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On the Propagation of Sound Waves in Narrow Conduits*

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The analysis of the propagation of sound waves in narrow tubes has usually been restricted to shapes yielding tractable mathematical expressions. A great number of practical applications do not fall within these categories and await a solution. An approximate solution of sufficient accuracy for narrow tubes of arbitrary shapes developed in this paper has been applied to a wire-filled tube. The theoretical predictions check satisfactorily with the experimental results. It is believed that this study will be useful in other similar applications.

I. INTRODUCTION

THE steady state problem of the propagation of sound waves in conduits has attracted attention for many years. Exact solutions, however, are limited to the few conduit shapes described by coordinate systems allowing the wave equation to be separable. In several practical applications it is required to investigate the propagation in narrow channels of configuration not included among the few that yield exact solutions. The available approximate solutions applicable to tubes of arbitrary shapes neglect the losses due to thermal conduction. The preponderance of these losses in narrow channels makes it imperative to develop a new solution which considers them in the discussion.

It was noticed by Rayleigh[1] that the results of Kirchoff's exact solution[2] for the propagation of sound, down tubes of circular sections, when dissipation is taken into account, could be deduced from an approximate formula developed by Helmholtz provided some constants in the latter formula are altered. These changes are necessary to account for thermal effects which are neglected in Helmholtz's derivation.

Rayleigh's observation suggests a method of determining an approximate solution applicable to narrow conduits of arbitrary shape. The relation which has been deduced in this manner is very simple to solve mathematically and has been found to be of reasonable accuracy. The study is restricted to propagation in dry air and at sonic frequencies, so that dissipation of energy can be explained to a good degree of approximation by classical factors, i.e., by viscosity, radiation, and thermal conduction.

II. STATEMENT OF THE PROBLEM

The conventional one-dimensional wave equation for a dissipationless medium for a periodic disturbance of frequency $\omega/2\pi$ is

$$d^2u/dx^2 = -(\omega/c)^2 u, \tag{2.1}$$

where u is the particle velocity and c is the velocity of propagation of sound. In a narrow conduit, the wave front is approximately plane and the dissipation at the walls accounts for a large portion of the total viscous losses. As a result, the previous equation can be changed to a new expression of the form:

$$d^2u/dx^2 = -(\omega/c)^2(1+\psi+j\phi)u, \tag{2.2}$$

where ψ is an accession to inertia due to the viscous drag and ϕ is a quantity related to the viscous resistance. Both ψ and ϕ are expected to depend on the frequency and shape of the tube. Expression (2.2) is similar only in *form* to Helmholtz's derivation.[1] The values of the parameters are, however, different. The velocity of propagation c, which was taken as an invariant quantity in Helmholtz's formula, is now made a function of both the frequency and the shape of the conduit. The latter step is to allow for thermal effects which are closely associated with the evaluation of the value of c.

From the preceding discussion it follows that the present study is reduced to an investigation of the dependence of the c, ψ, and ϕ functions on both the frequency and shape of the tube; these functions are subsequently referred to as correction terms.

III. EVALUATION OF THE CORRECTION TERMS

The dependence of the magnitude of the velocity of propagation c on the frequency of the vibrations is deduced from the thermodynamic relations for the gas. Using to that effect the first law of thermodynamics, it follows that:

$$\rho C_v(d\theta/dt) = -\rho P(dv/dt) + K\nabla^2\theta,$$

where θ is the instantaneous excess temperature of the gas above the mean T_0, v is the instantaneous volume of the gas, ρ is the density of the gas, P is the total pressure, K is the coefficient of thermal conductivity, and C_v is the specific heat at constant volume. Dividing both sides of the equation by ρC_v, it is readily found that

$$d\theta/dt = \beta(ds/dt) + \nu\nabla^2\theta. \tag{3.1}$$

In the above relation ν, the thermometric conductivity, stands for $K/\rho C_v$, β is $P/\rho C_v$, and s is the condensation. The solution of Eq. (3.1) will indicate the manner in which the temperature θ is distributed across a section of

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[1] Lord Rayleigh, *Theory of Sound* (Macmillan, London, 2nd ed. 1929), Vol. II, p. 328 ff.

[2] Kirchoff, Pogg. Annalen **134**, 177 (1868).

the conduit. To simplify the analysis, s is assumed to be constant across a section of the tube. This approximation is permissible, since both the pressure and the density of the gas are expected to vary slowly over any transverse section of the conduit. For a periodic time dependence $e^{j\omega t}$, (3.1) is rewritten as

$$\nabla^2(\theta/\beta s)=j\omega/\nu[(\theta/\beta s)-1]. \tag{3.2}$$

The above equation will have for solution an expression of the form,

$$(\theta/\beta s-1)=F(\omega/\nu,\ \xi,\ \eta,\ a,\ b)$$

or

$$\theta=\beta s(1+F), \tag{3.3}$$

where F is a function depending on the geometry of the conduit referred to some coordinate system (ξ, η) and a, b are two arbitrary constants. These two constants must make the function F assume the value of (-1) at the walls of the conduit described by $f(\xi, \eta)=0$. This special choice of a and b is imposed by the boundary conditions which require the excess temperature θ to vanish at the walls.

The average temperature fluctuation $\bar{\theta}$ across the conduit of sectional area S is evaluated by means of the relation

$$\begin{aligned}\bar{\theta}&=1/S\int\int_S \theta d\xi d\eta,\\ &=\beta s(A+jB).\end{aligned} \tag{3.4}$$

A and B are functions of ω/ν, a and b. Combining Eq. (3.4) with the general gas relation,

$$\delta P/P+\delta V/V=\delta T/T, \tag{3.5}$$

in which δP is made equal to the acoustic pressure p and δT to the mean fluctuation $\bar{\theta}$, it is found that the result can be written in the form

$$p=s\rho c_0{}^2\Lambda. \tag{3.6}$$

Λ is a function of ω/ν, a and b whose exact evaluation is not an easy matter when the boundaries assume complicated shapes. The difficulty of calculating Λ exactly lies chiefly in solving (3.2). The coefficient Λ, however, has been computed by Daniels[3] for some simple cases. When the results of his calculations are plotted against $(f)^{\frac{1}{2}}S/P$ (P being the perimeter of the conduit), it is noticed that $|\Lambda|$ is nearly independent of the shape of the conduit. The mathematical formulation of this remark is,

$$|\Lambda|=\Omega[(f)^{\frac{1}{2}}S/P]+\epsilon\Phi\ (\text{shape}), \tag{3.7}$$

where $\Omega[(f)^{\frac{1}{2}}S/P]$ is an intermediate curve (Fig. 1) through the set of curves calculated by Daniels and $\epsilon\Phi$ (shape) indicates a small deviation from the mean to suit a particular configuration. One can consequently infer that Λ is approximately given by a universal relation, when the independent variable is $(f)^{\frac{1}{2}}S/P$.

When $c_0{}^2\Lambda$ of Eq. (3.6) is replaced by c^2, then c is the velocity of propagation of sound inside the conduit. Since Λ tends to $C_v/C_p=1/\gamma$ for vanishingly small frequencies and to unity as the frequency is raised, the velocity c for these two extreme cases tends, respectively, to Newton's value and to Laplace's value. In the former case the expansions of the gas are isothermal and in the latter, purely adiabatic. The frequency interval, in which the transition occurs from one state to the other, is determined by the geometry of the conduit (i.e., by the value of the ratio S/P).

The curve of Fig. 1 plays also an important role in the evaluation of the ψ and ϕ terms. For values of $[(f)^{\frac{1}{2}}S/P]<0.1$ the curve predicts an isothermal state. It is then plausible to assume that a great fraction of the gas is in contact with the walls of the conduit. Friction then has such a "hold" on the vibrating mass that the inertia effect can be neglected in comparison with the viscous forces. The wave equation is then deduced from the three relations $-\partial p/\partial x=R_0u$; $\partial u/\partial x=-\partial\rho/\partial t$; $p=\rho c^2 s$, where R_0 is the flow resistance per unit length. The above ultimately leads to:

$$\partial^2u/\partial x^2=(jR_0u\omega/P). \tag{3.8}$$

The correction terms are then:

$$\psi=-1;\quad \varphi=R_0\rho/\omega;\quad c^2=P/\rho. \tag{3.9}$$

When $[(f)^{\frac{1}{2}}S/P]>0.1$, the above conditions no longer hold. The extent of the inertia effect can be best judged by examining Kirchoff's exact solution[1] for the propagation of sound waves between parallel walls. The mathematical analysis of this case is considerably simplified because of the reduction of the formulation to a one-dimensional dependence. The velocity distribution in a section at right angles to the wall has been calculated using Kirchoff results. The distribution is shown in Fig. 2. The rapidity of the alteration of the distribution with the frequency is worthy of notice. The clinging layer theory postulated in Helmholtz's solution is seen

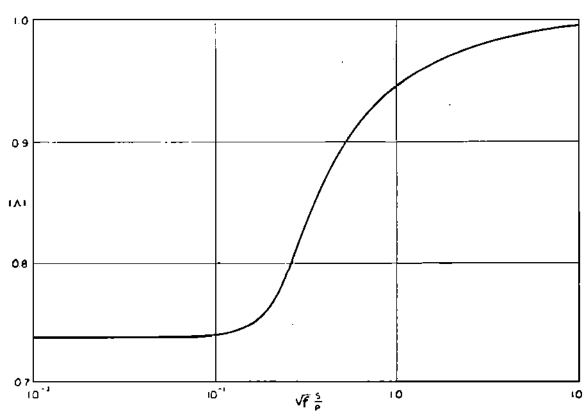

FIG. 1. Transition from isothermal to adiabatic state.

[3] F. B. Daniels, J. Acous. Soc. Am. 19, 569 (1947).

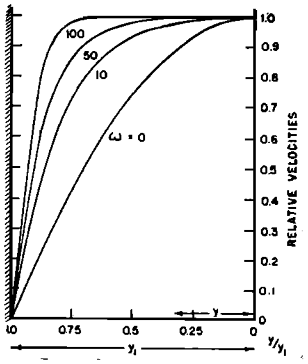

FIG. 2. Distribution of the velocity between parallel walls (Kirchoff formula).

to be very quickly reached and can be considered valid for a large range of frequencies. Helmholtz's values can then be taken for two of the correction terms, thus:

$$\psi=(P/S)(-\mu/2\omega\rho)^{\frac{1}{2}},\quad \phi=-j(P/S)(-\mu/2\omega\rho)^{\frac{1}{2}},\tag{3.10}$$

while c is calculated from the chart of Fig. 1 and by means of the relation $c=c_0(\Lambda)^{\frac{1}{2}}$.

IV. INPUT IMPEDANCE OF NARROW CONDUIT

A. Semi-Infinite Tube

The specific input impedance Z_0 of an infinitely long tube can be readily evaluated. For a time dependence of the form $e^{j\omega t}$ the solution of the differential Eq. (2.2) is $u=Ae^{j\omega t-mx}$, where

$$m^2=-(\omega/c)^2(1+\psi+j\phi).\tag{4.1}$$

Making use of the well-known relations,

$$p=\rho c^2 s \quad\text{and}\quad \partial u/\partial x=-\partial s/\partial t,$$

the specific impedance is deduced to be

$$Z_0=\rho c(-jmc/\omega).\tag{4.2}$$

It was shown in the preceding section that two sets of formulas are to be used to determine the correction terms. In the first set, the frequency of the vibrations is so low that conditions are approximately that of a steady flow of air in the tube. The other case covers nearly all the audible frequency range.

Using the appropriate values for the exponent m, it follows that for the first range of frequencies

$$Z_0=\rho c(1-j)(R_0/\omega\rho)/\sqrt{2}\tag{4.3}$$

and for the range of frequencies satisfying the "clinging layer" theory,

$$Z_0=\rho c[1+(1-j)(P/S)(\mu/2\omega\rho)^{\frac{1}{2}}]^{\frac{1}{2}}.\tag{4.4}$$

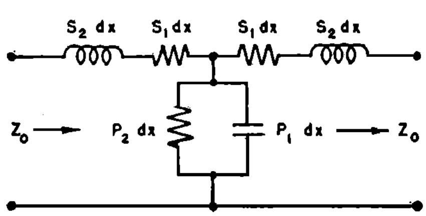

FIG. 3. Electrical analogue of an infinitesimal length of a tube, in which a sound wave is propagated with losses (for isothermal case $S_2=0$; $P_2=\infty$).

The propagation of sound waves in a tube as formulated in Eq. (2.2) can be represented by an electrical analogue, the leaky transmission line. In this manner it will be possible to adopt many of the techniques used in transmission line theory.

The equivalent T-network representation of an infinitesimal length dx of a leaky line having the characteristic impedance and propagation constant given by (4.1) and (4.2) is well known and is shown in Fig. 3. For isothermal changes of the gas (first case) the series arm of the equivalent T is:

$$\begin{aligned} Z_1&=2Z_0\tanh(mdx/2),\\ &=R_0dx,\\ &=2S_1 \text{ per unit length}\end{aligned}\tag{4.5}$$

and the shunt arm is given by:

$$\begin{aligned} Z_2&=(Z_0/\sinh mdx),\\ &=\rho c^2/j\omega dx,\end{aligned}$$

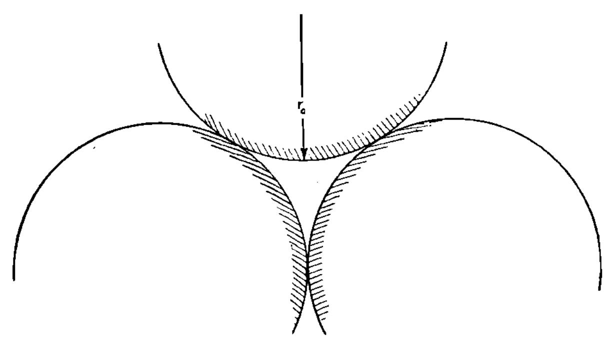

FIG. 4. Cross section of an interstitial channel.

or

$$1/Z_2=(j\omega dx/\rho c^2)=j\omega P_1 \text{ per unit length.}\tag{4.6}$$

For non-isothermal changes it is similarly found for the series arm of the equivalent circuit:

$$\begin{aligned} Z_1&=\{(P/S)(\mu\omega\rho/2)^{\frac{1}{2}}+j\omega[\rho+(P/S)(\mu\rho/2\omega)^{\frac{1}{2}}]\}dx,\\ &=2(S_1+j\omega S_2) \text{ per unit length,}\end{aligned}\tag{4.7}$$

and for the shunt arm:

$$Z_2=(\rho c^2/j\omega)dx.$$

If $c^2=|c^2|(\cos\xi+j\sin\xi)$, then the above expression can be rewritten as:

$$\begin{aligned}\frac{1}{Z_2}&=\left(\frac{\omega\sin\xi}{\rho|c^2|}+\frac{j\omega\cos\xi}{\rho|c^2|}\right)dx,\\ &=(P_2+j\omega P_1) \text{ per unit length.}\end{aligned}\tag{4.8}$$

B. Tube of Finite Length

For a tube of length l, connected at one end to a sound source and having the other left open to the atmosphere, the end conditions are closely approximated by an impedance of ρc per unit area of channel cross section. The input impedance is then:

$$Z_{in}=Z_0\coth(al+\sigma+j(bl+\tau)), \tag{4.9}$$

where $m=a+jb$ is the propagation constant defined by (4.1) and $\sigma+j\tau=\coth^{-1}(\rho c/Z_0)$ is the hyperbolic argument of the terminating impedance.

V. EXPERIMENTAL VERIFICATION

The preceding theoretical discussion was tested on a wire-filled tube. This device, formed by filling a conduit with straight round wires, so that sound can be transmitted along the interstitial channels between the wires, has been used often as a high acoustical impedance.[4]

A wire-filled tube was constructed by inserting inside a tube 4.45 cm in diameter wires of 0.0457 cm radius and 15.24 cm long. In a tightly packed tube, the section of a channel has the shape of a curved triangle as shown in Fig. 4. The theoretical number of wires which can be inserted inside the 4.45 cm tube to make the interstitial channels similar to Fig. 4, is 2300. Because of the practical difficulty of preventing the wires from twisting as they are rammed inside the tube, only 1860 could be inserted.

An elementary numerical calculation indicates that the S/P ratio for the interstitial channels is 2.33×10^{-4}. Referring to Fig. 1, it is found that the latter number definitely sets the character of the expansions as isothermal. Formula (4.3) is then to be used.

The precalculated value of flow resistance of the whole wire-filled tube is 30 rayls/cm* (see Appendix); the measured resistance was found to be 25 rayls/cm. The acoustical input impedance has been computed by means of the relation:

$$Z=Z_{in}/(A\times1860),$$

where A is the cross-sectional area of a single interstitial channel and Z_{in} is the impedance defined by expression (4.3). The result of the calculations is shown in Fig. 5. The input impedance has been measured also by a short tube method[5] and the results of the measurements have been plotted on the same curve of Fig. 5. The scattering in the measurements is mainly due to the random shape of the channels, some of them having a large enough area to produce noticeable fluctuations in the impedance. In general, agreement between theory and experiment is satisfactory, and it is hoped that the approximate formulas which have been developed in this study will be useful in other similar applications.

ACKNOWLEDGMENT

The author is indebted to Professor F. V. Hunt for his valuable advice and criticism in connection with this investigation. The assistance of Miss Jane Eager with numerical calculations is also gratefully acknowledged.

APPENDIX

Determination of Flow Resistance

By definition, the flow resistance is the reciprocal of the delivery from a pipe per unit pressure gradient. The velocity distribution across a section of the pipe, for steady laminar flow, satisfies the

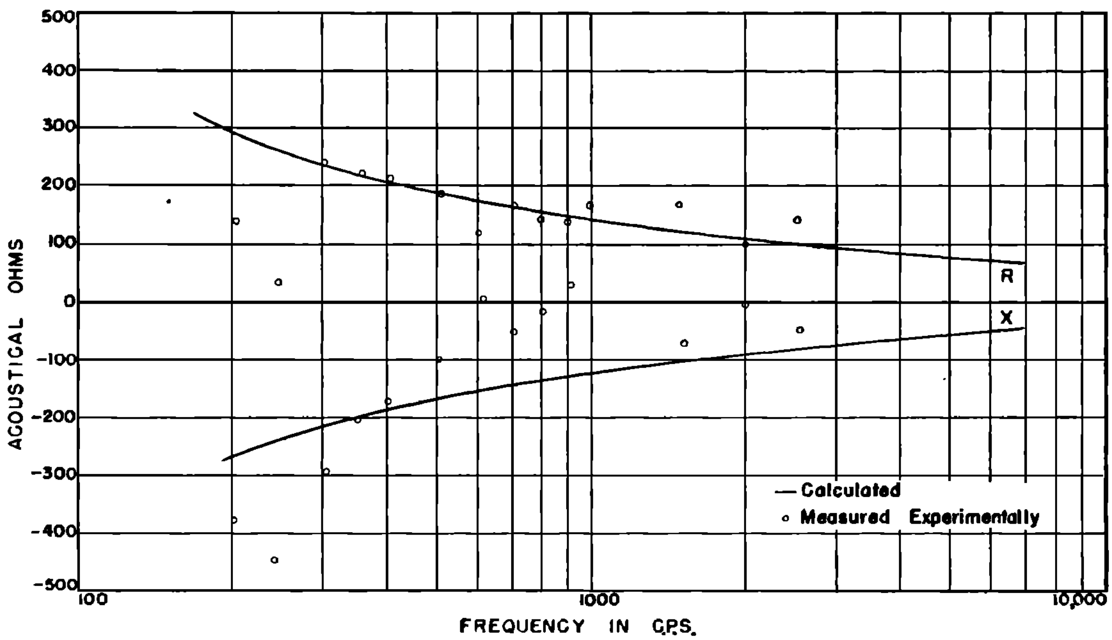

FIG. 5. Input impedance of wire-filled tube.

[4] F. V. Hunt, J. Acous. Soc. Am. 10, 216 (1939); L. L. Beranek, *ibid.* 12, 3 (1940); C. T. Morrow, *ibid.* 19, 645 (1947).
* The unit rayls has been defined by Beranek (J. Acous. Soc. Am. 19, 556, 1947) as dyne-sec cm^{-3}.
[5] O. K. Mawardi, J. Acous. Soc. Am. 21, 84 (1949).

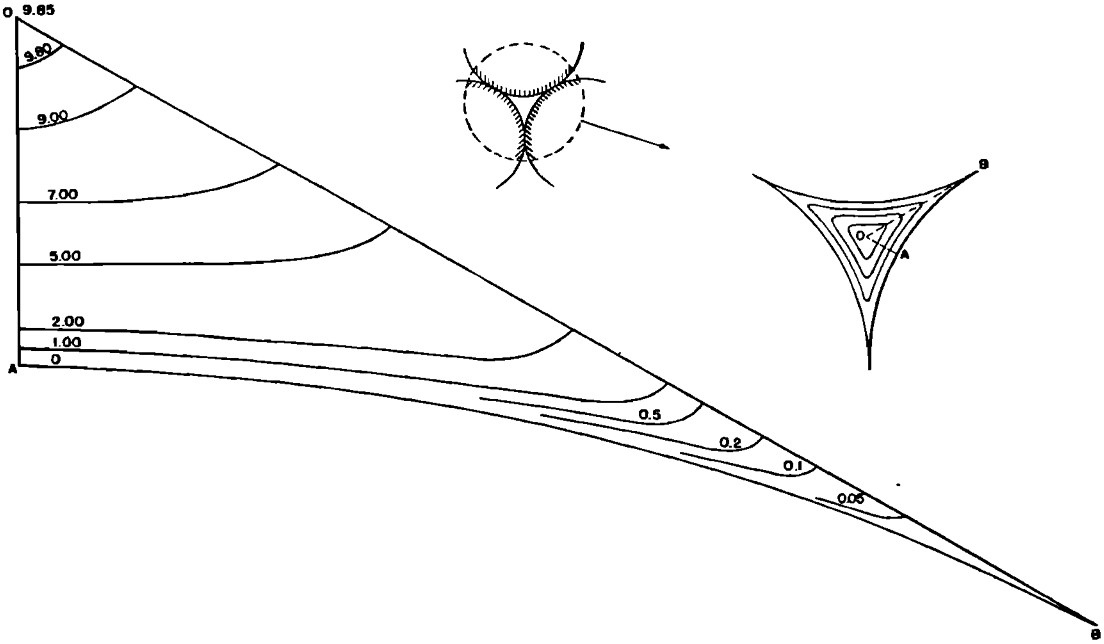

FIG. 6. Relative distribution of the velocities in a channel.

relation:

$$\nabla^2 u(x, y) = -1/\mu(\partial p/\partial l). \tag{1}$$

The boundary conditions imposed on the velocity u, require u to vanish at the walls. The delivery from the pipe is evaluated from

$$Q = \int\int_S u dy dx, \tag{2}$$

where S is the area of the cross section. The flow resistance is then

$$R_0 = (\partial p/\partial l)/Q.$$

The formal solution of (1) is not easy because of the complex shape of the boundaries. As an alternative, the relaxation method of Southwell[6] has been used to evaluate the distribution of the velocity in the channel. The result is shown in Fig. 6, where one-sixth of the conduit only has been drawn. The numbers inscribed indicate relative velocities. The discharge as defined by (2) has been evaluated by numerical integration and has been found to be:

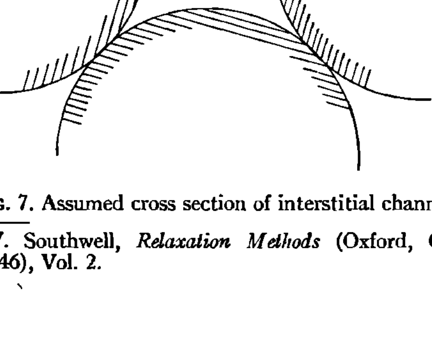

FIG. 7. Assumed cross section of interstitial channels.

[6] R. V. Southwell, *Relaxation Methods* (Oxford, Clarendon Press, 1946), Vol. 2.

$(r_0^4/1380\mu)$ per unit pressure gradient. The flow resistance of the wire-filled tube is then:

$$(1380\mu/r_0^4)\times 15.24\times 1/2300 = 374 \text{ rayls},$$

which is exceedingly high compared to the measured value of 25. The reason for this discrepancy is mainly due to the insufficient number of wires which have been packed in the tube. As the flow resistance of a duct approximately decreases with the square of the area, the change in the number of wires from 2300 to 1860 will appreciably decrease the flow resistance.

As a first approximation it will be assumed that the interstitial channels are formed by the empty space between four wires touching, so that the shape of a channel is similar to that shown in Fig. 7.

The accurate evaluation of the flow resistance by the relaxation method or other is a very tedious process; instead, an empirical formula due to Greenhill[7] is used. In this formula it is proved that the discharge of a viscous fluid from a pipe under steady laminar flow is proportional to the torsional rigidity of a homogeneous elastic cylinder of the same cross section. The latter rigidity has been found by Saint Venant (as quoted by Love[8]) to be expressed to a good approximation by replacing the section of the prism by an ellipse of the same area and moment of inertia I. The rigidity consequently is $nA^4/4\pi^2 I$, where n is related to the elastic constant of the substance. The flow resistance given by Greenhill analogy formula is then:

$$4(4\pi^2 I\mu/A^4) \text{ per channel per cm.}$$

For a configuration similar to that shown in Fig. 4, $A = 0.1613r_0^2$ and $I = 6.958\times 10^{-4} r_0^4$, the corresponding flow resistance is then: 440 rayls which is 15 percent off the theoretical value representing a satisfactory approximation. When channel shapes as indicated in Fig. 7 are used, then the new values are: $A = 0.8584r_0^2$ and $I = 0.3752r_0^4$. Hence the flow resistance is:

$$r\left(\frac{4\pi^2\times 0.3752r_0^4}{(0.8584r_0^2)^4}\right)\times 15.24\times\frac{1}{2300} = 30 \text{ rayls},$$

which is within 20 percent of the measured value.

[7] A. G. Greenhill, Proc. Lond. Math. Soc. **13**, 43 (1881).

[8] A. E. Love, *Elasticity* (Dover Publications, New York, 1944), p. 324.