Direct and inverse scattering in the time domain for a dissipative wave equation. I. Scattering operators

G. Kristensson
Division of Electromagnetic Theory, Royal Institute of Technology, S-100 44 Stockholm, Sweden

R. J. Krueger
Applied Mathematical Sciences, Ames Laboratory—United States Department of Energy, Iowa State University, Ames, Iowa 50011

(Received 8 August 1985; accepted for publication 30 January 1986)

This is the first part of a series of papers devoted to direct and inverse scattering of transient waves in lossy inhomogeneous media. The medium is assumed to be stratified, i.e., it varies only with depth. The wave propagation is modeled in an electromagnetic case with spatially varying permittivity and conductivity. The objective in this first paper is to analyze properties of the scattering operators (impulse responses) for the medium and to introduce the reader to the inverse problem, which is the subject of the second paper in this series. In particular, imbedding equations for the propagation operators are derived and the corresponding equations for the scattering operators are reviewed. The kernel representations of the propagation operators are shown to have compact support in the time variable. This property implies that transmission and reflection data can be extended from one round trip to arbitrary time intervals. The compact support of the propagator kernels also restricts the admissible set of transmission kernels consistent with the model employed in this paper. Special cases of scattering and propagation kernels that can be expressed in closed form are presented.

I. INTRODUCTION

The propagation of waves in lossy media can be modeled in a number of ways, depending on the features of the propagation that are of interest. This series of papers will deal with linear wave propagation in an inhomogeneous medium that is characterized by dissipation and phase velocity profiles that are independent of the frequency of the wave. A precise model for such propagation is given in Sec. II of this paper. This model involves one-dimensional electromagnetic wave propagation in the time domain in a medium that is characterized by spatially varying permittivity and conductivity profiles.

This series of papers presents a time domain approach to wave propagation that yields a unified theory for both direct and inverse scattering. The basis for this approach is in the splitting/invariant imbedding techniques that have been exploited in earlier work. Specifically, these techniques apply to time domain reflection and transmission operators for a given scattering medium.

For the convenience of the reader, the pertinent features of previous work in this area will be explicitly displayed when necessary. The present paper, Part I, deals with the direct scattering problem; i.e., given the dissipation and phase velocity profiles, determine the scattering operators (or impulse responses) for the medium. These are operators that can be used to map any transient normally incident field over to the resulting scattered fields. Various properties of these operators are developed, and it is shown how they can be utilized to "extend" scattering data.

A subsequent paper,[1] Part II, deals with the full inverse problem; i.e., given the scattering operators for a medium, determine both the dissipation and phase velocity profiles for the medium. Since a number of results derived in Part I are not used in Part II, the reader who is primarily interested in the inverse problem can proceed to Part II after reading this introduction and Sec. II of the present paper, in which notation is established and a precise statement of the problem is given. Results from Part I that are used in the inverse problem are summarized at the beginning of Part II. Some numerical examples showing scattering operators (as well as inversion procedures) will be given in Part II.

Section III of the present paper reviews the integrodifferential equations satisfied by the kernels of the scattering operators and relates these kernels to the propagator kernels for the medium. A reciprocity result is also derived. Integrodifferential equations for the propagator kernels are derived in Sec. IV. In Sec. V a result that can be used to characterize transmission data is developed. This result is also used to extend reflection and transmission data from a single round trip time trace to a time trace of arbitrary length. Section VI is a summary of the work in Part I. Appendix A supplies technical details used in Sec. V. Closed form expressions for scattering and propagator kernels in special cases are given in Appendix B. Finally, operator equations for the propagators are shown in Appendix C.

To put the present results in their proper context, some details regarding previous work are now given. Corones and Krueger[2] and Davison[3] developed a system of integrodifferential equations for the reflection and transmission operators. Those studies displayed the time domain behavior of these operators. However, it was also shown that those results could be interpreted in the frequency domain. In that case, a Riccati differential equation for the reflection coefficient was obtained.

In later work, Corones *et al.* used the reflection operator equation as the basis for an inversion algorithm in nondissipative[4–7] as well as dissipative media.[8–10] (In the dissipative

 0022-2488/86/061667-16$02.50

case, *a priori* information about phase velocity or dissipation is required.) Bruckstein *et al.*[11] have given a partial review of Riccati equation techniques in the frequency domain and shown their relation to certain integral equation approaches to inversion.

There are several well-documented solutions to the full inverse problem (i.e., simultaneous reconstruction of both dissipation and phase velocity profiles). Such solutions, however, use completely different methods than the techniques presented here. Weston[12–14] was the first to use the full scattering matrix in the time domain to solve the one-dimensional dissipative inverse problem. He applied a Riemann function approach to develop a system a Gel'fand–Levitan-type equations whose solution yielded the desired profiles. The data for this problem consisted of the time domain reflection and transmission operators. These results were generalized by Krueger[15–17] to include more realistic material profiles. This had the effect of also reducing the data requirements in the problem, although transmission data were still required. More will be said about this in Part II. Jaulent[18,19] pursued a frequency domain approach to dissipative inverse problems in a variety of settings. The problems considered involved a complex potential with a linear dependence on frequency, and the required data consisted of reflection and transmission coefficients.

A model of dissipative wave propagation, which is more physically motivated than that used in this paper or in any of the above-referenced papers, is possible. Such a model is obtained by appealing to the underlying constitutive relation in the problem. In the frequency domain, this implies a certain dispersion relation, whereas in the time domain, this implies the existence of a memory function for the medium. The methods used in the present series of papers also have been applied to direct and inverse scattering problems in electromagnetic[20] and viscoelastic[21] media, which are characterized by such a memory function.

II. STATEMENT OF THE PROBLEM

In this section some notation is introduced and a precise statement of the inverse problem is given. The geometry of the problem is shown in Fig. 1. An inhomogeneous slab occupies the region $0<z<L$. This medium is assumed to be stratified so that the permittivity and conductivity are functions of depth z only. A homogeneous, lossless medium is situated on either side of this slab.

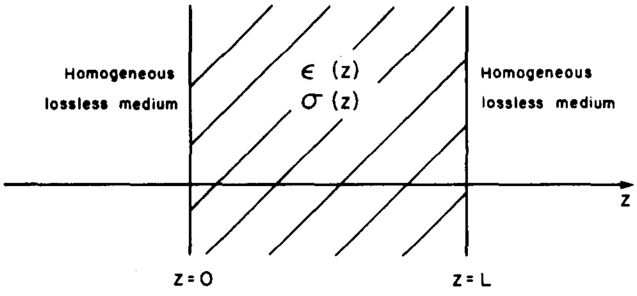

FIG. 1. The geometry of the inhomogeneous medium.

An electromagnetic plane wave is launched in the region exterior to the slab. This impinges normally on the medium, giving rise to an electric field $E(z,t)$ within the slab, with E satisfying

$$E_{zz}(z,t) - c^{-2}(z)E_{tt}(z,t) - b(z)E_t(z,t) = 0, \quad (2.1)$$

where

$$c^{-2}(z) = \epsilon(z)\mu_0, \quad b(z) = \sigma(z)\mu_0, \quad (2.2)$$

and μ_0 is the permeability in vacuum, $\sigma(z)$ is the conductivity, and $\epsilon(z)$ is the permittivity. The analysis becomes simpler if the phase velocity $c(z)$ is continuously differentiable within the slab. This will be assumed throughout this paper. It is further assumed that the phase velocity is continuous (although not necessarily smooth) at the boundary of the slab. Thus, in the regions exterior to the slab the phase velocity is given by

$$\begin{aligned} c(z) &= c(0^+), \quad z\leqslant 0,\\ c(z) &= c(L^-), \quad z\geqslant L \end{aligned} \quad (2.3)$$

(where the $\pm$ superscript denotes the limit from the right and the limit from the left, respectively). These assumptions insure that E and E_z are everywhere continuous. This precludes the existence of impulsive echoes in the scattered fields.

Now if the incident plane wave is launched in the region $z<0$, then the general solution of Eq. (2.1) in the region to the left of the slab is

$$E(z,t) = E^i_+(t - z/c(0)) + E^r_+(t + z/c(0)), \quad z<0. \quad (2.4)$$

Here, E^i_+ and E^r_+ denote the incident and reflected fields, respectively. The subscript "+" denotes the fact that the incident field is propagating in the $+z$ direction. In addition, a transmitted field is produced in the region to the right of the slab. This has the form

$$E(z,t) = E^t_+(t - l - (z-L)/c(L)), \quad z>L, \quad (2.5)$$

where

$$l = \int_0^L c^{-1}(z)dz. \quad (2.6)$$

The incident and scattered fields are related by the scattering operators (i.e., reflection and transmission operators) for the slab. These are integral operators represented by

$$\begin{aligned} E^r_+(t) &= \int_0^t \widetilde{R}^+(t-t')E^i_+(t')dt', \\ E^t_+(t) &= \tilde{t}^+E^i_+(t) + \int_0^t \widetilde{T}^+(t-t')E^i_+(t')dt', \end{aligned} \quad (2.7)$$

where

$$\tilde{t}^+ = \left[\frac{c(L)}{c(0)}\right]^{1/2} \exp\left[-\frac{1}{2}\int_0^L b(z)c(z)dz\right]. \quad (2.8)$$

In Eq. (2.7) the functions $\widetilde{R}^+$ and $\widetilde{T}^+$ are the reflection and transmission kernels, respectively, for incidence from the left. Notice that the lower limit of integration in (2.7) has been chosen to be 0, which is equivalent to assuming that the incident wave front first impinges on the slab at $t=0$. Notice also that the time variable t in Eq. (2.7) does not represent physical time, but rather a characteristic variable for Eq.

(2.1) outside the slab [cf. also Eqs. (2.4) and (2.5)].

The existence of the scattering operators in Eq. (2.7) can be verified in a number of ways, one of which is shown in Ref. 15. In particular, these operators are independent of the incident field used in the scattering experiment and depend only on the properties of the slab. Furthermore, velocity mismatch effects have been taken out of the problem by the assumption that c is continuous (although not necessarily smooth) at $z=0$ and $z=L$. Hence, in comparing the form of the operators given in Eq. (2.7) with those in Ref. 15, the constants c_0 and c_l in Ref. 15 must be set equal to 1.

A second pair of reflection and transmission operators describe scattering experiments for incident fields impinging on the medium from the right. In this case the general solution of Eq. (2.1) in the region $z>L$ is

$$E(z,t)=E^i_-\,(t+(z-L)/c(L)) + E^r_-\,(t-(z-L)/c(L)),\quad z>L, \tag{2.9}$$

where E^i_- and E^r_- are the incident and reflected fields, respectively. To the left of the slab the transmitted field is given by

$$E(z,t)=E^t_-\,((t-l+z/c(0)),\quad z<0. \tag{2.10}$$

These fields are again related by scattering operators for the slab, which are represented by

$$\begin{aligned} E^r_-\,(t) &= \int_0^t \widetilde{R}^-(t-t')E^i_-\,(t')dt', \\ E^t_-\,(t) &= \tilde{t}^- E^i_-\,(t) + \int_0^t \widetilde{T}^-(t-t')E^i_-\,(t')dt', \end{aligned} \tag{2.11}$$

where

$$\tilde{t}^- = \left[\frac{c(L)}{c(0)}\right]^{-1/2} \exp\left[-\frac{1}{2}\int_0^L b(z)c(z)dz\right]. \tag{2.12}$$

Again in Eq. (2.11) it is assumed that $t=0$ corresponds to the time the wave front first impinges on the slab at $z=L$. Notice that if $E^i_\pm\,(t)=\delta(t)$ (where δ is the Dirac delta), then from Eqs. (2.7) and (2.11), it follows that $E^r_\pm\,(t)=\widetilde{R}(t)$ and $E^t_\pm\,(t)=t^\pm\delta(t)+\widetilde{T}(t)$. Hence, the scattering kernels $\widetilde{R}^\pm$, $\widetilde{T}^\pm$ are the impulse responses for the medium.

The inverse problem considered in this series of papers is that of determining both $\epsilon(z)$ and $\sigma(z)$ (as well as L) for the slab through the use of scattering experiments performed on the slab. More precisely, the scattering data used in the reconstruction of ϵ and σ consist of finite time traces of both reflection kernels, $\widetilde{R}^\pm(t)$, and one of the transmission kernels, say, $\widetilde{T}^+(t)$ for $0<t<2l$. Here, $2l$ [with l defined by Eq. (2.6)] represents the time it takes a signal to travel one complete round trip through the medium.

The data used in this formulation of the problem are a deconvolution of Eq. (2.7) and (2.11). The effect of imperfect deconvolution can be studied (at least numerically) by means of the inversion algorithms presented in Part II.

At this point a transformation of dependent and independent variables in Eq. (2.1) is made. This transformation is not necessary for the implementation of the inversion algorithms given in Part II. However, it does result in a simpler-looking analysis (compare with Ref. 10) and numerical scheme. Begin with the change of coordinates,

$$\begin{aligned} x &= x(z) = \int_0^z \frac{c^{-1}(z')dz'}{l}, \\ s &= t/l, \\ u(x,s) &= E(z,t), \end{aligned} \tag{2.13}$$

where x is normalized travel time and s is normalized time. In these new coordinates the slab occupies the region $0\leqslant x\leqslant 1$ and a round trip time trace is described by $0<s<2$. Equation (2.1) becomes

$$u_{xx}-u_{ss}+A(x)u_x+B(x)u_s=0, \tag{2.14}$$

where

$$A(x) = -\frac{d}{dx}\ln c(z(x)), \tag{2.15}$$

$$B(x) = -lb(z(x))c^2(z(x)), \tag{2.16}$$

and ln denotes the natural logarithm function. The coefficient functions A and B vanish outside of the interval [0,1] and are continuous on the interval (0,1), with possible finite discontinuities at $x=0$ and $x=1$. Typical plots of A and B are shown in Fig. 2.

It follows from the compact support of A and B that for $x<0$ and $x>1$, solutions of (2.14) reduce to right and left moving waves. These are readily related to the physical fields. In particular, scattering operators again exist for Eq.

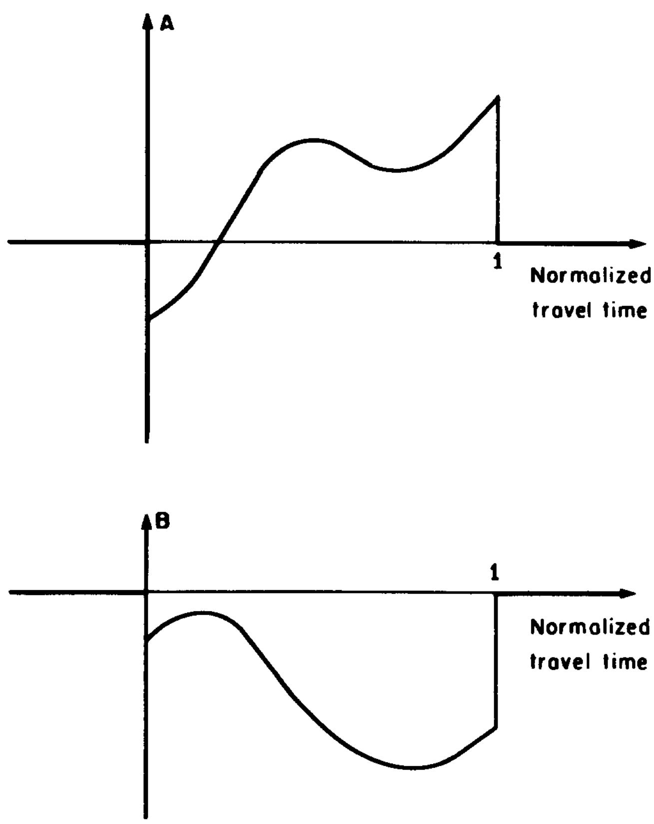

FIG. 2. Profile functions A and B.

(2.14). For a right moving incident wave $u^i_+(s-x)$, launched in the region $x<0$, it can be shown that the reflected and transmitted fields are given by

$$u^r_+(s) = \int_0^s R^+(0,1,s-s')u^i_+(s')ds', \tag{2.17}$$

$$u^t_+(s) = t^+(0,1)u^i_+(s) + \int_0^s T^+(0,1,s-s')u^i_+(s')ds', \tag{2.18}$$

while a left moving incident wave, $u^i_-(s+x-1)$, in the region $x>1$ produces reflected and transmitted fields

$$u^r_-(s) = \int_0^s R^-(0,1,s-s')u^i_-(s')ds', \tag{2.19}$$

$$u^t_-(s) = t^-(0,1)u^i_-(s) + \int_0^s T^-(0,1,s-s')u^i_-(s')ds', \tag{2.20}$$

where

$$t^\pm(0,1) = \exp\left[\mp\frac{1}{2}\int_0^1 \{A(x)\mp B(x)\}dx\right]. \tag{2.21}$$

The reflection and transmission kernels in (2.17)–(2.20) are related to the physical kernels in (2.7) and (2.11) via

$$\begin{aligned} R^\pm(0,1,s) &= l\widetilde{R}^\pm(ls),\\ T^\pm(0,1,s) &= l\widetilde{T}^\pm(ls). \end{aligned} \tag{2.22}$$

Notice that these transformed kernels reference the end points of the scattering medium. This is because in later sections of this paper, scattering kernels for subsections of the original medium will be considered. Observe that the independent variable in (2.18) and (2.20) can be thought of as a characteristic variable.

Finally, it is necessary to define a second set of operators for the scattering problems relevant to (2.14). These are propagation operators[22] for the medium and are used to express the incident and reflected fields in terms of the transmitted field. They are given by (see Ref. 15)

$$u^i_\pm(s) = [t^\pm(0,1)]^{-1}u^t_\pm(s) + \int_0^s W^\pm(0,1,s-s')u^t_\pm(s')ds', \tag{2.23}$$

$$u^r_\pm(s) = [t^\pm(0,1)]^{-1}\int_0^s V^\pm(0,1,s-s')u^t_\pm(s')ds'. \tag{2.24}$$

Notice that the "W" operator is just the inverse of the corresponding "T" operator. Consequently, the kernels $W^\pm(0,1,s)$ in Eq. (2.23) are just the resolvent kernels for the functions $T^\pm(0,1,s)$. The explicit relation between these kernels is

$$[t^\pm(0,1)]^{-1}T^\pm(0,1,s) + t^\pm(0,1)W^\pm(0,1,s) + \int_0^s T^\pm(0,1,s-s')W^\pm(0,1,s')ds' = 0. \tag{2.25}$$

The end points of the (transformed) slab are explicitly displayed in the arguments of $R^\pm(0,1,s)$ and $T^\pm(0,1,s)$, and as well as in $V^\pm(0,1,s)$ and $W^\pm(0,1,s)$. In the next section both of the end points of the slab are allowed to vary (see Fig. 3) and in this more general case $R^\pm(x,y,s)$ and $T^\pm(x,y,s)$ denote the reflection and transmission kernels, respectively, for the subregion of the slab with end points at x and y, with $0\leqslant x\leqslant y\leqslant 1$. A similar notation holds for $V^\pm(x,y,s)$ and $W^\pm(x,y,s)$. It should be stressed that it is only the kernels corresponding to $x=0$ and $y=1$ that are physically obtainable.

III. EQUATIONS FOR THE SCATTERING KERNELS

In the preceding section the physical reflection and transmission kernels were introduced. These are the data that are obtained from a scattering experiment. Throughout the remainder of this paper, the transformed problem given in (2.14) will be studied, and in particular the kernels on the left-hand side of (2.22) will be referred to as the physical scattering kernels (since they are easily obtained from the physical data).

The dependence of the scattering kernels on the parameters x and y (which are the end points of the subregion $[x,y]$) will be reviewed in this section. It is intuitively clear that this dependence is related to the material properties of the slab. Relations to this effect are developed in detail in Ref. 2. For the convenience of the reader and for completeness the main results of that reference are given here:

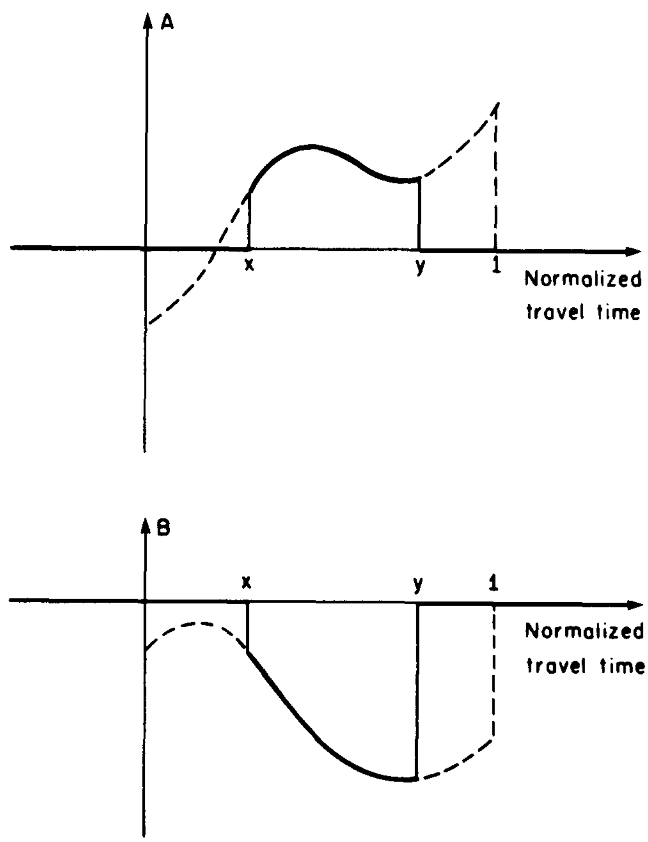

FIG. 3. Profile functions A and B for the subregion $[x,y]$. The dashed lines indicate the omitted portions of the physical region.

$$\begin{aligned}R_x^+(x,y,s) &= 2R_s^+(x,y,s) - B(x)R^+(x,y,s)\\ &\quad -\tfrac{1}{2}[A(x)+B(x)]\\ &\quad \times\int_0^s R^+(x,y,s')R^+(x,y,s-s')\,ds',\quad s>0,\\ R^+(y,y,s) &= 0,\quad s>0,\\ R^+(x,y,0^+) &= -\tfrac{1}{4}[A(x)-B(x)],\quad x<y;\end{aligned} \tag{3.1}$$

$$\begin{aligned}T_x^+(x,y,s) &= \tfrac{1}{2}[A(x)-B(x)]T^+(x,y,s)\\ &\quad -\frac{1}{2}[A(x)+B(x)]\Big\{t^+(x,y)R^+(x,y,s)\\ &\quad +\int_0^s T^+(x,y,s')R^+(x,y,s-s')\,ds'\Big\},\quad s>0,\\ T^+(y,y,s) &= 0,\quad s>0;\end{aligned} \tag{3.2}$$

$$\begin{aligned}T_x^-(x,y,s) &= -\tfrac{1}{2}[A(x)+B(x)]\\ &\quad \times\Big\{T^-(x,y,s)+t^-(x,y)R^+(x,y,s)\\ &\quad +\int_0^s T^-(x,y,s')R^+(x,y,s-s')\,ds'\Big\},\quad s>0,\\ T^-(y,y,s) &= 0,\quad s>0;\end{aligned} \tag{3.3}$$

$$\begin{aligned}R_x^-(x,y,s) &= \tfrac{1}{2}[A(x)+B(x)]\Big\{t^+(x,y)T^-(x,y,s-2(y-x))\\ &\quad +t^-(x,y)T^+(x,y,s-2(y-x))\\ &\quad +\int_0^{s-2(y-x)} T^+(x,y,s')\\ &\quad \times T^-(x,y,s-2(y-x)-s')\,ds'\Big\},\\ &\quad s>2(y-x),\\ R^-(y,y,s) &= 0,\quad s>0;\end{aligned} \tag{3.4}$$

$$\begin{aligned}R_y^-(x,y,s) &= -2R_s^-(x,y,s)+B(y)R^-(x,y,s)\\ &\quad -\tfrac{1}{2}[A(y)-B(y)]\\ &\quad \times\int_0^s R^-(x,y,s')R^-(x,y,s-s')\,ds',\quad s>0,\\ R^-(x,x,s) &= 0,\quad s>0,\\ R^-(x,y,0^+) &= \tfrac{1}{4}[A(y)+B(y)],\quad x<y;\end{aligned} \tag{3.5}$$

$$\begin{aligned}T_y^+(x,y,s) &= -\tfrac{1}{2}[A(y)-B(y)]\\ &\quad \times\Big\{T^+(x,y,s)+t^+(x,y)R^-(x,y,s)\\ &\quad +\int_0^s T^+(x,y,s')R^-(x,y,s-s')\,ds'\Big\},\quad s>0,\\ T^+(x,x,s) &= 0,\quad s>0;\end{aligned} \tag{3.6}$$

$$\begin{aligned}T_y^-(x,y,s) &= \tfrac{1}{2}[A(y)+B(y)]T^-(x,y,s)\\ &\quad -\frac{1}{2}[A(y)-B(y)]\Big\{t^-(x,y)R^-(x,y,s)\\ &\quad +\int_0^s T^-(x,y,s')R^-(x,y,s-s')\,ds'\Big\},\quad s>0,\\ T^-(x,x,s) &= 0,\quad s>0;\end{aligned} \tag{3.7}$$

$$\begin{aligned}R_y^+(x,y,s) &= -\tfrac{1}{2}[A(y)-B(y)]\Big\{t^-(x,y)T^+(x,y,s\\ &\quad -2(y-x))+t^+(x,y)T^-(x,y,s-2(y-x))\\ &\quad +\int_0^{s-2(y-x)} T^-(x,y,s')\\ &\quad \times T^+(x,y,s-2(y-x)-s')\,ds'\Big\},\quad s>2(y-x),\\ R^+(x,x,s) &= 0,\quad s>0;\end{aligned} \tag{3.8}$$

where

$$t^\pm(x,y) = \exp\Big\{\mp\frac{1}{2}\int_x^y [A(x')\mp B(x')]\,dx'\Big\}. \tag{3.9}$$

Equations (3.1)–(3.8) are the imbedding equations for the slab, obtained from continuously imbedding scattering kernels for subintervals of the slab into a family of scattering kernels. In particular, these equations display the change in the scattering kernels due to variations in one of the end points of the imbedded slab. As seen from above, these equations are in general nonlinear and of integrodifferential type. Note that two of the equations, Eqs. (3.1) and (3.5), are both equations for a single unknown kernel. The other six equations couple different kernels together. With each of the equations above there is also a boundary condition for the case when $x=y$. This corresponds to a slab of zero thickness. In Eqs. (3.1) and (3.5), there are also two auxiliary conditions relating the early time behavior of the reflection kernels $R^\pm(x,y,s)$ to the properties of the slab (see also Fig. 4). Equations (3.1)–(3.8) are written in a slightly different form than in Ref. 2 due to the particular representation of the scattering operators given in Eq. (2.17)–(2.20).

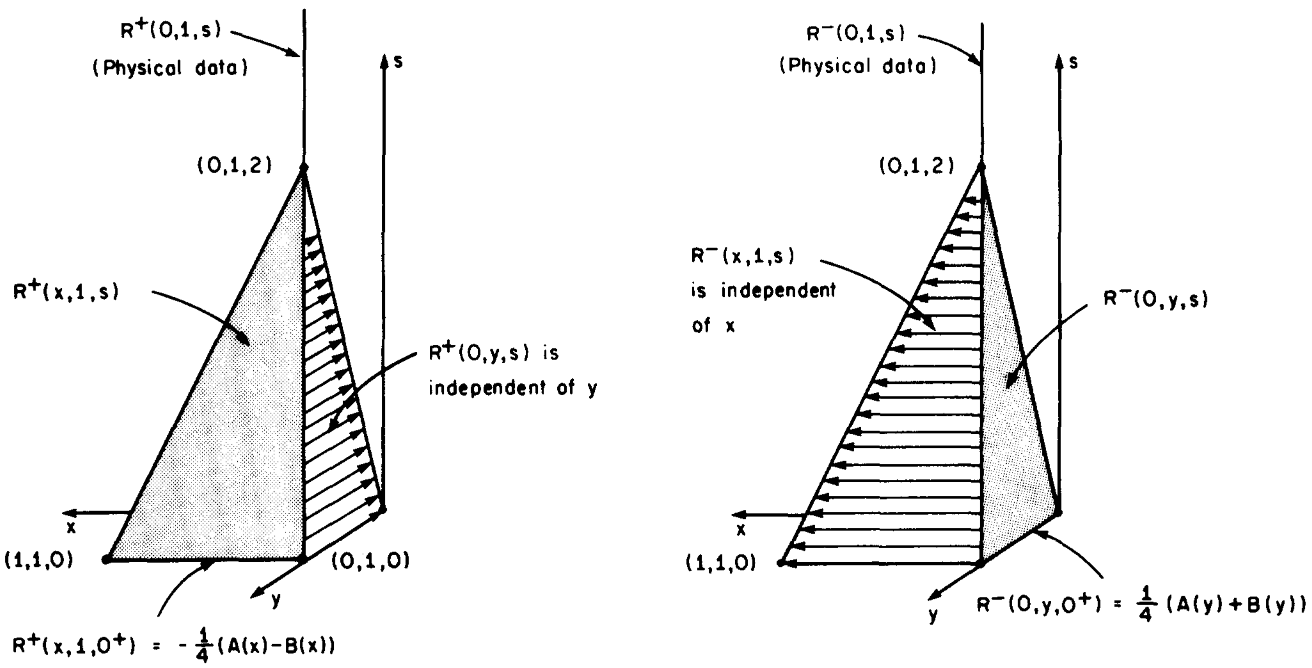

FIG. 4. A portion of the domain $R^{\pm}(x,y,s)$. The entire domain is $0<x<y<1$, $s>0$. The region inside the tetrahedron is the domain of $R^{\pm}(x,y,s)$ for s limited to one round trip in the subregion $[x,y]$.

The reflection kernels $R^{\pm}(x,y,s)$ are discontinuous across the plane $s=2(y-x)$. These discontinuities are associated with the echo of the wave front from the rear interface. Again referring to Ref. 2, the jumps in the kernels along that plane are

$$\begin{aligned}
&[R^{+}(x,y,s)]_{s=2(y-x)^-}^{s=2(y-x)^+} \\
&\quad=\frac{1}{4}[A(y)-B(y)]\exp\left\{\int_x^y B(x')dx'\right\}, \\
&[R^{-}(x,y,s)]_{s=2(y-x)^-}^{s=2(y-x)^+} \qquad\qquad (3.10)\\
&\quad=-\frac{1}{4}[A(x)+B(x)]\exp\left\{\int_x^y B(x')dx'\right\}.
\end{aligned}$$

In Ref. 2 it is also shown that the reflection kernels $R^{\pm}(x,y,s)$ satisfy (see also Fig. 4)

$$\begin{aligned}
R^{+}(x,y,s)&=R^{+}(x,x+s/2^{+},s), \quad s<2(y-x), \\
& \qquad\qquad\qquad\qquad (3.11)\\
R^{-}(x,y,s)&=R^{-}(y-s/2^{-},y,s), \quad s<2(y-x).
\end{aligned}$$

These relations state that the reflected field is independent of the position of the rear interface of the slab for times less than one round trip through the subregion $[x,y]$. The properties of the reflection kernels $R^{\pm}(x,y,s)$ given by Eqs. (3.10) and (3.11) will be used in the inversion algorithm presented in Part II of this series of papers.

In the transmission kernel equations given above, there is a distinction between the T^{+} and T^{-} kernels. Now assume that there is a relation between the kernels $T^{\pm}(x,y,s)$ of the following form:

$$T^{+}(x,y,s)=f(x,y)T^{-}(x,y,s), \qquad (3.12)$$

where $f(x,y)$ is an unknown function to be determined. A relation of this kind is suggested by the fact that a reciprocity result should exist for the transmission operators. Equations (3.2), (3.3), (3.6), (3.7), and (3.12) imply that $f(x,y)$ must satisfy

$$\begin{aligned}
f_x(x,y)&=A(x)f(x,y), \\
f_y(x,y)&=-A(y)f(x,y), \qquad (3.13)\\
f(x,y)&=t^{+}(x,y)/t^{-}(x,y).
\end{aligned}$$

These three equations are consistent and it is therefore convenient to introduce a single transmission kernel $T(x,y,s)$ defined by

$$T(x,y,s)=T^{+}(x,y,s)/t^{+}(x,y)=T^{-}(x,y,s)/t^{-}(x,y). \qquad (3.14)$$

In what follows, this new definition of the transmission kernel will be the one that is used and from now on there are only three different kinds of scattering kernels, i.e., $R^{\pm}(x,y,s)$ and $T(x,y,s)$. It is easy to see that with this new definition of the transmission kernel the Eqs. (3.2), (3.3), (3.6), and (3.7) can be replaced by two simpler ones,

$$\begin{aligned}
T_x(x,y,s)&=-\frac{1}{2}[A(x)+B(x)]\Big\{R^{+}(x,y,s) \\
&\quad+\int_0^s T(x,y,s')R^{+}(x,y,s-s')ds'\Big\}, \\
s&>0, \qquad (3.15)\\
T_y(x,y,s)&=-\frac{1}{2}[A(y)-B(y)]\Big\{R^{-}(x,y,s) \\
&\quad+\int_0^s T(x,y,s')R^{-}(x,y,s-s')ds'\Big\}, \\
s&>0. \qquad (3.16)
\end{aligned}$$

The resolvent equation (2.25) for $W^{\pm}$ generalizes to the subregion $[x, y]$ in the obvious way. Now since the $T^{\pm}$ are simply related to a single kernel T, it follows that $W^{\pm}$ can be related to a single kernel W. Specifically,

$$W(x,y,s) = t^{+}(x,y) W^{+}(x,y,s) = t^{-}(x,y) W^{-}(x,y,s). \tag{3.17}$$

The resolvent equation for $W(x,y,s)$ now reads

$$T(x,y,s) + W(x,y,s) + \int_0^s T(x,y,s-s') W(x,y,s')ds' = 0. \tag{3.18}$$

The equations for the $V^{\pm}(x,y,s)$ kernels are

$$R^{\pm}(x,y,s) = V^{\pm}(x,y,s) + \int_0^s T(x,y,s-s') V^{\pm}(x,y,s')ds', \tag{3.19}$$

which follows (in the "+" case) from inserting Eqs. (2.17) and (2.18) into Eq. (2.24) and using (3.14). Finally, Eq. (3.19) can be solved for $V^{\pm}$ by using the fact that W is the resolvent kernel for T. This yields

$$V^{\pm}(x,y,s) = R^{\pm}(x,y,s) + \int_0^s R^{\pm}(x,y,s-s') W(x,y,s')ds'. \tag{3.20}$$

Exact representations of the kernels $R^{\pm}$, T, W, and $V^{\pm}$ can be obtained in the special case when $A(x)$ and $B(x)$ are constants. This is done by using the Laplace transform in the variable s. Details are provided in Appendix B.

IV. THE W AND $V^{\pm}$ EQUATIONS

In this section, the dynamics of the kernels W and $V^{\pm}$ are derived. The definition of the resolvent $W(x,y,s)$ of the transmission kernel $T(x,y,s)$ is given by Eq. (3.18). Differentiation of this equation with respect to the left end point x gives

$$T_x(x,y,s) + W_x(x,y,s) + \int_0^s W_x(x,y,s') T(x,y,s-s')ds' + \int_0^s W(x,y,s') T_x(x,y,s-s')ds' = 0. \tag{4.1}$$

Now use the imbedding equation for the transmission kernel T given by Eq. (3.15) and the definition of the resolvent in Eq. (3.18) to get

$$W_x(x,y,s) - \tfrac{1}{2}[A(x) + B(x)] R^{+}(x,y,s) + \int_0^s W_x(x,y,s') T(x,y,s-s')ds' = 0, \tag{4.2}$$

which can be simplified to

$$W_x(x,y,s) = \frac{1}{2}[A(x) + B(x)] \left\{ R^{+}(x,y,s) + \int_0^s W(x,y,s') R^{+}(x,y,s-s')ds' \right\}, \quad s>0. \tag{4.3}$$

This equation gives the variation of the resolvent $W(x,y,s)$ as the left-hand side of the slab is varying. Note that this equation is very similar to the equation for the variation in the transmission kernel $T(x,y,s)$, given by (3.15).

The equation for a variation of the right-hand end point is similar to the derivation above and the result is

$$W_y(x,y,s) = \frac{1}{2}[A(y) - B(y)] \left\{ R^{-}(x,y,s) + \int_0^s W(x,y,s') R^{-}(x,y,s-s')ds' \right\}, \quad s>0. \tag{4.4}$$

A direct comparison between Eqs. (4.3), (4.4), and (3.20) shows that

$$W_x(x,y,s) = \tfrac{1}{2}[A(x) + B(x)] V^{+}(x,y,s), \tag{4.5}$$

$$W_y(x,y,s) = \tfrac{1}{2}[A(y) - B(y)] V^{-}(x,y,s). \tag{4.6}$$

The two pairs of equations for $V^{\pm}(x,y,s)$ now can be derived quite easily. Differentiate $V^{+}(x,y,s)$ in Eq. (3.20) once with respect to x and once with respect to s and use Eqs. (3.1) and (4.5) to obtain

$$\begin{aligned} V_x^{+}(x,y,s) &= 2V_s^{+}(x,y,s) - B(x) V^{+}(x,y,s) \\ &\quad + \tfrac{1}{2}[A(x) - B(x)] W(x,y,s), \quad s>0 \\ V^{+}(y,y,s) &= 0, \quad s>0, \\ V^{+}(x,y,0^{+}) &= -\tfrac{1}{4}[A(x) - B(x)], \quad x<y. \end{aligned} \tag{4.7}$$

Similarly, by differentiating $V^{-}(x,y,s)$ in Eq. (3.20) with respect to y and s the following equation is obtained by the use of Eqs. (3.5) and (4.6):

$$\begin{aligned} V_y^{-}(x,y,s) &= -2V_s^{-}(x,y,s) + B(y) V^{-}(x,y,s) \\ &\quad + \tfrac{1}{2}[A(y) + B(y)] W(x,y,s), \quad s>0, \\ V^{-}(x,x,s) &= 0, \quad s>0, \\ V^{-}(x,y,0^{+}) &= \tfrac{1}{4}[A(y) + B(y)], \quad x<y. \end{aligned} \tag{4.8}$$

Notice that the two Eqs. (4.7) and (4.8) do not contain any convolution integral, but couple $V^{\pm}$ with W.

The two final equations for the kernel $V^{\pm}(x,y,s)$ are derived by a differentiation with respect to the other end point in Eq. (3.20). The dynamics of V_y^{+} and V_x^{-} in the interval $0<s<2(y-x)$ are now easily obtained by the use of Eqs. (3.11), (4.5), and (4.6). This results in

$$\begin{aligned} V_y^{+}(x,y,s) &= \tfrac{1}{2}[A(y) - B(y)] \times \int_0^s R^{-}(x,y,s') V^{+}(x,y,s-s')ds' \\ &= \tfrac{1}{2}[A(y) - B(y)] \times \int_0^s R^{+}(x,y,s') V^{-}(x,y,s-s')ds', \\ &\quad 0<s<2(y-x), \end{aligned} \tag{4.9}$$

$$V_x^{-}(x,y,s) = \tfrac{1}{2}[A(x) + B(x)] \times \int_0^s R^{+}(x,y,s') V^{-}(x,y,s-s')ds'$$

$$= \tfrac{1}{2}[A(x)+B(x)] \times \int_0^s R^-(x,y,s')V^+(x,y,s-s')ds',$$

$$0<s<2(y-x), \tag{4.10}$$

where the last equality in each of these equations comes from the identity

$$\int_0^s R^+(x,y,s')V^-(x,y,s-s')ds' = \int_0^s R^-(x,y,s')V^+(x,y,s-s')ds', \quad s>0, \tag{4.11}$$

which is easily obtained from Eq. (3.20).

The equations (4.9) and (4.10) are not valid for $s>2(y-x)$. In the derivation of the corresponding equations valid for $s>2(y-x)$, the following integral is encountered:

$$\begin{aligned}\int_{2(y-x)}^s R_y^+(x,y,s')W(x,y,s-s')ds' &= -\tfrac{1}{2}[A(y)-B(y)]t^+(x,y)t^-(x,y)\{W(x,y,s-2(y-x))\\ &\quad+2\int_0^{s-2(y-x)} T(x,y,s')W(x,y,s-2(y-x)-s')ds'\\ &\quad+\int_0^{s-2(y-x)}\left[\int_0^{s'} T(x,y,s'')T(x,y,s'-s'')ds''\right]W(x,y,s-2(y-x)-s')ds'\Big\}, \quad s>2(y-x). \end{aligned}\tag{4.12}$$

This equation can be obtained by use of Eqs. (3.8), (3.10), and (3.14).

It is now straightforward to combine Eqs. (3.20) and (4.12) and repeatedly use the resolvent equation (3.18) to get

$$\begin{aligned}V_y^+(x,y,s) = -\tfrac{1}{2}[A(y)-B(y)]\Big\{&t^+(x,y)t^-(x,y)T(x,y,s-2(y-x))\\ &-\int_0^s R^+(x,y,s')V^-(x,y,s-s')ds'\Big\}, \quad s>2(y-x).\end{aligned}\tag{4.13}$$

The equation for V_x^- is derived similarly. The result is

$$\begin{aligned}V_x^-(x,y,s) = -\tfrac{1}{2}[A(x)+B(x)]\Big\{&t^+(x,y)t^-(x,y)T(x,y,s-2(y-x))\\ &-\int_0^s R^-(x,y,s')V^+(x,y,s-s')ds'\Big\}, \quad s>2(y-x).\end{aligned}\tag{4.14}$$

Equations (4.13) and (4.14) can be simplified considerably with use of results presented in the next section. The consequences of this simplification will be discussed at the end of Sec. V. An alternative derivation of the results in this section is obtained by considering the dynamics of the propagator matrix for Eq. (2.14). This is carried out in Appendix C.

V. THE EXTENSION OF DATA

The $W(x,y,s)$ and the $V^\pm(x,y,s)$ kernels all share the important feature that they have compact support. More precisely, for times larger than one round trip in the subregion $[x,y]$, i.e., $s>2(y-x)$, these kernels are identically zero,

$$W(x,y,s)=0, \quad s>2(y-x), \tag{5.1}$$

$$V^\pm(x,y,s)=0, \quad s>2(y-x). \tag{5.2}$$

These relations are derived in Appendix A.

Now suppose scattering data are known for one round trip through the subregion occupying $[x,y]$. Then Eqs. (5.1) and (5.2) can be used to extend that data beyond one round trip. To be more explicit, consider the end points x and y to be fixed for the moment, and assume that $T(x,y,s)$ is known for times $0<s<2(y-x)$. Equation (3.18), which is a Volterra equation of the second kind for the kernel $W(x,y,s)$, then can be solved for $W(x,y,s)$, $0<s<2(y-x)$. The kernel $W(x,y,s)$ is thus known for all $s>0$ due to Eq. (5.1).

Now assume that $s>2(y-x)$ and rewrite the resolvent Eq. (3.18), using Eq. (5.1), in the following form:

$$T(x,y,s)+\int_{2(y-x)}^s W(x,y,s-s')T(x,y,s')ds' = G(x,y,s) = \begin{cases} -\int_{s-2(y-x)}^{2(y-x)} W(x,y,s-s')T(x,y,s')ds', & 2(y-x)<s<4(y-x),\\ 0, & s>4(y-x).\end{cases}\tag{5.3}$$

Notice that the function $G(x,y,s)$ is known as a function of s for fixed values of x and y with the assumptions made above. Equation (5.3) is a Volterra equation of second kind for $T(x,y,s)$ for $s>2(y-x)$ and this equation can be solved for the unknown $T(x,y,s)$, $s>2(y-x)$. Equation (5.3) thus provides a tool for extending the data $T(x,y,s)$, $0<s<2(y-x)$, to the time interval $s>2(y-x)$.

The extension of the reflection data now follows quite similarly, with the exception that both reflection and transmission data have to be known for one round trip. More precisely, assume that $R^{\pm}(x,y,s)$ and $T(x,y,s)$ are known for $0<s<2(y-x)$. Then from Eq. (3.19) and Eq. (5.2) above, $R^{\pm}(x,y,s)$, $s>2(y-x)$, is expressed as

$$R^{\pm}(x,y,s)=\int_0^{2(y-x)} T(x,y,s-s')V^{\pm}(x,y,s')ds', \quad s>2(y-x). \tag{5.4}$$

However, $V^{\pm}(x,y,s)$, $0<s<2(y-x)$, is related to $R^{\pm}(x,y,s)$ by Eq. (3.20),

$$V^{\pm}(x,y,s)=R^{\pm}(x,y,s)+\int_0^{s} W(x,y,s-s')R^{\pm}(x,y,s')ds', \quad 0<s<2(y-x). \tag{5.5}$$

Combining these last two equations gives

$$R^{\pm}(x,y,s)=\int_0^{2(y-x)} T(x,y,s-s')\left[R^{\pm}(x,y,s')+\int_0^{s'} W(x,y,s'-s'')R^{\pm}(x,y,s'')ds''\right]ds', \quad s>2(y-x). \tag{5.6}$$

Notice that in this last equation, reflection data $R^{\pm}(x,y,s)$ are used only for times less than one round trip, i.e., $0<s<2(y-x)$, while transmission data, $T(x,y,s)$, are used for all $s>0$. However, for times beyond one round trip the transmission data can be extended with the technique discussed above in Eq. (5.3). These ideas will be exploited in a special context in Part II.

An alternate approach to the extension of the reflection data is to rewrite Eq. (3.20) for $s>2(y-x)$ and use Eq. (5.2) to obtain

$$R^{\pm}(x,y,s)+\int_{2(y-x)}^{s} W(x,y,s-s')R^{\pm}(x,y,s')ds'$$
$$=\begin{cases} -\displaystyle\int_{s-2(y-x)}^{2(y-x)} W(x,y,s-s')R^{\pm}(x,y,s')ds', & 2(y-x)<s<4(y-x), \\ 0, & s>4(y-x). \end{cases}$$

Thus far, the compact support of the kernels $V^{\pm}$ has not been used in the imbedding equations derived in Sec. IV. Now using the fact that $V^{\pm}$ vanish identically for $s>2(y-x)$ reduces Eq. (4.13) and (4.14) to the following new representation of the transmission kernel T for $s>2(y-x)$:

$$t^{+}(x,y)t^{-}(x,y)T(x,y,s-2(y-x))$$
$$=\int_0^{2(y-x)} V^{+}(x,y,s')R^{-}(x,y,s-s')ds'=\int_0^{2(y-x)} V^{-}(x,y,s')R^{+}(x,y,s-s')ds', \quad s>2(y-x). \tag{5.7}$$

VI. SUMMARY AND CONCLUSIONS

In this paper some mathematical tools for transient wave propagation in lossy media have been introduced. This work primarily focuses on the direct scattering problem and the properties of the scattering operators. However, many of the equations developed in the present paper are of importance for the inverse algorithm presented in Part II (See Ref. 1).

Reciprocity is shown to imply that the two transmission kernels $T^{\pm}$ are proportional to each other, see Eq. (3.14). This property simplifies the analysis considerably and also reduces the number of independent imbedding equations for the scattering kernels.

The propagator kernels for the medium are also introduced and some of their properties are exploited. In Sec. IV the new imbedding equations for these kernels are derived in Eqs. (4.3), (4.4), and (4.7)–(4.10).

One of the main results in this paper, the compact support of the propagator kernels, has several consequences. In Sec. V this property is shown to provide a way to extend transmission data from one round trip to arbitrary time, see Eq. (5.3). This extension is also possible for the reflection data provided transmission data are available, see Eq. (5.6).

The compact support of W provides an important limitation of the functional behavior of the transmission kernel T. To be an admissible transmission kernel T for the model considered in this paper, its resolvent W also must have compact support. This observation provides an important characterization of the transmission kernel T. Furthermore, only data for times less than one round trip are needed for this characterization, due to the extension of data discussed above. This implies that all information available in the transmission kernel is contained in the time interval up to one round trip and that if it is admissible or not is based upon the functional behavior in this finite interval. Unfortunately, the compact support of $V^{\pm}$ doe not imply any simple characterizations for $R^{\pm}$. However, from the new imbedding equations for $V^{\pm}$ and the compact support of $V^{\pm}$ a new representation of the transmission kernel T is obtained, see Eq. (5.7), which relates T to $V^{\pm}$ and $R^{\pm}$.

ACKNOWLEDGMENTS

This work was performed at Ames Laboratory, U.S.D.O.E., Iowa State University, and was supported by the Applied Mathematical Sciences subprogram of the Office of Energy Research, U.S. Department of Energy, under Contract No. W-7405-ENG-82, and by the Office of Naval Research Contract No. N0014-83-K-0038. One of the authors (G.K.) would like to thank Ames Laboratory for their hospitality during his visit in the United States. He is also grateful to the National Swedish Board for Technical Development (STUF) for partial support.

APPENDIX A: COMPACT SUPPORT OF W AND $V^{\pm}$

In this section it is shown that the kernels W and $V^{\pm}$ have compact support. This fact was introduced in Sec. V [Eqs. (5.1) and (5.2)]. The arguments given below suffice to show that the kernels $W(x,y,s)$ and $V^{\pm}(x,y,s)$ vanish for $s>2(y-x)$. The compact support then follows from causality, which implies that these kernels also vanish for $s<0$.

In the model problem given in Eq. (2.1), the velocity $c(z)$ is assumed to be continuous at the end points of the slab, $z=0$ and $z=L$. For the sake of proving a stronger result, which should be useful in later work, this assumption will be relaxed. Thus, $c(z)$ can have finite jump discontinuities at $z=0$ and $z=L$. This generalization alters the transformed problem given by Eq. (2.14) in that u_x is no longer continuous at $x=0$ and $x=1$. Instead, u_x satisfies the relations

$$\begin{aligned} c_0 u_x(0^-,s) &= u_x(0^+,s), \\ c_1 u_x(1^+,s) &= u_x(1^-,s), \end{aligned} \tag{A1}$$

where

$$\begin{aligned} c_0 &= c(0^+)/c(0^-), \\ c_1 &= c(L^-)/c(L^+). \end{aligned} \tag{A2}$$

It is now shown that for this more general problem, $W(0,1,s)$ vanishes for $s>2$. (The arguments given below clearly generalize to any subregion $[x,y]$ of the slab.) In order to produce the desired result, an explicit formula for W will be derived.

Being by expressing the solution u of Eq. (2.14) in terms of transmission data $u^t_+(s)$ via

$$\begin{aligned} u(x,s) = \tfrac{1}{2}[t^+(x,1)]^{-1}\Big\{&(c_1+1)u^t_+(s-x) \\ &-(c_1-1)u^t_+(s+x-2)\exp\Big[\int_x^1 B(s')ds'\Big] \\ &+\int_x^{2-x} u^t_+(s-s')N(x,s')ds'\Big\}, \end{aligned} \tag{A3}$$

for $0<x<1$. Equation (A3) is derived in Ref. 15. The function $N(x,s)$ is related to the Riemann function for Eq. (2.14) and satisfies

$$N_{xx}-N_{ss}+B(x)(N_x+N_s)+D_+(x)N=0,\quad 0<x<1, \tag{A4}$$

and boundary conditions for $0<x<1$,

$$\begin{aligned} 2N(x,x) &= c_1B(1^-)-A(1^-)-(c_1+1)\int_x^1 D_+(s')ds', \\ 2N(x,2-x) &= \Big[c_1B(1^-)-A(1^-)+(c_1-1)\int_x^1 D_-(s')ds'\Big] \\ &\quad\times\exp\Big[\int_x^1 B(s')ds'\Big], \end{aligned} \tag{A5}$$

where

$$D_{\pm}(x)=\tfrac{1}{4}(B^2-A^2)+\tfrac{1}{2}(-A'\pm B'). \tag{A6}$$

The prime in Eq. (A6) denotes differentiation with respect to x.

Differentiate Eq. (A3) with respect to x and set $x=0$ in the resulting equation. Rewrite the left-hand side in terms of $u^{i\prime}_+$ and $u^{r\prime}_+$ (differentiation with respect to the argument) using

$$u_x(0^+,s)=c_0[-u^{i\prime}_+(s)+u^{r\prime}_+(s)]. \tag{A7}$$

Now integrate this equation from 0 to s, using the assumption that $u^i_+(0)=u^r_+(0)=u^t_+(0)=0$, and obtain

$$\begin{aligned} 2c_0[-u^i_+(s)+u^r_+(s)] &= [t^+(0,1)]^{-1}\Big\{-(c_1+1)u^t_+(s) \\ &\quad-(c_1-1)u^t_+(s-2)\exp\Big[\int_0^1 B(s')ds'\Big] \\ &\quad+\int_0^s u^t_+(s')F(s-s')ds'\Big\}, \end{aligned} \tag{A8}$$

where

$$\begin{aligned} F(s) = a+bH(s-2)+\int_0^s[&N_x(0,s') \\ &-\tfrac{1}{2}(A-B)|_{0^+}N(0,s')]H(2-s')ds' \end{aligned} \tag{A9}$$

and

$$\begin{aligned} a &= -\tfrac{1}{2}(c_1+1)(A-B)|_{0^+}-N(0,0), \\ b &= \tfrac{1}{2}(c_1-1)(A+B)|_{0^+}\exp\Big[\int_0^1 B(s')ds'\Big]-N(0,2), \\ H(s) &= \text{Heaviside function} = \begin{cases}0, & s<0,\\ 1, & s>0.\end{cases} \end{aligned}$$

Now evaluate Eq. (A3) at $x=0^+$ and rewrite the left-hand side as $u^i_+(s)+u^r_+(s)$. Use Eq. (A8) to eliminate $u^r_+(s)$ from the resulting equation and thus obtain

$$\begin{aligned} u^i_+(s) = [t^+(0,1)]^{-1}\Big\{&\frac{(c_0+1)(c_1+1)u^t_+(s)}{4c_0} \\ &-\frac{(c_0-1)(c_1-1)u^t_+(s-2)\exp[\int_0^1 B(s')ds']}{4c_0} \\ &+\int_0^s u^t_+(s')W(0,1,s-s')ds'\Big\}, \end{aligned} \tag{A10}$$

where the W kernel is given by

$$W(0,1,s)=[c_0N(0,s)H(2-s)-F(s)]/4c_0. \tag{A11}$$

Equation (A10) is the generalization of Eq. (2.23) when

$c(z)$ is discontinuous as $z=0$ and $z=L$. Notice from Eqs. (A11) and (A9) that $W(0,1,s)$ is constant for $s>2$, a fact which also follows from the domain of dependence arguments. To evaluate that constant, set

$$f(x)=\int_x^{2-x}\left[N_x(x,s')-\frac{1}{2}(A-B)\Big|_x N(x,s')\right]ds' \tag{A12}$$

so that, from Eq. (A11),

$$W(0,1,s)=[k-f(0)]/4c_0,\quad s>2, \tag{A13}$$

where

$$k=N(0,0)+N(0,2)+\tfrac{1}{2}(c_1+1)(A-B)|_{0^+} -\frac{1}{2}(c_1-1)(A+B)\Big|_{0^+}\exp\left[\int_0^1 B(s')ds'\right]. \tag{A14}$$

The constant k is known from the boundary conditions Eq. (A5), so it remains to determine $f(0)$.

Differentiate Eq. (A12) with respect to x and eliminate the N_{xx} term by using Eq. (A4). Upon performing the s' integrations, it follows that

$$f'(x)+\tfrac{1}{2}(A+B)|_x f(x)=g(x), \tag{A15}$$

where

$$g(x)=-\frac{d}{dx}N(x,x)-\frac{d}{dx}N(x,2-x) +\tfrac{1}{2}(A+B)|_x N(x,x)+\tfrac{1}{2}(A-3B)|_x N(x,2-x). \tag{A16}$$

Solving Eq. (A15) yields

$$f(0)=-\int_0^1 g(x)\exp\left[\frac{1}{2}\int_0^x[A(x')+B(x')]dx'\right]dx. \tag{A17}$$

Using Eqs. (A17) and (A5), a tedious calculation now shows that

$$f(0)=k. \tag{A18}$$

Hence, $W(0,1,s)=0$ for $s>2$.

Fortunately, this calculation does not need to be repeated to verify the compact support of the $V^\pm$ kernels. Instead observe that if Eq. (A8) is used to eliminate $u'_+(s)$ from Eq. (A3), then

$$V^+(0,1,s)=-W(0,1,s),\quad s>2. \tag{A19}$$

In a similar manner it follows that

$$V^-(0,1,s)=-W(0,1,s),\quad s>2. \tag{A20}$$

APPENDIX B: EXACT REPRESENTATIONS OF THE SCATTERING AND PROPAGATOR KERNELS FOR CONSTANT *A*(*x*) AND *B*(*x*)

For constant $A(x)$ and $B(x)$ the scattering kernels $R^\pm$, T, W, and $V^\pm$ can be determined analytically. Throughout this appendix it is therefore assumed that the function $A(x)=A$ and $B(x)=B$, for $0<x<1$, where A and B are real constants. For the convenience of the reader the basic equations [see Eqs. (3.1), (3.5), (3.15), (4.3), (4.7), and (4.8)] that are used in this appendix are repeated here:

$$\begin{aligned}
&R_x^+(x,y,s)\\
&\quad=2R_s^+(x,y,s)-BR^+(x,y,s)\\
&\qquad-\frac{1}{2}(A+B)\int_0^s R^+(x,y,s')R^+(x,y,s-s')ds',\\
&\qquad s>0,
\end{aligned} \tag{B1}$$

$$R^+(y,y,s)=0,\quad s>0,$$
$$R^+(x,y,0^+)=-\tfrac{1}{4}(A-B),\quad x<y;$$

$$\begin{aligned}
&R_y^-(x,y,s)\\
&\quad=-2R_s^-(x,y,s)+BR^-(x,y,s)\\
&\qquad-\frac{1}{2}(A-B)\int_0^s R^-(x,y,s')R^-(x,y,s-s')ds',\\
&\qquad s>0,
\end{aligned} \tag{B2}$$

$$R^-(x,x,s)=0,\quad s>0,$$
$$R^-(x,y,0^+)=\tfrac{1}{4}(A+B),\quad x<y;$$

$$\begin{aligned}
&T_x(x,y,s)\\
&\quad=-\tfrac{1}{2}(A+B)\Big[R^+(x,y,s)\\
&\qquad+\int_0^s T(x,y,s')R^+(x,y,s-s')ds'\Big],\quad s>0,
\end{aligned} \tag{B3}$$

$$T(y,y,s)=0,\quad s>0;$$

$$\begin{aligned}
&W_x(x,y,s)\\
&\quad=\tfrac{1}{2}(A+B)\Big[R^+(x,y,s)\\
&\qquad+\int_0^s W(x,y,s')R^+(x,y,s-s')ds'\Big],\quad s>0,
\end{aligned} \tag{B4}$$

$$W(y,y,s)=0,\quad s>0;$$

$$\begin{aligned}
&V_x^+(x,y,s)\\
&\quad=2V_s^+(x,y,s)-BV^+(x,y,s)\\
&\qquad+\tfrac{1}{2}(A-B)W(x,y,s),\\
&V^+(y,y,s)=0,\quad s>0,
\end{aligned} \tag{B5}$$

$$V^+(x,y,0^+)=-\tfrac{1}{4}(A-B),\quad x<y;$$

$$\begin{aligned}
&V_x^-(x,y,s)\\
&\quad=-2V_s^-(x,y,s)+BV^-(x,y,s)\\
&\qquad+\tfrac{1}{2}(A-B)W(x,y,s),\\
&V^-(x,x,s)=0,\quad s>0,
\end{aligned} \tag{B6}$$

$$V^-(x,y,0^+)=\tfrac{1}{4}(A+B),\quad x<y.$$

These equations can be solved by a Laplace transformation in the "time" variable s. The Laplace transform of a function f is indicate $\hat f$ or $\Lambda[f]$ and the transformed time variable is denoted by p. In this notation the x and y dependence of transformed functions are suppressed for convenience. Equation (B1) transforms into the Riccati equation

$$\begin{aligned}
&\hat R_x^+(p)-(2p-B)\hat R^+(p)-\tfrac{1}{2}(A-B)\\
&\quad+\tfrac{1}{2}(A+B)\hat R^{+2}(p)=0,\\
&\hat R^+(p)=0,\quad x=y,
\end{aligned} \tag{B7}$$

with solution

$$\hat{R}^{+}(p) = -\frac{(1-e^{-\beta})\gamma}{(A+B)(1+\delta e^{-\beta})}$$
$$= -\frac{\gamma}{(A+B)}\left\{1-(1+\delta)\sum_{n=1}^{\infty}(-\delta)^{n-1}e^{-n\beta}\right\}, \quad \text{(B8)}$$

where

$$\alpha = \tfrac{1}{2}(A^2-B^2)^{1/2},$$
$$\beta = (y-x)((2p-B)^2+4\alpha^2)^{1/2}, \quad \text{(B9)}$$
$$\gamma = ((2p-B)^2+4\alpha^2)^{1/2}-2p+B,$$
$$\delta = \gamma^2/4\alpha^2.$$

It is now rather straightforward to invert each term in the bracket in Eq. (B8) with the use of the identity

$$\hat{f}((p^2+\alpha^2)^{1/2}) = \hat{f}(p) - \Lambda\left\{\alpha\int_0^s f(u)\,J_1(\alpha(s^2-u^2)^{1/2}) \times (s^2-u^2)^{-1/2}u\,du\right\}(p). \quad \text{(B10)}$$

The final result is

$$R^{+}(x,y,s) = -\tfrac{1}{2}(A-B)e^{Bs/2}\sum_{n=0}^{\infty}(-1)^n H(s-2n(y-x)) \times\Big\{S_n(s-2n(y-x)) - 2n(y-x)\alpha^2\int_{2n(y-x)}^{s} S_n(s'-s) \times\frac{J_1(\alpha(s'^2-4n^2(y-x)^2)^{1/2})}{\alpha(s'^2-4n^2(y-x)^2)^{1/2}}ds'\Big\}, \quad \text{(B11)}$$

where

$$S_n(s) = [(2n-1)J_{2n-1}(\alpha s) + (2n+1)J_{2n+1}(\alpha s)]/\alpha s, \quad n=1,2,3,..., \quad \text{(B12)}$$
$$S_0(s) = J_1(\alpha s)/\alpha s,$$

where J_n is the Bessel function of order n. Some plots of R^{+} are shown in Fig. 5.

The reflection kernel from the right-hand side $R^{-}(x,y,s)$ is easily obtained by replacing A with $-A$ in the equations above. With the same definitions of $S_n(s)$ as above, the result is

$$R^{-}(x,y,s) = \tfrac{1}{2}(A+B)e^{Bs/2}\sum_{n=0}^{\infty}(-1)^n H(s-2n(y-x)) \times\Big\{S_n(s-2n(y-x)) - 2n(y-x)\alpha^2\int_{2n(y-x)}^{s} S_n(s'-s) \times\frac{J_1(\alpha(s'^2-4n^2(y-x)^2)^{1/2})}{\alpha(s'^2-4n^2(y-x)^2)^{1/2}}ds'\Big\}. \quad \text{(B13)}$$

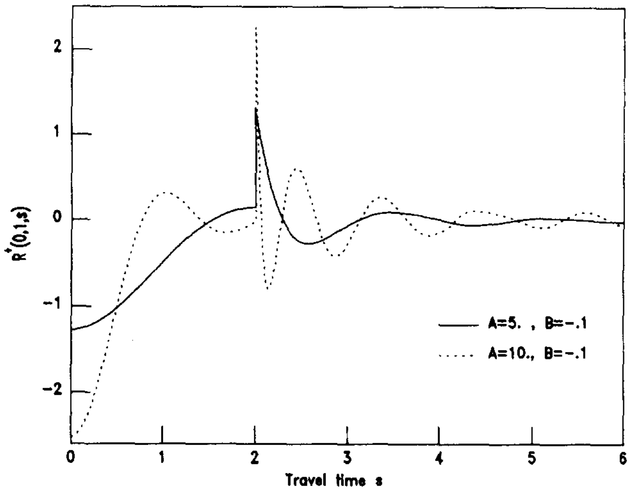

FIG. 5. The reflection kernel $R^{+}(0,1,s)$ for three round trips in the medium. Two examples with constant A and B profiles are shown.

The transmission kernel $T(x,y,s)$ is obtained from the Laplace transform of Eq. (B3). The solution in the transformed time variable p is

$$\hat{T}(p) = -1 + \frac{1+\delta}{1+\delta e^{-\beta}}e^{-\gamma(y-x)/2}$$
$$= e^{-\gamma(y-x)/2}(1+\delta) - 1 + e^{-\gamma(y-x)/2}(1+\delta)\sum_{n=1}^{\infty}(-\delta)^n e^{-n\beta}. \quad \text{(B14)}$$

The inversion of this equation leads to rather cumbersome algebra. The following identity is of great help:

$$\hat{f}(a+(p^2+\alpha^2)^{1/2}-p) - \hat{f}(a) = \Lambda\left\{-\alpha^2\int_0^{\infty} e^{-au}\frac{J_1(\alpha(s^2+2us)^{1/2})}{\alpha(s^2+2us)^{1/2}}uf(u)du\right\}(p). \quad \text{(B15)}$$

The result of the inversion is

$$T(x,y,s) = -\alpha^2 e^{Bs/2}\sum_{n=0}^{\infty}(-1)^n H(s-2n(y-x)) \times\Big\{P_n(s-2n(y-x)) - 2n(y-x)\alpha^2\int_{2n(y-x)}^{s} P_n(s-s') \times\frac{J_1(\alpha(s'^2-4n^2(y-x)^2)^{1/2})}{\alpha(s'^2-4n^2(y-x)^2)^{1/2}}ds'\Big\}, \quad \text{(B16)}$$

where

$$P_n(s) = \left[\left(\frac{\partial}{\alpha\,\partial u}\right)^{2n} + \left(\frac{\partial}{\alpha\,\partial u}\right)^{2n+2}\right] \times\left[u\frac{J_1(\alpha(s^2+2us)^{1/2})}{\alpha(s^2+2us)^{1/2}}\right]_{u=2(y-x)}$$
$$= 2w^{-2n-4}(\alpha s)^{2n}\{\alpha^2 s[4n(y-x) \times(s+y-x)-s^2]J_{2n+2}(w)$$

$$+w[(y-x)^2\alpha^2 s-2n(2n+1)$$

$$\times(s+2(y-x))]J_{2n+1}(w)\} \quad \text{(B17)}$$

and

$$w=\alpha(s^2+2s(y-x))^{1/2}.$$

Plots of T are shown in Fig. 6.

The solution to Eq. (B4) in the transformed time variable p is

$$\widehat{W}(p)=-1+\frac{1+\delta e^{-\beta}}{1+\delta}e^{\gamma(y-x)/2}. \quad \text{(B18)}$$

The inversion of this equation gives

$$W(x,y,s)=\tfrac{1}{2}H(2(y-x)-s)e^{Bs/2}\alpha^2[2(y-x)-s]$$

$$\times\frac{I_1(\alpha(2(y-x)s-s^2)^{1/2})}{\alpha(2(y-x)s-s^2)^{1/2}}. \quad \text{(B19)}$$

Proceeding to the V^+ equation, the result in the transformed variable p is

$$\widehat{V}^+(x,y,p)=-\tfrac{1}{2}(A-B)((2p-B)^2+4\alpha^2)^{-1/2}$$

$$\times e^{\gamma(y-x)/2}[1-e^{-\beta}], \quad \text{(B20)}$$

with inverse

$$V^+(x,y,s)=-\tfrac{1}{4}(A-B)H(2(y-x)-s)e^{Bs/2}$$

$$\times I_0(\alpha(2(y-x)s-s^2)^{1/2}). \quad \text{(B21)}$$

The corresponding result for $V^-(x,y,s)$ is easily obtained by replacing A with $-A$ in the equation above. The result is

$$V^-(x,y,s)=\tfrac{1}{4}(A+B)H(2(y-x)-s)e^{Bs/2}$$

$$\times I_0(\alpha(2(y-x)s-s^2)^{1/2}). \quad \text{(B22)}$$

The compact support of the kernels W and $V^\pm$, which is a general property derived in Appendix A, is clearly seen in Eqs. (B19), (B21), and (B22). Typical examples of the kernels W and V^+ are shown in Figs. 7 and 8.

Now having the explicit expressions for the kernels $R^\pm$, T, W, and $V^\pm$ for the case when A and B are constants, it is instructive to verify some of the basic equations in this paper for this special case. For example, the jump in the kernels $R^\pm$ along the plane $s=2(y-x)$ given by Eq. (3.10) is easily verified from Eqs. (B11) and (B13). It is also easy to verify Eq. (3.11).

Equations (4.9) and (4.10) can also be verified in this special case of constant A and B. These equations are equivalent to the Bessel function identity

$$\frac{d}{dx}J_0((t^2-2xt)^{1/2})$$

$$=\int_0^t J_0((t'^2-2xt')^{1/2})\frac{J_1(t-t')}{t-t'}\,dt'. \quad \text{(B23)}$$

This identity can be proved by showing that both sides are the same entire function in x.

APPENDIX C: PROPAGATOR DYNAMICS

In this appendix the dynamics of the kernels V^+ and W are derived in an alternate manner that does not utilize the dynamics of $R^\pm$ and T. This derivation depends on a contin-

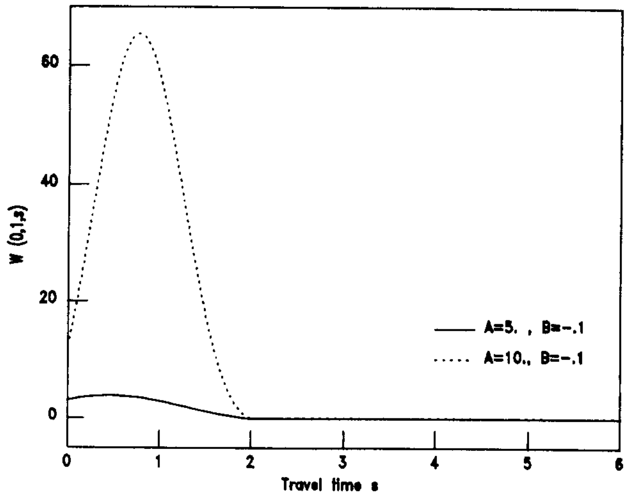

FIG. 7. The same as Fig. 5 but showing the propagation kernel $W(0,1,s)$.

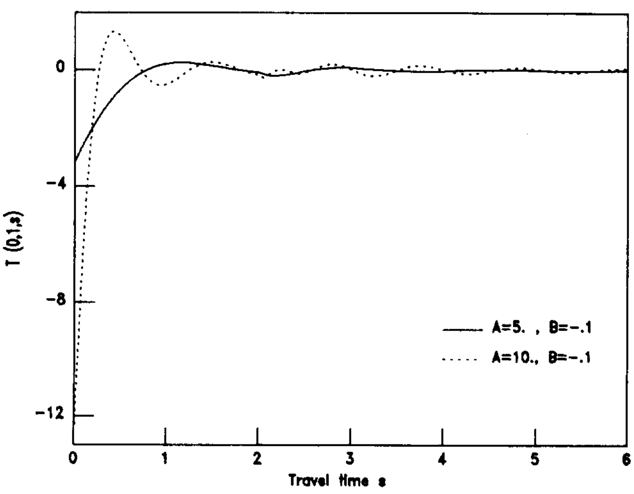

FIG. 6. The same as Fig. 5 but showing the transmission kernel $T(0,1,s)$.

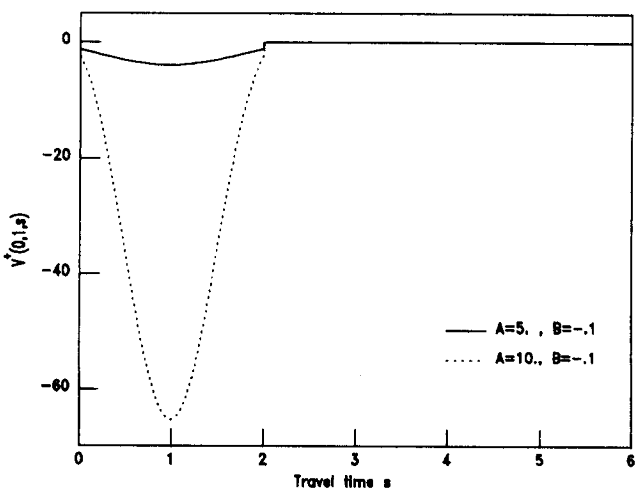

FIG. 8. The same as Fig. 5 but showing the propagation kernel $V^+(0,1,s)$.

uous representation of the propagation operator.

For the sake of convenience, the transformed problem [Eq. (2.14)] will be the starting point for this analysis, although the derivation could be carried out in terms of the physical variables that appear in Eq. (2.1). The analysis presented here is similar in spirit to that given elsewhere[3,5,10] for derivation of the scattering operator equations. The independent variable x in Eq. (2.14) will be replaced by the dummy variable z, since x is used to denote the end point of a subregion $[x, y]$. The variable z should not be confused with that appearing in Eq. (2.1). Begin by introducing a splitting[23] of the field $u(z,s)$ in Eq. (2.14), defined by

$$u^{\pm}(z,s) = \tfrac{1}{2}[u(z,s) \mp \partial_s^{-1} u_z(z,s)], \tag{C1}$$

where

$$\partial_s^{-1} u_z(z,s) = \int_{-\infty}^{s} u_z(z,s')ds'.$$

In a homogeneous medium, this splitting merely reduces the field $u(z,s)$ into right moving (+) and left moving ($-$) waves. More generally, Eq. (C1) is a change of basis from $(u,u_z)^T$ to $(u^+,u^-)^T$ for Eq. (2.14). In this new basis, Eq. (2.14) becomes

$$\frac{\partial}{\partial z}\begin{bmatrix} u^+(z,s) \\ u^-(z,s) \end{bmatrix} = \begin{bmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{bmatrix}\begin{bmatrix} u^+(z,s) \\ u^-(z,s) \end{bmatrix}$$

$$\equiv D(z)\begin{bmatrix} u^+(z,s) \\ u^-(z,s) \end{bmatrix}, \tag{C2}$$

where

$$\begin{aligned} \alpha(z) &= -\frac{1}{2}[A(z) - B(z)] - \frac{\partial}{\partial s}, \\ \beta(z) &= \tfrac{1}{2}[A(z) + B(z)], \\ \gamma(z) &= \tfrac{1}{2}[A(z) - B(z)], \\ \delta(z) &= -\frac{1}{2}[A(z) + B(z)] + \frac{\partial}{\partial s}. \end{aligned} \tag{C3}$$

Now consider a subregion $[x, y]$ of the original slab. Let $P(x, y)$ denote the propagator for the subregion $[x, y]$; i.e., P is a 2×2 matrix of operators that maps the field at y over to the field at x,

$$\begin{bmatrix} u^+(x,s) \\ u^-(x,s) \end{bmatrix} = P(x,y)\begin{bmatrix} u^+(y,s) \\ u^-(y,s) \end{bmatrix}. \tag{C4}$$

Now differentiate Eq. (C4) with respect to x to obtain

$$\frac{\partial}{\partial x}\begin{bmatrix} u^+(x,s) \\ u^-(x,s) \end{bmatrix} = \frac{\partial P(x,y)}{\partial x}\begin{bmatrix} u^+(y,s) \\ u^-(y,s) \end{bmatrix}. \tag{C5}$$

Use Eq. (C2) (evaluated at $z = x$) and (C4) to express the left-hand side of (C5) in terms of the fields at y:

$$D(x)P(x,y)\begin{bmatrix} u^+(y,s) \\ u^-(y,s) \end{bmatrix} = \frac{\partial P(x,y)}{\partial x}\begin{bmatrix} u^+(y,s) \\ u^-(y,s) \end{bmatrix}. \tag{C6}$$

Since $(u^+(y,s),u^-(y,s))^T$ can be chosen arbitrarily, it follows that

$$\frac{\partial P(x,y)}{\partial x} = D(x)P(x,y). \tag{C7}$$

It can similarly be shown that

$$\frac{\partial P(x,y)}{\partial y} = -P(x,y)D(y). \tag{C8}$$

Having now obtained differential equations (C7) and (C8) for the propagator, a representation for the entries of P is required. In order to use a representation compatible with that in Sec. III, let $u^{\pm}$ be represented by

$$u^+(z,s) = \begin{cases} u^i_+(s - z + x), & z < x, \\ u^t_+(s - z + x) + u^r_-(s - z + y), & z > y, \end{cases} \tag{C9}$$

$$u^-(z,s) = \begin{cases} u^r_+(s + z - x) + u^t_-(s + z - y), & z < x, \\ u^i_-(s + z - y), & z > y. \end{cases} \tag{C10}$$

The fields on the right-hand sides of Eqs. (C9) and (C10) are related by [cf. Eqs. (2.17)–(2.20), (2.23), and (2.24) for the special case $x = 0$ and $y = 1$]

$$\begin{aligned} u^r_{\pm}(s) &= [\mathscr{R}^{\pm}(x,y)u^i_{\pm}(\cdot)](s) \\ &= \int_{-\infty}^{s} R^{\pm}(x,y,s-s')u^i_{\pm}(s')ds', \end{aligned} \tag{C11}$$

$$\begin{aligned} u^t_{\pm}(s) &= [\mathscr{T}^{\pm}(x,y)u^i_{\pm}(\cdot)](s) \\ &= t^{\pm}(x,y)\Big[u^i_{\pm}(s) \\ &\quad + \int_{-\infty}^{s} T(x,y,s-s')u^i_{\pm}(s')ds'\Big], \end{aligned} \tag{C12}$$

$$\begin{aligned} u^r_+(s) &= [\mathscr{V}^+(x,y)u^t_+(\cdot)](s) \\ &= [t^+(x,y)]^{-1}\int_{-\infty}^{s} V^+(x,y,s-s')u^t_+(s')ds', \end{aligned} \tag{C13}$$

$$\begin{aligned} u^i_+(s) &= [\mathscr{W}^+(x,y)u^t_+(\cdot)](s) \\ &= [t^+(x,y)]^{-1}\Big[u^t_+(s) \\ &\quad + \int_{-\infty}^{s} W(x,y,s-s')u^t_+(s')ds'\Big]. \end{aligned} \tag{C14}$$

The relations Eqs. (3.14) and (3.17) have been used in Eqs. (C12) and (C14), respectively, and $t^{\pm}(x,y)$ is defined in Eq. (3.9). In Eqs. (C11)–(C14) it is assumed that the fields are quiescent prior to some finite time s_0, although s_0 is not necessarily zero.

It is also convenient to introduce a shift operator Q, whose action on a function of the s variable is defined by

$$Q(y,x)f(s) = f(s + x - y).$$

Repeated applications of Q have the obvious interpretation:

$$\begin{aligned} Q^2(y,x)f(s) &= Q(y,x)[Q(y,x)f(s)] \\ &= Q(y,x)f(s + x - y) \\ &= f(s + 2x - 2y), \\ Q(x,y)Q(y,x)f(s) &= f(s). \end{aligned}$$

Still confining attention to the subregion $[x, y]$, it now follows that

$$\begin{aligned} u^+(x,s) &= u^i_+(s) \\ &= [\mathscr{W}^+(x,y)u^t_+(\cdot)](s) \\ &= [\mathscr{W}^+(x,y)\{u^+(y,\cdot) - u^r_-(\cdot)\}](s+y-x) \\ &= Q(x,y)[\mathscr{W}^+(x,y)u^+(y,\cdot)](s) \\ &\quad - Q(x,y)[\mathscr{W}^+(x,y)\mathscr{R}^-(x,y)u^-(y,\cdot)](s), \end{aligned} \tag{C15}$$

and

$$\begin{aligned} u^-(x,s) &= u^r_+(s) + u^t_-(s+x-y) \\ &= [\mathscr{V}^+(x,y)u^t_+(\cdot)](s) + Q(y,x) \\ &\quad \times [\mathscr{T}^-(x,y)u^-(y,\cdot)](s) \\ &= Q(x,y)[\mathscr{V}^+(x,y)u^+(y,\cdot)](s) \\ &\quad - Q(x,y)[\mathscr{V}^+(x,y)\mathscr{R}^-(x,y,)u^-(y,\cdot)](s) \\ &\quad + Q(y,x)[\mathscr{T}^-(x,y)u^-(y,\cdot)](s). \end{aligned} \tag{C16}$$

Using Eqs. (C15) and (C16), the propagator can be written in the explicit form

$$P(x,y) = Q(x,y)\begin{bmatrix} \mathscr{W}^+(x,y) & -\mathscr{W}^+(x,y)\mathscr{R}^-(x,y) \\ \mathscr{V}^+(x,y) & Q^2(y,x)\mathscr{T}^-(x,y) - \mathscr{V}^+(x,y)\mathscr{R}^-(x,y) \end{bmatrix}. \tag{C17}$$

In order to pass from the operator equations (C7) and (C8) to equations involving the kernels W and V^+, it is easier to consider these equations at the level of Eq. (C6). Setting $u^-(y,s) \equiv 0$ yields

$$\begin{aligned} &Q(y,x)\frac{\partial}{\partial x}[Q(x,y)\mathscr{W}^+(x,y)u^+(y,\cdot)](s) \\ &\quad = [(\alpha(x)\mathscr{W}^+(x,y) + \beta(x)\mathscr{V}^+(x,y))u^+(y,\cdot)](s), \\ &Q(y,x)\frac{\partial}{\partial x}[Q(x,y)\mathscr{V}^+(x,y)u^+(y,\cdot)](s) \\ &\quad = [(\gamma(x)\mathscr{W}^+(x,y) + \delta(x)\mathscr{V}^+(x,y))u^+(y,\cdot)](s). \end{aligned}$$

Expressing these equations in terms of kernels produces Eqs. (4.5) and (4.7). Similarly, applying Eq. (C8) to $(u^+(y,s),0)^T$ yields Eqs. (4.4), (4.9), and (4.13) and verifies that the jump in $V^+(x,y,s)$ at $s = 2(y-x)$ is the same as that in $R^+(x,y,s)$, as given in Eq. (3.10). This last fact also follows from Eq. (3.20).

With the notation established above, it is now easy to compare the action of the propagator matrix $P(x,y)$ with that of the scattering matrix $S(x,y)$. Here S relates the $\pm$ components of u according to

$$\begin{bmatrix} u^+(y,s) \\ u^-(x,s) \end{bmatrix} = S(x,y)\begin{bmatrix} u^+(x,s) \\ u^-(y,s) \end{bmatrix},$$

and is represented by

$$S(x,y) = \begin{bmatrix} \mathscr{T}^+(x,y) & \mathscr{R}^-(x,y) \\ \mathscr{R}^+(x,y) & \mathscr{T}^-(x,y) \end{bmatrix}.$$

It can be shown (see Ref. 3, 5, or 10) that S satisfies

$$\begin{aligned} \frac{\partial S}{\partial x} &= -\begin{bmatrix} \mathscr{T}^+(x,y) & 0 \\ \mathscr{R}^+(x,y) & I \end{bmatrix}\begin{bmatrix} \alpha(x) & \beta(x) \\ -\gamma(x) & -\delta(x) \end{bmatrix} \\ &\quad \times \begin{bmatrix} I & 0 \\ \mathscr{R}^+(x,y) & \mathscr{T}^-(x,y) \end{bmatrix}, \\ \frac{\partial S}{\partial y} &= -\begin{bmatrix} I & \mathscr{R}^-(x,y) \\ 0 & \mathscr{T}^-(x,y) \end{bmatrix}\begin{bmatrix} \alpha(y) & \beta(y) \\ -\gamma(y) & -\delta(y) \end{bmatrix} \\ &\quad \times \begin{bmatrix} \mathscr{T}^+(x,y) & \mathscr{R}^-(x,y) \\ 0 & I \end{bmatrix}, \end{aligned} \tag{C18}$$

where 0 and I denote the zero and identity operators, respectively. Equations (3.1)–(3.8) can be obtained from the operator equations (C18) by rewriting the latter in terms of the representations (C11) and (C12).

[1] G. Kristensson and R. J. Krueger, "Direct and inverse scattering in the time domain for a dissipative wave equation. II. Simultaneous reconstruction of dissipation and phase velocity profiles," J. Math. Phys. **27**, 1683 (1986).

[2] J. P. Corones and R. J. Krueger, "Obtaining scattering kernels using invariant imbedding," J. Math. Anal. Appl. **95**, 393 (1983).

[3] M. Davison, "A general approach to splitting and invariant imbedding techniques for linear wave equations," Ames Laboratory preprint, to appear in J. Math. Anal. Appl.

[4] J. P. Corones, M. E. Davison, and R. J. Krueger, "Direct and inverse scattering in the time domain via invariant imbedding equations," J. Acoust. Soc. Am. **74**, 1535 (1983).

[5] J. P. Corones, M. E. Davison, and R. J. Krueger, "Wave splittings, invariant imbedding and inverse scattering," in *Inverse Optics, Proceedings of the SPIE*, Vol. 413, edited by A. J. Devaney (SPIE, Bellingham, WA, 1983), pp. 102–106.

[6] J. P. Corones, "Wave splitting and invariant imbedding in direct and inverse scattering," in *Wave Propagation in Homogeneous Media and Ultrasonic Nondestructive Evaluation*, AMD-Vol. 62, edited by G. C. Johnson (ASME, New York, 1984), pp. 31–35.

[7] J. P. Corones, R. J. Krueger, and C. R. Vogel, "The effects of noise and bandlimiting in a one dimensional time dependent inverse scattering technique," in *Review of Progress in Quantitative Nondestructive Evaluation*, Vol. 4, edited by D. O. Thompson and D. E. Chimenti (Plenum, New York, 1985), pp. 551–558.

[8] J. P. Corones, M. E. Davison, and R. J. Krueger, "The effects of dissipation in one-dimensional inverse problem," in Ref. 5, pp. 107–114.

[9] J. P. Corones, R. J. Krueger, and V. H. Weston, "Some recent results in inverse scattering theory," in *Inverse Problems of Acoustic and Elastic Waves*, edited by F. Santosa, Y. Pao, W. Symes, and C. Holland (SIAM, Philadelphia, PA, 1985), pp. 65–81.

[10] J. P. Corones, M. E. Davison, and R. J. Krueger, "Dissipative inverse problems in the time domain," in *Inverse Methods in Electromagnetic Imaging*, NATO ASI series, Series C, Vol. 143, edited by W-M. Boerner (Reidel, Dordrecht, 1985), pp. 121–130.

[11] A. M. Bruckstein, B. C. Levy, and T. Kailath, "Differential methods in inverse scattering," SIAM J. Appl. Math. **45**, 312 (1985).

[12] V. Weston, "On the inverse problem for a hyperbolic dispersive partial differential equation," J. Math. Phys. **13**, 1952 (1972).

[13] V. Weston and R. J. Krueger, "On the inverse problem for a hyperbolic

dispersive partial differential equation. II," J. Math. Phys. **14**, 406 (1973).
[14]V. Weston, "On inverse scattering," J. Math. Phys. **15**, 209 (1974).
[15]R. J. Krueger, "An inverse problem for a dissipative hyperbolic equation with discontinuous coefficients," Quart. Appl. Math. **34**, 129 (1976).
[16]R. J. Krueger, "An inverse problem for an absorbing medium with multiple discontinuities," Quart. Appl. Math. **36**, 235 (1978).
[17]R. J. Krueger, "Numerical aspects of a dissipative inverse problem," IEEE Trans. Antennas Propag. **AP-29**, 253 (1981).
[18]M. Jaulent, "Inverse scattering problems in absorbing media," J. Math. Phys. **17**, 1351 (1976).
[19]M. Jaulent, "Inverse scattering problems for LCRG transmission lines," J. Math. Phys. **23**, 2286 (1982).
[20]R. S. Beezley and R. J. Krueger, "An electromagnetic inverse problem for dispersive media," J. Math. Phys. **26**, 317 (1985).
[21]E. Ammicht, J. P. Corones, and R. J. Krueger, "Direct and inverse scattering for viscoelastic media," preprint.
[22]B. L. N. Kennet, *Seismic Wave Propagation in Stratified Media* (Cambridge U. P., Cambridge, 1983).
[23]J. P. Corones, "Bremmer series that correct parabolic approximations," J. Math. Anal. Appl. **50**, 361 (1975).