Journal of Sound and Vibration (1984) **93**(4), 473–480

WALL EFFECTS ON SOUND PROPAGATION IN TUBES

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(*Received* 12 *January* 1983)

Numerical solutions have been obtained for the exact equations describing the propagation of periodic axisymmetric waves in a rigid cylindrical tube. Results were obtained for air over a range of conditions corresponding to shear wave numbers ($s = R\sqrt{\rho\omega/\mu}$) from 0·2 to 5000 and reduced frequencies ($k = \omega R/a$) from 0·01 to 6. For conciseness and convenient application, the results for the attenuation and phase shift coefficients are given in the form of simple polynomials for the ranges $5 \leqslant s \leqslant 5000$ and $0{\cdot}01 \leqslant k \leqslant 6$. This range covers virtually all values of tube diameter and sound frequency likely to be met in practical situations that are consistent with a continuum gas model.

1. INTRODUCTION

The effect of thermoviscous action at the wall of a rigid circular tube in which there are small amplitude oscillations of a fluid is one of the classical problems of acoustics. The attenuation and dispersion resulting has been of great interest to scientists and engineers for the last century or so. Kirchhoff was the first to provide a complete solution to the problem. A convenient full description of this has been given by Rayleigh [1]. Kirchhoff's solution was, however, in the form of a complex transcendental equation which has been found difficult to interpret for practical situations. Much of the subsequent work on this subject has been aimed at providing simpler solutions that can be readily applied to practical applications. Many workers have developed approximate analytical solutions valid for limited ranges of tube dimensions, frequency or fluid properties. This work has been comprehensively reviewed and added to by Weston [2] and more recently by Tijdeman [3]. There have been a number of numerical solutions to the problem also, but only those by Shields *et al.* [4] and Tijdeman [3] have treated the full Kirchhoff equations.

All previously published solutions have provided results for only a limited range of conditions of engineering interest. For example, in the most complete solution so far [3], it was shown that the propagation constant Γ could be completely specified by an equation of the form $\Gamma(\gamma, \sigma, s, k) = 0$, where γ is the ratio of specific heats, σ the square root of the Prandtl number, s the shear wave number, k the reduced frequency, Γ', the real part of Γ corresponding to the attenuation per unit length along the tube and Γ'', the imaginary part of Γ corresponding to the phase-shift per unit length along the tube (a complete list of symbols is given in the Appendix). Tijdeman presented results obtained numerically for the following conditions: $k \ll 1$, $0{\cdot}2 \leqslant s \leqslant 100$; $0{\cdot}025\pi \leqslant k \leqslant 0{\cdot}50\pi$, $0{\cdot}2 \leqslant s \leqslant 100$. In physical terms, these ranges correspond to relatively low frequencies in tubes of relatively small diameter. Many practical applications correspond to much larger shear wave numbers and larger reduced frequencies.

In the present work then, Tijdeman's approach is extended to permit solution over the more useful range for practical purposes of $0{\cdot}01 \leqslant k \leqslant 6$, $0{\cdot}2 \leqslant s \leqslant 5000$. In common with previous solutions, the theoretical basis to the solution is Kirchhoff's formulation. This is presented in summary form in section 2 where the underlying assumptions are

0022-460X/84/080473+08 $03.00/0

critically discussed in terms of practical applications. The numerical solution is presented in section 3 where the extensions necessary to Tijdeman's approach to permit solution at high values of both shear wave number and reduced frequency are described. The numerical results are presented in section 4 and compared with previous solutions where these are valid. The numerical solutions involve the evaluation of ordinary and modified Bessel functions of the first kind of arbitrary complex argument in the process of solving Kirchhoff's complicated transcendental equation describing the problem. To facilitate simple calculation of attenuation effects in practical applications, curves have been fitted to the results of the numerical solution to the exact equations, and these are presented in section 5.

2. THEORETICAL BACKGROUND

Kirchhoff's formulation of the problem of sound propagation in tubes was based on the following assumptions: (1) homogeneous (continuum) medium; (2) small amplitude periodic disturbances (laminar flow equations); (3) infinitely long tube; (4) axisymmetric disturbance. With these assumptions, the basic fluid equations of momentum and mass continuity can be manipulated into the form [3]

$$\mathrm{i}Z\left(Z-\mathrm{i}\frac{s^2}{k^2}\right)^{-1/2}\left(\frac{1}{\chi_1}-\frac{1}{\chi_2}\right)\frac{\mathrm{J}_1(a_1)}{\mathrm{J}_0(a_1)}$$
$$+\left(\frac{\gamma k^2}{\sigma^2 s^2}-\mathrm{i}\frac{1}{\chi_1}\right)(Z-\chi_1)^{1/2}\frac{\mathrm{J}_1(a_2)}{\mathrm{J}_0(a_2)}-\left(\frac{\gamma k^2}{\sigma^2 s^2}-\mathrm{i}\frac{1}{\chi_2}\right)(Z-\chi_2)^{1/2}\frac{\mathrm{J}_1(a_3)}{\mathrm{J}_0(a_3)}=0, \tag{2.1}$$

where $a_1=k(Z-\mathrm{i}s^2/k^2)^{1/2}$, $a_2=k(Z-\chi_1)^{1/2}$ and $a_3=k(Z-\chi_2)^{1/2}$, and χ_1 and χ_2 are, respectively, the small and large roots of

$$1+\chi\left\{1+\mathrm{i}\frac{k^2}{s^2}\left(\frac{4}{3}+\frac{\gamma}{\sigma^2}\right)\right\}+\mathrm{i}\frac{\gamma k^2}{\sigma^2 s^2}\left(\frac{1}{\gamma}+\mathrm{i}\frac{4k^2}{3s^2}\right)\chi^2=0,$$

where $Z=\Gamma^2$, the square of the propagation constant, which is related to the general solution to the pressure fluctuations by

$$p=(A\,\mathrm{e}^{\Gamma\xi}+B\,\mathrm{e}^{-\Gamma\xi})\,\mathrm{e}^{\mathrm{i}\omega t}, \tag{2.2}$$

in which $\xi=\omega x/a$ is the non-dimensional axial co-ordinate. Both A and B are functions of radius, but only axial variations are of concern in this work. The problem is then reduced to finding the zero of equation (2.1) in terms of s and k for a given gas (γ and σ^2).

A clear understanding of the physical significance of both the shear wave number and the reduced frequency is necessary for the correct interpretation of the solution of equation (2.1). The definition of the reduced frequency can be rewritten so as to highlight its importance as a measure of the wavelength compared with the tube radius: i.e., $k=\omega R/a=2\pi R/\lambda$, where λ is the wavelength. The assumption of axisymmetry will only be guaranteed for frequencies less than the cut-off frequency for the first asymmetric cross mode excitation. For a rigid circular tube this cut-off frequency is given by [5] $f=0{\cdot}5861a/2R$, which in turn corresponds to a cut-off wavelength of $\lambda=3{\cdot}412R$, or, in terms of the reduced frequency, $k=1{\cdot}841$. Cross mode disturbances would quickly attenuate below this value. However, there are many practical situations where it is desirable to know propagation characteristics in tubes at values of k greater than this. For this reason it was decided to obtain solutions for values of k up to 6, which would cover most situations of practical interest. The results obtained would apply only to axisymmetric disturbances and as such would provide a lower bound for attenuation of the longitudinal disturbance in a tube. If transverse disturbances were present these would remove further energy from the longitudinal component. This extension to the range of k has been validated by measurements of sound attenuation in a tube with k ranging up to 4·1 [6]. There was excellent agreement between measured attenuation and that predicted from the solution of equation (2.1) provided propagation was planar.

A further limitation to the validity of the solution is that the medium is a continuum. The mean free path for a gas at normal temperature and pressure is of the order 10^{-7} m [7]. From the definitions of s and k then, the solution would only be valid when $(2\pi\nu/a)(s/k)^2 \gg 10^{-7}$, which for air at 273 K and 1 atm reduces to the requirement $s/k \gg 0{\cdot}5$. As a practical criterion, the results presented in the present work are limited to $s/k \geqslant 5$. Tijdeman [3] used a Newton–Raphson method to find the zero of equation (2.1) for air over the range of values of s and k given in section 1. Only about two-thirds of the results presented by Tijdeman conform to the above restriction. Here, results are presented for a much wider range of s and k consistent with the assumptions listed above, except that wavelengths shorter than the duct diameter are permitted for the reasons given above.

3. NUMERICAL ANALYSIS

Following Tijdeman, the zero of equation (2.1) was found by using an iterative Newton–Raphson technique. The algorithm used by Tijdeman to evalute the Bessel functions was developed by Simons [8] and relies upon an integral representation of the Bessel functions in which the trapezoidal rule is used. The accuracy of the answers given by this method decreases with increasing function argument for ordinary Bessel functions of order zero and one. Since evaluations for large values of s and k were of interest in this work, the Bessel functions were instead evaluated by using the method developed by Scarton [9] in which the ordinary Bessel function was first expressed in terms of the modified Bessel function. This in turn was expressed in terms of an asymptotic series expansion for relatively large function arguments while an ascending series expansion was employed for smaller arguments.

This procedure worked well with high precision provided the complex argument was small enough to avoid computational overflow when the series was evaluated. (For example, this overflow occurred for arguments of real part greater than 87 on a computer that accepted real numbers up to 10^{38}.) This limitation provided a severe restriction on the values of s and k that could be used. This drawback was overcome by introducing a scaling term in the asymptotic series evaluation so that, with this modification, Scarton's procedure could be used to evaluate Bessel functions of argument with real part of magnitude up to about twice the maximum integer representation permissible in the computer used.

4. RESULTS

Solutions for the propagation constant were obtained for air ($\gamma = 1{\cdot}4$, $\sigma^2 = 0{\cdot}71$) by using the method described in section 2. Both the attenuation coefficient and the phase shift coefficient agreed to at least the accuracy quoted by Tijdeman [3] over the limited range of s and k treated in that earlier work (see section 1).

As stated earlier, the lower limits of s and k for which solutions were obtained in this work were 0·2 and 0·01, respectively. Tijdeman [3] has given a detailed discussion of solutions at low values of s and k. With $k = 0{\cdot}01$, the propagation constant was indistinguishable from the "low reduced frequency" solution given by Tijdeman. Here, the main interest is in results at higher values for s and k. Hence, further discussion will be confined to results corresponding to values of s greater than 5.

With this modification to the lower limit for s, a more complete presentation of the results for the attenuation coefficient is given in Figure 1, which shows Γ' as a function of s and k. For comparison purposes, the well known and widely used approximate solution by Kirchhoff for wide tubes is also shown. The curve for $s/k = 5$ marks the limit of continuum theory and hence the validity of the solution.

The results show that the attentuation coefficient is more sensitive to the reduced frequency at lower values of shear wave number. For $s > 1000$, the solution is only weakly

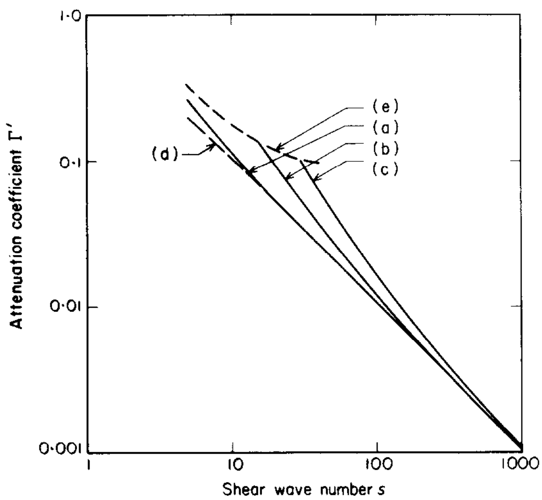

Figure 1. Attenuation coefficient *vs.* shear wave number. (a) $k \ll 1$; (b) $k = 3$; (c) $k = 6$; (d) Kirchhoff's approximate solution [3]; (e) continuum limit ($s/k = 5$).

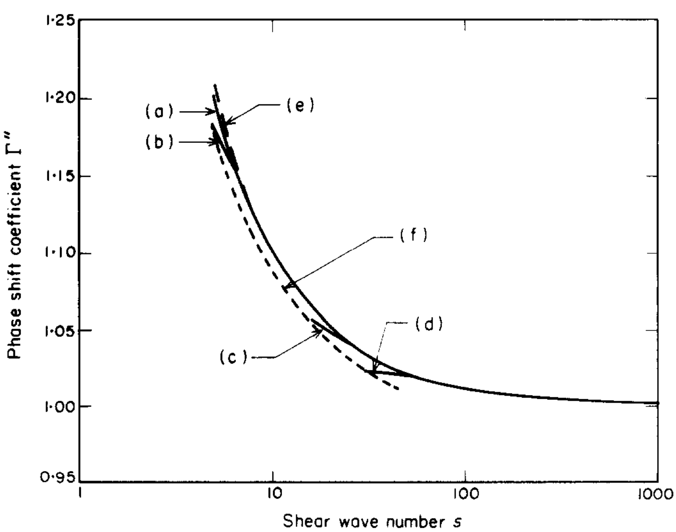

Figure 2. Phase shift coefficient *vs.* shear wave number. (a) $k \ll 1$; (b) $k = 1$; (c) $k = 3$, (d) $k = 6$; (e) Kirchhoff's approximate solution from [3]; (f) continuum limit ($s/k = 5$).

dependent on k. Kirchhoff's solution is a good approximation when $s > 40$ and k is small, but is inaccurate at other values of s and k.

In a similar fashion, results for the phase shift coefficient are shown in Figure 2. It can be seen that this coefficient shows a dependence on k at low values of s but is insensitive to the reduced frequency over most of the range of shear wave number considered: for $s > 100$, variations in the value of Γ'' for a given value of s are no more than one figure in the 5th decimal digit. It can also be seen that Kirchhoff's solution is a good approximation for $s > 100$ or, provided that k is small, for $s > 10$ but inaccurate at other values.

5. ALGEBRAIC APPROXIMATIONS

A convenient way of representing the numerical results described in section 4 is by means of algebraic relationships linking the important variables, of the kind given by the approximate analytical solutions mentioned in section 2. For this reason, the numerical results obtained for air ($\gamma = 1{\cdot}4$, $\sigma^2 = 0{\cdot}71$) were treated to curve fitting procedures to yield simple functional relations valid for a wide range of values of s and k.

The approximate analytical solutions showed that the attenuation constant (Γ') can be related to the shear wave number and reduced frequency through terms involvng $1/s$, $(1/s)^2$, $(1/s)^3$, and $(k/s)^2$, depending on the range of s and k of interest. This then provided a starting point for a least squares best fit to the data for the attenuation coefficient generated by using the (exact) numerical solution. However, numerical experimentation showed that a more complex dependence on k was present and good accuracy of representation was achieved by using an equation of the form

$$\Gamma' = A_1(1/s) + A_2(1/s)^2 + A_3(1/s)^3 + A_4(k/s)^2 + A_5(k/s)^4. \tag{5.1}$$

For agreement between this curve and the exact solution to be better than 1% over the entire range being considered, it was necessary to have two sets of coefficients. With the break point at $s = 35$ the coefficients obtained are shown in Table 1.

TABLE 1

Curve fit coefficients for Γ'

	A_1	A_2	A_3	A_4	A_5	Maximum error (%)
$5 \leq s < 35$	1·03973	1·09164	0·945891	1·58455	0·530622	0·62
$35 \leq s \leq 5000$	1·04117	1·26675	−4·74691	1·53507	2·35661	0·24

Equation (5.1) can therefore be used with air ($\gamma = 1{\cdot}4$, $\sigma^2 = 0{\cdot}71$) to predict the real part of the propagation constant to better than 1% accuracy.

A similar approach was tried for the imaginary part of the propagation constant (phase shift coefficient). The approximate analytical solutions were again reviewed in order to identify those terms in s and k whch would effect Γ'' but this process was more difficult than for Γ'. Not only were the terms in s of much more complex form but no approximate solution contained any term in k. However, the exact solution indicated a small dependence on k. In these circumstances more numerical experimentation was necessary to establish a satisfactory fit to the exact results. The outcome of this is

$$\Gamma'' = B_1 + B_2(1/s) + B_3(1/s)^2 + B_4(1/s)^3 + B_5(k/s)^3 + B_6(k/s)^4. \tag{5.2}$$

TABLE 2

Curve fit coefficients for Γ''

	B_1	B_2	B_3	B_4	B_5	B_6	Maximum error (%)
$5 \leq s < 35$	1·00055	1·02646	0·212165	−1·89499	−1·13760	−2·07118	0·11
$35 \leq s \leq 5000$	1·00000	1·04357	−0·206666	0·609034	−0·509937	−5·27103	0·01

Once again a break point was selected at $s = 35$ and the appropriate coefficients and accuracy of representation are shown in Table 2.

Values predicted by using equations (5.1) and (5.2) are compared with the exact solutions in Table 3. Also shown for comparison are several of the approximate analytical solutions shown by Tijdeman to cover the present range of interest. Specifically, these are the solutions of Weston [2] for the "Wide–Narrow" and "Wide–Very Wide" cases and the solution of Kirchoff. Equations (5.1) and (5.2) can be seen to have consistently high accuracy over the full range of s and k considered in contrast to the approximate solutions used for comparison which have limited ranges of accuracy.

TABLE 3

(*a*) *Comparison of attenuation coefficients* (Γ')

s	k	Exact	Equation (5.1)	Weston [2] wide–narrow	Kirchhoff wide	Weston [2] wide–very wide
5	0·01	0·25831	0·25919	0·26932	0·20856	0·20856
10	0·01	0·11584	0·11584	0·11728	0·10428	0·10428
20	0·01	0·05493	0·05483	0·05511	0·05214	0·05214
35	0·01	0·03069	0·03062	0·03073	0·02979	0·02979
50	0·01	0·02129	0·02129	0·02131	0·02086	0·02086
100	0·01	0·01054	0·01053	0·01054	0·01043	0·01043
500	0·01	0·00209	0·00209	0·00209	0·00209	0·00209
1000	0·01	0·00104	0·00104	0·00104	0·00104	0·00104
2000	0·01	0·00052	0·00052	0·00052	0·00052	0·00052
5000	0·01	0·00021	0·00021	0·00021	0·00021	0·00021
20	3·0	0·09042	0·09075	0·05511	0·05214	0·08571
35	3·0	0·04205	0·04229	0·03073	0·02979	0·04076
50	3·0	0·02680	0·02685	0·02131	0·02086	0·02623
100	3·0	0·01190	0·01192	0·01054	0·01043	0·01177
500	3·0	0·00214	0·00214	0·00209	0·00209	0·00214
1000	3·0	0·00106	0·00106	0·00104	0·00104	0·00106
2000	3·0	0·00053	0·00052	0·00052	0·00052	0·00052
5000	3·0	0·00021	0·00021	0·00021	0·00021	0·00021
35	6·0	0·07773	0·07764	0·03073	0·02979	0·07364
50	6·0	0·04399	0·04389	0·02131	0·02086	0·04234
100	6·0	0·01607	0·01609	0·01054	0·01043	0·01580
500	6·0	0·00231	0·00231	0·00209	0·00209	0·00230
1000	6·0	0·00110	0·00110	0·00104	0·00104	0·00110
2000	6·0	0·00054	0·00053	0·00052	0·00052	0·00053
5000	6·0	0·00021	0·00021	0·00021	0·00021	0·00021

(*b*) *Comparison of phase shift coefficients* (Γ'')

s	k	Exact	Equation (5.2)	Weston [2] wide–narrow	Kirchhoff wide	Weston [2] wide–very wide
5	0·01	1·20121	1·19917	1·23714	1·20856	1·26351
10	0·01	1·10349	1·10342	1·10709	1·10428	1·11642
20	0·01	1·05204	1·05217	1·05245	1·05214	1·05501
35	0·01	1·02978	1·03001	1·02985	1·02979	1·03071
50	0·01	1·02085	1·02079	1·02087	1·02086	1·02130
100	0·01	1·01043	1·01042	1·01043	1·01043	1·01054
500	0·01	1·00209	1·00209	1·00209	1·00209	1·00209
1000	0·01	1·00104	1·00104	1·00104	1·00104	1·00104
2000	0·01	1·00052	1·00052	1·00052	1·00052	1·00052
5000	0·01	1·00021	1·00021	1·00021	1·00021	1·00021
20	3·0	1·04799	1·04728	1·05245	1·05214	1·05501
35	3·0	1·02917	1·02918	1·02985	1·02979	1·03071
50	3·0	1·02066	1·02062	1·02087	1·02086	1·02130
100	3·0	1·01041	1·01040	1·01043	1·01043	1·01054
500	3·0	1·00209	1·00209	1·00209	1·00209	1·00209
1000	3·0	1·00104	1·00104	1·00104	1·00104	1·00104
2000	3·0	1·00052	1·00052	1·00052	1·00052	1·00052
5000	3·0	1·00021	1·00021	1·00021	1·00021	1·00021
35	6·0	1·02268	1·02249	1·02985	1·02979	1·03071
50	6·0	1·01878	1·01882	1·02087	1·02086	1·02130
100	6·0	1·01022	1·01024	1·01043	1·01043	1·01054
500	6·0	1·00208	1·00209	1·00209	1·00209	1·00209
1000	6·0	1·00104	1·00104	1·00104	1·00104	1·00104
2000	6·0	1·00052	1·00052	1·00052	1·00052	1·00052
5000	6·0	1·00021	1·00021	1·00021	1·00021	1·00021

6. CONCLUSIONS

The exact equations describing the propagation of periodic axisymmetric waves in a rigid tube have been solved numerically. The solutions were shown to be valid for $s/k > 5$ and for unlimited values of k provided axial symmetry is maintained. Results were obtained for air with values of shear wave number up to 5000 and reduced frequency up to 6. Over most of this range the attenuation coefficient was found to be sensitive to the reduced frequency. The phase shift coefficient was found to be weakly dependent on the reduced frequency. The results for the attenuation and phase shift coefficients can be represented by simple polynomials which predict values to better than 0·7% accuracy over a wide range of conditions of practical interest.

REFERENCES

1. LORD RAYLEIGH 1896 *Theory of Sound Volume II.* London: Macmillan, second edition. See pp. 319–326.
2. D. E. WESTON 1953 *Proceedings of the Physical Society London* **B66**, 695–709. The theory of the propagation of plane sound waves in tubes.
3. H. TIJDEMAN 1975 *Journal of Sound and Vibration* **39**, 1–33. On the propagation of sound waves in cylindrical tubes.
4. F. D. SHIELDS, K. P. LEE and W. J. WILEY 1965 *Journal of the Acoustical Society of America* **37**, 724–729. Numerical solution for sound velocity and absorption in cylindrical tubes.

5. P. M. MORSE and K. U. INGARD 1968 *Theoretical Acoustics.* New York: McGraw-Hill. See pp. 509–514.
6. D. J. MEE 1982 *B. E. Thesis, University of Queensland.* Sound attenuation in cylindrical tubes.
7. S. CHAPMAN and T. G. COWLING 1960 *The Mathematical Theory of Non-Uniform Gases.* Cambridge University Press. See pp. 91–92.
8. J. L. SIMONS 1972 *NLR Report MP* 72018*U.* On the calculation of Bessel functions of the first kind of integer order for complex arguments.
9. H. A. SCARTON 1971 *Journal of Computational Physics* **8**, 295–299. Double precision Fortran subroutines to compute both ordinary and modified Bessel functions of the first kind and of integer order with arbitrary complex argument: $J(x+iy)$, $I(x+iy)$.

APPENDIX: LIST OF SYMBOLS

a	undisturbed speed of sound
f	frequency
J_n	Bessel function of the first kind of order n
k	$=\omega R/a$, reduced frequency
p	pressure amplitude of acoustic signal
R	tube radius
s	$=R\sqrt{\rho\omega/\mu}$, shear wave number
t	time
x	axial coordinate
Z	$=\Gamma^2$, square of propagation constant
γ	ratio of specific heats
Γ	$=\Gamma'+i\Gamma''$, propagation constant
Γ'	attenuation per unit ξ
Γ''	phase shift per unit ξ
λ	wavelength
μ	fluid viscosity
ν	kinematic viscosity
ξ	$=\omega x/a$, dimensionless axial co-ordinate
ρ	mean density
σ	square root of Prandtl number
ω	radian frequency