An Improved Transmission Line Model for Visco-Thermal Lossy Sound Propagation 3828 (P8.6)

Joseph Maisano
Laboratory of Electromagnetism and Acoustics
Department of Electrical Engineering
Swiss Federal Institute of Technology
CH-1015 Lausanne, Switzerland

Presented at the 96th Convention 1994 February 26 - March 01 Amsterdam

AES

AN AUDIO ENGINEERING SOCIETY PREPRINT

MAISANO Joseph

Laboratory of Electromagnetism and Acoustics, Departement of Electrical Engineering, Swiss Federal Institute of Technology, CH-1015 Lausanne, Switzerland

An improved transmission line model for visco-thermal lossy sound propagation

Abstract

A transmission line model for lossy sound propagation has been obtained by solving the state law of air and the Navier-Stokes, mass conservation, Fourier heat equations. The series impedance and shunt admittance of general sound propagation has been established in order to obtain the acoustic equivalent elements representing the visco-thermal effects. Acoustic equivalent elements are given for sound propagation in various structures such as holes, cavities, or ducts present in every miniaturized transducer and in particular in integrated microphones and earphones.

1 Introduction

The main difficulty in designing electro-acoustical transducers is to express the losses in their acoustical structures such as holes, cavities or apertures. The exactness of the simulated characteristics of the transducer, in particular its sensitivity and its noise, will solely depend on the precision of this modelling. This paper presents the way to model any acoustical structure taking into account the complete visco-thermal losses.

Classical models for sound propagation only consider reactive effects [2]. In general a transmission line model for sound propagation is essentially represented with an acoustic mass as series impedance and an acoustic compliance as shunt admittance. Taking into account the losses in a non ideal fluid due to viscosity only, the basic model can be improved and represented by figure 1. The resistor expresses losses due to friction of the different speed layers in the fluid.

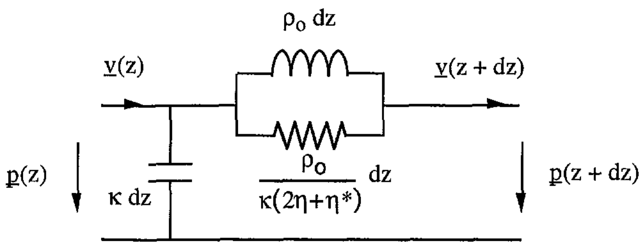

figure 1

2 Sound propagation equations

To improve the sound propagation model by taking into account thermal losses, an equation representing heat conduction in the fluid is required. Thermal and viscous losses in a general fluid are therefore described by four equations representing properly the sound propagation. The small amplitude variations of the acoustical variables lead to linearization of these equations.

The Lamb equation or commonly called Navier-Stokes equation represents the movement conservation of a fluid particle

$$\rho_0 \cdot \frac{\partial \vec{v}'}{\partial t} = -\overrightarrow{\mathrm{grad}}\,p' + (\eta+\eta^*) \cdot \overrightarrow{\mathrm{grad}}\left(\mathrm{div}\,\vec{v}'\right) + \eta \cdot \overrightarrow{\mathrm{lap}}\,\vec{v}' \qquad (1)$$

The conservation of energy leads to the expression of a general heat-conduction equation or generalized Fourier equation taking into account the variation of the density of the fluid induced by the temperature gradient

$$\frac{\partial T'}{\partial t} - \frac{\lambda}{\rho_0 \cdot C_v} \cdot \mathrm{lap}\,T' = \frac{p_0}{\alpha_v \rho_0^2 C_v} \cdot \frac{\partial \rho'}{\partial t} \qquad (2)$$

The mass conservation is represented by the continuity equation

$$\frac{\partial \rho'}{\partial t} + \rho_0 \mathrm{div}\,\vec{v}' = 0 \qquad (3)$$

The state law of the fluid determines the interaction between the thermodynamical variables of the fluid

$$\frac{\partial T'}{\partial t} = -\frac{\beta_p T_0}{\rho_0} \cdot \frac{\partial \rho'}{\partial t} + \frac{\alpha_v T_0}{p_0} \cdot \frac{\partial p'}{\partial t} \qquad (4)$$

3 Line transmission model for sound propagation

The assumption of sound propagation in miniature acoustical structures allows to consider a unidimensional sinusoidal propagation only along z axis, defined by a complex propagation vector $\underline{k}$. The propagation variables can then be represented with phasor notation

$$\underline{\varphi}(z, t) = \underline{\varphi}_0 \cdot e^{(j\omega t - \underline{k}z)} \qquad (5)$$

where z represents the position along the propagation axis, t the time, ω the frequency, and $\underline{\varphi}_0$ the complex phasor in z = 0 and at t = 0.

With this notation, the previous equations can be rewritten

$$j\omega\underline{T} - \frac{\lambda}{\rho_0 \cdot C_v} \cdot \underline{k}^2 \underline{T} = j\omega \frac{p_0}{\alpha_v \rho_0^2 C_v} \cdot \underline{\rho} \tag{1'}$$

$$j\omega\underline{\rho} + \rho_0 \frac{\partial \underline{v}}{\partial z} = 0 \tag{2'}$$

$$\underline{T} = -\frac{\beta_p T_0}{\rho_0} \cdot \underline{\rho} + \frac{\alpha_v T_0}{p_0} \cdot \underline{p} \tag{3'}$$

$$j\omega\rho_0 \cdot \underline{v} = -\frac{\partial \underline{p}}{\partial z} + (2\eta + \eta^*) \cdot \underline{k}^2 \underline{v} \tag{4'}$$

The elimination of T and ρ in (1') to (4') yields

$$j\omega \frac{\alpha_v}{\beta_p \cdot p_0} \cdot \left[1 - \frac{\lambda}{j\omega\rho_0 C_v} \cdot \underline{k}^2\right] \cdot \underline{p} = -\left[\left(1 + \frac{p_0}{\alpha_v \beta_p \rho_0 C_v T_0}\right) - \frac{\lambda}{j\omega\rho_0 C_v} \cdot \underline{k}^2\right] \cdot \frac{\partial \underline{v}}{\partial z} \tag{6}$$

If we denote the specific heat ratio [1] as

$$\gamma = 1 + \frac{p}{\rho \alpha_v \beta_p C_v T} \tag{7}$$

and the isothermal compressibility as,

$$\kappa = \frac{1}{\rho} \cdot \left(\frac{\partial \rho}{\partial p}\right)_{T = cste} = -\frac{1}{\rho} \cdot \left(\frac{\partial \rho}{\partial T}\right)_{p = cste} \left(\frac{\partial T}{\partial p}\right)_{\rho = cste} = \frac{\alpha_v}{\beta_p \cdot p} \tag{8}$$

the equations (4') and (6) can be rewritten as

$$\left[j\omega\rho_0 - \left(2\eta + \eta^*\right) \cdot \underline{k}^2\right] \cdot \underline{v} = -\frac{\partial \underline{p}}{\partial z} \tag{9}$$

$$\frac{j\omega\kappa \cdot \left[1 - \frac{\lambda}{j\omega\rho_0 C_v} \cdot \underline{k}^2\right]}{\left[\gamma - \frac{\lambda}{j\omega\rho_0 C_v} \cdot \underline{k}^2\right]} \cdot \underline{p} = -\frac{\partial \underline{v}}{\partial z} \tag{10}$$

or, in a more concise manner,

$$\underline{Z}' \cdot \underline{v} = - \frac{\partial \underline{p}}{\partial z} \tag{11}$$

$$\underline{Y}' \cdot \underline{p} = - \frac{\partial \underline{v}}{\partial z} \tag{12}$$

where $\underline{Z}'$ is the series impedance and $\underline{Y}'$ the shunt admittance of the transmission line representation of the lossy sound propagation, with

$$\underline{Z}' = j\omega\rho_0 - (2\eta + \eta^*) \cdot \underline{k}^2 \tag{13}$$

$$\underline{Y}' = \frac{j\omega\kappa \cdot \left[1 - \frac{\lambda}{j\omega\rho_0 C_v} \cdot \underline{k}^2\right]}{\left[\gamma - \frac{\lambda}{j\omega\rho_0 C_v} \cdot \underline{k}^2\right]} \tag{14}$$

Equations (11) and (12), can be combined by a matrix representation as

$$\frac{d}{dz}\begin{pmatrix} \underline{p}(z) \\ \underline{v}(z) \end{pmatrix} = \begin{pmatrix} 0 & -\underline{Z}' \\ -\underline{Y}' & 0 \end{pmatrix} \begin{pmatrix} \underline{p}(z) \\ \underline{v}(z) \end{pmatrix} \tag{15}$$

The eigenvalues are $\underline{k}$ and $-\underline{k}$ with

$$\underline{k}^2 = \underline{Y}' \cdot \underline{Z}' \tag{16}$$

and the eigenvectors are then

$$\begin{pmatrix} 1 \\ \underline{Y}_c \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -\underline{Y}_c \end{pmatrix} \tag{17}$$

$$\text{with } \underline{Y}_c = \frac{1}{\underline{Z}_c} = \frac{\underline{k}}{\underline{Z}'} = \frac{\underline{Y}'}{\underline{k}} = \sqrt{\frac{\underline{Y}'}{\underline{Z}'}} \tag{18}$$

Equation (16) allows us to write the propagation phasor as

$$\frac{j\omega\kappa \cdot \left[1 - \frac{\lambda}{j\omega\rho_0 C_v} \cdot \underline{k}^2\right]}{\left[\gamma - \frac{\lambda}{j\omega\rho_0 C_v} \cdot \underline{k}^2\right]} \cdot \left[j\omega\rho_0 - (2\eta + \eta^*) \cdot \underline{k}^2\right] = \underline{k}^2 \tag{19}$$

which gives the fourth order equation with complex coefficients

$$\frac{\lambda}{j\omega\rho_0 C_v}\left[1+j\omega\kappa\left(2\eta+\eta^*\right)\right]\cdot\underline{k}^4-\left[\gamma+j\omega\kappa\left(\frac{\lambda}{C_v}+\left(2\eta+\eta^*\right)\right)\right]\cdot\underline{k}^2-\omega^2\rho_0\kappa=0 \tag{20}$$

Solving this equation yields the value of the propagation phasor as

$$\underline{k}^2=\frac{\gamma+j\omega\kappa\left(\frac{\lambda}{C_v}+\left(2\eta+\eta^*\right)\right)\pm\sqrt{\left[\gamma+j\omega\kappa\left(\frac{\lambda}{C_v}+\left(2\eta+\eta^*\right)\right)\right]^2-\frac{4j\omega\kappa\lambda}{C_v}\left[1+j\omega\kappa\left(2\eta+\eta^*\right)\right]}}{2\frac{\lambda}{j\omega\rho_0 C_v}\left[1+j\omega\kappa\left(2\eta+\eta^*\right)\right]} \tag{21}$$

The sign in relation (21) is chosen to satisfy the isothermal case, which requires, when $\gamma = 1$, the propagation vector to satisfy (19), i. e.

$$j\omega\kappa\cdot\left[j\omega\rho_0-\left(2\eta+\eta^*\right)\cdot\underline{k}^2\right]=\underline{k}^2 \tag{22}$$

or

$$\underline{k}^2=\frac{-\omega^2\rho_0\kappa}{1+j\omega\kappa\left(2\eta+\eta^*\right)} \tag{23}$$

For $\gamma = 1$, the argument of the square root in (21) becomes

$$\left[1+j\omega\kappa\left(\left(2\eta+\eta^*\right)-\frac{\lambda}{C_v}\right)\right]^2 \tag{24}$$

Only the minus sign in equation (21) satisfies the condition (23) and corresponds to the physical solution of the propagation phasor.

The square root in (21) can be expressed by

$$\sqrt{\left[\gamma^2-\omega^2\kappa^2\left(\left(2\eta+\eta^*\right)-\frac{\lambda}{C_v}\right)^2\right]+2j\omega\kappa\left[\gamma\left(\left(2\eta+\eta^*\right)+\frac{\lambda}{C_v}\right)-\frac{2\lambda}{C_v}\right]} \tag{25}$$

The extraction of a complex square root is expressed by

$$\sqrt{a+j\cdot b}=\sqrt{\frac{a+r}{2}}+\frac{j\cdot b}{\sqrt{2(a+r)}} \quad (26)$$

where a and b are real values and r is the modulus of $a + jb$, defined by $r=\sqrt{a^2+b^2}$

The square root (25) can then be expressed by

$$\sqrt{\frac{a+r}{2}}+\frac{2j\omega\kappa\left[\gamma\left((2\eta+\eta^*)+\frac{\lambda}{C_v}\right)-\frac{2\lambda}{C_v}\right]}{\sqrt{2(a+r)}} \quad (27)$$

with

$$a=\gamma^2-\omega^2\kappa^2\left((2\eta+\eta^*)-\frac{\lambda}{C_v}\right)^2 \quad (28)$$

and

$$r=\sqrt{\left[\gamma^2-\omega^2\kappa^2\left((2\eta+\eta^*)-\frac{\lambda}{C_v}\right)^2\right]^2+4\omega^2\kappa^2\left[\gamma\left((2\eta+\eta^*)+\frac{\lambda}{C_v}\right)-\frac{2\lambda}{C_v}\right]^2} \quad (29)$$

The propagation phasor is then rewritten

$$\underline{k}^2=\frac{\gamma+j\omega\kappa\left(\frac{\lambda}{C_v}+(2\eta+\eta^*)\right)-\left[\sqrt{\frac{a+r}{2}}+\frac{2j\omega\kappa\left[\gamma\left((2\eta+\eta^*)+\frac{\lambda}{C_v}\right)-\frac{2\lambda}{C_v}\right]}{\sqrt{2(a+r)}}\right]}{2\frac{\lambda}{j\omega\rho_0 C_v}\left[1+j\omega\kappa(2\eta+\eta^*)\right]} \quad (30)$$

or

$$\underline{k}^2=\frac{\frac{j\omega\rho_0 C_v}{2\lambda}[\gamma-R]-\frac{\omega^2\rho_0\kappa C_v}{2\lambda}\left[(2\eta+\eta^*)\cdot\left(1-\frac{\gamma}{R}\right)+\frac{\lambda}{C_v}\left(1+\frac{1}{R}\frac{(\gamma-1)}{R}\right)\right]}{1+j\omega\kappa(2\eta+\eta^*)} \quad (31)$$

if we assume that

$$R = \gamma \cdot \sqrt{\frac{a+r}{2}} \tag{32}$$

The value of the square root in R varies between 0 and 1. For low frequencies (f < 100kHz), and sound propagation in dry air under normal conditions, it is nearly equal to unity and $R = \gamma$. For higher frequencies R gives a frequency dependence to the acoustical components of the structures.

With the low frequency assumption, the propagation phasors then yields

$$\underline{k}^2 = \frac{-\dfrac{\omega^2 \rho_0 \kappa}{\gamma}}{1 + j\omega\kappa(2\eta+\eta^*)} \tag{33}$$

It is to be noted that this propagation phasor replaced in the expressions of the series impedance (13) and shunt admittance (14) determines the equivalent circuit for sound propagation shown in figure 1.

4 Series impedance and shunt admittance

The series impedance and the shunt admittance are then given by equations (13) and (14).

The replacement of propagation phasor taken from (32) into these relations gives the value of series impedance

$$\underline{Z}' = \frac{j\omega\rho_0\left[1 - \frac{C_v}{2\lambda}(2\eta+\eta^*)\cdot(\gamma-R)\right] - \omega^2\rho_0\kappa(2\eta+\eta^*)\cdot\left[\frac{1}{2} - \frac{1}{2R} + \frac{(\gamma-1)}{2R} - \frac{C_v}{2\lambda}(2\eta+\eta^*)\cdot\left(1-\frac{\gamma}{R}\right)\right]}{1 + j\omega\kappa(2\eta+\eta^*)} \tag{34}$$

or

$$\underline{Z}' = \frac{j\omega\rho_0\upsilon - \omega^2\rho_0\kappa(2\eta+\eta^*)\cdot\left[\frac{1}{2}\left(1+\frac{\gamma}{R}\right) - \frac{\upsilon}{R}\right]}{1 + j\omega\kappa(2\eta+\eta^*)} \tag{35}$$

with

$$\upsilon = 1 - \frac{C_v}{2\lambda}(2\eta + \eta^*)\cdot(\gamma - R) \tag{36}$$

The impedance of the circuit represented on figure 2 is given by the next formula

$$\underline{Z}' = \frac{j\omega(L_1' + L_2') - \omega^2 \frac{L_1' L_2'}{R_2'}}{1 + j\omega \frac{L_1'}{R_2'}} \tag{37}$$

The identification with the coefficients of relation (35) yields

$$L_1' = \rho_0 \left[\upsilon\left(1 + \frac{1}{R}\right) - \frac{1}{2}\left(1 + \frac{\gamma}{R}\right) \right] \tag{38}$$

$$L_2' = \rho_0 \left[\frac{1}{2}\left(1 + \frac{\gamma}{R}\right) - \frac{\upsilon}{R} \right] \tag{39}$$

$$R_2' = \frac{\rho_0}{\kappa(2\eta + \eta^*)} \left[\upsilon\left(1 + \frac{1}{R}\right) - \frac{1}{2}\left(1 + \frac{\gamma}{R}\right) \right] \tag{40}$$

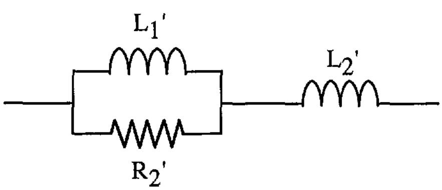

figure 2

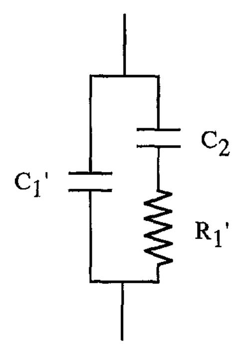

figure 3

The value of shunt admittance is given by (14), and can be rewritten as

$$\underline{Y}' = \frac{j\omega\kappa\cdot\left[j\omega\rho_0 C_v - \lambda\cdot \underline{k}^2\right]}{\left[j\omega\rho_0 C_v \gamma - \lambda\cdot \underline{k}^2\right]} \tag{41}$$

After replacing the propagation phasor, the shunt admittance becomes

$$\underline{Y}' = \frac{j\omega\kappa\cdot\left(\frac{2-\gamma+R}{\gamma+R}\right) - \omega^2\kappa^2\left[(2\eta+\eta^*)\frac{1}{R}\frac{\lambda}{C_v}\frac{1}{\gamma+R}\left(1+\frac{1}{R}\frac{\gamma-1}{R}\right)\right]}{1 + j\omega\kappa\cdot\left[(2\eta+\eta^*)\left(\frac{1}{R}+2\frac{\gamma-1}{\gamma+R}\right)+\frac{\lambda}{C_v}\frac{1}{\gamma+R}\left(1+\frac{1}{R}\frac{\gamma-1}{R}\right)\right]} \tag{42}$$

The extraction of the components of the shunt admittance is done in the same manner with the value of the admittance of figure 3 given by

$$\underline{Y}' = \frac{j\omega\left(C_1' + C_2'\right) - \omega^2 R_1' C_1' C_2'}{1 + j\omega R_1' C_1'} \tag{43}$$

The identification yields

$$R_1' = \frac{1}{2R(\gamma-1)} \frac{\left[(2\eta+\eta^*)\left[R(\gamma-1)+\gamma(1+R)\right]-\frac{\lambda}{C_v}\left[1+R-(\gamma-1)\right]\right]^2}{(2\eta+\eta^*)\left[(1+R)(R-\gamma)\right] + \frac{\lambda}{C_v}\left[1+R-(\gamma-1)\right]} \tag{44}$$

$$C_1' = \kappa\cdot\frac{(2\eta+\eta^*)(\gamma+R) - \frac{\lambda}{C_v}\left[1+R-(\gamma-1)\right]}{(2\eta+\eta^*)\left[R(\gamma-1)+\gamma(1+R)\right]-\frac{\lambda}{C_v}\left[1+R-(\gamma-1)\right]} \tag{45}$$

$$C_2' = 2\kappa\cdot\frac{\gamma-1}{\gamma+R}\frac{(2\eta+\eta^*)\left[(1+R)(R-\gamma)\right] + \frac{\lambda}{C_v}\left[1+R-(\gamma-1)\right]}{(2\eta+\eta^*)\left[R(\gamma-1)+\gamma(1+R)\right]-\frac{\lambda}{C_v}\left[1+R-(\gamma-1)\right]} \tag{46}$$

The complete line-transmission model is finally shown by figure 4, the values of the components being given by formulas (38) to (40) and (44) to (46).

As a means of verification, the adiabatic condition applied to the enhanced model enables us to retrieve the classical model given in figure 1.

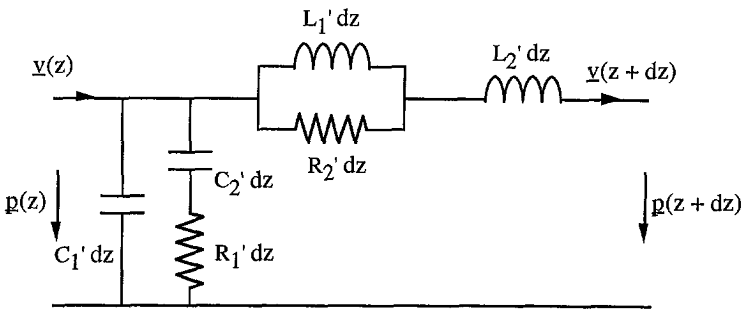

figure 4

With the assumption of low frequency, $R = \gamma$ and the previous elements become

$$L_1' = \frac{\rho_0}{\gamma} \tag{47}$$

$$L_2' = \rho_0\left(\frac{\gamma-1}{\gamma}\right) \tag{48}$$

$$R_2' = \frac{\rho_0}{\kappa(2\eta+\eta^*)\gamma} \tag{49}$$

$$R_1' = \frac{\left[(2\eta+\eta^*)\gamma^2-\frac{\lambda}{C_v}\right]^2}{\frac{\lambda}{C_v}\gamma(\gamma-1)} \tag{50}$$

$$C_1' = \kappa\cdot\frac{(2\eta+\eta^*)\gamma-\frac{\lambda}{C_v}}{(2\eta+\eta^*)\gamma^2-\frac{\lambda}{C_v}} \tag{51}$$

$$C_2' = \kappa\cdot\frac{\frac{\gamma-1}{\gamma}}{(2\eta+\eta^*)\frac{C_v}{\lambda}\gamma^2-1} \tag{52}$$

This means the effects of heat conduction only influence the shunt admittance and the viscosity of the fluid interferes with the value of the compliances.

5 Equivalent elements

Equivalent circuits were established for different structures present in integrated transducers such as miniaturized microphones on silicon or earphones and especially the holes, ducts and cavities.

The solution of the system (15) is finally, by use of the values of eigenvectors and eigenvalues obtained in (16) and (17),

$$\begin{pmatrix} \underline{p}(z) \\ \underline{v}(z) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \underline{Y}_c & -\underline{Y}_c \end{pmatrix} \begin{pmatrix} e^{-kz} & 0 \\ 0 & e^{kz} \end{pmatrix} \begin{pmatrix} \underline{p}_+ \\ \underline{p}_- \end{pmatrix} \tag{53}$$

where $\underline{p}_+$ and $\underline{p}_-$ are the backward and the forward waves respectively depending on the boundary conditions.

The transfer function of such a transmission line of length L is given [4] by

$$\begin{pmatrix} \underline{p}_1 \\ \underline{v}_1 \end{pmatrix} = \begin{pmatrix} \cosh \underline{k}L & \underline{Z}_c \cdot \sinh \underline{k}L \\ \underline{Y}_c \cdot \sinh \underline{k}L & \cosh \underline{k}L \end{pmatrix} \begin{pmatrix} \underline{p}_2 \\ -\underline{v}_2 \end{pmatrix} \tag{54}$$

with $\underline{p}_1$, $\underline{p}_2$, $\underline{v}_1$ and $\underline{v}_2$ defined as on figure 5.

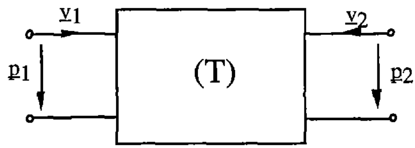

figure 5

The total input impedance is obtained as a function of the load impedance $\underline{Z}_L = \frac{\underline{p}_2}{\underline{v}_2}$ by

$$\underline{Z}_t(z{=}0) = \underline{Z}_C \cdot \frac{\underline{Z}_L + \underline{Z}_C \cdot \tanh \underline{k}L}{\underline{Z}_C + \underline{Z}_L \cdot \tanh \underline{k}L} \tag{55}$$

This relation can be rewritten

$$\underline{Z}_t = \underline{Z}_C \cdot \frac{\underline{Z}_L + \underline{Z}_C \cdot \tanh \underline{k}L + \underline{Z}_L \cdot \tanh^2 \underline{k}L - \underline{Z}_L \cdot \tanh^2 \underline{k}L}{\underline{Z}_C + \underline{Z}_L \cdot \tanh \underline{k}L} \tag{56}$$

or

$$\underline{Z}_t = \underline{Z}_C \cdot \tanh \underline{k}L + \frac{1}{\cosh^2 \underline{k}L} \cdot \frac{1}{\frac{1}{\underline{Z}_L} + \frac{\tanh \underline{k}L}{\underline{Z}_C}} \tag{57}$$

By using the values of the open and closed duct given by (60) and (61), it can be rewritten,

$$\underline{Z}_t = \underline{Z}_{to} + \frac{1}{\cosh^2 \underline{k}L} \cdot \frac{1}{\frac{1}{\underline{Z}_L} + \frac{1}{\underline{Z}_{tc}}} \tag{58}$$

And yields the equivalent circuit shown by figure 6, for the input impedance of a duct loaded by any other acoustical load.

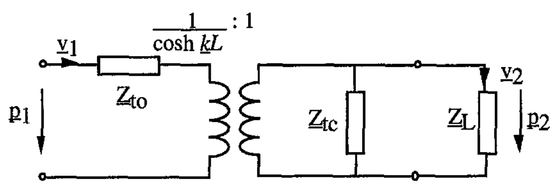

figure 6

If the dimensions L are smaller than the wavelength, the argument $\underline{k}L$ of the hyperbolic cosine function can be neglected and yields

$$\underline{Z}_t = \underline{Z}_{to} + \frac{1}{\frac{1}{\underline{Z}_L} + \frac{1}{\underline{Z}_{tc}}} \tag{59}$$

The impedance can then be represented by figure 7

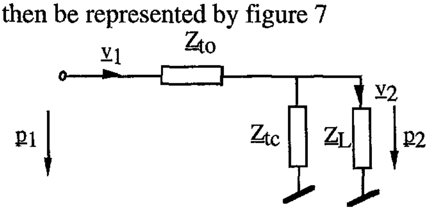

figure 7

It means that any acoustical structure opened on any duct can be modelled as if it were simultaneously a closed and an open duct.

6 Open and closed ducts

Setting some boundary conditions for a duct opened on both sides or a duct closed on one side leads to more simple values for input impedance.

The open duct is defined with a null acoustical pressure at its end or $\underline{Z}_L = 0$, its input impedance, given by (55), becomes

$$\underline{Z}_{to} = \underline{Z}_C \cdot \tanh \underline{k}L \qquad \text{or} \qquad \underline{Z}_{tc} = \frac{\underline{Z}' \cdot L}{\underline{k} \cdot L} \cdot \tanh \underline{k}L \tag{60}$$

The closed duct is defined with a null velocity at its end, equivalent to a load $\underline{Z}_L = \infty$. Its input impedance is therefore given by

$$\underline{Z}_{tc} = \underline{Z}_C \cdot \frac{1}{\tanh \underline{k}L} \qquad \text{or} \qquad \underline{Z}_{tc} = \frac{\underline{k} \cdot L}{\underline{Y}' \cdot L} \cdot \coth \underline{k}L \tag{61}$$

Expressing the Taylor series for $\frac{\tanh \underline{k}L}{\underline{k}L}$ and $\underline{k}L \cdot \coth \underline{k}L$ the previous expressions can be written

$$\underline{Z}_{to} = \underline{Z}' \cdot L \cdot \left(1 - \frac{(\underline{k}L)^2}{3} + 2 \cdot \frac{(\underline{k}L)^4}{15} + \ldots\right) \tag{62}$$

and

$$\underline{Z}_{tc} = \frac{1}{\underline{Y}' \cdot L} \cdot \left(1 + \frac{(\underline{k}L)^2}{3} - \frac{(\underline{k}L)^4}{45} + \ldots\right) \tag{63}$$

When the dimensions are smaller than the wavelength, the previous assumptions lead to

$$\underline{Z}_{to} = \underline{Z}' \cdot L \tag{64}$$

and

$$\underline{Z}_{tc} = \frac{1}{\underline{Y}' \cdot L} \tag{65}$$

7 Equivalent acoustical elements

The equivalent acoustical elements are given by transformation of the specific impedance into acoustical impedance by multiplying this impedance by the area S of the ducts.

The acoustical impedances are given by the next formula and can be represented by figure 8 for the open duct or hole shown in figure 9

$$\underline{Z}_{to} = \underline{Z}' \cdot \frac{L}{S} \quad (66)$$

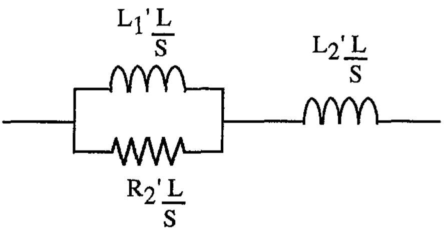

figure 8

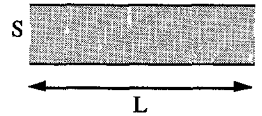

figure 9

Figure 10 represents the equivalent circuit for the closed duct or cavity shown in figure 11 and given by

$$\underline{Z}_{tc} = \frac{1}{\underline{Y}' \cdot L \cdot S} \quad (67)$$

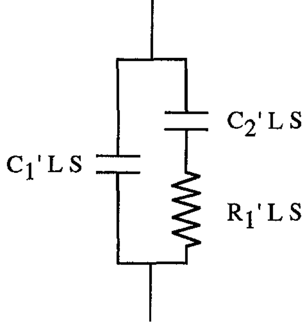

figure 10

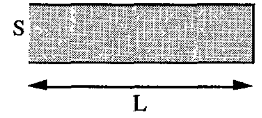

figure 11

8 Conclusion

This research improves the classical theory of acoustical structures modelisation by equivalent components in a scheme, taking into account visco-thermal losses and heat transfers.

A complete model with frequency dependent components has been established for very high frequency. For frequencies under 100 kHz, approximations are given with constant value components.

The method for interconnecting the components of the acoustical structures was given and it has been demonstrated that any structure is represented as if it were simultaneously a duct and a cavity.

9 Appendix

Thermodynamical variables

p	acoustical pressure
ρ	dynamic density
$\vec{v}$	acoustical speed
τ	temperature

Physical constants

ρ_o	static density
κ	isothermal compressibility
$\beta_p = -\frac{\rho}{T}\left(\frac{\partial T}{\partial \rho}\right)_{p=cste}$	inverse of coefficient of volume expansion
$\alpha_v = \frac{p}{T}\left(\frac{\partial T}{\partial p}\right)_{\rho=cste}$	inverse of coefficient of pressure expansion
λ	thermal conductivity
C_p	specific heat at constant pressure
C_v	specific heat at constant volume
$\gamma = \frac{C_p}{C_v}$	ratio of specific heat capacities
η	static viscosity
η^*	bulk viscosity

10 References

[1] L. Borel, *Thermodynamique et energetique*, Presses Polytechniques Romandes, Lausanne, 1984

[2] M. Rossi, *Acoustics and Electroacoustics*, Artech House, 1988

[3] F. E. Gardiol, *Lossy Transmission Lines*, Artech House, 1987