setting $Z_i = Z_0$ the solution is

$$Z_m = Z_{12}^2(Z_{11}-1)^{-1} - Z_{22}. \tag{71}$$

For the $\lambda = 4a$ corner considered above, (71) yields $Z_m = 2.50 - j0.31 = 2.52 \angle -7.14°$.

In order to obtain the impedance Z_m at the output terminals 2, a reactance X is shunted across the line 2 at a distance d from the plane $Y=0$, d being sufficiently large to mitigate the possibility of higher order mode interaction between the reactance X and the corner itself. The problem of solving for d and X when Z_m is specified was considered in reference 1.

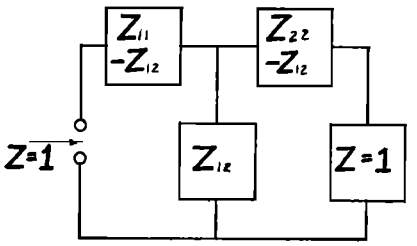

FIG. 9. Equivalent circuit for reflection at right-angle bend.

Experimental techniques for measuring equivalent impedances are analogous to those described for plane discontinuities.[1]

THE JOURNAL OF THE ACOUSTICAL SOCIETY OF AMERICA VOLUME 19, NUMBER 4 JULY, 1947

The Equivalent Circuit for a Bifurcated Cylindrical Tube

JOHN W. MILES
Engineering Department, University of California, Los Angeles, California

(Received March 14, 1947)

Using the impedance concepts developed in earlier papers, the problem of the diffraction of sound, caused by a bifurcation of a cylindrical tube, is solved. The equivalent circuit elements are shown to be related to the analogous changes of cross section. It is shown that the transmitted power is divided between the two tubes resulting from the bifurcation in proportion to their areas and without frequency distortion or reflection, and that the bifurcation may, therefore, be regarded as taking place in a virtual plane parallel to, but somewhat removed from, the geometrical plane. A formula is given to locate this plane.

The results are applied to a concentric-circular bifurcation and to a rectangular-tube bifurcated parallel to one of its walls. Numerical results are given which may be applied both to the circular change of cross section and to the concentric bifurcation of a circular tube.

NOTATION

a, b—dimensions of rectangular tube.
a_n, b_n—amplitudes of individual modes.
c—characteristic velocity of sound in medium.
d—distance of virtual plane behind geometrical plane of bifurcation.
e_n—eigenvalue in circular tube.
f—normalized longitudinal velocity.
h_n—propagation constant of nth mode.
i, j—free indices.
$k = (2\pi/\lambda)$—wave number.
p—pressure.
u, v, z—cylindrical coordinates.
w—longitudinal velocity.
B—susceptance.
G—Green's function.
I_0—transmission-line current.
J_n—Bessel function, 1st kind, nth order.
N_n—Bessel function, 2nd kind, nth order.
S—cross-sectional area.
V_0—transmission-line voltage.
$Y_n = 1/Z_n$—admittance of nth mode.
$Y_{ij} = jB_{ij}$—admittance element in equivalent circuit.
α—ratio of linear dimensions.
β_n—dimensionless eigenvalue in circular tube.
γ—ratio of areas.
δ_n^m—Kronecker delta $= 1$ if $m = n$, $= 0$ if $m \neq n$.
λ—wave-length.
ϕ_n—eigenfunction.
ρ_0—density of medium.
$\eta = (2\pi d/\lambda)$—radian measure of d.

INTRODUCTION

A PROBLEM of some interest is the scattering of sound when a small tube is inserted in a larger tube in order to make measurements of the local fields. This problem

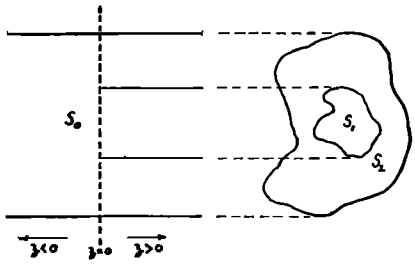

FIG. 1. Bifurcation of cylindrical tube of arbitrary cross section.

is a special case of the bifurcation of a cylindrical tube in a plane perpendicular to the axis.

In the general case a cylindrical tube of area S_0 is suddenly divided into two cylindrical tubes of areas S_1 and S_2 such that $S_1+S_2\equiv S_0$ as shown in Fig. 1. It will be assumed that this discontinuity defines the plane $z=0$ for a system of cylindrical coordinates (u, v, z). The notation and concepts to be used are those developed in the author's earlier papers on the same subject.[1] In particular, it is assumed that the frequency is sufficiently low so that only the principal wave propagates freely.

VELOCITIES AND PRESSURES

The longitudinal velocities in the tubes 0, 1, 2 are

$$w^i(u, v, z)=(a_0{}^ie^{\mp jkz}+b_0{}^ie^{\pm jkz})\phi_0{}^i+\sum_n b_n{}^i\phi_n{}^i(u, v)\exp[\pm jh_n{}^iz], \quad i=0, 1, 2, \tag{1}$$

$$b_n{}^i=-\delta_n{}^0a_0{}^i+\int_{S_i} w^i\phi_n{}^idS, \tag{2}$$

where the top and bottom signs apply for negative z (region 0) and positive z (regions 1, 2), respectively. The pressures are given by

$$p^i(u, v, z)=\pm Y_0{}^i(a_0{}^ie^{\mp jkz}-b_0{}^ie^{\pm jkz})\phi_0{}^i\mp\sum_n Y_n{}^ib_n{}^i\phi_n{}^i(u, v)\times\exp[\pm jh_n{}^iz]. \tag{3}$$

The summations in (1), (2) are taken over all the modes which may be excited at the discontinuity (excepting the principal mode) and are, therefore (most generally), doubly infinite.

The principal waves of velocity and pressure are represented by the transmission-line voltages and currents.

$$V_0{}^i(z)=(a_0{}^ie^{\mp jkz}+b_0{}^ie^{\pm jkz})(\phi_0{}^i)^{-1}, \tag{4}$$

$$I_0{}^i(z)=Y_0{}^i(a_0{}^ie^{\mp jkz}-b_0{}^ie^{\pm jkz})(\phi_0{}^i). \tag{5}$$

THE EQUIVALENT CIRCUIT

In the reference plane $z=0$, (4) and (5) reduce to

$$V_0{}^i=(a_0{}^i+b_0{}^i)(\phi_0{}^i)^{-1}=\int_{S_i} w^idS \tag{6}$$

$$I_0{}^i=Y_0{}^i(a_0{}^i-b_0{}^i)\phi_0{}^i, \tag{7}$$

where (6) follows from (2); moreover, since $S_1+S_2\equiv S_0$,

$$V_0{}^0=V_0{}^1+V_0{}^2. \tag{8}$$

The requirement that the pressure be continuous at $z=0$ yields, from (3) and (7)

$$I_0{}^0+I_0{}^i=\int_{S_0} G_0(u, v, u', v')w(u', v')dS'+\int_{S_i} G_i(u, v, u', v')w(u', v')dS' \quad (u, v) \text{ in } S_i;\quad i=1, 2 \tag{9}$$

$$G_i(u, v; u', v')=\sum_n Y_n{}^i\phi_n{}^i(u, v)\phi_n{}^i(u', v'), \quad i=0, 1, 2. \tag{10}$$

The simplest equivalent circuit which will satisfy the foregoing relations is shown in Fig. 2a and has the circuit equations

$$I_0{}^0+I_0{}^1=Y_{11}V_0{}^1+Y_{12}V_0{}^2, \tag{11a}$$

$$I_0{}^0+I_0{}^2=Y_{12}V_0{}^1+Y_{22}V_0{}^2. \tag{11b}$$

Now, if $w(u, v)$ is written

$$w_i=V_0{}^if_i(u, v) \tag{12}$$

in regions $i=1, 2$, (9) becomes

$$Y_{ii}=\int_{S_i}(G_0+G_i)f_idS', \quad u, v \text{ in } S_i, \tag{13}$$

$$Y_{ij}=\int_{S_i} G_0f_idS', \quad u, v \text{ in } S_j. \tag{14}$$

[1] John W. Miles, "The analysis of plane discontinuities in cylindrical tubes," J. Acous. Soc. Am. 17, Part I, 259–272, Part II, 272–285 (1946).

Moreover, substituting (12) in (6),

$$1=\int_{S_i} f_i dS. \tag{15}$$

Multiplying (13), (14) by $f_j dS$, integrating over S_j, and dividing by (15) to normalize, yields

$$Y_{ij}=\frac{\int_{S_i}\int_{S_j} f_i[G_0+\delta_j{}^i G_i]f_j dS dS'}{\left[\int_{S_i} f_i dS\right]\left[\int_{S_j} f_j dS\right]}. \tag{16}$$

For $i=j$ (16) may be recognized as the variational expression for the admittance caused by a change of cross section from S_0 to S_i (cf. reference (1), Eq. (60), where $\phi_0{}^1$ is replaced by unity), and is, therefore, an absolute minimum for the true field; for $i\neq j$ (16) is still a variational expression, but it is not necessarily a minimum for the true field. The solution for $i=j$ is given in Eqs. (61)–(71) of reference (1), whence, if the $\phi_{pq}{}^3$ are taken as the $\phi_p{}^i$, the subject results can be generalized to read

$$f_i=\frac{1}{S_i}+\sum_p A_p\phi_p{}^i, \tag{17}$$

$$B_0{}^{ij}=\sum_n B_n{}^0 R_{n0}{}^i R_{n0}{}^j, \tag{18}$$

$$C_p{}^{ij}=\sum_n B_n{}^0 R_{n0}{}^i R_{np}{}^j, \tag{19}$$

$$D_{pp'}{}^{ij}=\sum_n B_n{}^0 R_{np}{}^i R_{np'}{}^j+\delta_j{}^i\delta_{p'}{}^p B_p{}^i, \tag{20}$$

$$R_{n0}{}^i=\frac{1}{S_i}\int_{S_i}\phi_n{}^0 dS, \tag{21}$$

$$R_{np}{}^i=\int_{S_i}\phi_n{}^0\phi_p{}^i dS, \tag{22}$$

$$B_{ij}=B_0{}^{ij}+\sum_p C_p{}^{ij}A_p{}^j, \tag{23}$$

$$Y=jB, \tag{24}$$

$$\sum_{p'} D_{pp'}{}^{ij}A_{p'}{}^i=-C_p{}^{ij}, \tag{25}$$

where j in (24) represents $(-1)^{\frac{1}{2}}$ but elsewhere is a subscript. (17)–(25) follow directly[1] for $i=j$, while for $i\neq j$, it is necessary to substitute (17) in (16) and vary with respect to either $A_p{}^i$ or $A_p{}^j$, with the results (23) and (25). When a more accurate solution is required, (25) must be solved (for $i=i$ and $j=j$); it should be noted that (23) gives two expressions for B_{12}, one of which may converge more rapidly than the other.

REDUCTION TO CHANGE OF CROSS SECTION

From the orthogonality of the eigenfunctions and the definition of G_0 (cf. (10))

$$\int_{S_1}\phi_0{}^0 G_0 dS+\int_{S_2}\phi_0{}^0 G_0 dS=0. \tag{26}$$

Multiplying (26) by $\phi_n{}^i dS$ ($i=1$ or 2), integrating over S_i, substituting $\phi_0{}^i=(S_0/S_i)^{\frac{1}{2}}\phi_0{}^0$, and comparing with (18) and (19) yields

$$\begin{aligned} B_0{}^{jj}&=\left(\frac{S_i}{S_j}\right)^2 B_0{}^{ii}, \quad B_0{}^{ij}=-\left(\frac{S_i}{S_j}\right)B_0{}^{ii},\\ C_p{}^{jj}&=\left(\frac{S_i}{S_j}\right)^2 C_p{}^{ii}, \quad C_p{}^{ij}=-\left(\frac{S_i}{S_j}\right)C_p{}^{ii}, \end{aligned} \tag{27}$$

from which it follows that

$$B_{jj}=\left(\frac{S_i}{S_j}\right)^2 B_{ii}, \quad B_{ij}=-\left(\frac{S_i}{S_j}\right)B_{ii}, \tag{28}$$

so that the problem of the bifurcated tube is reduced to that of a simple change of cross

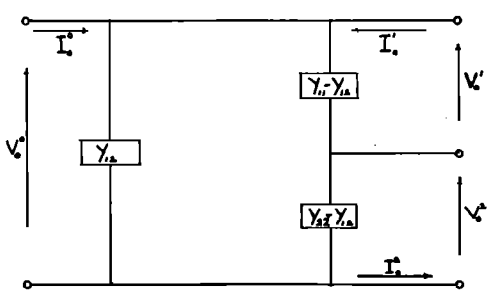

FIG. 2a. Equivalent circuit of bifurcated tube.

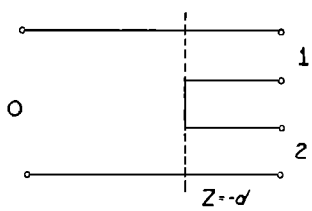

FIG. 2b. Equivalent circuit referred to virtual plane.

section, since (cf. (16)) B_{ii} is the susceptance due to a change of cross section from S_0 to S_i.

REFLECTION AND TRANSMISSION

It is profitable to consider the implications of (27), (28). Let

$$\gamma = S_1/S_0 = Y_0^0/Y_0^1, \qquad (1-\gamma) = S_2/S_0 = Y_0^0/Y_0^2, \tag{29}$$

so that, from (27), (28),

$$B_0^{12} = -\left(\frac{\gamma}{1-\gamma}\right)B, \quad B_0^{22} = \left(\frac{\gamma}{1-\gamma}\right)^2 B, \qquad B_0^{11} = B. \tag{30}$$

Now the admittance seen at the input (0) terminals of the equivalent circuit of Fig. 2a when the output terminal pairs (1 and 2) are terminated in Y_T^1 and Y_T^2 is given by

$$Y_T^0 = jB_{12} + \frac{[j(B_{11}-B_{12})+Y_T^1][j(B_{22}-B_{12})+Y_T^2]}{[j(B_{11}+B_{22}-2B_{12})+(Y_T^1+Y_T^2)]}. \tag{31}$$

For the relations of (30), (31) reduces to

$$Y_T^0 = \frac{1}{Z_T^0} = \frac{jB\left[\left(\frac{\gamma}{1-\gamma}\right)^2 Y_T^1 + Y_T^2\right] + Y_T^1 Y_T^2}{[jB(1-\gamma)^{-2} + Y_T^1 + Y_T^2]}, \tag{32}$$

and if $Y_T^{1,2} = Y_0^{1,2}$ (32) reduces to $Y_T^0 = Y_0^0$. Accordingly, to the degree of accuracy to which (27), (28) are valid, the bifurcation of a cylindrical tube introduces no reflection, provided that the two tubes resulting from the bifurcation are correctly terminated. Moreover, since the relations (30) possess the same frequency variation, the power is divided between the two tubes in the ratio of their areas without distortion.

VIRTUAL PLANE OF BIFURCATION

From the foregoing results it may be inferred that there is a reference plane where the equivalent circuit susceptances vanish, in which case the equivalent circuit reduces simply to the three transmission lines in series, as shown in Fig. 2b; and the impedance seen by the tube 0 would be

$$Z_0^{T'} = Z_1^{T'} + Z_2^{T'}, \tag{33}$$

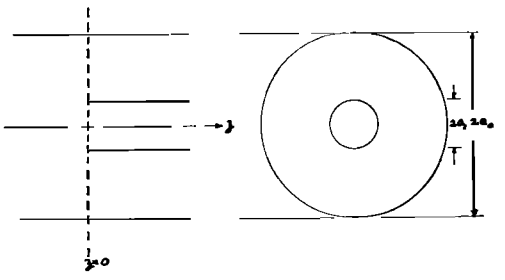

FIG. 3. Bifurcation of circular tube.

where the primes indicate that the impedances are evaluated in the new reference plane. Thus, the new reference plane is a "virtual plane of bifurcation," inasmuch as all measurements (made at a distance) will indicate that the bifurcation occurs in this plane; in particular, if one of the tubes is introduced in order to take measurements, the quantities will be measured in the virtual plane and not at the mouth of the tube.

Assume that the virtual plane is specified by $z=-d$ (i.e., lies a distance d ahead of the plane of bifurcation) and introduce the parameter

$$\eta = 2\pi d/\lambda = kd. \tag{34}$$

In order to solve for η it is necessary to transform each of the impedances Z_T^i (plane $z=0$) to $Z_T^{i'}$ (plane $z=-d$), which transformation is effected by

$$\frac{Z_T^{i'}}{Z_0^i} = \frac{j\sin\eta + (Z_T^i/Z_0^i)\cos\eta}{\cos\eta + j(Z_T^i/Z_0^i)\sin\eta} \tag{35}$$

and substitute in (33), which may then be solved for η. Carrying out these steps (where Z_T^0 is given by (32)) yields

$$\tan\eta = \left(\frac{\gamma}{1-\gamma}\right)\left(\frac{B}{Y_0^0}\right). \tag{36}$$

APPLICATION TO CIRCULAR TUBE

The most important practical application of the foregoing appears to be concentric circular tubes; i.e., where a tube of radius a_0 with the axis z contains a smaller concentric tube of radius a_1 which terminates (with an open end) at $z=0$ and extends along the positive z axis, as shown in Fig. 3. The equivalent change of cross section in a circular tube is treated in (141)–(147) of reference (1); hence, if $i=1$ specifies the smaller circular tube:

$$\phi_0^{0,1}=\pi^{-\frac{1}{2}}(a_{0,1})^{-1}=S_{0,1}^{-\frac{1}{2}}, \tag{37}$$

$$\phi_n^{0,1}(r)=\phi_0^{0,1}[J_0(e_n^{0,1}r)/J_0(e_n^{0,1}a_{0,1})], \tag{38}$$

$$J_1(e_n^{0,1}a_{0,1})=0, \tag{39}$$

$$B_n^{0,1}=k[(e_n^{0,1})^2-(k)^2]^{-\frac{1}{2}}\rho_0 c, \tag{40}$$

$$R_{n0}^1=S_0^{-\frac{1}{2}}\cdot 2\left[\frac{J_1(e_n^0a_1)}{(e_n^0a_1)J_0(e_n^0a_0)}\right], \tag{41}$$

$$R_{np}^1=\left[1-\left(\frac{e_p^1}{e_n^0}\right)^2\right]^{-1}2\left(\frac{a_1}{a_0}\right)\times\left[\frac{J_1(e_n^0a_1)}{(e_n^0a_1)J_0(e_n^0a_0)}\right]. \tag{42}$$

The eigenfunctions for the area S_2 ($r=a_1$ to $r=a_0$) are

$$\phi_0^2=S_2^{-\frac{1}{2}}=\pi^{-\frac{1}{2}}[(a_0)^2-(a_1)^2]^{-\frac{1}{2}}, \tag{43}$$

$$\phi_n^2(r)=[N_1(e_n^2a_1)J_0(e_n^2r)-J_1(e_n^2a_1)N_0(e_n^2r)]\cdot\left\{(a_0)^2\pi[J_0(e_n^2a_0)N_1(e_n^2a_1)-N_0(e_n^2a_0)J_1(e_n^2a_1)]^2+\frac{2(a_1)^2}{(e_n^2a_1)}\right\}^{-\frac{1}{2}}, \tag{44}$$

$$N_1(e_n^2a_1)J_1(e_n^2a_0)-J_1(e_n^2a_1)N_1(e_n^2a_0)=0, \tag{45}$$

where $N_1(x)$ is Neumann's solution to Bessel's equation.[2] $\phi_n^2(r)$ is defined so that its normal derivative (i.e., with respect to r) vanishes at $r=a_1$, while (37) insures that the normal derivative vanishes at $r=a_0$ and thus determines the eigenvalues e_n^2. The R_{np}^2 are then given by (22) as

[2] E. Jahnke and T. Emde, *Tables of Functions* (Dover Publications, New York, 1943).

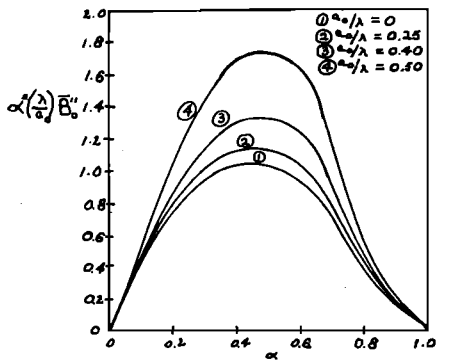

FIG. 4. Susceptance due to circular change of cross section from Eq. (48).

$$R_{np}^2=S_0^{-1}(e_p^2)^{-1}\left[1-\left(\frac{e_p^2}{e_n^0}\right)^2\right]^{-1}\times\left[\frac{J_1(e_n^0a_1)}{(e_n^0a_1)J_0(e_n^0a_0)}\right]\cdot\left\{(e_p^2)^2[J_0(e_p^2a_0)N_1(e_p^2a_1)-N_0(e_p^2a_0)J_1(e_p^2a_1)]^2+2\frac{(e_p^2a_1)}{(a_0)^2}\right\}^{-\frac{1}{2}}. \tag{46}$$

For purposes of calculation the B_0^{ij} will generally be sufficiently accurate. It is convenient to normalize with respect to

$$Y_0=\rho_0c/S_0. \tag{47}$$

Substituting (32) and (33) in (18):

$$\frac{B_0^{11}}{Y_0^1}=\bar{B}_0^{11}=4\sum_1^\infty\left[\left(\frac{e_n^0}{k}\right)^2-1\right]^{-\frac{1}{2}}\times\left[\frac{J_1(e_n^0a_1)}{(e_n^0a_1)J_0(e_n^0a_0)}\right]^2. \tag{48}$$

B_0^{12} and B_0^{22} are given by (27), (28).

NUMERICAL CALCULATIONS

In order to carry out numerical calculations, it is convenient to introduce the parameters

$$\alpha=(a_1/a_0), \tag{49}$$

$$\beta_n=e_n^0a_0, \tag{50}$$

$$\theta=2\pi a_0/\lambda=ka_0, \tag{51}$$

whence

$$\bar{B}_0^{11}=4\sum_1^\infty\left[\left(\frac{\beta_n}{\theta}\right)^2-1\right]^{-\frac{1}{2}}\left[\frac{J_1(\beta_n\alpha)}{(\beta_n\alpha)J_0(\beta_n)}\right]^2. \tag{52}$$

For actual computation (52) may be rewritten

$$\alpha^2\left(\frac{\lambda}{a_0}\right)\bar{B}_0{}^{11}=8\pi\sum_1^\infty\frac{1}{\beta_n}\left[\frac{J_1(\beta_n\alpha)}{\beta_nJ_0(\beta_n)}\right]^2$$
$$+8\left(\frac{\pi}{\theta}\right)\sum_1^\infty\left\{\left[\left(\frac{\beta_n}{\theta}\right)^2-1\right]^{-\frac{1}{2}}-\left(\frac{\theta}{\beta_n}\right)\right\}$$
$$\times\left[\frac{J_1(\beta_n\alpha)}{\beta_nJ_0(\beta_n)}\right]^2 \quad (53)$$

so that if $\alpha^2(\lambda/a_0)\bar{B}_0{}^{11}$ is plotted, the result will be relatively insensitive to variations in θ, so that, except for small α or θ near β_1, the first term in (53) (i.e., in the second series) is sufficient. $\bar{B}_0{}^{11}$ is plotted in Fig. 4 for values of (a_0/λ), of 0, 0.25, 0.40, and 0.50 (0.61 is the upper frequency limit for pure plane-wave propagation), and for α ranging from 0 to 1. The result as given by (53) is useful for a concentric change of cross section, but for an actual bifurcation, the virtual plane, given by (36), is more useful; in terms of (47), it is specified by

$$\tan\eta=\left(\frac{1}{1-\alpha^2}\right)\left(\frac{a_0}{\lambda}\right)\left[\alpha^2\left(\frac{\lambda}{a_0}\right)\bar{B}_0{}^{11}\right]. \quad (54)$$

If (a_0/λ) is small, (54) can be approximated by

$$\frac{d}{a_0}=\frac{1}{2\pi(1-\alpha^2)}\left[\alpha^2\left(\frac{\lambda}{a_0}\right)\bar{B}_0{}^{11}\right]. \quad (55)$$

APPLICATION TO A RECTANGULAR TUBE

Consider next a rectangular tube bounded by $x=0$, a, $y=0$, b_0, and bifurcated by the plane $y=b_1$ extending from $z=0$ to $z=\infty$. The analogous change of cross section was treated in Part II of reference 1, and the susceptance due to the change of cross section is given approximately by II (132), (135) as

$$\frac{B_{11}}{Y_0{}^0}=4\left(\frac{b_0}{\lambda}\right)\left[1-\left(\frac{2b_0}{\lambda}\right)^2\right]^{-\frac{1}{2}}L(\alpha), \quad (56)$$

$$L(\alpha)=\log\left[\left(\frac{1-\alpha^2}{4\alpha}\right)\left(\frac{1+\alpha}{1-\alpha}\right)^{\frac{1}{2}(\alpha+1/\alpha)}\right], \quad (57)$$
$$\alpha=b_1/b_0.$$

From (27), (28), B_{12}, B_{22} are given (to a good approximation) by

$$\frac{B_{12}}{Y_0{}^0}=-\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{B_{11}}{Y_0{}^0}\right), \quad (58)$$

$$\frac{B_{22}}{Y_0{}^0}=\left(\frac{\alpha}{1-\alpha}\right)^2\left(\frac{B_{22}}{Y_0{}^0}\right), \quad (59)$$

while, from (36) and (56), the virtual bifurcation plane is specified by

$$\tan\left(\frac{2\pi d}{\lambda}\right)$$
$$=4\left(\frac{b_0}{\lambda}\right)\left[1-\left(\frac{2b_0}{\lambda}\right)^2\right]^{-\frac{1}{2}}\left(\frac{\alpha^2}{1-\alpha^2}\right)L(\alpha). \quad (60)$$

THE JOURNAL OF THE ACOUSTICAL SOCIETY OF AMERICA VOLUME 19, NUMBER 4 JULY, 1947

The Growth of Auditory Sensation

W. A. MUNSON
Bell Telephone Laboratories, Inc., Murray Hill, New Jersey
(Received May 26, 1947)

The integration of sensation with respect to time was studied experimentally by means of tones of short duration. Loudness tests were made on sounds persisting from 0.005 to 0.2 second and covering a wide range of levels. The observed increase in magnitude of a sensation as the duration time is increased is attributed to the integration characteristic of the central nervous system, and an equivalent electrical circuit is derived. The circuit analogy is then used in the computation of loudness as a function of the duration of the stimulus.

IT has been known for many years that the loudness of a sound depends in part upon how long it persists. In 1929 Békésy[1] published data showing that loudness increases rapidly for a brief interval after the onset of a tone, then gradually approaches a steady value, and finally decreases slowly if the duration is long. The relatively long time required for the tone to reach

[1] G. V. Békésy, "Zur Theorie des Hörens," Physik. Zeits. **30**, 115 (1929).