Some Aspects of the Theory of Room Acoustics

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An exact solution for the decay of sound in a rectangular room is obtained; assuming that each wall is uniformly covered with absorbing material, which may differ from wall to wall. It is concluded, from recent experimental measurements, that the boundary conditions for the sound field are correctly expressed in terms of the effective normal impedance of the wall material. The sound is analyzed into its component normal modes of vibration, and the reverberation times and frequencies of the different normal modes are calculated as functions of the wall impedances and their phase angles. Curves are given for these quantities for a wide range of the parameters involved. The effect of the absorbing material in distorting the sound field is shown, and several other interesting points are brought out in the discussion: that waves which travel "parallel" to a wall are absorbed by the wall, but are not absorbed as much as are waves striking at more oblique angles; that it is sometimes possible to *increase* the reverberation time for a standing wave by *decreasing* a wall's effective acoustic resistance; etc.

THE problem of the decay of sound in a room is pleasantly simple to analyze approximately and surprisingly complicated to solve exactly. As long as one confines the analysis to first-order effects, considers only average sound intensities, and uses only moderately absorbing materials, the first-order formulas of Sabine or of Eyring are satisfactory. If one wishes, however, to investigate in detail the distribution of sound energy in the room, particularly with very absorbent material present, or if one wishes to make careful measurements of the absorbing qualities of the materials, the problem becomes so complicated that it can be solved at present only for particularly simple configurations.

One important cause of the difficulties is that the distribution of the sound field throughout the room is not only determined by the shape of the room; it is also distorted by the presence of absorbing material on the walls, and the distorting effect is greater the more absorbing the material. This is only too apparent to the experimental investigator of room acoustics, and it must be taken into account in any thoroughgoing theoretical analysis.

In the present paper a start is made toward a detailed theoretical analysis of room acoustics. A particularly simple case is studied; that of a rectangular room with uniform coverage of absorbing material on each wall, with only one of the walls being very absorbent. The aim will be to take the data from a single, relatively simple measurement of the absorbing property of a material; and to try to predict, from these data, what will be the acoustic properties of a room having the material spread uniformly over one or more of the walls. The theoretical results obtained will be correlated with experimental data in other[1] papers. The arrangement analyzed is too simplified to be of great use in practical acoustic design. Nevertheless simple cases of this sort must be understood in detail, and checked experimentally, before more complicated problems can be attacked. We shall see that even in this case the results are not always simple.

THE NORMAL IMPEDANCE

The first question to be settled is the nature of the property of the wall material which is responsible for the absorption; the physical quantity whose measurement will make it possible to predict the acoustic properties of the material under various conditions. It is not the purpose of this paper to make a detailed analysis of the mechanism of sound absorption in the wall material, of the nature of the work begun by Monna[2] and others. We are here interested

[1] To appear in this Journal. Some preliminary experimental confirmations of the theory outlined here were reported by F. V. Hunt and by N. B. Bhatt at the Symposium on Absorption Coefficients at the last meeting of the Acoustical Society.

[2] A. Monna, Physica **5**, 129 (1938), Rev. d'acoustique **7**, 126 (1938). See also V. Kühl and E. Meyer, Berl. Ber., Phys.

primarily in the sound field in the room, and need study the wall material only enough to determine the form of boundary condition which will represent the actual conditions at the wall adequately as far as the sound field in the room is concerned. The physical quantity representing the property of the material cannot be the usual absorption coefficient α, either averaged over angle of incidence or not. For if α is not averaged it depends on the angle of incidence of the wave, and if it is averaged it depends on the nature of the averaging; and in either case it depends on the size and shape of the room. The absorption coefficient is therefore not a suitable primary property, for its value depends on the nature of the incident wave, as well as on the nature of the material.

Recent experimental results[3] have indicated that the proper physical quantity which measures the absorbing qualities of the material is the substance's *normal acoustic impedance*, the ratio of pressure to normal air velocity at the surface of the material. The experiments indicate that this quantity depends only on the material and not on the incident wave (except for the variation with frequency). Of course further detailed experiments, with other materials, may show that even the normal impedance varies with angle of incidence of the wave: in which case we will have to use another, more deep-seated, physical quantity to measure the material's absorption. Until such time as experiment forces us to complicate the picture, however, it seems worth while to develop a theory of absorption in terms of the normal acoustic impedance Z, and to assume tentatively that Z is a function only of the frequency of the incident wave, and not of its angle of incidence. It has been shown elsewhere[4] that the reflection coefficient for a free plane wave striking an infinite plane surface of the material at an angle θ to the normal is

$$R_{\text{free}}=\left|\frac{\zeta\cos\theta-1}{\zeta\cos\theta+1}\right|^2,\quad \zeta=(Z/\rho c),\qquad (1)$$

where R is the ratio between the reflected and incident intensities, ρ is the density of air and c the velocity of sound in air. This formula has been approximately confirmed by two series of measurements,[3] which have also indicated that in many cases (though not in all cases) the impedance Z is real.

It should be noted here that Eq. (1) is strictly true only for free waves and for an infinite surface of material. We shall see later in this paper that it is only approximately true for the standing waves in a room of finite size; indeed, in certain special cases it is not at all applicable.

Before turning to the detailed analysis of the relation between the decay of sound in a room and the normal impedances of its walls, it will be useful to discuss the relation between the normal impedance concept and the sound refraction analysis discussed by Monna[2] and others. When sound strikes a wall, part of the intensity is reflected and part transmitted into the wall. The transmitted part consists partly of air vibrations in the pores of the wall and partly of vibrations of the material of the wall itself. For our purposes, however, it can be considered to be an average wave traveling through the material, presumably being attenuated as it penetrates. Since the wall pores are not usually isotropic, the average wave motion will not always be isotropic, the velocity parallel to the wall surface being, in general, different from that normal to the surface inside the wall.

We shall express these velocities in terms of two average indices of refraction, the wave velocity in the wall, tangential to the surface being $v_t=c/n_t$ (where c is the velocity of sound in air), and that normal to the surface being $v_n=c/n_n$. If there is attenuation in the wall, n_n and perhaps also n_t will be complex quantities. The wave equation for the pressure in this average wave inside the wall is then

$$\frac{1}{n_n^2}\frac{\partial^2 p}{\partial x^2}+\frac{1}{n_t^2}\left(\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}\right)=(1/c^2)\frac{\partial^2 p}{\partial t^2},$$

where the wall surface is taken to be the y, z plane. An analysis of the refraction and reflection of waves at this surface, using the boundary condition that the pressure and normal velocity be continuous in value at the surface, shows

Math. Kl. 26 (1932). An interesting analysis of the effect of a vibrating plate wall on the sound in a room, using methods somewhat similar to the present analysis, has just appeared, by R. Rogers, J. Acous. Soc. Am. 10, 280 (1939).

[3] F. J. Willig, J. Acous. Soc. Am. 10, 257(A) (1938); F. V. Hunt, J. Acous. Soc. Am. 10, 216 (1938).

[4] P. M. Morse, *Vibration and Sound* (McGraw-Hill), p. 304.

that the ratio between reflected and incident amplitudes is

$$D=\frac{(Q/n_n)\cos\theta-(1-(1/n_t^2)\sin^2\theta)^{\frac{1}{2}}}{(Q/n_n)\cos\theta+(1-(1/n_t^2)\sin^2\theta)^{\frac{1}{2}}}$$

for a free plane wave striking an infinite plane surface at an angle of incidence θ. The quantity Q is the ratio of the effective density of the wall material to the density of air, and is therefore quite large. The corresponding reflection coefficient is then $|D|^2=R$.

The above formula is a more general one than that given in Eq. (1), reducing to Eq. (1) when n_t becomes exceedingly large, and when n_n is equal to $(Q\rho c/Z)=(\rho_m c/Z)$, where ρ_m is the density of the wall material, and c is the velocity of sound in air. Therefore the materials which show experimentally a dependence of R on θ of the sort given by Eq. (1) must have an effective tangential velocity of sound considerably less than $c(n_t\gg 1)$, and an effective normal velocity greater than (c/Q) (so that Q/n_n be larger than unity, as it usually is). The normal velocity is then related to the normal impedance by the equation $v_n=(Z/\rho_m)$, indicating that the characteristic normal impedance of the wall, Z, is equal to its density times its normal wave velocity. Of course the actual phenomenon of sound absorption in the wall is much more complicated than a pair of average velocities can express; but for the purpose of studying the behavior of the sound field in the room, the two constants n_t and n_n/Q suffice to fix the boundary conditions; and, conversely, measurements of sound in the room can determine only these two constants.

Experimental data on the dependence of R on θ have not been taken for very many materials. Those which have been measured show curves indicating values of (Q/n_n) between 2 and 15 (some are complex), and values of n_t so much larger than unity as to be indistinguishable experimentally from infinity. It therefore seems justifiable to consider the absorbing property of a wall material to be adequately represented by a normal acoustic impedance Z, independent of the angle of incidence of the sound wave, but dependent on frequency. This impedance can be measured by a suitable modification of the Fay-Hall[5] impedance bridge, for instance. It is the purpose of the rest of this paper to show how the acoustic properties of a simple room can be computed if the normal impedances of its walls are known.

The Standing Waves of Sound in a Room

For our detailed analysis we will assume a rectangular room, with three joining edges along the positive parts of the three coordinate axes, and with dimensions l_x, l_y and l_z. Each wall will be uniformly covered with absorbing material: that on the wall in the plane $x=0$ having normal impedance Z_{x1}, and that on the wall in the plane $x=l_x$ having the impedance Z_{x2} (these two walls will be called the x walls), and so on. The pressure distribution in a single standing wave (either in forced or free oscillation) is

$$\begin{aligned} p_n&=X(x)Y(y)Z(z),\\ X&=\cosh\left[(\pi x/l_x)(-\kappa_x+j\mu_x)+\psi_x\right] \end{aligned} \tag{2}$$

and the factors Y and Z are similar to X, except that x is changed to y or z. The values of the constants κ, μ and ψ are to be determined by the boundary conditions.

For instance, at the $x=0$ plane the pressure is $ZY\cosh\psi_x$, and the normal velocity just in front of the wall is

$$\begin{aligned} u_{x=0}&=-(1/2\pi j\nu\rho)(\partial p/\partial x)_{x=0}\\ &=-\left(\frac{-\kappa_x+j\mu_x}{2j\nu\rho l_x}\right)\sinh\psi_x\cdot Y\cdot Z, \end{aligned}$$

where ν is either the driving frequency in forced vibrations, or is the natural frequency if the wave is in free vibration. The boundary condition that $p_{x=0}=-Z_{x1}u_{x=0}$ then corresponds to the equation

$$j\coth(\psi_x)=(\zeta_{x1}/\eta_x)(-\kappa_x+j\mu_x),$$

where $\zeta_{x1}=Z_{x1}/\rho c$ is the ratio of the normal impedance of the wall to the characteristic impedance of the air, and where $\eta_x=2l_x\nu/c=2l_x/\lambda$ is the ratio of the room length to the half-wavelength, and is a quantity proportional to the frequency. The six boundary conditions at the six

[5] R. D. Fay and W. M. Hall, J. Acous. Soc. Am. 10, 259(A) (1938).

walls result in three equations from which the quantities κ_x, κ_y, κ_z, μ_x, μ_y and μ_z can be determined. The equation for the x walls is

$$\pi j(\mu_x+j\kappa_x)+\coth^{-1}[(\zeta_{1x}/\eta_x)(\mu_x+j\kappa_x)] +\coth^{-1}[(\zeta_{2x}/\eta_x)(\mu_x+j\kappa)]=0. \quad (3)$$

Due to the periodicity of the hyperbolic cotangent with μ, we have an infinity of possible solutions of each of the three equations; the lowest value corresponding to a wave with no pressure nodes perpendicular to the x axis, and the n_xth corresponding to one with n_x pressure nodes perpendicular to the x axis, and so on. The integers n_x, n_y, n_z therefore serve to label the particular standing wave under examination. Those waves for which $n_y=n_z=0$ will be said to be "normally incident" on the x-walls; those for which $n_x=0$ are "grazing incidence" waves for the x-walls, and so on. The reason for the quotation marks will become apparent later.

As soon as the values of the κ's and μ's are known, the acoustic properties of the standing wave can be computed. For instance, the frequency factor for free vibration can be expressed by the exponential $e^{j\omega_n t-k_n t}$, where substitution in the wave equation shows that

$$(\omega_n+jk_n)^2=(\pi c)^2[(\mu_x+j\kappa_x/l_x)^2 +(\mu_y+j\kappa_y/l_y)^2+(\mu_z+j\kappa_z/l_z)^2].$$

In the great majority of cases we can neglect k^2 compared to ω^2, in which case the frequency of free vibration of the standing wave is

$$\nu_n=\frac{\omega_n}{2\pi}=\frac{c}{2}\left[\frac{\mu_x{}^2-\kappa_x{}^2}{l_x{}^2}+\frac{\mu_y{}^2-\kappa_y{}^2}{l_y{}^2}+\frac{\mu_z{}^2-\kappa_z{}^2}{l_z{}^2}\right]^{\frac{1}{2}} \quad (4)$$

and the attenuation constant, giving the rapidity of decay of the wave, is

$$k_n=\left(\frac{\pi^2c^2}{\omega_n}\right)[(\mu_x\kappa_x/l_x{}^2)+(\mu_y\kappa_y/l_y{}^2)+(\mu_z\kappa_z/l_z{}^2)]$$
$$=(c/4)[(1/l_x)(\delta_{x1}+\delta_{x2}) +(1/l_y)(\delta_{y1}+\delta_{y2})+(1/l_z)(\delta_{z1}+\delta_{z2})], \quad (5)$$

where $\quad (\delta_{x1}+\delta_{x2})=(4\pi\mu_x\kappa_x/\eta_x), \quad$ etc.

In this formula, in $\eta_x=(2\nu l_x/c)$ etc., we use here the frequency ν_n of free vibration.

Comparison with the usual equations[6] for the decay constant k_n shows that the quantity $(4\pi\mu_x\kappa_x/\eta_x)$ plays the same role in the expression for k as the quantities $(\alpha_{x1}+\alpha_{x2})\cos\theta_x$ do in the free wave analysis. We shall see later that in many cases this quantity $(4\pi\mu_x\kappa_x/\eta_x)$ breaks into two terms; one, which can be called δ_{x1}, depending only on the $x1$ wall, and the other, δ_{x2}, depending only on the $x2$ wall In these cases the analogy with the free wave case is particularly close.

The quantities δ will be called *damping coefficients*, and are the correct expressions to use in the formula for k_n, instead of the expressions $\alpha\cos\theta$ which are obtained when the waves are considered to be undistorted plane waves, uniformly distributed throughout the room. Even when δ_{x1} is not a function of the $x1$ wall alone, the sum $(\delta_{x1}+\delta_{x2})$, a function of the properties of both x-walls, can be considered to be a combined damping constant for both x-walls.

In the case of forced vibration, where the source function is $q(x, y, z)e^{j\omega t}$, the usual expansion in normal modes[7] shows that the steady-state pressure in the room is given by the series

$$p_s=\sum_n\frac{j\omega\rho c^2B_n}{\omega^2-(\omega_n+jk_n)^2}p_ne^{j\omega t}.$$

The amplitude of the nth standing wave is therefore equal to a product of p_n (see Eq. (2)), giving the distribution in space of the nth wave, times a constant B_n dependent on the placing of the source, times a resonance term

$$\frac{j\rho\omega c^2}{\omega^2-(\omega_n+jk_n)^2}\simeq\frac{-\rho c^2}{2k_n+j(\omega-(\omega_n{}^2/\omega))}, \quad (6)$$

where ω_n is given by Eq. (3) and k_n by Eq. (4). In this case, however, the quantities $\eta_x=(2\nu l_x/c)$, etc., in k_n are in terms of the frequency ν of the source. The constant k_n is the quantity used in steady-state measurements of room acoustics.[6]

Therefore, to investigate the acoustic properties of a room with uniform coverage, we need to compute the behavior of the quantities $\delta_1+\delta_2=(4\pi\kappa\mu/\eta)$ and $\mu^2-\kappa^2$ for each of the wall pairs, as functions of the wall impedances $\zeta=(Z/\rho c)$, for each of the standing waves in the room. The acoustic response of the room, and its reverbera-

[6] F. V. Hunt, J. Acous. Soc. Am. **10**, 223 (1938). H. Cremer and L. Cremer, Akust. Zeits. **2**, 6 (1937).

[7] Reference 4, page 315.

tion time, will be composite quantities, obtained by taking the corresponding values for each standing wave, weighted by its source coefficient B_n and its resonance denominator, and combining them in the proper manner. This will be done in subsequent papers.[1] The final result will depend as much on the shape of the room as on the nature of the absorbing material.

Calculation of the Damping Coefficients

Equations (3) indicate that the quantities $(\mu^2-\kappa^2)$ and $4\pi\mu\kappa$ are functions of the complex variables $(\eta/\zeta_m)=(2\nu l/c)(\rho c/Z_m)$, $(m=1,\ 2)$, which are proportional to the frequency and inversely proportional to the normal impedances of the two walls. It is also apparent that the two quantities for x depend only on the properties of the two x-walls; and likewise for y and for z; so that the calculations can be carried out for each of the three wall pairs separately, and the results combined at the finish.

The equations to be solved are each of the form

$$\pi j(\mu+j\kappa)+\coth^{-1}[(\zeta_1/\eta)(\mu+j\kappa)] +\coth^{-1}[(\zeta_2/\eta)(\mu+j\kappa)]=0.$$

The impedances Z_m are sometimes complex, with phase angles φ_m. In such a case we can write $\zeta_m=\gamma_m e^{i\varphi_m}$, where γ_m is the magnitude of $(Z_m/\rho c)$. Solutions of this equation can be obtained in the form of series in powers of the quantity (η/γ), useful for calculations for low frequencies, small rooms and large Z's (small absorption); or in series in powers of the reciprocal (γ/η), useful for high frequencies, large rooms and small Z's (large absorption). One computes a series for $(\mu+j\kappa)^2$ and then obtains series for $(\mu^2-\kappa^2)$ and $2\mu\kappa$ in terms of the frequency parameter η, and ζ_1 and ζ_2, the impedance parameters for the pair of walls under consideration. The intermediate range of the variables must be calculated by numerical or graphical methods.

The solutions are labeled with the number n, giving the number of pressure nodes parallel to the wall pair concerned, in the particular standing wave studied. The series solutions for different values of n, for the case where $(\eta/n\gamma)$ (or η/γ for $n=0$) is small compared to unity for both walls, the series are:

$n=0$

$$\begin{aligned}
\mu_0+j\kappa_0&=(1+j)[(\eta/2\pi\zeta_1\zeta_2)(\zeta_1+\zeta_2)]^{\frac{1}{2}}\\
&\quad+(1-j)[\omega\eta^3/72\zeta_1^3\zeta_2^3(\zeta_1+\zeta_2)]^{\frac{1}{2}}\\
&\qquad\times(\zeta_1^2-\zeta_1\zeta_2+\zeta_2^2)+\cdots,\\
\mu_0^2-\kappa_0^2&=\frac{\eta}{\pi}\left(\frac{1}{\gamma_1}\sin\varphi_1+\frac{1}{\gamma_2}\sin\varphi_2\right)\\
&\quad+\frac{\eta^2}{3}\left(\frac{1}{\gamma_1^2}\cos 2\varphi_1-\frac{1}{\gamma_1\gamma_2}\cos(\varphi_1+\varphi_2)\right.\\
&\qquad\left.+\frac{1}{\gamma_2^2}\cos 2\varphi_2\right)+\cdots,\\
\delta_{01}+\delta_{02}&=(4\pi\mu_0\kappa_0/\eta)\\
&=2\left(\frac{1}{\gamma_1}\cos\varphi_1+\frac{1}{\gamma_2}\cos\varphi_2\right)\\
&\quad-\frac{2\pi}{3}\left(\frac{1}{\gamma_1^2}\sin 2\varphi_1-\frac{1}{\gamma_1\gamma_2}\sin(\varphi_1+\varphi_2)\right.\\
&\qquad\left.+\frac{1}{\gamma_2^2}\sin 2\varphi_2\right)+\cdots,
\end{aligned}\tag{7}$$

$n>0$

$$\begin{aligned}
\mu_n+j\kappa_n&=n+\frac{j\eta}{\pi n}\left(\frac{1}{\zeta_1}+\frac{1}{\zeta_2}\right)\\
&\quad+\frac{\eta^2}{\pi^2 n^3\zeta_1^2\zeta_2^2}(\zeta_1+\zeta_2)^2+\cdots,\\
\mu_n^2-\kappa_n^2&=n^2+\frac{2\eta}{\pi}\left(\frac{1}{\gamma_1}\sin\varphi_1+\frac{1}{\gamma_2}\sin\varphi_2\right)\\
&\quad+\frac{\eta^2}{\pi^2n^2}\left(\frac{1}{\gamma_1^2}\cos 2\varphi_1+\frac{2}{\gamma_1\gamma_2}\cos(\varphi_1+\varphi_2)\right.\\
&\qquad\left.+\frac{1}{\gamma_2^2}\cos 2\varphi_2\right)+\cdots,\\
\delta_{n1}+\delta_{n2}&=4\left(\frac{1}{\gamma_1}\cos\varphi_1+\frac{1}{\gamma_2}\cos\varphi_2\right)\\
&\quad-\frac{2\eta}{\pi n^2}\left(\frac{1}{\gamma_1^2}\sin 2\varphi_1+\frac{2}{\gamma_1\gamma_2}\sin(\varphi_1+\varphi_2)\right.\\
&\qquad\left.+\frac{1}{\gamma_2^2}\sin 2\varphi_2\right)+\cdots.
\end{aligned}\tag{8}$$

The series for $(\eta/n\gamma_1)$ small and $(\eta/n\gamma_2)$ large are

$$\mu_n+j\kappa_n=n+\tfrac{1}{2}+\frac{j}{\pi}\left[\frac{\eta}{\zeta_1(n+\frac{1}{2})}+\frac{\zeta_2(n+\frac{1}{2})}{\eta}\right]+\frac{1}{\pi^2(n+\frac{1}{2})^2}\left[\frac{\eta^2}{\zeta_1^2(n+\frac{1}{2})^2}-\frac{\zeta_2^2(n+\frac{1}{2})^2}{\eta^2}\right]+\cdots,$$

$$\mu_n^2-\kappa_n^2=(n+\tfrac{1}{2})^2+\frac{2}{\pi}\left[\frac{\eta}{\gamma_1}\sin\varphi_1-(n+\tfrac{1}{2})^2\frac{\gamma_2}{\eta}\sin\varphi_2\right]+\frac{1}{\pi^2}\left[\frac{\eta^2\cos 2\varphi_1}{\gamma_1^2(n+\frac{1}{2})^2}-2\frac{\gamma_2}{\gamma_1}\cos(\varphi_2-\varphi_1)-3\frac{\gamma_2^2}{\eta^2}(n+\tfrac{1}{2})^2\cos 2\varphi_2\right]+\cdots, \tag{9}$$

$$\delta_{n1}+\delta_{n2}=4\left[\frac{1}{\gamma_1}\cos\varphi_1+(n+\tfrac{1}{2})^2\frac{\gamma_2}{\eta^2}\cos\varphi_2\right]-\frac{2}{\pi}\left[\frac{\eta\sin 2\varphi_1}{\gamma_1^2(n+\frac{1}{2})^2}+\frac{2\gamma_2}{\eta\gamma_1}\sin(\varphi_2-\varphi_1)+3\frac{\gamma_2^2}{\eta^3}(n+\tfrac{1}{2})^2\sin 2\varphi_2\right]+\cdots.$$

The series for both $(\eta/n\gamma_1)$ and $(\eta/n\gamma_2)$ large are

$$\mu_n+j\kappa_n=n+1+\frac{j}{\pi\eta}(n+1)(\zeta_1+\zeta_2)+\frac{n+1}{\pi^2\eta^2}(\zeta_1^2+2\zeta_1\zeta_2+\zeta_2^2)+\cdots,$$

$$\mu_n^2-\kappa_n^2=(n+1)^2-\frac{2}{\pi\eta}(n+1)^2(\gamma_1\sin\varphi_1+\gamma_2\sin\varphi_2)-\frac{3}{\pi^2\eta^2}(n+1)^2[\gamma_1^2\cos 2\varphi_1+2\gamma_1\gamma_2\cos(\varphi_1+\varphi_2)+\gamma_2^2\cos 2\varphi_2]+\cdots, \tag{10}$$

$$\delta_{n1}+\delta_{n2}=(4/\eta^2)(n+1)^2(\gamma_1\cos\varphi_1+\gamma_2\cos\varphi_2)-\frac{6}{\pi\eta^3}(n+1)^2[\gamma_1^2\sin 2\varphi_1+2\gamma_1\gamma_2\sin(\varphi_1+\varphi_2)+\gamma_2^2\sin 2\varphi_2]+\cdots.$$

In all these formulas, $\eta=(2\nu l/c)=(2l/\lambda)$ where l is the distance between the pair of walls under consideration. The normal acoustic impedance $Z_1=\rho c\zeta_1=\rho c\gamma_1 e^{j\varphi_1}$, and similarly for the second wall. To determine the distribution of the wave in the room, the frequency of the wave, its decay time and resonance response, one inserts these series results into Eqs. (2), (4) and (5). Contour plots of the damping constants $(\delta_1+\delta_2)$ for $\varphi_1=\varphi_2=0$, are given as functions of γ_1 and γ_2, for $n=0$ and $n=1$, in Fig. 1. We note that the damping constant has a maximum value, and that the coefficient for $n=0$ is smaller than that for $n=1$.

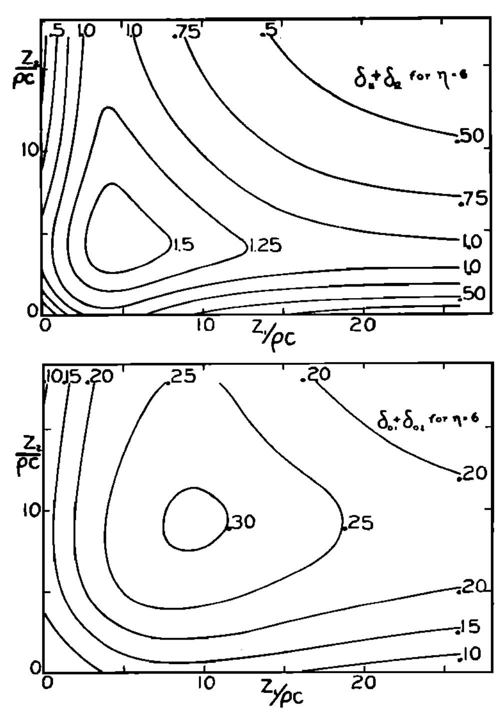

FIG. 1. Contours for the damping coefficients δ, for a normal mode, due to a pair of opposite walls having real impedances Z_1 and Z_2; for frequency parameter $\eta=6$. Values are given of the sum of the δ's for both walls, as functions of the impedance parameters γ_1 and γ_2. Lower contour plot is for "grazing incidence" waves; upper plot for waves with one pressure node parallel to walls.

When one of the walls in a pair is stiff (η/γ_1 very small), the formulas simplify, δ_1 and δ_2 can be separated, δ_1 being equal to $(4/\gamma_1)\cos\varphi_1$ (or to one-half this for $n=0$), and the series for δ_2 and $(\mu^2-\kappa^2)$ become:

(η/γ_2) small, $n=0$,

$$\mu_0{}^2-\kappa_0{}^2=\frac{\eta}{\pi\gamma_1}\sin\varphi_1+\left[\frac{\eta}{\pi\gamma_2}\sin\varphi_2+\frac{\eta^2}{3\gamma_2{}^2}\cos 2\varphi_2-\frac{4\pi}{45}\frac{\eta^3}{\gamma_2{}^3}\sin 3\varphi_2+\cdots\right]$$
$$\delta_2=\frac{2}{\gamma_2}\cos\varphi_2-\frac{2\pi}{3}\frac{\eta}{\gamma_2{}^2}\sin 2\varphi_2-\frac{8\pi^2}{45}\frac{\eta^2}{\gamma_2{}^3}\cos 3\varphi_2+\cdots, \tag{11}$$

$(\eta/n\gamma_2)$ small, $n>0$

$$\mu_n{}^2-\kappa_n{}^2=n^2+\frac{2\eta}{\pi\gamma_1}\sin\varphi_1+\left[\frac{2\eta}{\pi\gamma_2}\sin\varphi_2+\frac{1}{\pi^2n^2}\frac{\eta^2}{\gamma_2{}^2}\cos 2\varphi_2+\frac{2}{3\pi^3n^4}(\pi^2n^2-3)\frac{\eta^2}{\gamma_2{}^3}\sin 3\varphi_2+\cdots\right]$$
$$\delta_2=\frac{4}{\gamma_2}\cos\varphi_2-\frac{2}{\pi n^2}\frac{\eta}{\gamma_2{}^2}\sin 2\varphi_2+\frac{4}{3\pi^2n^4}(\pi^2n^2-3)\frac{\eta^2}{\gamma_2{}^3}\cos 3\varphi_2+\cdots, \tag{12}$$

$(\gamma_2/\eta)(n+\frac{1}{2})$ small

$$\mu_n{}^2-\kappa_n{}^2=(n+\tfrac{1}{2})^2+\frac{2\eta}{\pi\gamma_1}\sin\varphi_1-\left\{\frac{2}{\pi}(n+\tfrac{1}{2})^2\frac{\gamma_2}{\eta}\sin\varphi_2+\frac{3}{\pi^2}(n+\tfrac{1}{2})^2\frac{\gamma_2{}^2}{\eta^2}\cos 2\varphi_2+\frac{2}{3\pi^3(n+\frac{1}{2})^4}[\pi^2(n+\tfrac{1}{2})^2-6]\times\frac{\gamma_2{}^3}{\eta^3}\sin 3\varphi_2+\cdots\right\}, \tag{13}$$

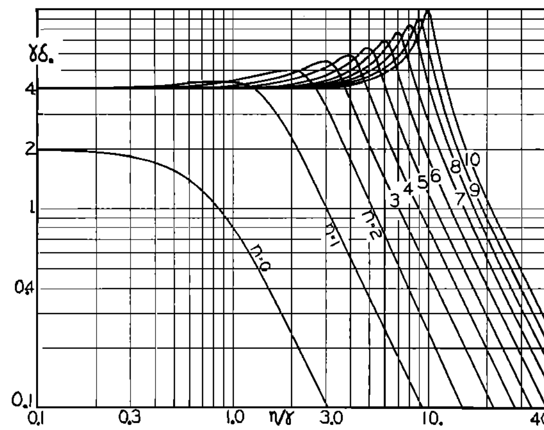

FIG. 2. Damping coefficient δ, for a normal mode, due to one wall when opposite wall is a poor absorber; plotted as function of the ratio of the frequency parameter $\eta=(2\nu l/c)$ to the impedance parameter $\gamma=(Z/\rho c)$, for real wall impedance Z. Curves are given for different values of n, the number of pressure nodes parallel to wall in the standing wave.

$$\delta_2=4(n+\tfrac{1}{2})^2\left(\frac{\gamma_2}{\eta^2}\right)\cos\varphi_2-\frac{6}{\pi}(n+\tfrac{1}{2})^2(\gamma_2{}^2/\eta^3)\sin 2\varphi_2+\frac{4}{2\pi^2}(n+\tfrac{1}{2})^2[\pi^2(n+\tfrac{1}{2})^2-6]\times(\gamma_2{}^3/\eta^4)\cos 3\varphi_2+\cdots.$$

Values of the damping constant δ_2 for this separable case are plotted as function of the variable (η/γ_2), for real values of ζ_2 in Fig. 2, and for

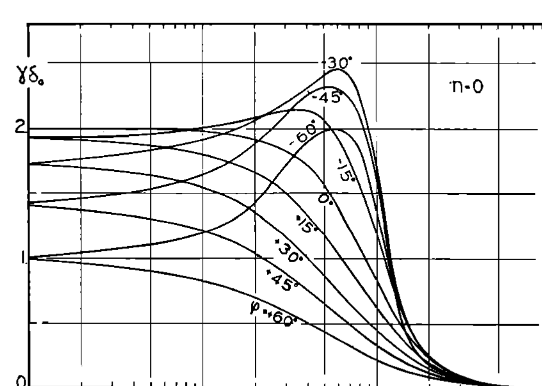

FIG. 3. Damping coefficients, for "grazing incidence" modes, due to a wall having impedance magnitude $\rho c\gamma$ and phase angle φ; plotted as function of (η/γ) for different values of φ.

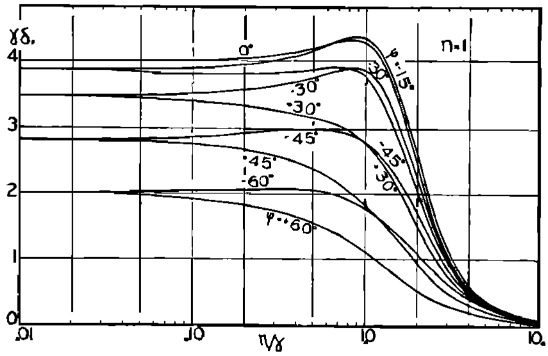

FIG. 4. Damping coefficients due to one wall for the normal modes having one pressure node parallel to the wall, as functions of (η/γ), for different values of φ.

different values of the impedance phase angle φ_2 in Figs. 3, 4 and 5. Figure 6 gives values[8] of $(\mu_n^2-\kappa_n^2)^{\frac{1}{2}}$ for φ_2 and $(1/\gamma_1)$ zero. From these the acoustic properties of most rooms can be calculated.

DISCUSSION

A number of conclusions of interest in acoustics can be obtained from these calculations. In the first place the damping coefficient for any wall pair for a given standing wave is an additive function of the constants for each separate wall only if one or both of the walls are not very absorbent (i.e., if γ is considerably larger than η). If both quantities (η/γ_1) and (η/γ_2) are large enough so that their squares cannot be neglected, the series is not additive, and the absorption of one wall affects that of the opposite one. Another way of stating this same fact is to say that the effect of having one wall live and one wall very absorbent may not be the same as the effect for both walls having an average value of the absorption.

Figure 1 shows some of these properties of the damping coefficients. When both γ_1 and γ_2 are very large (both opposite walls live) the δ's are small and separable. In this region the δ's for $n=0$ (the "grazing incidence" waves) are approximately equal to one-half the value of the δ's for n larger than zero. As γ_1 and γ_2 diminish, $(\delta_1+\delta_2)$ increases until it comes to a maximum. In this region the δ's are not separable, and the

[8] I am much indebted to Mr. N. B. Bhatt and Mr. R. L. Brown for valuable aid in the calculation of some of these curves.

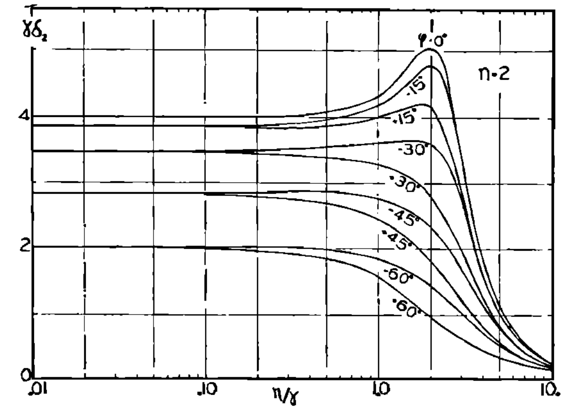

FIG. 5. Damping coefficients due to one wall for the normal modes having two pressure nodes parallel to the wall.

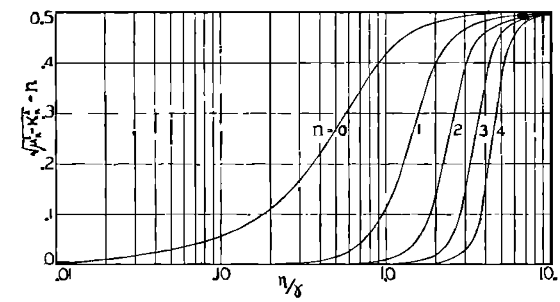

FIG. 6. Frequency coefficients $(\mu_n^2-\kappa_n^2)^{\frac{1}{2}}$, for a normal mode, due to one wall when opposite wall is a poor absorber, plotted as function of (η/γ), for real wall impedance $\rho c\gamma$. Curves are given for different values of n, the number of pressure nodes parallel to the wall, in the standing wave.

value of their sum for $n=0$ is considerably smaller than the values for $n>0$. If the γ's are decreased enough, the damping coefficients will again drop off in value. In most cases, there is an optimum set of values of γ_1 and γ_2 for maximum damping; making the walls either softer or harder will diminish the damping coefficients.

Another interesting point is brought out when we attempt to compare the free wave equation for damping coefficient

$$\delta_{\text{free}}=(1-R)\cos\theta \underset{\zeta\text{ real}}{=} (4/\gamma)\left(1+\frac{1}{\gamma\cos\theta}\right)^{-2},$$

obtained from Eq. (1), with those for the part of $(4\pi\mu_n\kappa_n/\eta)$ due to wall 2 in Eqs. (11), (12) and (13). In the first place, there is no angle of incidence θ in the latter formulas, since these are for standing waves whose angles are determined by the boundary conditions. The nearest approach to an angle of incidence is given by the equation

$$\cos\theta=\frac{1}{\eta}(\mu^2-\kappa^2)^{\frac{1}{2}}\simeq(n/\eta)$$

corresponding to the requirement that $\cos\theta$ is the ratio of the wave-length to twice the distance between nodal surfaces parallel to the wall in question.

For normal incidence Eq. (12) shows that η must be approximately equal to n, an integer; or, to put it another way, for any value of $\eta=(2\nu l/c)$ there is a maximum allowed value of n, n_{max}, which is not larger than η. For this nearly normal incidence the damping coefficient is approximately equal to $(4/\gamma_2)$ (for ζ real) which is the value required by the free wave formula when γ is large. More oblique angles are obtained by choosing values of n less than n_{max}, for the same value of η. Not all values of the angle of incidence can be obtained, however, for the boundary requirements specify that only integral values of n can occur. One cannot, therefore, plot a continuous curve of damping coefficient against angle of incidence for a given frequency; all that is possible is to obtain discrete values of δ_n for the various allowed values of $\cos\theta=(\mu^2-\kappa^2)^{\frac{1}{2}}/\eta$.

Figure 7 shows this behavior for two cases. The circles show the correct values of δ and the solid line the corresponding values for δ_{free}. The circles approach closer and closer to the lines as γ is increased. In every case, however, the correct value of δ is larger than the free wave would allow. In fact, for the case $\gamma=5$, the damping coefficient has a maximum value for $n=3$. There is some evidence that experimental data matches the circles better than the solid line.[9]

This excess over the free wave, uniform distribution value is another aspect of the distortion of the wave by the absorbing material. A moderately "stiff" wall tends to "pull" the sound wave toward it, causing the pressure amplitude for each standing wave to be somewhat larger near the wall than it is elsewhere. This tendency continues as the wall impedance decreases until a certain optimum impedance is reached, whose value depends on η and on n. Any further decrease in wall impedance will then cause the sound wave to recede from the wall; and eventually, for a very "soft" wall (or a very high frequency) the standing wave will have a pressure *node* at the wall instead of a loop. This case corresponds to that of an organ pipe with open end. These tendencies are illustrated in Fig. 8, where the amplitude of the factor dependent on x in the expression for p_n is plotted for $n=1$ against (x/l_x), for one perfectly hard wall at $x=0$ and one absorbing wall at $x=l_x$. When (η/γ) is small, the factor is nearly equal to a cosine curve, with pressure loops at the two walls and a node midway between. As (η/γ) increased, the loop at the absorbing wall first increases in amplitude and then diminishes, changing to a pressure node for (η/γ) large enough. The node, originally at the midpoint,

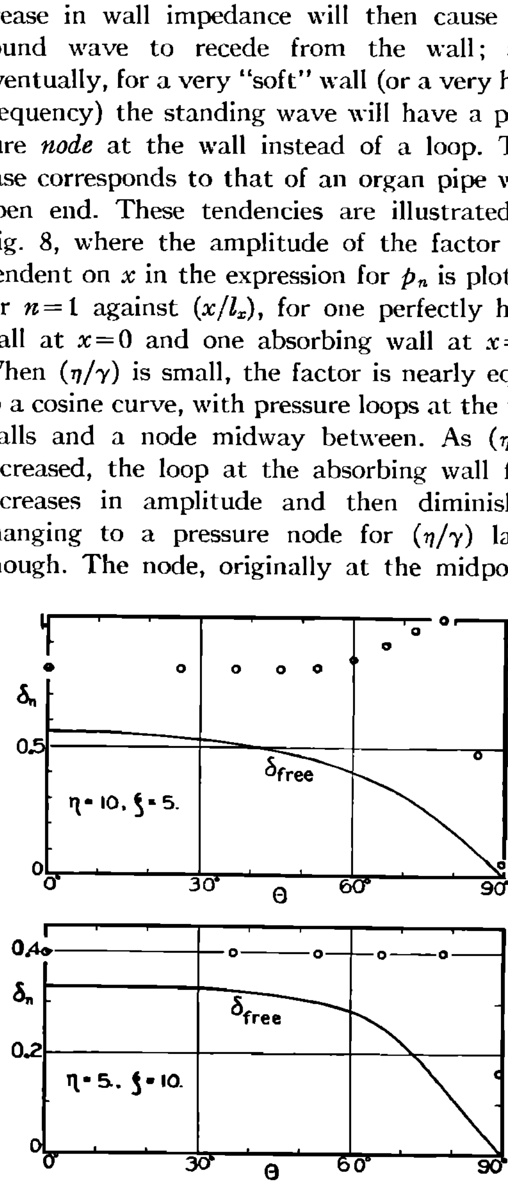

FIG. 7. Damping coefficients δ for standing waves, for two pairs of values of frequency parameter η and impedance parameter $\zeta=\gamma$ (for $\varphi=0$), as function of angle of incidence θ of wave on wall. Circles show allowed values of δ and of θ as given by exact theory. Solid line gives values obtained by making the approximation that the incident wave is an undistorted plane wave.

[9] The data given by F. V. Hunt, J. Acous. Soc. Am. **10**, 226 (1938) are plotted as $\delta/\cos\theta$, except for the case of $\theta=90°$, where δ is given. If the points in his Fig. 7 are multiplied by $\cos\theta$ except for the "grazing incidence" point, it will be seen that they follow the general trend of the circles in Fig. 7 of the present paper rather better than they do the solid curve. In fact the data seem to fit a set of circles for $\gamma=12$ quite satisfactorily, including the case of $\theta=90°$. Incidentally, it should be noted that the curve in Hunt's Fig. 9 for δ for grazing incidence is, within the accuracy of the data, just one-half as high as the curve for δ for normal incidence, given in his Fig. 6.

blurs out and moves away from the absorbing wall. The x term in the expression for the frequency of the standing wave starts out, for (η/γ) small, as that for a wave in a pipe closed at both ends; and ends up, for (η/γ) large, as that for a wave in a pipe closed at one end and open at the other.

Another manifestation of this alternate attraction of the wave toward, and then recession from the wall as (η/γ) is increased, is evident in the curves for δ shown in Fig. 2. The damping coefficient δ_n first increases to a maximum at (η/γ) roughly equal to n, and thereafter rapidly decreases as (η/γ) is further increased. The tendency to form a node at the wall can be considered to begin at $(\eta/\gamma)\simeq n$ for the nth wave.

Another very important point indicated in Fig. 6, and also in Eq. (11), is that the effective damping coefficient for "grazing incidence" waves (those for $n=0$) is *not* zero, as might be expected from the equation for δ_{free}. This does not mean that the equation for δ_{free} is wrong, but simply that it does not apply inside a room; for the angle of incidence θ for the waves for $n=0$ is *not* exactly 90°. No waves can be true grazing incidence waves in a finite room with absorbing walls. A little consideration will show why this must be so. A true grazing incidence wave has no component of air velocity normal to the wall. This contradicts our boundary conditions, for there is a pressure fluctuation at the wall, and since the wall has a finite normal impedance, there must be some normal velocity.

For low frequencies (more specifically, for small values of (η/γ), Eq. (11) shows that the damping coefficient for "grazing incidence" $(n=0)$ is just half that for normal incidence. This factor of one-half has been derived theoretically before by more approximate methods,[10] and has since been verified by several experimental measurements.[1, 9] The present calculations indicate that when the damping coefficient for normal incidence is less than 0.4 (i.e., γ is greater than 10), and η is less than 10, then all standing waves except the "grazing incidence" ones have damping coefficients equal to the normal coefficient, and the ones for $n=0$ have coefficients equal to one-half the normal coefficient.

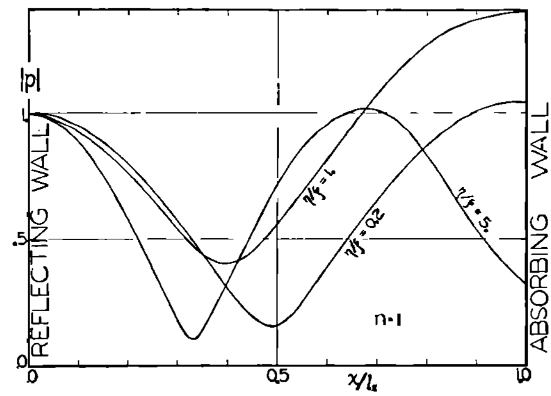

FIG. 8. Pressure amplitude of standing wave as function of distance x between two parallel walls of a room, for a mode having one pressure node perpendicular to x; when one wall is rigid and the other has an effective acoustical impedance $\rho c\zeta$ which is real. Curves show distortion of wave due to absorbing wall, for three different values of ζ.

When this is the case, the damping of any combination of standing waves in a room can be built up fairly easily out of a set of exponentials e^{-kt}. The exponential factor k for all waves which do *not* graze any wall (n_x, n_y, n_z all greater than zero) is obtained by using the normal damping coefficients for each wall in the expression

$$k=\frac{c}{4l_xl_yl_z}\left[(\delta_{1x}+\delta_{2x})l_yl_z+(\delta_{1y}+\delta_{2y})l_xl_z+(\delta_{1z}+\delta_{2z})l_xl_y\right]. \quad (14)$$

In these cases the δ's play the same role as the average absorption coefficients in Sabine's formula. However, the grazing incidence waves have a different exponential factor; those grazing the x-walls and not grazing the others will have a factor k similar to that given in Eq. (14), except that the term for the x-walls will be multiplied by $(\frac{1}{2})$; and so on. The pressure decay formula will have seven different exponential factors, and if one wall is somewhat softer than the other five, the decay times of several of the factors may be considerably longer than the others. We must therefore expect that pressure decay curves for a combination of several standing waves (excited by a warble-tone, for instance) *cannot* give true straight line plots on a decibel scale. Except in unusual cases, *only* individually ex-

[10] Reference 4, page 309. See also L. Brillouin, Rev. d'acoustique 5, 99 (1936).

cited standing waves will have straight line decay curves.

At higher frequencies, or for smaller γ's, the effective damping coefficient for $n=0$ decreases in value, approaching zero as (η/γ) becomes infinite (i.e., as the waves become effectively "free"). This is due to the receding of the pressure wave from the soft wall, as has been mentioned earlier. As (η/γ) increases, one after another of the standing waves (for larger and larger values of n) recedes from the wall, and its damping coefficient reduces in size. The standing wave whose angle of incidence is near zero ("near-normal" incidence) still have pressure loops near the wall and still are strongly absorbed. The waves whose angles are near grazing have pressure nodes near the wall and are poorly absorbed.

In such cases the decay curve for a combination of waves is quite complicated in form. The near-normal incidence waves will be very rapidly damped out, leaving the much more slowly decaying waves which "graze" the soft wall. In such cases the measured "reverberation time" for the combination will depend almost entirely on the decay of the "grazing incidence" waves, which are only slightly affected by the absorbing material. In certain cases, in fact, making one "soft" wall still softer will actually *increase* the effective "reverberation time" for the warble-tone sound.

This difference in distribution of sound energy, the strongly absorbed waves having large amplitudes near the soft wall, and the near-grazing waves having small amplitude there, may explain why the sound near a very absorbent wall decays more rapidly than the sound at some distance from the wall.

The analysis given in this paper seems to be adequate to explain the contemporary experimental results for the acoustics of rooms with uniform coverage on each wall. The results obtained enable one to calculate the acoustic properties of the standing waves in such a room in terms of the normal acoustic impedances of the wall materials; provided only that the impedances do not change with angle of incidence (i.e., provided the effective index of refraction for waves in each wall, parallel to the surface, is much larger than unity). Before the results can be applied to the usual practical problems of room acoustics, however, the damping constant for each standing wave must be combined to give an average absorption coefficient for all waves excited by a given source: and some simple method must be devised to calculate the dependence of this average absorption coefficient on the room arrangement and on the nature and position of the source. The analysis must also be extended to rooms with non-uniform coverage of absorbing material on the walls, where diffraction effects will enter.[11]

Both of these extensions of the theory are being attacked.

[11] P. M. Morse and P. Rubinstein, J. Acous. Soc. Am. **10**, 258 (1938); Phys. Rev. **54**, 895 (1938).