CPAE **8 B**, Doc. 673.

[116] E. Schrödinger, Über eine bemerkenswerte Eigenschaft der Quantenbahnen eines einzelnen Elektrons; Zeitschr. Physik **12**, 13–23 (1922); partial English tranlation in [102], pp. 87–93.

[117] F. London, Quantenmechanische Deutung der Theorie von Weyl; Zeitschr. Physik **42**, 375–389 (1927). English translation in [102], pp. 94–106.

[118] H. Weyl, Elektron und Gravitation. I.; Zeitschr. Physik **56**, 330–352 (1929); reprinted in HWGA III, paper 85, pp. 245–267. English Translation in [102], pp. 121–144.

[119] H. Weyl, Geometrie und Physik; Die Naturwiss. **19**, 49–58 (1931); reprinted in HWGA III, paper 93, pp. 336–345.

[120] T. Kaluza, Zum Unitätsproblem der Physik; Sitzungsber. Königl. Preuß. Akad. Wiss. Berlin **1921**, 966–972. English translation in [102], pp. 53–58.

[121] HWGA II (see [106]) contains most of Weyl's papers on the subject.

[122] H. Weyl, Space – Time – Matter, translation of the 4th German edition from 1921 by H. L. Brose (Methuen, London, 1922); the latest German edition is:
Raum, Zeit, Materie: Vorlesungen über allgemeine Relativitätstheorie, 7. Aufl., hrsg. von J. Ehlers (Heidelberger Taschenbücher **251**, Springer, Berlin etc., 1988).
Weyl discussed his ideas also in his Barcelona/Madrid lectures:
H. Weyl, Mathematische Analyse des Raumproblems: Vorlesungen gehalten in Barcelona und Madrid (Springer, Berlin, 1923).

[123] H. Weyl, Reine Infinitesimalgeometrie; Mathem. Zeitschr. **2**, 384–411 (1918) here §4., III. Reprinted in HWGA II, paper 30, pp. 1–28.

[124] A. Einstein, Über eine naheliegende Ergänzung des Fundamentes der allgemeinen Relativitätstheorie; Sitzungsber. Königl. Preuß. Akad. Wiss. Berlin **1921**, pp. 261–264. Reprinted in CPAE **7** (The Berlin Years: Writings 1918–1921), Doc. 54.

[125] J. A. Schouten, Ricci-Calculus, 2nd Ed. (Die Grundlehren der Mathem. Wiss. in Einzeldarst. **10**, Springer-Verlag, Berlin etc., 1954) contains an extensive list of references on conformal geometries.

[126] R. M. Wald, General Relativity (The University of Chicago Press, Chicago and London,1984) here Appendix D.

[127] R. Penrose, Conformal treatment of infinity; in: Relativity, Groups and Topology, Lectures delivered at Les Houches during the 1963 Summer School of Theoretical Physics, ed. by C. DeWitt and B. DeWitt (Gordon and Breach Science Publishers, New York etc.,1964) here pp. 563–584;
B. Carter, Black hole equilibrium states; in: Black Holes, Les Houches Summer School, August 1972, ed. by C. DeWitt and B. S. DeWitt (Gordon and Breach Science Publishers, New York etc., 1973) here pp. 57–214;
S. W. Hawking and G. F. R. Ellis, The large scale structure of space-time (At the University Press, Cambridge, 1973) here Sect. 6.9;
C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation (Freeman and Co., San Francisco, 1973) here Chap. 34;
R. Wald, [126] here Chap. 11.

[128] J. Frauendiener, Conformal Infinity; Living Reviews, http://www.livingreviews.org/lrr-2004-1.

[129] See [127], especially Hawking and Ellis, here Chap. 6; and
Wald, [126] Chap. 8.

[130] E. Noether, Invariante Variationsprobleme. (F. Klein zum fünfzigjährigen Doktorjubiläum.); Nachr. Königl. Gesellsch. Göttingen, Math. Physik. Klasse **1918**, 235–257; available from GDZ: http://gdz.sub.uni-goettingen.de; reprinted in: Emmy Noether, Collected Papers, hrsg. von N. Jacobson (Springer. Berlin etc., 1983) here paper 13, pp. 248–270; English translation of Noether's article by M. A. Tavel, in: Transport Theory and Statistical Physics **1**, 183–207 (1971).

[131] The history of conservation laws before and after E. Noether's seminal paper has been discussed in the following conference contribution:
H. A. Kastrup, The contributions of Emmy Noether, Felix Klein and Sophus Lie to the modern concept of symmetries in physical systems; in: Symmetries in Physics (1600–1980), 1st Intern. Meeting on the History of Scient. Ideas, Sant Feliu de Guíxols, Catalonia, Spain, Sept. 20–26, 1983, ed. by M.G. Doncel, A. Hermann, L. Michel, and A. Pais (Seminari d'Història de les Ciències, Universitat Autònoma de Barcelona, Bellaterra (Barcelona) Spain, 1987) here pp. 113–163.

[132] E. Bessel-Hagen, Über die Erhaltungssätze der Elektrodynamik; Mathem. Ann. **84**, 258–276 (1921); available from GDZ: http://gdz.sub.uni-goettingen.de/.

[133] F. Engel, Über die zehn allgemeinen Integrale der klassischen Mechanik; Nachr. Königl. Gesellsch. Göttingen, Math. Physik. Klasse **1916**, 270–275; available from GDZ: http://gdz.sub.uni-goettingen.de/.

[134] F. Engel, Nochmals die allgemeinen Integrale der klassischen Mechanik; Nachr. Königl. Gesellsch. Göttingen, Math. Physik. Klasse **1917**, 189–198; available from GDZ: http://gdz.sub.uni-goettingen.de/.

[135] H. Poincaré, Sur le problème des trois corps et les équations de la dynamique; Acta Mathem. **13**, 1–270 (1890) here pp. 46–52; reprinted in: Œuvres de Henri Poincaré, t. **VII**, ed. by J. Lévy (Gauthier-Villars, Paris, 1952) here paper 21, pp. 262–479; available from UMDL: http://quod.lib.umich.edu/u/umhistmath/.

[136] L. D. Landau and E. M. Lifshitz, Mechanics (Vol. 1 of *Course of Theoretical Physics*, Pergamon Press, Oxford etc.,1960) here §10.

[137] More on scale and conformal symmetries in non-relativistic and relativistic systems is contained in the (unpublished but available) notes of my guest lectures given during the summer term 1971 at the University of Hamburg and at DESY. See also
V. de Alfaro, S. Fubini, and G. Furlan, Conformal invariance in quantum mechanics; Il Nuovo Cim. (Ser. 11) **34 A**, 569–612 (1976).

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The connection between the trace of the energy-momentum tensor and the possible scale invariance of the system is more subtle than Pauli appears to suggest: A free massless scalar field $\varphi(x)$ in 4 space-time dimensions has the simple Lagrangean density $L = (\partial_\mu\varphi\,\partial^\mu\varphi)/2$ which yields the symmetrical canonical energy-momentum tensor (see Eq. (113)) $T_{\mu\nu} = \partial_\mu\varphi\,\partial_\nu\varphi - \eta_{\mu\nu}\,L$ with the non-vanishing trace $-\partial_\mu\varphi\,\partial^\mu\varphi$. Nevertheless the action integral is invariant under $\delta x^\mu = \gamma\,x^\mu$, $\delta\varphi = -\gamma\,\varphi$ which, according to Eq. (116), implies the conserved current $s^\mu(x) = T^\mu_{\ \nu}x^\nu + \varphi\,\partial^\mu\varphi$.
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Y. Nambu and G. Jona-Lasinio, Dynamical model of elementary particles based on an analogy with superconductivity. I; Phys. Rev. **122**, 345–358 (1961); II.: Phys. Rev. **124**, 246–254 (1961).

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H. J. Borchers and G. C. Hegerfeldt, The structure of space-time transformations; Comm. math. Phys. **28**, 259–266 (1972).

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[179] This remark does not apply to J. Wess and W. Thirring who already in 1962 invited me to Vienna for a seminar on the subject of my Ph.D. Thesis. But otherwise:
Once, when I was about to receive my Ph.D. 1962 in Munich, I was offered an Assistent position under the condition that I quit my work on the conformal group (I refused). The acceptance was similar reserved during my years in Berkeley (1964/65) and Princeton (1965/66):
When in 1966 – at a Midwest Conference on Theoretical Physics in Bloomington – Sidney Coleman heard me talk to Rudolf Haag about my interest in the conformal group, he commented: "I shall never work on this!" A few years later he wrote several papers on the subject! Also, later Arthur Wightman told me that he and his colleagues in Princeton did not appreciate the possible importance of the conformal group when I was there.
I was very grateful to Eugene Wigner (1902–1995) for his invitation to come to Princeton in order to work on scale and conformal invariance, but he was not much interested in asymptotic symmetries at very high energies and wanted the continuous 2-particle energy spectrum to be analysed in the framework of scale invariance etc. This was then done in the Ph.D. thesis of A.H. Clark, Jr.: The Representation of the Special Conformal Group in High Energy Physics; Princeton University, Department of Physics, 1968.

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After the XVth Intern. Conf. on High Energy Physics from Aug. 26–Sept. 4, 1970, in Kiev, where I had talked about work on dilatation and conformal symmetries by students of mine and by myself (see the list on pp. 730 and 731 of the Proceed.) I had visited Sascha Polyakov and Sascha Migdal in Moscow, where we had lively discussions on these symmetries and other topics (see acknowledgments in Polyakov's paper and in the first of the two following ones by Migdal).
Polyakov discussed conformal invariance of 3-point correlation functions in three space dimensions and their critical indices near phase transitions; The subject was soon taken up in
G. Parisi and L. Peliti, Calculation of critical indices; Lett. Nuovo Cim. (Ser. 2) **2**, 627–628 (1971).

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P. Goddard and D. Olive, Kac-Moody and Virasoro algebras in relation to quantum physics; Intern. Journ. Mod. Phys. **1**, 303–414 (1986); see also [198].

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D. H. Mayer, Wightman distributions on conformal space; Journ. Math. Phys. **18**, 456–464 (1977).

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for a considerably expanded discussion of those causality problems see:
I. E. Segal, Mathematical Cosmology and Extragalactic Astronomy (Academic Press, New York etc., 1976) especially Chaps. II and III.

[232] As to the global structure of AdS_4 see the discussion by Hawking and Ellis, [127], here pp. 131–134. That of AdS_5 is discussed in [249], Part 1.

[233] I mention only some of the later papers, from which earlier ones may be traced back:
W. Rühl, Distributions on Minkowski space and their connection with analytic representations of the conformal group; Comm. math. Phys. **27**, 53–86 (1972);
W. Rühl, On conformal invariance of interacting fields; Comm. math. Phys. **34**, 149–166 (1973).
G. Mack, All unitary ray representations of the conformal group $SU(2,2)$ with positive energy; Comm. math. Phys. **55**, 1–28 (1977).
V. K. Dobrev, G. Mack, V. B. Petkova, S. G. Petrova, and I. T. Todorov, Harmonic Analysis on the n-Dimensional Lorentz Group and Its Application to Conformal Quantum Field Theory (Lecture Notes in Physics **63**; Springer, Berlin etc., 1977).
I. T. Todorov et al., [197].

A. W. Knapp and B. Speh, Irreducible unitary representations of $SU(2,2)$; Journ. Funct. Analysis **45**, 41–73 (1982).
E. Angelopoulos, The unitary irreducible representations of $\overline{SO}_0(4,2)$; Comm. math. Phys. **89**, 41–57 (1983).
An earlier unknown paper is:
A. Wyler, On the conformal groups in the theory of relativity and their unitary representations; Arch. Ration. Mech. and Anal. **31**, 35–50 (1968).
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