Close inspection of Eq. (9) showed that only one value of k'a, namely

k'a=0

satisfies it. This indicates that in the limit, as the wall thickness of the hollow tube approaches zero, the only possible mode of torsional wave propagation is the zeroth mode, as would be expected.

For completeness, Fig. 1 shows the values of the characteristic numbers k'a plotted against the ratio of the tube inner radii to tube outer radii for the cases calculated. This shows that torsional wave phase velocities increase with decreasing wall thickness of tubes for all modes except the primary, and also increase with the higher modes of motion.

The values presented in Tables I through IV were calculated by use of various tables of Bessel functions which are readily available, but particular use was made of the Bureau of Standards¹¹ and the Royal Society¹² tables for larger values of the arguments than are commonly tabulated. The tabular values presented are calculated to the nearest 0.001 units, to which they are believed accurate.

¹¹ Tables of Bessel-Clifford Functions of Orders Zero and One (U. S. Dept. of Commerce, Natl. Bur. Standards Appl. Math. Series, February, 1953), No. 28.

¹² L. Fox, A Short Table for Bessel Functions of Integer Orders and Large Arguments (Royal Society Shorter Mathematical Tables No. 3, Cambridge University Press, 1954).

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Wave Propagation through Fluid Contained in a Cylindrical, Elastic Shell*

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A study is presented of the propagation of axisymmetric waves through compressible, inviscid fluid contained in a cylindrical, elastic shell. The dependence of the phase velocity as a function of frequency on four dimensionless parameters of the system is discussed and illustrated graphically.

1. INTRODUCTION

N analysis is presented of the problem of the propagation of pressure waves through inviscid fluid contained in a cylindrical, elastic shell. The essential character of the phenomenon is the interaction of the compressible fluid and the elastic shell in the transmission of waves along the axis. The study is restricted to waves which have axial symmetry and purely sinusoidal variation along the axis.

The motion of the tube is described by means of equations developed by Lin and Morgan¹ and the velocity-frequency curves obtained in this reference for waves traveling in an empty tube are used to represent certain limiting positions of the dispersion curves for the coupled system.

The equations of motion for the thin-walled tube take account of the effects of transverse shear and rotatory inertia which may be expected to be of importance at high frequencies.

To facilitate the analysis of the phenomenon and to gain a clearer understanding thereof the two limiting

cases for which the tube is either rigid or "infinitely flexible" (i.e., the tube is absent and the fluid column has a free boundary), respectively, are examined first. Attention is then devoted to waves with very small frequencies when only two modes are found to exist. The nature of the modes and their dependence on the physical parameters of the system are investigated. Finally, the general case of fluid in an elastic shell is studied for a wide range of frequency. In dimensionless form the velocity-frequency relation depends on the four dimensionless parameters h/a, the ratio of shell wall-thickness to shell radius; ρ/ρ_t , the ratio of fluid density to solid density; c_f/c_p , the ratio of the sound speed in the fluid to the speed of longitudinal waves in an infinite elastic solid, and the Poisson's ratio ν .

It is seen that all modes except two are intermediate between the limiting cases of a rigid tube and an infinitely flexible tube over most or all of the frequency range. Over the remainder of the range (if it exists) each mode resembles an empty tube mode. All these modes have minimum cut-off frequencies. The two exceptional modes are the two which exist at all frequencies and they may be considered to correspond to the 1st empty tube mode and the longitudinal sound wave propagating through fluid in a rigid tube with uniform velocity over each cross section. This "correspondence" is not one-to-one, the motions of the

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sity. ¹ T. C. Lin and G. W. Morgan, J. Appl. Mech. Trans. Am. Soc. Mech. Engrs. 23, No. 2, 255 (1956).

general system being the result of coupling of the fluid and the solid motions.

Dispersion curves are computed and plotted for one set of values of the dimensionless parameters corresponding to water contained in a brass tube. Curves for two other sets of values are drawn; these are not computed, however, their shape being predicted approximately on the basis of their correspondence to the dispersion curves for the limiting cases.

Work on certain aspects of this problem has been carried on by a number of investigators2-6 usually for relatively low frequencies, or with attention directed toward the lowest mode only, or without allowing for the influence of Poisson's ratio. A recent study by Thomson⁷ extends to high frequencies, but differs from the present one in two respects which are discussed in detail later on. The first difference arises in the formulation of the equations of motion for the tube, particularly the inclusion in the present paper of transverse shear and rotatory inertia, the former being of importance at high frequencies. The second difference concerns the results of the analyses, those obtained here differing from those of Thomson for all values of the phase velocity greater than approximately $2c_f$ in the example for which computations are carried out. This difference will be seen to arise primarily from Thomson's erroneous conclusion that only one mode exists at very small frequencies. An investigation by Biot⁸ of the propagation of waves in a cylindrical bore through an elastic material of infinite extent filled with fluid affords an interesting comparison with the present analysis.

2. FLUID MOTION

Consider a circular tube filled with compressible, inviscid fluid. Let x, r, and θ be the cylindrical coordinates with the x axis along the axis of the tube. The thickness h of the tube wall is assumed to be small compared with the mean wall radius a. For axially symmetric vibrations the pressure p in the fluid is independent of θ and satisfies the wave equation

$$\frac{\partial^2 p}{\partial t^2} = c_f^2 \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} \right), \qquad (2.1)$$

where t is the time and c_f is the speed of sound in the fluid. c_f is related to the bulk modulus K_f and the density ρ of the fluid by the equation

$$c_f^2 = K_f / \rho. \tag{2.2}$$

The components of the displacement of the fluid, (u,w) in the (x,r) directions are related to the pressure ϕ by the linearized, hydrodynamic equations

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial p}{\partial x} = 0,$$

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial p}{\partial r} = 0.$$
(2.3)

The component of the displacement in the θ direction is zero because of symmetry.

For plane waves propagating in the x direction p, uand w may be assumed in the form

$$p = P(r) \exp\left[\frac{\omega}{i-}(x-ct)\right],$$

$$u = U(r) \exp\left[\frac{\omega}{i-}(x-ct) + \frac{i\pi}{2}\right],$$

$$w = W(r) \exp\left[\frac{\omega}{i-}(x-ct)\right],$$
(2.4)

where ω is the circular frequency and c is the phase velocity. The phase angle $\pi/2$ is introduced into the expression for u for convenience. Substituting the first of Eqs. (2.4) into Eq. (2.1) and applying the condition of regularity at r=0 to the solution of the resulting ordinary differential equation, one obtains

$$P(r) = p_0 J_0(kr),$$
 (2.5)

where

$$k^{2} = \omega^{2} \left[\frac{1}{c_{f}^{2}} - \frac{1}{c^{2}} \right], \qquad (2.6)$$

and p_0 is an arbitrary constant which denotes the pressure at the axis of the tube. Substitution of Eqs. (2.4) and (2.5) into Eq. (2.3) yields

$$U(r) = \frac{p_0}{\rho\omega c} J_0(kr), \qquad (2.7)$$

$$W(r) = -\frac{p_0 k}{\rho \omega^2} J_1(kr),$$
 (2.8)

where J_0 and J_1 are Bessel functions of the first kind. Eqs. (2.4), (2.5), and (2.8) give

$$\frac{p}{w} = \frac{P(r)}{W(r)} = -\frac{\rho \omega^2}{k} \frac{J_0(kr)}{J_1(kr)}.$$
 (2.9)

So far the three parameters k, c, and ω are related by Eq. (2.6) only. In order to find a relation between any two of the parameters, another equation is required. This is obtained from the boundary conditions imposed

² H. Lamb, Mem. Proc. Manchester Lit. Phil. Soc. 42, 9 (1898).

⁴ H. Lamb, Mem. Froc. Manchester Lit. Phil. Soc. 42, 9 (1898).
⁵ Fay, Brown, and Fortier, J. Acoust. Soc. Am. 19, 850 (1947).
⁴ W. Jacobi, J. Acoust. Soc. Am. 21, 120 (1949).
⁵ P. M. Morse, Vibration and Sound (McGraw-Hill Book Company, Inc., New York, 1948), p. 305.
⁶ M. L. Baron and H. H. Bleich, J. Appl. Mech., Trans. Am. Soc. Mech. Engrs. 21, No. 2, 178 (1954).
⁷ W. T. Thomson, "Transmission of pressure waves in liquid filled tubes" Proc. First U. S. Natl. Congr. Appl. Mech., 927 (Chicago, 1951). (Chicago, 1951). ^a M, A. Biot, J. Appl. Phys. 23, 997 (1952).

on the fluid motion and will be discussed in the next section.

3. BOUNDARY CONDITIONS

Before discussing the boundary conditions which must be applied to the motion of the fluid in an elastic tube, it is useful to consider two extreme cases. One is that of a rigid tube and the other is that of an infinitely flexible tube, i.e., a cylinder of fluid with a free surface. The results for these simple cases will serve as a guide for the more complicated case of an elastic tube.

So far as the fluid is concerned, a measure of the rigidity of the tube wall is supplied by the ratio p/w at the inner surface of the wall, or, for small disturbances (within linear theory) and for small h/a, at the surface r=a. For a rigid tube,

$$\left(\frac{p}{w}\right)_{r=a} = \infty. \tag{3.1}$$

It follows from Eq. (2.9) that

 $ka = \alpha_n$, where $J_1(\alpha_n) = 0$, $n = 0, 1, 2, \cdots$. (3.2)

The root k=0 for the denominator of the right-hand side of Eq. (2.9) is included in Eqs. (3.2), since $\alpha_0=0$ and $a\neq 0$. Elimination of k between Eqs. (3.2) and (2.6) yields the characteristic equation relating c and ω for the propagation of pressure waves through fluid contained in a rigid tube:

$$\frac{1}{c^2} = \frac{1}{c_f^2} - \frac{\alpha_n^2}{\omega^2 a^2}.$$
 (3.3)

To each value of α_n there corresponds one mode. For an unattenuated wave both c and ω are real. It follows from Eq. (3.3) that

$$\omega^2 \ge \omega_{rn}^2 \equiv \frac{c_f^2 \alpha_n^2}{a^2}, \qquad (3.4)$$

where ω_{rn} is the cut-off frequency for the *n*th mode of vibrations in a rigid tube. Hence the *n*th mode exists only at frequencies higher than ω_{rn} .

Similarly, for an infinitely flexible tube,

$$\left(\frac{p}{w}\right)_{r=a} = 0. \tag{3.5}$$

It follows from Eq. (2.9) that

$$ka = \beta_n$$
, where $J_0(\beta_n) = 0$, $n = 1, 2, \cdots$ (3.6)

The characteristic equation is obtained from Eqs. (3.6) and (2.6) by eliminating k:

$$\frac{1}{c^2} = \frac{1}{c_f^2} - \frac{\beta_n^2}{\omega^2 a^2}.$$
 (3.7)

For an unattenuated wave one has the cut-off frequency for vibrations of a free-boundary fluid column

$$\omega_{fn}^2 \equiv c_f^2 \beta_n^2 / a^2, \qquad (3.8)$$

below which the nth mode does not exist.

The variation of c/c_f with $\omega a/c_f$ as given by Eqs. (3.3) and (3.7) is shown in Fig. 1 for the first few values of α_n and β_n . Except for the 0th mode, which has a constant phase velocity $c=c_f$ and exists at all frequencies, all modes begin at their cut-off frequencies ω_{rn} or ω_{fn} , respectively, and have phase velocities which decrease monotonically from $c=\infty$ to $c=c_f$ as ω increases from ω_{rn} or ω_{fn} to infinity.

We now turn to the case of an elastic tube. The boundary conditions to be imposed on the motion of a fluid in a thin elastic tube have been considered by various investigators.^{2-4,7} Usually the assumptions made are such as to restrict the analyses to waves whose length is greater than, or at least of the order of, the radius of the tube. As the frequency is increased (or the wavelength is decreased) two effects become increasingly important. These are the inertia connected with the rotation of the elements of the tube wall, and the influence of shear strain on the bending terms, the latter being the more important. These factors have been neglected in previous treatments of the problem. Since we are interested in a wide range of frequencies, we shall make use of the analysis given in reference 1 where both of these factors are taken into account.

Approximate equations governing the motion of the tube are [from Eqs. (8), (9), and (28), reference 1]

$$\rho_{t} \frac{\partial^{2} \bar{u}_{t}}{\partial t^{2}} = \frac{Eh}{1 - \nu^{2}} \left(\frac{\partial^{2} \bar{u}_{t}}{\partial x^{2}} + \frac{\nu}{a} \frac{\partial \bar{u}_{t}}{\partial x} \right)$$

$$\rho_{t} \frac{\partial^{2} \bar{w}_{t}}{\partial t^{2}} = \frac{\partial V}{\partial x} - \frac{Eh}{1 - \nu^{2}} \left(\frac{\bar{w}_{t}}{a^{2}} + \frac{\nu}{a} \frac{\partial \bar{u}_{t}}{\partial x} \right) + p_{t} \left(1 - \frac{h}{2a} \right)$$

$$\rho_{t} \frac{h^{3}}{12} \frac{\partial^{2} \alpha}{\partial t^{2}} = \frac{Eh^{3}}{12(1 - \nu^{2})} \frac{\partial^{2} \alpha}{\partial x^{2}} + V$$

$$V = KGh \left(\frac{\partial \bar{w}_{t}}{\partial x} - \alpha \right)$$
(3.9)

where ρ_t , E, and ν are the density, Young's modulus, and Poisson's ratio, respectively, of the tube material, V is the transverse shear force, K is a constant, a little less than one, introduced to improve the approximate expression for V, p_t is the fluid pressure on the wall, and \bar{u}_t , \bar{w}_t , α are defined by the approximate relations

$$u_{t}(\mathbf{r}, x, t) = \bar{u}_{t}(x, t) + (a - \mathbf{r})\alpha(x, t), w_{t}(\mathbf{r}, x, t) = \bar{w}_{t}(x, t),$$
(3.10)

 u_t and w_t denoting the axial and radial displacement, respectively, of a point on the wall. If we look for solutions in the form of traveling waves with circular frequency ω and phase velocity c, then the following relation may be derived from Eqs. (3.9):

$$\left(1 - \frac{h}{2a}\right) \frac{p_{\iota}^{*}}{\omega_{\iota}^{*}} = 1 - \omega^{*2} - \frac{\nu^{2}}{1 - c^{*2}} + \frac{\xi \omega^{*4} (1 - Nc^{*2})}{c^{*2} [c^{*2} + \xi \eta \omega^{*2} (1 - Nc^{*2})]}$$
(3.11)

where

$$\omega^* = \frac{a\omega}{c_p}, \quad c_p^2 = \frac{E}{\rho_t(1-\nu^2)}, \quad c^* = \frac{c}{c_p}$$
$$\omega^* = \frac{\bar{w}_t}{a}, \quad p^* = \frac{a\rho_t}{\rho_t c_p^2 h}, \quad (3.12)$$
$$\xi = \frac{h^2}{12a^2}, \quad \eta = \frac{2}{(1-\nu)K},$$

and N is a constant "tracer," which is equal to one, and which is introduced to indicate the terms which are contributed by rotatory inertia. Setting N=0 is tantamount to neglecting this effect. Similarly, the influence of transverse shear is associated with the constant η and may be neglected by setting $\eta=0$.

For small disturbances and for small h/a we may apply the boundary conditions for the fluid motion at r=a and hence set

$$P(a) \exp\left[\frac{\omega}{i-(x-ct)}\right] = p_t$$

$$W(a) \exp\left[\frac{\omega}{i-(x-ct)}\right] = \bar{w}_t.$$
(3.13)

Making use of the relation (2.9) for P/W, introducing the dimensionless expressions defined by Eqs. (3.12), and neglecting the ratio h/2a on the left side of Eq. (3.11), the following characteristic equation is obtained from application of the boundary conditions

$$1 - \frac{1}{\omega^{*2}} + \frac{\nu^2}{\omega^{*2}(1 - c^{*2})} - \frac{\xi \omega^{*2}(1 - Nc^{*2})}{c^{*2}[c^{*2} + \xi \eta \omega^{*2}(1 - Nc^{*2})]} = \frac{\alpha^2 J_0(ka)}{ka J_1(ka)}, \quad (3.14)$$

where

$$\alpha^2 = \rho a / \rho_t h. \tag{3.15}$$

The parameter k is related to c and ω by Eq. (2.6), which, on using Eqs. (3.12), may be written as

$$k^{2}a^{2} = \omega^{*2} \left(\frac{1}{\gamma^{2}} - \frac{1}{c^{*2}} \right), \qquad (3.16)$$

where

$$\gamma^2 = c_f^2 / c_p^2. \tag{3.17}$$

Elimination of ka between Eqs. (3.14) and (3.16) yields

$$1 - \frac{1}{\omega^{*2}} + \frac{\nu^2}{\omega^{*2}(1 - c^{*2})} - \frac{\xi \omega^{*2}(1 - Nc^{*2})}{c^{*2}[c^{*2} + \xi \eta \omega^{*2}(1 - Nc^{*2})]} = \frac{\alpha^2 J_0 \left[\omega^* \left(\frac{1}{\gamma^2} - \frac{1}{c^{*2}} \right)^* \right]}{\omega^* \left(\frac{1}{\gamma^2} - \frac{1}{c^{*2}} \right)^* J_1 \left[\omega^* \left(\frac{1}{\gamma^2} - \frac{1}{c^{*2}} \right)^* \right]}, \quad c^{*2} \ge \gamma^2 \qquad (3.18)$$

and

$$1 - \frac{1}{\omega^{*2}} + \frac{\nu^2}{\omega^{*2}(1 - c^{*2})} - \frac{\xi \omega^{*2}(1 - Nc^{*2})}{c^{*2}[c^{*2} + \xi \eta \omega^{*2}(1 - Nc^{*2})]} = -\frac{\alpha^2 I_0 \left[\omega^* \left(\frac{1}{c^{*2}} - \frac{1}{\gamma^2}\right)^* \right]}{\omega^* \left(\frac{1}{c^{*2}} - \frac{1}{\gamma^2}\right)^* I_1 \left[\omega^* \left(\frac{1}{c^{*2}} - \frac{1}{\gamma^2}\right)^* \right]}, \quad c^{*2} \le \gamma^2$$
(3.19)

where I_0 and I_1 are Bessel functions of imaginary argument. Equation (3.18) or (3.19) gives the relation between c^* and ω^* in terms of the four independent parameters α , ξ , γ , and ν , the parameter η being a function of ν given by Eq. (3.12) in which K is a known constant.

4. CASE OF LOW FREQUENCIES

It is of interest to study the vibrations when $\omega^{*2} \ll 1$. This corresponds to the case of long waves. On neglecting second and higher powers of ω^* with respect to 1, Eqs. (3.18) and (3.19) may be reduced to a quadratic in c^{*2} :

4 . 1-

$$(1+2\alpha^{2}\gamma^{2})c^{*4}-(1-\nu^{2}+\gamma^{2}+2\alpha^{2}\gamma^{2})c^{*2} +\gamma^{2}(1-\nu^{2})=0. \quad (4.1)$$

The discriminant of Eq. (4.1) is

$$\Delta^2 = (1 - \nu^2 - \gamma^2 + 2\alpha^2 \gamma^2)^2 + 8\alpha^2 \gamma^4 \nu^2, \qquad (4.2)$$

which is always positive since the parameters α , γ , and ν are all real. The two roots of Eq. (4.1) may

easily be shown to be positive. They are

$$c_{1}^{*2} = \frac{(1 - \nu^{2} + \gamma^{2} + 2\alpha^{2}\gamma^{2}) + \Delta}{2(1 + 2\alpha^{2}\gamma^{2})},$$

$$c_{2}^{*2} = \frac{(1 - \nu^{2} + \gamma^{2} + 2\alpha^{2}\gamma^{2}) - \Delta}{2(1 + 2\alpha^{2}\gamma^{2})}.$$
(4.3)

Hence, when $\omega^{*2} \ll 1$ the waves are unattenuated and there are in general two distinct modes. These coincide in the limiting case $\gamma^2 = 1 - \nu^2$ and $\alpha = 0$, since then $\Delta = 0$. In general $\Delta \ge 0$, so that $c_1^{*2} \ge c_2^{*2}$ and it may be shown that $c_1^{*2} \le \gamma^2$ when $\gamma^2 \ge 1$, $c_1^{*2} \ge \gamma^2$ when $\gamma^2 \le 1$, $c_2^{*2} \le \gamma^2$ for all γ . Approximate expressions for the phase velocities may be derived for certain ranges of the parameters α and γ . They are when $\alpha^2 = 0(1)$, $\gamma^2 \ll 1$:

$$c_{1}^{*2} \approx 1 - \nu^{2} + 2\nu^{2}\alpha^{2}\gamma^{2},$$

$$c_{1}^{*2} \approx \frac{E}{\rho_{t}} \left(1 + 2\nu^{2} \frac{K_{f}a}{Eh} \right),$$

$$(4.4a)$$

$$c_{2}^{2} \approx \gamma^{2} \left(1 - \frac{1}{1 - \nu^{2}} \right),$$

$$c_{2}^{2} \approx \frac{K_{f}}{\rho} \left(1 - 2 \frac{K_{f}a}{Eh} \right),$$

$$(4.4b)$$

when $\alpha^2 \ll 1$, $\gamma^2 \gg 1$:

$$c_1^{*2} \approx \frac{\gamma^2}{1+2\alpha^2 \gamma^2},$$
(4.5a)

$$c_{1}^{2} \approx \frac{K_{f}}{\rho \left[1 + 2(1 - \nu^{2}) \frac{K_{f}a}{Eh} \right]},$$

$$c_{2}^{*2} \approx (1 - \nu^{2})(1 - 2\nu^{2}\alpha^{2}),$$

$$c_{2}^{2} \approx \frac{E}{\rho_{t}} \left(1 - 2\nu^{2} \frac{\rho a}{\rho_{t}h} \right)$$
(4.5b)

when $\alpha^2 \gamma^2 \gg 1$, $\gamma^2 \ll 1$:

$$c_{1}^{*2} \approx 1 - \frac{\nu^{2}}{2\alpha^{2}\gamma^{2}},$$

$$c_{1}^{*2} \approx \frac{E}{\rho_{t}(1-\nu^{2})} \left[1 - \frac{\nu^{2}}{2(1-\nu^{2})} \frac{Eh}{K_{f}a} \right]$$

$$c_{2}^{*2} \approx \frac{1-\nu^{2}}{2\alpha^{2}} \left(1 - \frac{1-\nu^{2}}{2\alpha^{2}\gamma^{2}} \right),$$

$$c_{2}^{*2} \approx \frac{Eh}{2\rho a} \left(1 - \frac{Eh}{2K_{f}a} \right)$$
(4.6b)





when
$$\alpha^{2} \gg 1, \gamma^{2} \gg 1$$
:
 $c_{1}^{*2} \approx 1 + \frac{\nu^{2}}{2\alpha^{2}},$

$$E\left(1 + \frac{\nu^{2}}{2} \frac{\rho_{t}h}{\rho_{a}}\right)$$
 $c_{1}^{2} \approx \frac{E\left(1 + \frac{\nu^{2}}{2} \frac{\rho_{t}h}{\rho_{a}}\right)}{\rho_{t}(1 - \nu^{2})}$
 $c_{2}^{*2} \approx \frac{1 - \nu^{2}}{2\alpha^{2}}\left(1 - \frac{\nu^{2}}{2\alpha^{2}}\right),$
 $c_{2}^{*2} \approx -\frac{Eh}{2\rho a}\left(1 + \frac{\nu^{2}}{2} \frac{\rho_{t}h}{\rho_{a}}\right)$
(4.7a)
(4.7a)
(4.7b)

when $\gamma^2 = 1 - \nu^2$, $\alpha \ll 1$:

$$c_{1}^{*2} \approx \frac{(1-\nu^{2})(1+2^{\frac{1}{2}}\nu\alpha),}{c_{1}^{2} \approx \frac{E}{\rho_{l}} \left[1+\nu \left(\frac{2K_{I}a}{Eh}\right)^{\frac{1}{2}} \right],$$

$$c_{2}^{*2} \approx \frac{1-\nu^{2}}{1+2^{\frac{1}{2}}\nu\alpha},$$

$$c_{2}^{*2} \approx \frac{E}{\rho_{l} \left[1+\nu \left(\frac{2\rho a}{\rho_{l}h}\right)^{\frac{1}{2}} \right]}.$$
(4.8b)



FIG. 2. c_1^* and c_2^* vs α for $\nu = 0.3$ and vanishing frequency.

When $\gamma = 1$ and α is arbitrary, Eqs. (4.3) become exactly

$$c_1^{*2} = 1, \quad c_1^{*2} = \frac{E}{\rho_t(1-\nu^2)}, \quad (4.9a)$$

$$c_{2}^{*2} = \frac{1 - \nu^{2}}{1 + 2\alpha^{2}},$$

$$c_{2}^{*2} = \frac{E}{\rho_{t} \left(1 + \frac{2\rho a}{\rho_{t} h}\right)} = \frac{Eh}{2\rho a [1 + (\rho_{t} h/2\rho a)]}.$$
(4.9b)

Curves showing the velocities c_1^* and c_2^* , as given by Eqs. (4.3), are shown in Fig. 2 as functions of α for $0 \le \alpha \le 1$ and of $1/\alpha$ for $1 \le \alpha \le \infty$, with $\nu = 0.3$ and for various values of γ .

The Eqs. (4.4) to (4.9) together with Fig. 2 give insight into the dependence of the nature of the waves on α and γ . When $\gamma > 1.4$, say, and α is very small, c_1 corresponds to a sound wave propagating through fluid contained in an essentially rigid tube. The same is true for c_2 when $\gamma < 0.6$, say. An increase in α , with γ held fixed, may be interpreted as being due to a decrease in ρ_i (from $\rho_i = \infty$ for $\alpha = 0$) and a simultaneous decrease in E in such a way that E/ρ_i , and hence c_p , remain constant. As α increases the decreasing stiffness of the tube lowers the velocity [Eqs. (4.5a) and (4.4b)]. The decrease in the mass of the tube which might be expected to tend to increase the velocity has much less effect. This becomes plausible when one recalls that for very small α the tube provides pressure on the fluid without moving and hence without contributing to the effective inertia of the system. As α continues to increase the c_1 mode represents a more definitely coupled motion of fluid and tube, with the radial motion of the tube gradually giving rise, due to the presence of Poisson's ratio, to axial motion also. For still larger α the motion is more easily interpreted by thinking of the increase in α as being due to an increase in ρ accompanied by a simultaneous increase in K_f with K_f/ρ constant. The wave propagation is governed primarily by the properties of the tube; it becomes independent of the stiffness of the fluid (K_f) , the latter being essentially incompressible [Eq. (4.7a)]. When the fluid density becomes infinite, the tube executes purely axial vibrations and the phase velocity is that of longitudinal waves in an infinite elastic medium.

The wave with velocity c_2 represents, when $\alpha^2 \gamma^2 \gg 1$ and $\alpha^2 \gg 1$, a well-known motion whose essential nature is an interaction of tube and fluid (see for example reference 9). The fluid is effectively incompressible so that the controlling stiffness is that of the tube, but the primary contribution to the inertia is derived from the fluid. If $\gamma^2 \ll 1$ the stiffness of the system is somewhat decreased by the flexibility of the fluid, Eq. (4.6b), the tube inertia being negligible; if $\gamma^2 \gg 1$, the inertia of the system is somewhat increased by the mass of the tube, Eq. (4.7b), the flexibility of the fluid being negligible. Comparison of Eqs. (4.7b) and (4.9b) shows that when $\alpha^2 \gg 1$, the phase velocities for $\gamma = 1$ and $\gamma^2 \gg 1$ differ only by a term of order $\nu^2/2\alpha^2$. Thus, when γ is near one or greater, the fluid may be treated as incompressible in this mode of vibration, provided α is sufficiently large.

The c_1 wave for $\gamma^2 \ll 1$ and the c_2 wave, $\gamma^2 \gg 1$, correspond when $\alpha = 0$ to a longitudinal wave in an empty tube. If the increase in α is due to a simultaneous increase in ρ and K_{f} , then it is seen that the tube motion becomes influenced by the stiffness and inertia of the fluid. Considering c_1 , the flexibility of the fluid is relatively great $(c_f < c_p)$, thus causing little fluid motion, so that the primary effect of larger ρ and K_f is that of increasing the stiffness of the system with consequent increase in the velocity [Eq. (4.4a)]. This tendency continues until the velocity reaches that of waves in an infinite elastic medium, the fluid then acting as a rigid core [Eq. (4.6a)]. For the c_2 mode, on the other hand, the fluid stiffness is relatively great $(c_f > c_p)$, thus causing considerable fluid motion, so that the influence of added inertia predominates and the velocity is lowered [Eq. (4.5b)]. With increased α the motion becomes more definitely coupled and approaches, when α becomes of order one, the motion previously discussed for $\alpha^2 \gamma^2 \gg 1$.

It is of interest to study the case when Poisson's

⁹G. W. Morgan and J. P. Kiely, J. Acoust. Soc. Am. 42, 323 (1954).

ratio ν equals zero because the consequent independence of the radial and axial motions of the tube simplifies the system. Equations (4.3) yield

$$c_{1}^{*2} = \frac{\gamma^{2}}{1 + 2\alpha^{2}\gamma^{2}} \} \text{ when } \alpha^{2} \leq \frac{1}{2} \left(1 - \frac{1}{\gamma^{2}} \right) \quad (4.10a)$$

$$c_{2}^{*2} = 1 \qquad (4.10b)$$

$$c_{1}^{*2} = 1$$

$$c_{2}^{*2} = \frac{\gamma^{2}}{1 + 2\alpha^{2}\gamma^{2}}$$
when $\alpha^{2} \ge \frac{1}{2} \left(1 - \frac{1}{\gamma^{2}} \right).$
(4.11a)
(4.11b)

Representative curves are shown in Fig. 3. The c_1^* curves for $\gamma > 1$ are very similar to those for $\nu \neq 0$ (Fig. 2) as long as α is sufficiently small so that c_1^* is somewhat greater than one. For larger α the curves begin to deviate from those for $\nu \neq 0$ because the tube vibration is purely radial when $\nu = 0$ and hence, as α increases, the motion cannot tend to a longitudinal wave in the tube as it does when $\nu \neq 0$. The velocity becomes one at $\alpha^2 = \frac{1}{2} \left[1 - (1/\gamma^2) \right]$. At this point the slope of the c_1^* curve is discontinuous and $c_1^*=1$ for all greater α . This part of the curve represents an entirely different wave, viz. a longitudinal tube vibration, which, in the absence of Poisson's ratio, is entirely independent of the fluid. The same vibration must exist also when $\alpha^2 < \frac{1}{2} [1 - (1/\gamma^2)]$. It is represented in that range by the c_2^* curve. Since $\alpha^2 \ge \frac{1}{2} [1 - (1/\gamma^2)]$ when $\gamma^2 \leq 1$, the c_1^* curves for $\gamma^2 \leq 1$ trace the entire line $c_1^* = 1$ and the corresponding c_2^* branch is absent. The c_2^* curve for $\gamma < 0.9$, say, is very similar to its counterpart for $\nu \neq 0$ because the inertia force due to axial motion of the tube does not play an important role in this mode even when $\nu \neq 0$. For $\gamma > 1$, the velocity c_2^* which starts at $(1-\nu^2)^{\frac{1}{2}}$ when $\nu \neq 0$ and begins to drop due to the influence of tube inertia, now



FIG. 3. c_1^* and c_2^* vs α for $\nu = 0$ and vanishing frequency; $c_1^* \ge 1, c_2^* \le 1.$

starts at one and remains constant up to $\alpha^2 = \frac{1}{2} [1 - (1/\gamma^2)]$. At this point it has a discontinuous slope and smoothly joins the corresponding c_1^* curve. The velocity decreases with increasing α and the curve approaches the corresponding curve for $\nu \neq 0$. Thus, for $\gamma > 1$, the curves which are given by c_1^* when $\alpha^2 \leq \frac{1}{2} [1 - (1/\gamma^2)]$ and by c_2^* when $\alpha^2 \geq \frac{1}{2} [1 - (1/\gamma^2)]$ represent a continuously changing mode of vibration from a sound wave through fluid in a rigid tube to a wave in which the essential features are hoop stiffness and fluid inertia.

Expanding Eqs. (4.3) in power series of ν , one obtains

$$\alpha^{2} < \frac{1}{2} \left(1 - \frac{1}{\gamma^{2}} \right), \quad c_{1}^{*2} = \begin{cases} \gamma^{2} \\ 1 + 2\alpha^{2}\gamma^{2} \\ \gamma^{2} \\ 1 + 2\alpha^{2}\gamma^{2} \\ 1 - \gamma^{2} + 2\alpha^{2}\gamma^{2} \\ \gamma^{2} \\ \gamma^{2}$$

$$\alpha^{2} = \frac{1}{2} \left(1 - \frac{1}{\gamma^{2}} \right), \quad \nu^{2} \ll 8\alpha^{2} \gamma^{4}, \quad c^{*}_{1}^{2} = 1 \pm 2^{\frac{1}{2}} \nu \alpha - \frac{\nu^{2}}{2\gamma^{2}} + O(\nu^{3}).$$
(4.14)

5. GENERAL VELOCITY-FREQUENCY RELATION

The velocity-frequency relation is given by Eqs. (3.18) and (3.19) and involves the parameters α , γ , ν , and ξ . Computations have been carried out for one set of values of these parameters corresponding to a brass tube filled with water. The values are the same as those employed by Thomson⁷ to permit comparison with his results. Using

 $\rho = 1.0 \text{ g/cm}^3$, $c_f = 1.43 \times 10^5 \text{ cm/sec}$, $\rho_t = 8.5 \text{ g/cm}^3$, $c_p = 3.59 \times 10^5 \text{ cm/sec}$, $h/a = \frac{1}{8}$, $\nu = 0.3$,

the parameters assume the following values:

$$\alpha = 0.97, \quad \gamma = 0.4, \quad \nu = 0.3, \quad \xi = 0.0013, \quad \eta = 3.21.$$

Figure 4 is a plot of the dispersion curves for the first few modes. As is to be expected from the results for small ω^* (Fig. 2) and from those for the limiting cases of a rigid and an infinitely flexible tube (free boundary) (Fig. 1), there exists only one mode for which $c^* < \gamma$. It is called the "0" mode because it corresponds to the 0 mode of Fig. 1 and it exists for all frequencies. There are an infinite number of modes with phase velocity $c^* > \gamma$. The 1st mode exists for all frequencies, the higher modes exist only above certain cut-off frequencies. The dashed curve indicates the 0th



FIG. 4. Phase velocity vs frequency for fluid in an elastic tube with $\alpha = 0.97$, $\xi = 0.0013$, $\gamma = 0.40$, and $\nu = 0.30$. Dashed curve neglects rotatory inertia and transverse shear.

mode when the transverse shear and rotatory inertia are neglected in formulating the equations of motion for the tube. The error is seen to be about 10% when ω^* is around two, which corresponds to a wavelength of approximately one-half the tube radius. Since $c^{*2} \ll 1$ for this mode, it is evident from Eq. (3.19) that the terms which involve the tracer N and owe their existence to rotatory inertia are small, so that the correction is primarily the result of taking into account the transverse shear, the effect of which is represented by the term involving η . The dashed curve corresponds to the (0) mode of Thomson.⁷ The modes whose velocities are greater than γ are not perceptibly influenced by transverse shear and rotatory inertia.



FIG. 5. Phase velocity vs frequency; $\alpha = 0.97$, $\xi = 0.0013$, $\gamma = 0.40$, $\nu = 0.3$.

Figure 5 shows the 0th and the 1st, 2nd, and 3rd modes together with the curves for an empty tube and for fluid in a rigid and an infinitely flexible tube. The 0th mode starts at $c^{*2} \approx \gamma^2 [1-2\alpha^2\gamma^2/(1-\nu^2)]$, [Eq. (4.4b)]. The motion is essentially that due to a plane sound wave in the fluid, but the velocity is somewhat reduced as a result of the flexibility of the tube wall. The coupling of fluid and tube vibration increases as ω^* approaches one (the frequency of hoop vibrations of the empty tube). For larger ω^* the coupled mode resembles that of flexural vibrations in an empty tube, but deviates from the empty tube mode for still

higher ω^* , the velocity approaching that of a fluid wave in a rigid tube $(c^* = \gamma)$, while the velocity of the empty tube mode approaches the higher value $1/\eta^3$. Since the velocity curve for the 0 mode does not rise beyond γ , the error caused by neglecting transverse shear (and rotatory inertia) begins to diminish beyond a certain value of ω^* . It can be shown that if the physical constants of the system are such that $1 > \gamma$ $> 1/\eta^3$ then the 0th dispersion curve continues to follow the curve for the empty tube as ω^* increases and the velocity approaches the asymptote $1/\eta^3$.

The first mode begins at $\omega^*=0$ with $c^{*2}\approx 1-\nu^2$ $+2\nu^2\alpha^2\gamma^2$ [Eq. (4.4a)] and is essentially a longitudinal wave propagating through the tube, the velocity being somewhat greater than that of a wave in an empty tube because of the additional stiffness contributed to the system by the fluid. The nature of the vibration changes as ω^* approaches one, the fluid and tube motion becoming very definitely coupled. The velocity continues to decrease, the curve lying between the line $c^* = \gamma$ and the 1st free boundary curve. This continues up to the value of ω^* at which the 1st mode curve of the empty tube crosses the 1st mode curve of the free boundary vibration. At this point the 1st mode curve of the coupled system crosses the other two curves and then continues to approach $c^* = \gamma$, always lying above the 1st free boundary curve. This "crossing" will be discussed further in connection with the higher modes. It occurs beyond the range of Fig. 5 but appears on Fig. 6 which shows a plot for $\gamma = 0.2$.

It should be noted that in the illustrative example of Fig. 5 the minimum velocity of the 1st empty tube mode (given approximately by $c^{*2} \approx [(1-\nu^2)/3]^{\frac{1}{2}}(h/a)$ (reference 1) lies below $c^* = \gamma$. If the contrary is true then the 1st mode curve for the coupled system will tend to follow the 1st empty tube mode curve more closely, possibly crossing the latter at its intersections with the 1st free boundary curve (if more than one such intersection exists) until, lying above the 1st free boundary curve, it approaches $c^*=\gamma$. The 0 mode will, in this case, deviate much less from the line $c^*=\gamma$ over the entire range of ω^* and will never resemble the 1st empty tube mode (Fig. 6).

All modes above the 1st have minimum cut-off frequencies ω^*_{cn} which may be found from Eq. (3.18) by allowing c^* to approach infinity. They are given by

$$\frac{J_0(\omega^*/\gamma)}{J_1(\omega^*/\gamma)} = \frac{\omega^{*2} - 1}{\alpha^2 \gamma \omega^*}.$$
(5.1)

Recall that for an infinitely flexible tube $J_0(\omega^*[1/\gamma^2 - 1/c^{*2}]^i) = 0$ while for a rigid tube $J_1(\omega^*[1/\gamma^2 - 1/c^{*2}]^i) = 0$ (Sec. 3). We can therefore identify the regions in Fig. 5 in which the right-hand side of Eqs. (3.18), and hence -p(a)/w(a), [Eq. (2.9)], are positive or negative, respectively. They are positive in the region between $c^* = \gamma$ and the 1st free boundary curve,



FIG. 6. Phase velocity vs frequency; $\alpha \approx 1$, $\xi = 0.001$, $\gamma = 0.20$ $\nu \approx 0.3$. Curves sketched, not computed; see Sec. 5.

negative between this curve and the 1st rigid tube curve, then positive again, etc. Since the 1st free boundary cut-off frequency ω_{f_1} given by the first root of $J_0(\omega^*/\gamma) = 0$ [Eq. (3.8)] is about 0.96, the cut-off frequency ω_{c2}^* of the 2nd coupled mode cannot be less than this value because otherwise the dispersion curve would have to lie to the left of the free boundary curve for large c^* , i.e., in a region in which the lefthand side of Eq. (5.1) is positive, while the right side would be negative. For slightly larger ω^* , $J_0(\omega^*/\gamma)/J_1(\omega^*/\gamma)$ is negative and so is the right side of Eq. (5.1), provided $\omega^* < 1$. Hence the first cut-off frequency ω_{c2} lies between those of the 1st free boundary mode and the second empty tube mode which has a cut-off frequency equal to one.

If ω^*_{I1} were greater than one (i.e., if $\beta_1\gamma > 1$), then ω^*_{c2} , again lying between the cut-off frequencies of the 1st free boundary and the 2nd empty-tube modes, would be greater than one (Fig. 7; $\gamma = 1.05$), for in this region both sides of Eq. (5.1) would be positive. To interpret these results, note that for all free boundary modes the vibration with cut-off frequency represents a purely radial oscillation of the fluid column independent of distance along the axis with zero pressure at r=a. Similarly the motion of the empty tube at the cut-off frequency is a pure hoop vibration. If the frequency for the empty tube is greater than that of



FIG. 7. Phase velocity *vs* frequency; $\alpha \approx 1$, $\xi \approx 0.001$, $\gamma = 1.05$, $\nu \approx 0.3$. Curves sketched, not computed; see Sec. 5.

the free boundary mode, then, in the combined system, the tube is relatively stiff and the influence of increased stiffness when the fluid column is surrounded by a tube outweighs that of increased inertia, so that the resulting frequency is greater than that of the free boundary mode, but less than that of the empty tube. The contrary argument holds when the situation is reversed.

Since in the present numerical example (Fig. 5) a second change of sign of the left side of Eq. (5.1) does not occur until $\omega^* > 1$, the cut-off frequencies for the higher modes (n > 2) must be greater than one; hence, from Eq. (5.1), the dispersion curves for large c^* must lie in a region in which $J_0(\omega^*/\gamma)/J_1(\omega^*/\gamma) > 0$, i.e., in a region bounded by a rigid tube curve on the left and a free boundary curve on the right; hence ω^*_{rn-2} $\leq \omega^*_{cn} \leq \omega^*_{fn-1}$ (n>2). For the higher modes $\omega^*_{cn} \gg 1$ so that the right side of Eq. (5.1) becomes large and hence the ω^*_{cn} approach the roots of $J_1(\omega^*/\gamma)$, that is, the cut-off frequencies for the rigid tube ω^*_{rn-2} [Eq. (3.4)]. This approach is the more rapid the smaller α . that is, the smaller the inertia of the fluid relative to that of the tube. On the contrary, when ω^*_{cn} is moderate, i.e., excluding the high modes, and $\alpha \gg 1$, then the right side of Eq. (5.1) will be very small and ω^*_{cn} approaches ω^*_{fn-1} .

Certain approximate expressions for the cut-off frequencies may be derived from Eq. (5.1). If ω^*/γ is not greater than 1, say, the Bessel functions may be replaced by the first few terms of their power series expansions. An approximate formula for ω^*_{c2} may then be derived. It is

$$\omega^{*2}{}_{c2} \approx \frac{1+2\alpha^{2}\gamma^{2}}{1+(\alpha^{2}/4)}, \quad \omega^{2}{}_{c2} \approx \frac{\frac{Eh}{(1-\nu^{2})a^{2}} + \frac{2K_{f}}{a}}{\rho_{t}h+(\rho a/4)} \quad (5.2)$$
$$\gamma > 2, \quad \alpha^{2} < \frac{1}{2} \quad \text{say.}$$

The radial velocity has no node, the distribution being essentially a straight line. The motion is a hoop vibration in which the fluid behaves like a simple spring contributing stiffness and inertia. For small α the frequency is close to one, the frequency of hoop vibrations of the empty tube. For large $\alpha^2 \gamma^2$ the principal contribution to the stiffness is derived from the fluid.

For sufficiently high modes, or for sufficiently small α , the Bessel functions may be expanded in power series about ω^*_{rn-2} or ω^*_{rn-1} . Subsequent use of the relations between Bessel functions of different orders leads to

$$\omega^{*}_{cn} \approx \omega^{*}_{rn-2} \left(1 + \frac{\alpha^{2} \gamma^{2}}{\omega^{*2}_{rn-2} - 1 - \alpha^{2} \gamma^{2}} \right);$$

$$n \geq 3, \quad \omega^{*}_{rn-2} > 1, \quad \left| \frac{\alpha^{2} \gamma \omega^{*}_{rn-2}}{\omega^{*2}_{rn-2} - 1 - \alpha^{2} \gamma^{2}} \right| \ll 1,$$
or

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$$\omega^{*}_{en} \approx \omega^{*}, _{n-1} \left(1 + \frac{\alpha^{2} \gamma^{2}}{\omega^{*2}_{rn-1} - 1 - \alpha^{2} \gamma^{2}} \right);$$

$$l > n \ge 2, \quad \omega^{*}_{r1} < 1, \quad \left| \frac{\alpha^{2} \gamma \omega^{*}_{rn-1}}{\omega^{*2}_{rn-1} - 1 - \alpha^{2} \gamma^{2}} \right| \ll 1 \quad (5.3)$$

where n=l defines that mode whose cut-off frequency lies between one and ω^*_{fl-1} (Fig. 6). It is the mode whose $c^* - \omega^*$ curve tends to follow the 2nd empty tube mode curve for $c^*>1$. Similarly, when α is large and ω^*_{cn} not too large, the Bessel functions may be expanded in series about ω^*_{n-1} . This leads to

..* ~..*. . 1+____

for

$$n \ge 2, \quad \left| \frac{\omega^{*_{fn-1}}(1 - \omega^{*_{fn-1}})}{\gamma(1 + \omega^{*_{fn-1}} + \alpha^{2}\omega^{*_{fn-1}})} \right| \ll 1.$$
(5.4)

 $1 - \omega^{*2}_{f_{n-1}}$

Ι

Equation (5.4) applies if ω^{*2}_{fn-1} happens to be sufficiently close to one, as in Fig. 5 when n=2 and α is of order one.

If γ is such that *m* free boundary cut-off frequencies are less than one, $m \ge 0$, then the coupled system will also have m cut-off frequencies whose magnitude is less than one. Vibrations whose frequency lies close to

1175

a free boundary cut-off frequency are essentially radial oscillations of a free boundary fluid column modified by the stiffness and inertia of the surrounding tube. When $\omega^*_{cn} < 1$ (i.e., $\omega^*_{fn-1} < 1$) the empty tube constitutes a relatively stiff system compared with the free boundary fluid column and hence the influence of tube stiffness outweighs that of tube inertia so that the frequency of the coupled system is greater than that of the free boundary system. The converse is true when $\omega^*_{en} > 1(\omega^*_{(n-1)} > 1)$. Vibrations whose frequency lies close to a rigid tube cut-off frequency are essentially radial fluid oscillations in a rigid tube modified by the small flexibility and the inertia of the tube. If one considers the motion of the tube to be a forced vibration subject to the pressure p, then one expects p(a) and w(a) to be out of phase when $\omega^*_{rn-2} > 1$ since $\omega^* = 1$ is the frequency of natural hoop vibrations of the tube. Equation (2.9) shows that this requires the left side of Eq. (5.1) to be positive and therefore $\omega^*_{cn} > \omega^*_{rn-2}$. The reverse argument applies when $\omega^*_{rn-1} < 1$, so that $\omega^*_{cn} < \omega^*_{rn-1}$.

Considering now the shape of the dispersion curves $(n \ge 2)$, it is seen that these intersect the free boundary curves at the points where the latter intersect the 2nd mode curve of the empty tybe. This statement holds for those modes whose $\omega^*_{cn} > 1$; in Fig. 5 this is the case for $n \ge 3$. When $\omega^*_{cn} < 1$ no such intersection occurs. The significance of the intersection points is that at each of these points a free boundary mode and the 2nd empty tube mode happen to have the same frequency and velocity. Since each of these motions is entirely independent of the other, p(a) being zero for both, the combined motion is a simple superposition of two independent motions having the same amplitude of w(a).

Equation (3.18) shows that intersections of the dispersion curves with those of the rigid tube modes, for which the right side of Eq. (3.18) becomes infinite, must occur when $c^*=1$. These intersections exist for all $n \ge 2$ (Fig. 5). Now $c^*=1$ represents a motion consisting of purely longitudinal oscillations of the tube, the proper pressure being applied at r=a to avoid all hoop strain. Hence, at their intersection points, the rigid tube dispersion curves and the line $c^*=1$ represent waves which have the same frequency and velocity and which involve no radial motion at r=a. The two motions can therefore be superposed by selecting a common value of p(a).

For $c^* < 1$, each dispersion curve lies to the right of a rigid tube curve until it reaches the frequency at which the free boundary curve on its right intersects the 1st mode curve for the empty tube. This intersection was previously mentioned in discussing the 1st mode and it takes place for this mode provided $\gamma < 1/\eta^{\frac{1}{2}}$. At the intersection point the two independent motions can be superposed by selecting a common amplitude of w(a), to give a combined motion. These intersections appear in Fig. 6. Beyond the intersection all curves $n \ge 1$ approach $c^* = \gamma$. It should be borne in mind that in view of the approximate nature of the theory (thin shell formulation and no viscosity) the results may not be valid at these high frequencies.

It is seen from Fig. 5 that the dispersion curves for $n \ge 3$ tend to follow the rigid tube curves over most of the range, except near $c^*=1$ where they follow the 2nd empty tube mode. The second mode curve tends to follow the 2nd empty tube mode for $c^*>1$ and the 1st rigid tube curve for $c^*<1$. It is seen from Eqs. (5.3) and (3.18) that these tendencies are more pronounced when $\alpha^2\gamma^2$ is small and for higher modes.

The above results are substantially different from those of Thomson.⁷ This is not immediately evident by comparison of Fig. 4 with the corresponding Fig. 3 of reference 7 because in the reference the plot is limited to the range $c^* \leq 0.8$ which excludes the s-shape portion of the dispersion curves for $n \ge 2$ as well as the low frequency portion of the first mode, and it is in this range in which the results differ. According to reference 7 the first mode does not exist at all frequencies, but has a certain lower cut-off frequency. Actually, as has been seen, this mode does exist at all frequencies. The frequency stated by Thomson to be the cut-off frequency for the first mode is actually the cut-off frequency for the second mode ($\omega^*_{c2} \approx 0.98$ in the present notation, $\omega R/c_0 = \omega^* c_2/\gamma \approx 2.48$ in Thomson's notation); similarly Thomson's cut-off frequency for the second mode belongs in reality to the third. Thus, in the absence of the low-frequency branch of the 1st mode, Thomson's results imply that the steep portion of the dispersion curve for that mode (Fig. 4) joins the upper steep branch $(c^*>1)$ of the 2nd mode, that the lower steep portion of the 2nd mode joins the upper steep one of the 3rd mode, and so on. This causes the transition region around $c^*=1$ in which the dispersion curve moves from the neighborhood of a rigid tube mode to the neighborhood of the next higher rigid tube mode to disappear altogether. The fallacy of this situation can also be seen from the fact that the dispersion curves would have to cross the line $c^* = 1$ at points which are not the intersections of that line with the rigid tube curves, and, a ain, the intersection with the 2nd empty tube mode dispersion curve would not take place at the latter's intersection with the free boundary curves. This is impossible.

While computations have been carried out only for one set of values of the parameters, (Fig. 5), the approximate shape of the velocity-frequency curves for other values can easily be predicted by means of the approximate formulae derived in Secs. 4 and 5 and by the reasoning exposed in this section. Figures 6 and 7 are two representative sketches of the velocityfrequency diagrams to be expected when $\gamma \ll 1$ and when $\gamma > 1$, respectively. The major difference between Figs. 5 and 6 is that, when γ is very small, the 0 mode is close to $c^*=\gamma$ over the entire range and some of the modes $n \ge 2$ (in this case one) have cut-off frequencies less than one. When $\gamma > 1$, Fig. 7 shows that the zero mode is entirely due to interaction between fluid and tube for small and moderate frequencies [see Eq. (4.9b)], and approaches the 1st empty tube mode with increasing ω^* . In this case the 1st mode also has a velocity which is less than γ . It is essentially constant and equal to the velocity of elastic waves in an infinite medium. The 2nd mode tends to follow either the 2nd empty tube mode or the free boundary mode, depending on the value of α . Each of the higher modes tends to follow a rigid tube mode.

It is of interest to consider the magnitude of ka [Eq. (3.16)] for a given mode. This parameter determines the shape of the radial distribution curves of the velocity components and the pressure. For waves propagating through fluid contained in a rigid tube or an infinitely flexible tube, ka is constant for each mode since it is a root of J_1 or J_0 , respectively. Hence, for each of these modes, the radial distribution of pressure and velocity components is the same at all frequencies. When the tube is elastic, it is seen from Figs. 5 to 7, that ka varies within a finite range for each mode; hence the shape of the radial distribution curves changes with frequency.

6. SUMMARY

There exists an infinite discrete set of modes. Two of these, the 0 and 1st modes, exist at all frequencies, the 0 mode having a phase velocity less than the speed of sound in the fluid (c_f) , the 1st mode having a phase velocity greater than, or less than, the speed of sound in the fluid, accordingly as the latter is less than, or greater than, the speed of longitudinal waves in an infinite elastic medium consisting of the tube material (c_p) .

Each of the higher modes has associated with it a cut-off frequency below which the mode does not exist. The motion at each of the cut-off frequencies is a purely radial vibration independent of position along the tube, (waves having infinite length and velocity). The dispersion curves tend to follow those pertaining either to the fluid-rigid tube system or to the empty tube system.

The curves for the 2nd and higher modes tend to exhibit a very large change in the frequency for a very small change in the velocity in the neighborhood of c_p . The shape of the curves and the significance of the waves they represent depend on the magnitudes of c_f/c_p , $\rho a/\rho_i h$, h/a and ν .

The influence of rotatory inertia is unimportant even at high frequencies; that of transverse shear is of importance only for the 0 mode where it tends to reduce the phase velocity.