J Comput Electron (2008) 7: 47–51
DOI 10.1007/s10825-008-0250-2

Conservation laws—a simple application to the telegraph equation

Uwe Nordbrock · Reinhold Kienzler

Published online: 1 May 2008

Abstract Conservation laws are a recognized tool in physical and engineering sciences. The classical procedure to construct conservation laws makes use of Noether's Theorem. It requires the existence of a Lagrange-function for the system under consideration. Two unknown sets of functions have to be determined. A broader class of such laws is obtained, if Noether's Theorem is applied together with the Bessel-Hagen extension, raising the number of sets of unknown functions to three. The same conservation laws can be obtained by using the Neutral-Action method with the advantage that only one set of unknown functions is required. Moreover, the Neutral-Action method is also applicable in the absence of a Lagrangian, since for this procedure only the governing differential equations are needed. By this, the Neutral-Action method appears to be the most useful tool in constructing conservation laws for systems with dissipation.

The intention of this paper is to give a reference for further research on this topic rather than showing some detailed calculation on a special subject. Thus, the Neutral-Action method is introduced in detail and is applied to a simple example (the telegraph equation) to show the usefulness of this procedure for further applications in electronics.

Keywords Conservation laws · Telegraph equation · Neutral-Action method

U. Nordbrock (✉) · R. Kienzler
Department of Production Engineering, University of Bremen, IW3-FB4, PO Box 330 440, 28334 Bremen, Germany
e-mail: uwenord@uni-bremen.de

R. Kienzler
e-mail: rkienzler@uni-bremen.de

1 Introduction

When investigating mathematical problems, the use of conservation laws is quite common in physical and engineering sciences. In electrodynamics the Maxwell-equations (in integral form) can be interpreted as conservation laws. Kirchhoff's nodal and loop laws in electronics are conservation laws for currents and voltages, respectively. Recently, a broader class of these conservation laws, so-called material conservation laws, were established [1–3]. In the application to the field of piezoelectricity has been discussed. Especially, the literature on these conservation laws in electronics is sparse. One reason might be that it is not always possible to find a physical interpretation of the conservation laws established. But even though, conservation laws form a reliable tool for testing new calculation schemes and numerical programming. They are employed in the discussion of global existence theorems, stability of solutions and others [4]. The classical method to construct conservation laws is based on Noether's Theorem. However, it can only be applied in certain restricted cases. In her classical approach, Noether [5] assumed that a Lagrangian function is available for the system of interest. This assumption excludes the application of these methods to systems for which a Lagrangian function does not exist, due to, e.g., dissipation. By using the Neutral-Action method [6], this requirement can be disregarded, since a given set of governing partial differential equations is sufficient to construct conservation laws. It, therefore, follows that a systematic treatment of dissipative systems in terms of conservation laws becomes possible. But even if a Lagrangian function is available, it can be shown that the Neutral-Action method delivers (with considerably less effort) the same results as Noether's method in combination with Bessel-Hagen's extension [3, 7, 8].

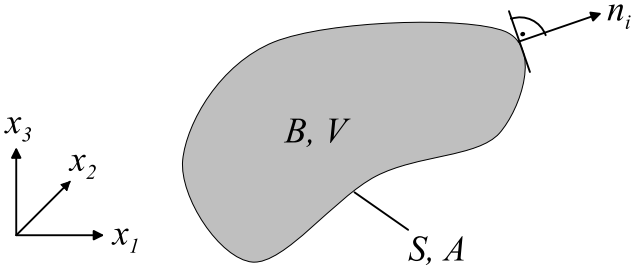

Fig. 1 Definition of conservation laws

2 Definition of conservation laws

A mechanical system is considered that is governed by a system of q differential equations

$$\Delta_\beta \left(x_i, v_\alpha, \frac{dv_\alpha}{dx_k} \right) = 0, \ \beta = 1, 2, \ldots, q \tag{1}$$

with $x_i \ i, k = 1, 2, \ldots, m$ (independent variables), $v_\alpha \ \alpha = 1, 2, \ldots, \mu$ (dependent variables), in which Δ_β denotes an operator acting on the independent and dependent variables, as well as on their derivatives, representing some differential equation. If any set of m associated functions

$$P_i, \quad i = 1, 2, \ldots, m \tag{2}$$

satisfies

$$\sum_{i=1}^{m} \frac{dP_i}{dx_i} = 0 \tag{3}$$

along solutions of (1), then (3) is denoted a conservation law with P_i being the conserved current. Equation (3) can be interpreted as a local formulation of a conservation law, because the divergence of the conserved current occurs.

A conservation law may also be written in integral form. Let B be a body (infinitesimal volume element dV), which is enclosed by a surface S (area element dA) and unit outward normal vector n_i (Fig. 1). By using the divergence theorem we can write

$$\int_B \sum_{i=1}^{m} \frac{dP_i}{dx_i} dV = \int_S \sum_{i=1}^{m} P_i n_i \, dA = 0 \tag{4}$$

leading to a conservation law in global form.

3 The Neutral-Action method

As mentioned above the methodology to establish conservation laws is different depending upon whether the system considered is Lagrangian or not. The classical way of constructing conservation laws for Lagrangian systems has been established by the mathematician Emmy Noether [5] and was extended by Bessel-Hagen [7] in (1921). It starts from the action integral defined as the integral of the Lagrangian over an arbitrary domain in the space of independent variables. Noether's theorem guarantees the existence of a conservation law, if transformations of the dependent and independent variables exist leaving the action integral invariant. Such transformations are called variational symmetries. A detailed elaboration on this topic can be found in [4] and [3]. The extension of Bessel-Hagen is based on the fact that the Lagrangian is not unique. It is rather possible to add a term which satisfies the Euler-Lagrange equations identically, thus leaving the equations of motion unchanged. Such an extension is also called "gauge function" or "Null-Lagrangian". By adding this "Null-Lagrangian" to the original Lagrange-function, further conservation laws can be derived. For systems without a Lagrangian, no procedure existed for a systematic construction of conservation laws, until the Neutral-Action method was advanced [6]. Most recently, this method has been applied to the subject of material or configurational mechanics [3] as well as for dynamics [9] and dispersive wave motion [10]. All what is required is the set of differential equations governing the system

$$\Delta_\beta \left(x_i, v_\alpha, \frac{dv_\alpha}{dx_k} \right) = 0. \tag{5}$$

First, the concept of the already mentioned "Null Lagrangian" will be introduced. In the following, let $E_\alpha()$ denote the Euler-operator (cf., e.g., [4])

$$E_\alpha() = \left[\frac{\partial}{\partial v_\alpha} - \sum_{i=1}^{m} \frac{d}{dx_i} \left(\frac{\partial}{\partial \frac{dv_\alpha}{dx_i}} \right) + \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{d^2}{dx_i dx_j} \left(\frac{\partial}{\partial \frac{d^2 v_\alpha}{dx_i \, dx_j}} \right) \right] (). \tag{6}$$

If a Lagrange function can be written as

$$\tilde{L} = \sum_{i=1}^{m} \frac{dF_i}{dx_i} \tag{7}$$

with

$$F_i = F_i(x_k, v_\alpha), \quad i, j, k = 1, \ldots, m \tag{8}$$

it follows that

$$\tilde{L} = \sum_{i=1}^{m} \frac{dF_i}{dx_i} \Longleftrightarrow E_\alpha(\tilde{L}) = 0, \tag{9}$$

i.e., this special Lagrange function satisfies the Euler-Lagrange equation identically. On the other hand, if the Euler-Lagrange equation is satisfied identically, then it follows that the associated Lagrange function is a divergence.

The proof is quiet simple. For convenience, we only prove this relationship for the first order Euler-Lagrange equation. It should be mentioned that F_i may also depend on the derivatives of the dependent variables $\frac{dv_\alpha}{dx_i}, \frac{d^2v_\alpha}{dx_i dx_j}, \ldots$. In this case the Euler-Lagrange equation has to be expanded [4]. Consider the Lagrangian $\tilde{L}$

$$\tilde{L}\left(x_i, v_\alpha, \frac{dv_\alpha}{dx_i}\right) = \sum_{i=1}^{m} \frac{dF_i(x_i, v_\alpha)}{dx_i} = \sum_{i=1}^{m} \frac{\partial F_i}{\partial x_i} + \sum_{i=1}^{m}\sum_{\alpha=1}^{\mu} \frac{\partial F_i}{\partial v_\alpha}\frac{dv_\alpha}{dx_i}. \tag{10}$$

The first order Euler-Lagrange equation is

$$\frac{\partial \tilde{L}}{\partial v_\alpha} - \sum_{i=1}^{m} \frac{d}{dx_i}\left(\frac{\partial \tilde{L}}{\partial \frac{dv_\alpha}{dx_i}}\right) = 0. \tag{11}$$

For the first term in (11) we find

$$\frac{\partial \tilde{L}}{\partial v_\alpha} = \frac{\partial}{\partial v_\alpha}\sum_{i=1}^{m}\frac{dF_i}{dx_i} = \sum_{i=1}^{m}\frac{\partial^2 F_i}{\partial v_\alpha \partial x_i} + \sum_{i=1}^{m}\sum_{\beta=1}^{\mu}\frac{\partial^2 F_i}{\partial v_\alpha \partial v_\beta}\frac{dv_\beta}{dx_i} \tag{12}$$

while for the second term it follows

$$\begin{aligned}\sum_{i=1}^{m}\frac{d}{dx_i}\left(\frac{\partial \tilde{L}}{\partial \frac{dv_\alpha}{dx_i}}\right) &= \sum_{i=1}^{m}\frac{d}{dx_i}\left(\frac{\partial}{\partial \frac{dv_\alpha}{dx_i}}\sum_{j=1}^{m}\frac{dF_j}{dx_j}\right)\\ &= \sum_{i=1}^{m}\frac{d}{dx_i}\left[\frac{\partial}{\partial \frac{dv_\alpha}{dx_i}}\left(\frac{\partial F_i}{\partial x_i} + \sum_{j=1}^{m}\sum_{\beta=1}^{\mu}\frac{\partial F_j}{\partial v_\beta}\frac{dv_\beta}{dx_j}\right)\right]\\ &= \sum_{i=1}^{m}\frac{d}{dx_i}\frac{\partial F_i}{\partial v_\alpha}\\ &= \sum_{i=1}^{m}\frac{\partial^2 F_i}{\partial x_i \partial v_\alpha} + \sum_{i=1}^{m}\sum_{\beta=1}^{\mu}\frac{\partial^2 F_i}{\partial v_\beta \partial v_\alpha}\frac{dv_\beta}{dx_i}.\end{aligned} \tag{13}$$

Because of permutability of second order derivatives, equations (12) and (13) yield the same result. Thus the Euler-Lagrange equation is satisfied identically. $\tilde{L}$ is called a "Null Lagrangian". Setting the variation of the action integral

$$\mathrm{A} = \int_B \tilde{L}\, dV \tag{14}$$

of such a Null Lagrangian to zero, one obtains

$$\delta\mathrm{A} = 0 \Longleftrightarrow E_\alpha(\tilde{L}) = 0, \tag{15}$$

where δA denotes the variation of the dependent variables. This means that the action integral A does not depend on the explicit functional form $g(x)$ inside the domain of integration but only on the values at the boundary of B. So the idea is to seek after characteristic functions f_β such that

$$\sum_{\beta=1}^{q} f_\beta \Delta_\beta = \sum_{i=1}^{m}\frac{dP_i}{dx_i}. \tag{16}$$

From (9) and (15), it follows

$$E_\alpha\left(\sum_{\beta=1}^{q} f_\beta \Delta_\beta\right) = E_\alpha\left(\sum_{i=1}^{m}\frac{dP_i}{dx_i}\right) = E_\alpha(\tilde{L}) = 0 \Longleftrightarrow \delta\mathrm{A} = 0 \tag{17}$$

with

$$\mathrm{A} = \int_B \tilde{L}\, dV = \int_B \sum_{\beta=1}^{q} f_\beta \Delta_\beta\, dV. \tag{18}$$

The characteristics f_β have to be determined from (17). The action integral behaves neutrally under its variation, so the formalism is called "Neutral-Action method".

4 Application to the telegraph-equation

The method is illustrated with a rather classical example of electronics. The wave equation with a damping term is a reasonable mathematical model for a variety of evolution processes in many areas of physics. One special application in electromagnetism is the so-called telegraph equation, in which positive damping occurs, which, in turn, corresponds to dissipation. Telegraph wires can be modeled as an electrical circuit, which consists of a resistor of resistance R and a coil of inductance L. Furthermore, it is supposed that current is getting lost from the wire to the ground, either through a resistor of conductance G or through a capacitor of capacitance C (Fig. 2).

The electric voltage $u(x, t)$ at position x and time t can be determined by solving the partial differential equation [11]

$$\frac{\partial^2 u}{\partial x^2} = RGu + (RC + LG)\frac{\partial u}{\partial t} + LC\frac{\partial^2 u}{\partial t^2} \tag{19}$$

which is referred to as the telegraph-equation. For the electric current, an analogous equation can be derived. If, for convenience, we set

$$()' = \frac{\partial}{\partial x}(),$$

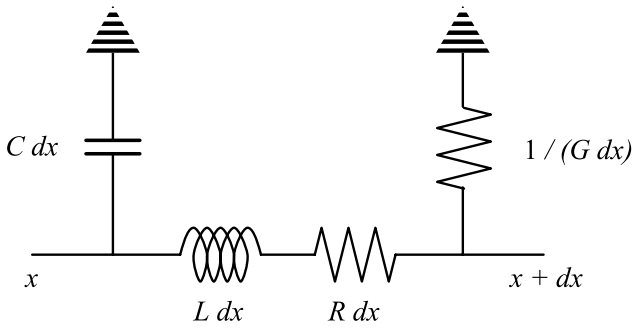

Fig. 2 Schematic of a telegraph wire

$$()^{\cdot} = \frac{\partial}{\partial t}()$$

and use the abbreviations

$$k_1 = LC,$$

$$k_2 = RC + LG,$$

$$k_3 = RG,$$

we can rewrite (19) as

$$\Delta = u'' - k_1\ddot{u} - k_2\dot{u} - k_3 u = 0. \tag{20}$$

In the following, we use this representation of the telegraph equation to discuss the construction of conservation laws in electronics.

5 Calculation of conservation laws

Since dissipation occurs in the telegraph equation, the classical methods in constructing conservation laws are not applicable. As mentioned above, the Neutral-Action method appears to be the most useful tool in constructing conservation laws for these kind of systems. The condition for the existence of a conservation law for the telegraph equation (20) requires the fulfillment of the equation

$$E(f\Delta) = \left[\frac{\partial}{\partial u} - \frac{d}{dx}\left(\frac{\partial}{\partial u'}\right) - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{u}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial}{\partial \ddot{u}}\right) + \frac{d^2}{dx^2}\left(\frac{\partial}{\partial u''}\right)\right](f\Delta) = 0. \tag{21}$$

In (21), the mixed second derivative of u is dropped, since (20) does not depend on $\dot{u}'$. Now, we have to specify the dependence of the characteristic f which we take to be

$$f = f(t, x). \tag{22}$$

A simple calculation involving equations (20)–(22) results in

$$E(f\Delta) = f'' - k_1\ddot{f} + k_2\dot{f} - k_3 f = 0. \tag{23}$$

For any f which satisfies (23), the associated conserved currents can be calculated using (16) with $q = 1$:

$$f\Delta = f(u'' - k_1\ddot{u} - k_2\dot{u} - k_3 u) = \frac{dP_t}{dt} + \frac{dP_x}{dx}. \tag{24}$$

The conserved currents are

$$P_t = k_2 fu - k_1(\dot{f}u - f\dot{u}) \tag{25}$$

and

$$P_x = f'u - fu' \tag{26}$$

which, in fact, fulfill $\dot{P}_t + P_x' = 0$. In particular, one such f can be found by the ansatz

$$f(t, x) = g(x)h(t) \tag{27}$$

leading to

$$g'' + K^2 g = 0 \tag{28}$$

and

$$\ddot{h} - \frac{k_2}{k_1}\dot{h} + \frac{k_3 + K^2}{k_1}h = 0 \tag{29}$$

with the solutions (for $K^2 > 0$)

$$g(x) = A\cos Kx + B\sin Kx \tag{30}$$

and

$$h(t) = Ce^{\lambda_1 t} + De^{\lambda_2 t} \tag{31}$$

with

$$\lambda_{1,2} = \frac{k_2 \pm \sqrt{k_2^2 - 4(K^2 + k_3)k_1}}{2k_1}.$$

Here A, B, C, D (only three beeing linearly independent) and K are arbitrary constants. The corresponding currents P_t and P_x are

$$P_t = \left[(k_2 u + k_1\dot{u})\left(Ce^{\lambda_1 t} + De^{\lambda_2 t}\right) - k_1\left(C\lambda_1 e^{\lambda_1 t} + D\lambda_2 e^{\lambda_2 t}\right)u\right] \times [A\cos Kx + B\sin Kx] \tag{32}$$

and

$$P_x = [K(-A\sin Kx + B\cos Kx)u - (A\cos Kx + B\sin Kx)u'\left]\left[Ce^{\lambda_1 t} + De^{\lambda_2 t}\right] \tag{33}$$

in which u is the solution of (20). Unfortunately an interpretation of this conservation law in electronical terms has not been found so far.

6 Conclusions

The classical procedure of constructing conservation laws is via Noether's theorem. It requires the existence of a Lagrangian for the system under consideration. Furthermore, this method demands the knowledge of infinitesimal transformations, which have to be calculated in a separate step. Further conservation laws can be obtained by using Bessel-Hagen's extension, since the equations of motion are left unchanged when a so called "gauge function" is added to the Lagrangian. This gauge functions have to be determined additionally. The same conservation laws as above can be obtained by using the Neutral-Action method, where only one set of unknown functions f_β have to be calculated. Moreover the Neutral-Action method can also be applied in the absence of a Lagrangian, since only the governing differential equations are required for this procedure. Thus it is possible to calculate conservation laws even for dissipative systems. A conservation law for such a dissipative system, the telegraph equation, has been derived with the Neutral-Action method. As regards the value und usefulness of conservation and balance laws in a general way, reference may be made to an evaluation of such laws by Olver [4]. It may suffice to mention here the applicability of conservation (and balance) laws in numerics. Being incorporated into various algorithms, the accuracy of the numerical results can be validated by checking whether or not the conservation laws are satisfied identically. If the equations are not satisfied, so-called spurious material nodal forces occur in finite-element calculations, which can be used to improve the finite-element mesh by shifting the nodes in such a way as to eliminate the spurious forces [12, 13]. It seems that with a systematic treatment even more conservation laws can be obtained for this problem. Studies along this line are in progress and will be dealt with in a forth-coming paper.

References

1. Maugin, G.A.: Material Inhomogeneities in Elasticity. Chapman & Hall, London (1993)
2. Gurtin, M.E.: Configurational Forces as Basic Concepts of Continuum Physics. Springer, New York (2000)
3. Kienzler, R., Herrmann, G.: Mechanics in Material Space. Springer, Berlin (2000)
4. Olver, P.J.: Applications of Lie Groups to Differential Equations. Springer, New York (1993)
5. Noether, E.: Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl. **2** (1918)
6. Honein, T., Chien, N., Herrmann, G.: Phys. Lett. A **155**, 1 (1991)
7. Bessel-Hagen, E.: Math. Ann. **84** (1921)
8. Chien, N., Honein, T., Herrmann, G.: Int. J. Solids Struct. **33** (1996)
9. Nordbrock, U., Kienzler, R.: J. Sound Vib. **259**, 4 (2003)
10. Herrmann, G., Kienzler, R.: Wave Motion **42** (2005)
11. Hinsch, H.: Elektronik. Springer, Berlin (1996)
12. Braun, M.: Proc. Est. Acad. Sci. Phys. Math. **46** (1997)
13. Müller, R., Maugin, G.A.: Comput. Mech. **29** (2002)