# **Regular** Contributions

## **Duality in Mechanics**

P. LE CORBEILLER AND YING-WA YEUNG Cruft Laboratory, Harvard University, Cambridge, Massachusetts (Received July 21, 1952)

Given an electric network  $E_v$  which has M meshes and P node-pairs, its electric dual  $E_i$  will have P meshes and M node-pairs and its classical mechanical analog  $M_I$  will have M+1 nodes, M independent node-pairs, and P independent node cycles. A second mechanical system  $M_v$ , the classical analog of  $E_v$ , will have M cycles and P node-pairs. If, for example, M=2, P=3, the systems  $E_v$  and  $M_v$ , analogs in the Firestone or "mobility" method, will be governed by two mesh equations, expressing that the algebraic sum of the voltages or velocities around any loop is zero; the systems  $E_i$  and  $M_f$ , also Firestone analogs, will satisfy two node equations, expressing that the algebraic sum of the currents or forces leaving any node is zero. These four sets of equations are identical, interchanging symbols suitably. The consideration of the four systems,  $E_v$ ,  $E_i$ ,  $M_v$ ,  $M_f$ , forming a complete set, shows the advantages of the Firestone over the classical system of analogies and suggests a systematic use of duality in mechanical as well as in electrical systems.

#### 1. INTRODUCTION

**`HE** purpose of this paper is to present a new way of introducing the "mobility" system of electromechanical analogies advocated in 1933 by Dr. F. A. Firestone.<sup>1</sup> Along the way we shall hit upon an interesting method of analyzing a mechanical system, first suggested by Dr. Firestone<sup>1,2</sup> and recently developed by Dr. H. M. Trent.<sup>3</sup> Our idea is to bring together two types of transformation much employed in system analysis and synthesis, electromechanical analogy and duality, and to make complete use of their possibilities. We shall assume that the reader is familiar with the "classical" system of analogy (force-emf, velocitycurrent) and with duality in electric networks (exchange of current and voltage, exchange of series and parallel connections), but not with Dr. Firestone's system. The symbols used will be for the most part those recommended jointly by the ASME and the AIEE.

## 2. THE FUNDAMENTAL FOUR-SYSTEMS SET

Consider an electric system  $E_v$  and the mechanical system  $M_f$  which is its classical analog. As simple examples we choose a series circuit (L, R, S) containing a voltage generator v(t) (Fig. 1), and its mechanical analog (M, D, K), acted upon by a force f(t) (Fig. 2). The justification for this correspondence is that the differential equations for the two systems are identical, with only a change in symbols (L-M, i-v, etc.):

$$L\frac{di}{dt} + Ri + Sq = v(t), \tag{1}$$

$$\frac{dv}{dt} + Dv + Kx = f(t). \tag{2}$$

The electric system  $E_v$  and the mechanical system  $M_f$  are thus two distinct examples of realization of

<sup>1</sup> F. A. Firestone, J. Acoust. Soc. Am. 4, 249 (1933). <sup>2</sup> F. A. Firestone, J. Appl. Phys. 9, 373 (1938). <sup>3</sup> H. M. Trent, "An Alternative Formulation of the Laws of Mechanics" (Naval Research Lab., September, 1950); Am. Soc. Mech. Eng. Paper No. 51-A30, (November, 1951).

one and the same mathematical equation. Inductance L in example  $E_v$  is the analog of mass M in example  $M_f$ , etc.

Consider next the same electric system  $E_v$  and another electric system  $E_i$ , corresponding to each other by *duality*. Here system  $E_i$  will consist of a current generator i(t) in parallel with three branches  $C, G(=R^{-1})$ and  $\Gamma(=L^{-1})$  (Fig. 3.). The justification for this second type of correspondence is that the differential equations for the two systems are identical, with only a change in symbols (L-C, i-v, etc.):

$$L\frac{di}{dt} + Ri + S\int idt = v(t), \qquad (1)$$

$$C\frac{dv}{dt} + Gv + \Gamma \int v dt = i(t).$$
(3)

The two electric systems  $E_v$ ,  $E_i$  and the mechanical system  $M_f$  are now three distinct examples of realization of one and the same mathematical equation. We can arrange their symbols in the following way:

$$\begin{array}{c} E_v \longleftrightarrow M_f \\ \uparrow \\ \downarrow \\ E_i \end{array}$$

The horizontal double arrow is for "classical analogy": the vertical double arrow is for "duality." Obviously a fourth system is missing (a fourth realization of the same equation!). This mechanical system  $M_v$  should be the classical analog of  $E_i$ , and should correspond to  $M_f$  by (mechanical) duality. We don't know yet what this system is, but it is easy to write its differential equation, deriving it from Eq. (2) in the same way Eq. (3) can be derived from Eq. (1). We obtain

$$\frac{1}{K}\frac{df}{dt} + \frac{f}{D} + \frac{1}{M}\int fdt = v(t).$$
(4)

System  $M_{\nu}$  must then consist of three mechanical elements K, D, M in series, through which the same



force f(t) will be acting; the sum of the three relative velocities df/Kdt, f/D, ffdt/M, must equal the velocity v(t) imposed at some point of the system. The system shown on Fig. 4 satisfies these conditions. It contains a mass M, a dashpot of viscous resistance D and a spring of stiffness K in series, and a crank-and-rod velocity generator, also in series. (In all our drawings we shall assume that no solid friction is present anywhere, and a viscous resistance will always be explicitly represented by a dashpot.)

The set of four systems  $E_v$ ,  $E_i$ ,  $M_f$ ,  $M_v$  is obviously a complete set (like two blue and red triangles and two blue and red circles). Until we had this complete set before our eyes we could not have a good understanding of the possibilities of the two transformations, analogy and duality.

#### 3. A NEW ANALYSIS OF MECHANICAL SYSTEMS

Equations (1) and (3) for the electrical systems are the simplest examples of Kirchhoff's Laws. Kirchhoff's voltage law states that the algebraic sum of the voltages around any closed circuit or mesh must be zero. System  $E_v$  has only one mesh and one unknown current i(t) (Fig. 1). In general there will be in an electric network a certain number M of independent meshes, and their equations will determine the M unknown mesh currents.

Kirchhoff's current law states that the algebraic sum of the currents leaving any node must be zero. System  $E_i$  has two nodes, A and the ground g (Fig. 3) (points at the same potential constitute a single node). Nodes A and g provide the same equation (3); the only unknown is the voltage across the *node-pair* Ag. In general there will be in an electric network a certain number P of independent node-pairs, and their equations will determine the unknown voltages across the P node-pairs.

Consider now the mechanical system  $M_f$  (Fig. 2). We may say that all the points in a mechanical system which are rigidly connected to each other constitute a single *node*. This system then has two nodes, the massless horizontal connecting rod AA and the ground gg. Equation (2) expresses that the algebraic sum of the forces applied to node A is zero. From the standpoint of classical dynamics (-Dv), (-Kx), and f(t)are applied forces, while -M(dv/dt) is an inertial or d'Alembert force. In general there will be in a mechanical system a certain number P of mobile nodes, in addition to the "fixed" ground g, and thus P independ-

ent node-pairs. In this paper we confine ourselves to the simple case where every mobile node in the system is defined in space by a single coordinate, for the moment a translational coordinate x. The configuration of the system will then be defined at any time by the values of P coordinates, and the P equations of the system will express the Newton-d'Alembert law that at every mobile node the algebraic sum of the applied forces and of the inertial force (if present) must be zero. These P equations will determine the unknown velocities of the mobile nodes with regard to ground.

The analysis of system  $M_*$  (Fig. 4) brings a surprise. Every term in Eq. (4) is a velocity. The equation expresses that the sum of the relative velocities of node A with regard to node B (across the spring), of B to C (across the dashpot), and of C (the mass) to the wall gg (which is here the reference system and thus serves the role of "ground") is equal to the velocity imposed to node A with regard to gg. Thus, if we consider the closed *loop* (gABCg), Eq. (4) expresses the correct if unfamiliar statement that the algebraic sum of the relative velocities of every node in the loop with regard to the one on the right must be zero.<sup>1,2</sup>

The generalization of this is obvious on the basis of what we know of duality and analogy. In such mechanical systems as will be considered in this paper there will be a certain number M of independent loops, and their equations  $(\Sigma v=0 \text{ around each loop})$  will determine M unknown forces,  $F_1, F_2, \ldots, F_M$ , the force  $F_k$  acting through all the elements connected in series around the kth loop.

It is well known that an electric network can be analyzed either on a mesh basis ( $\Sigma v=0$ , mesh currents unknown) or on a node-pair basis ( $\Sigma i=0$ , voltages across node-pairs unknown). In most cases one of these methods will lead to fewer equations than the other. It appears that a mechanical system of the type considered here can be analyzed either on a node-pair basis ( $\Sigma f=0$ , relative velocities unknown), or on a loop basis ( $\Sigma v=0$ , forces unknown). We may call for short the Newton-d'Alembert equations  $\Sigma f=0$  the dynamical or the force equations, and the new equations  $\Sigma v=0$ the kinematic or the velocity equations of the system.

This new, circuital formulation of the laws of mechanics, in contrast with the classical, nodal formulation, has been emphasized in a recent paper by Dr.



FIG. 2. A mechanical system  $M_f$ , the "classical" analog of the electric circuit  $E_v$  in Fig. 1. AA is a rigid massless rod constrained to remain horizontal. D is a dashpot.

H. M. Trent, using a different approach from that of the present paper.<sup>3</sup>

## 4. A PHILOSOPHICAL ASIDE

The reader may wonder how it is possible to derive the dynamical behavior of a system from kinematic equations. This question arises because we are accustomed in dynamics to a mistaken emphasis on forces, traceable to a metaphysical attitude current in Newton's times. From this emphasis on force considered as the "cause" of motion, there followed in electricity an emphasis on electromotive force considered as the "cause" of the current flow. To one without such preconceptions, it should be clear that in Ohm's law E=IR, as well as in Newton's law f=ma, the general gas law PV = nRT, or any natural law whatsoever, there is no cause and no effect, there are only variable physical quantities permanently connected by a mathematical equation. For instance, whenever any two of the quantities E, I, R are known at any time t, Ohm's law allows us to find the third one; if I and Rhappen to be known, it would make little sense to call them the "cause" of E, and call E their "effect." The



same remark applies throughout the field of classical physics.

In the analysis of electric circuits we make use of three such laws or relations, which are (using p for short either for d/dt or for  $j\omega$  in the alternating steady-state)

$$v=q/C=i/Cp, v=Ri, v=Lpi.$$
 (5)

Similarly in the analysis of mechanical systems we make use of three relations, relative to a spring, a dashpot, and a mass:

$$f = Kx = Kv/p, f = Dv, f = Ma = Mpv.$$
 (6)

Each one of these relations is characteristic of a single mechanical element, and relates the force *through* the element to the velocity *across* it. It is basically indifferent (when K, D, and M are given) whether we solve them for the force or for the velocity and combine these in the dynamical or in the kinematic equations. Forces and velocities will be connected in either case.

#### 5. THE "MOBILITY" SYSTEM OF ELECTROMECHANICAL ANALOGIES

A remark forces itself upon us at this stage of the argument. Any one of the four systems  $E_v$ ,  $E_i$ ,  $M_j$ ,  $M_v$ 



FIG. 4. A mechanical system  $M_{e_i}$  the classical analog of the electric circuit  $E_i$  in Fig. 3. The dashpot casing and the mass glide without friction on the plane gg'.

which make up the complete set can be described equally well by node equations or by mesh equations, and these equations are identical for the four systems if we disregard the changes in symbols. Let us assume, for example, that the electric network  $E_v$  has two meshes and three node-pairs. Then it will be most simply described by two mesh equations,  $\Sigma v = 0$  (mesh currents unknown). Then  $E_i$ , which corresponds to  $E_v$  by duality, will be most simply described by the same two equations, which are now node equations,  $\Sigma i=0$  (voltages across node-pairs unknown). Then  $M_f$ , the classical analog of  $E_v$ , will be most simply described by the same two equations, which are here  $\Sigma f = 0$  for every node (relative velocities unknown); and  $M_{\nu}$ , the dual of  $M_{f}$ , will be most simply described by the same two equations, which are here loop equations,  $\Sigma v=0$ (forces unknown). (Let the reader check this on the systems of Fig. 6, and on the corresponding equations.)

Of course, the four systems are basically equivalent; they are just four examples of realization of the same set of equations. However, if on a less abstract plane we ask ourselves which of the two mechanical systems  $M_f$ ,  $M_v$ , most "resembles" the electrical system  $E_v$ , it is clear that  $M_{v}$  is the unequivocal answer, since  $E_{v}$ and  $M_{v}$  both have two meshes and three node-pairs and are described by the same two mesh equations. Similarly  $E_i$  and  $M_f$  are most alike, since they both have two node-pairs and three meshes and are described by the same two node-pair equations. The classical analogy makes the meshes of  $E_{n}$  correspond to the node-pairs of  $M_{f}$ , an awkward situation due to the topological innocence of our forebears. The only electromechanical analogy suited to this enlightened age is that between  $E_{v}$  and  $M_{v}$ , and between  $E_{i}$  and  $M_{f}$ ; and that is the one advocated by Dr. Firestone (who calls it the "mobility" system, references 1, 2).

We, therefore, adopt this analogy, and in consequence present the array of our four systems or examples in the new and better form:





FIG. 5. Four systems, two mechanical,  $M_v$ ,  $M_f$ , two electrical,  $E_v$ ,  $E_i$ , forming a complete set. Two systems in the same row correspond by the Firestone system of analogy; two systems in the same column correspond by duality.

The horizontal double arrows are for electromechanical analogy (Firestone, of course); the vertical ones for duality, either between electrical or between mechanical systems, a topological transformation exchanging node-pairs and meshes. The order of rows or columns in the array is indifferent. Dotted diagonals indicate "classical" analogy. It is clear that classical analogy is doing two things at a time: exchanging node-pairs and meshes *and* exchanging mechanical and electrical quantities (going back to the triangles and circles quoted in Sec. 2, it makes a blue triangle the analog of a red circle).

We submit that any one interested in the analysis or synthesis of electrical and mechanical systems should always as a matter of routine divide his sheet of paper into four quarters and put down in each one the diagram



FIG. 6. Four examples of a one-section low-pass filter. (Arrows  $f_1, f_2$  should point downwards.)

and equations of one of the examples of the complete set. There may be constructional reasons why in a given case  $M_f$  should be preferred to  $M_v$  or vice-versa, and the same for  $E_i$ ,  $E_v$ . The display of the complete set enables one to choose the most convenient realization.

We shall immediately follow our own advice and bring together the examples of Figs. 1 to 4 (Fig. 5). We are now using the same graphical symbols for alternating constant-velocity and constant-force generators as for constant-voltage and constant-current generators, a circle and an oblong, respectively, with two terminals each; this makes mechanical diagrams very close copies of their electrical analogs. The same symbol v is used for velocity and voltage, which happens to fit the Firestone "mobility" system of analogy.

#### 6. CORRESPONDENCE BETWEEN MECHANICAL AND ELECTRICAL QUANTITIES

There are now *four* sets of mechanical and electrical quantities to be considered, which correspond two by two either horizontally (by "analogy") or vertically (by duality). We accordingly divide our paper into



FIG. 7. (a) An ideal 1:1 transformer, closed on a capacitor with one plate grounded; (b) its mechanical analog.

four quarters. In those on the left we put mechanical quantities, in those on the right electrical quantities. When two mechanical elements correspond to each other by duality we place them symmetrically with regard to the horizontal (mirror) line. We do the same for electrical quantities (see, for instance, the pair velocity-force, the pair capacity-inductance, etc.). We thus obtain the following *complete* correspondence scheme:

We note that coordinate and momentum are dual, as also their analogs magnetic flux and electric charge. Duality consists essentially, rather than in the exchange of v and i (or v and f) as in elementary presentations, in the exchange of the coordinate and momentum of each particle. It is actually the simplest example of a transformation which is of importance in analytical dynamics<sup>4</sup> and in modern physics.

#### 7. LOW-PASS FILTER

We shall give now the four "examples" of a onesection low-pass filter, a simple system defined by two

<sup>4</sup> H. Goldstein, *Classical Mechanics* (Addison-Wesley Press, Inc., Cambridge, 1950), p. 245.

{ velocity } force mass viscosity coeff. stiffness f momentum coordinate	$\begin{cases} v \\ f \\ M \\ D \\ K \\ p = Mv \\ x = K^{-1}f \end{cases}$	voltage current capacity conductance recipr. inductance electric charge magnetic flux	$ \begin{array}{c} v \\ i \\ C \\ G = R^{-1} \\ \Gamma = L^{-1} \\ \dot{q} = Cv \\ \phi = Li \end{array} $
{ momentum coordinate recipr. mass recipr. visc. coeff. compliance velocity force	$ \begin{array}{c} p \\ x \\ M^{-1} \\ D^{-1} \\ K^{-1} \\ v \\ f \end{array} $	electric charge magnetic flux susceptance resistance inductance voltage current	$ \begin{array}{c} q\\ \phi\\S=C^{-1}\\R\\L\\v\\i\end{array} $

TABLE I. Correspondence scheme between mechanical and electrical quantities, and between "duals."

equations, as further application of the preceding remarks (Fig. 6). It will be seen that all four sets of two equations are identical.

It is interesting to observe that by adding the force equations (A) and (B) of system  $M_f$  we obtain

$$-f + M_1 p v_A + M_2 p v_B + D v_B = 0.$$

This can be considered as the equation  $\Sigma f = 0$  relative to "ground." It thus appears that in establishing the P force equations of a mechanical system, the ground can be substituted for any one of the P other nodes in the system, as we know to be the case in electricity (communication of Dr. Trent).

## 8. SPECIAL CASE OF A MASS

The point-masses in a dynamical system require a different treatment from that of the other five basic mechanical and electrical elements.

A spring and a dashpot have two ends (nodes, terminals) and in the two equations.

$$f = Kv/p, \quad f = Dv, \tag{6'}$$



FIG. 8. Four systems forming a complete set, one of which requires the adjunction of an ideal 1:1 transformer.



FIG. 9. Four systems, two rotational,  $M_{\omega}$ ,  $M_{q}$ , two electrical,  $E_{v}$ ,  $E_{i}$ , forming a complete set.

the symbol v represents the relative velocity of the two ends at time t (with suitable sign conventions); its value is therefore independent of the reference system. A point-mass or particle, however, constitutes a single node, and the "fundamental equation of dynamics,"

$$f = Ma = Mpv = Mp^2x, \qquad (6'')$$

holds only when x is the coordinate of that particle or node in an "inertial" reference system. It would not be true, for instance, if x represented the distance of the particle in question to some other particle, itself in oscillatory motion with regard to the laboratory floor or walls (which are usually a satisfactory inertial reference system). In order to remind ourselves of this very important point, we shall follow a suggestion of Dr. Trent and draw an interrupted line from each mass-point or particle to ground, in the direction of its coordinate X.

It follows that in the strict analog of a given mechanical system, every capacitor should have one of its plates connected to ground. The two equations.

$$f = M pv, \quad i = C pv \tag{7}$$

will then represent equivalent physical situations, v and v representing the velocity of a particle and the voltage of a specific capacitor plate, both with regard to a ground permanently at rest.

On a given electric diagram it may happen that neither plate of a capacitor can be grounded. If however it is necessary to do so, one can replace each capacitor by one winding of an *ideal 1:1 transformer*, close the other winding on the capacitor, and ground one of the plates (Fig. 7(a)). This will bring no change in the equations of the system. It is then the arrangement, the mechanical analog of which we want to find.

The mechanical arrangement on Fig. 7(b) answers the question. It consists of three massless, rigid bars, 1a3, 2b4, and ab. Bars 13 and 24 have the same length, and a, b are their middle points. All hinges are frictionless. Node 4 or g is fixed with regard to the ground or walls; node 3 is the mass M to which force f is to be applied. Nodes 1, 2 can have independent but small vertical displacements, hence bar ab remains vertical. The forces acting on all four nodes are equal, and the velocity of 2 with regard to 1 is equal to the (absolute) velocity of 3 with regard to g, choosing positive directions as indicated.5.6

Figure 8 shows four "examples" of a simple system, one of which contains a mechanical 1:1 transformer.

It should be noticed that this difficulty is not special to the Firestone type of mechanical-electrical analogy. In the classical analogy it is not possible to draw the system  $M_v$ , corresponding to  $E_i$ , without making use of a 1:1 mechanical transformer. However, the difficulty is harder to analyze than in the Firestone system because of the complication resulting from the interchange of node-pairs and meshes.

#### 9. MECHANICAL ROTATIONAL SYSTEMS

To avoid lengthy repetition of most of the above material, the treatment of rotational systems will be explained on the four examples of a single system (Fig. 9). Particles free to move on a line are now replaced by particles free to rotate around an axis, that is, free to move on a circumference centered on the axis. Such a particle A is defined by the angle of its radius vector OA with a fixed direction Og which serves as reference system or "ground," and by a moment of *inertia*  $J_A$  (which replaces the mass). Several particles, rigidly connected to a single shaft or sleeve, constitute a single node. (Thus, a whole disk or flywheel is one node. with a specific moment of inertia J; the coordinate of the node is the variable angle of any radius OB or OA, painted on the flywheel, with the fixed direction Og.)

One can form with the distinct mobile directions OA, OB . . . , and the "ground" Og, as many independent loops  $(gAB \dots g)$  as there are independent meshes in the analog electric network. The analog of Kirchhoff's voltage laws is that  $\Sigma \omega$  must be zero around every loop, where  $\omega$  is the angular velocity of every

direction in a loop with regard to the one on the right. These are the M kinematic equations of the system.

There will be in the rotational system as many independent mobile nodes as there are independent node-pairs in the analog electric network. Several torques  $q_i$  may be acting on the same node, as well as an inertial or d'Alembert torque  $(-J\check{\theta})$ . The analog of Kirchhoff's current laws is that  $\Sigma q$  must be zero for every mobile node. These are the P dynamical equations of the system.

The equations in Fig. 9 will make these condensed explanations clear.

#### **10. CONCLUSION**

The above presentation of electromechanical analogies emphasizes the role of duality in the analysis of mechanical as well as of electrical systems. This point of view enables one to carry over to a broad class of mechanical systems the results of the research done on the mesh and node equations of electric networks by a number of workers, notably by Mr. Gabriel Kron.<sup>7</sup>

The Firestone or "mobility" system of analogies is usually presented in the following way: First it is shown to be a "possible" system, just as workable as the classical one; then one remarks that electrical circuits derived on the mobility basis resemble their mechanical analogs very closely, which is obviously an advantage. No explanation is given of why it should be so, nor do we know whether a third type of analogy might not be possible and perhaps be preferable to the two others.

Once we realize that mechanical systems can be analyzed, just like electric networks, on either a mesh or a node-pair basis, we can apply the notion of duality to mechanical systems, and it follows that two electric and two mechanical systems, dual two by two, form in a sense a complete set. The Firestone electromechanical analogy is then most naturally defined as the one in which an electric node corresponds to a mechanical node, an electric mesh to a mechanical loop. The consideration of a complete set of four systems is believed to be original with the present paper.

It should be noted that we have only covered a certain class of mechanical systems, that in which the displacement of every particle is defined by one coordinate only. If mechanical particles with two or three coordinates are considered, entirely new possibilities appear, as the example of a gyro plainly shows.

<sup>&</sup>lt;sup>6</sup> H. F. Olson, *Dynamical Analogies* (D. Van Nostrand Company, Inc., New York, 1943). <sup>6</sup> H. M. Trent, *Practical Aspects of Electromechanical Analogies*, N.R.L. Engineering Colloquium, June 26, 1947 (unpublished memorandum).

<sup>&</sup>lt;sup>7</sup>G. Kron, Tensor Analysis of Networks (John Wiley & Sons, Inc., New York, 1939).