The effect of wall elasticity on the properties of a Helmholtz resonator

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(Received 14 August 1990; accepted for publication 28 March 1991)

The effect of compliant walls on the properties of a Helmholtz resonator is examined. The effective stiffness of the resonator is decreased by the wall compliance, while the effective mass is unchanged to leading order. The radiation resistance is also decreased due to a cancellation between the radiation from the cavity opening and the radiation from the cavity walls. This leads to a reduction in the cross section at resonance by the factor $[Z_s/(Z_s + Z_f^0)]^4$, where Z_s is the wall impedance and Z_f^0 is the total fluid loading on the breathing mode of the cavity.

PACS numbers: 43.20.Ks, 43.20.Tb, 43.20.Rz, 43.55.Ev

LIST OF SYMBOLS

a	Cavity radius
ρ	Density of the fluid
c	Acoustic wave speed
k	Acoustic wave number
A_h	Hole area
θ_0	Polar angle subtended by the hole
Ω_h	Solid angle subtended by the hole
Ω_s	Solid angle subtended by the shell
Z_s	Shell impedance
Z_h	Effective hole impedance
Z_f^l	$\rho c(ka)^{-2}[j_l'(ka)h_l'(ka)]^{-1}$
Z_k	$\xi_0 Z_f^0$
Z_m	$\Sigma_{l=1}^{\infty} Z_f^l \xi_l^2/\xi_0$
Z_t	$Z_s + Z_f^0 - Z_k - \xi_0(1-\xi_0)^{-1}Z_m$
Z_{eff}	$Z_h + Z_k(Z_s/Z_t) + Z_m[Z_s + Z_f^0/(1-\xi_0)]/Z_t$
ξ_l	$(2l+1)^{-1/2}\frac{1}{2}[P_{l-1}(\cos\theta_0) - P_{l+1}(\cos\theta_0)]$
p_{eff}	$P_{\text{rig}}^0[Z_s + Z_m/(1-\xi_0)]/Z_t + \delta p(Z_s + Z_f^0)/Z_t$
p_{rig}	Field in the presence of rigid sphere
δp	$\Sigma_{l=1}^{\infty} p_{\text{rig}}^{l0}\xi_l/\xi_0$
$H(x)$	Heaviside function
P_l	Legendre polynomial
Y_{lm}	Spherical harmonic
$j_l(h_l)$	Spherical Bessel(Hankel) function[1]

INTRODUCTION

The use of Helmholtz resonators as sound absorbers in air has been studied in great detail.[2–5] Nevertheless, the effects of wall elasticity on the properties of a Helmholtz resonator have not been reported. Apparently, such resonators have been primarily considered for in air applications and the effects of wall elasticity would typically be unimportant.

Near the fundamental mode, the dominant forces on the cavity walls are due to a buildup of pressure inside the enclosure; the motion of most interest is therefore the breathing mode. As a simple model to examine the leading-order effects of the wall elasticity, consider a spherical cavity with a uniform radial velocity determined by the average pressure difference across the cavity wall,

$$Z_s v_s = \int_{\Omega_s} \frac{d\Omega}{\Omega_s}(p_{\text{in}} - p_{\text{out}}). \tag{1}$$

The geometry is shown in Fig. 1. The external pressure field p_i is incident on the resonator. The fields p_{in} and p_{out} are the pressure inside and outside the enclosure, respectively. The hole is located in the region $\theta < \theta_0$ in spherical coordinates. We assume that the shell impedance is stiffnesslike as appropriate for a shell below the ring frequency.[7]

The velocity of the fluid at the hole is assumed equal to a constant v_h, and to satisfy,

$$Z_h v_h = \int_{\Omega_h} \frac{d\Omega}{\Omega_h}(p_{\text{in}} - p_{\text{out}}). \tag{2}$$

Determining the mean fluid velocity in this way actually gives somewhat better results in air than the physically attractive choice[3] $v_h \propto (\cos\theta - \cos\theta_0)^{-1/2}$ containing the appropriate singularity at the edge of the hole.[4] The imped-

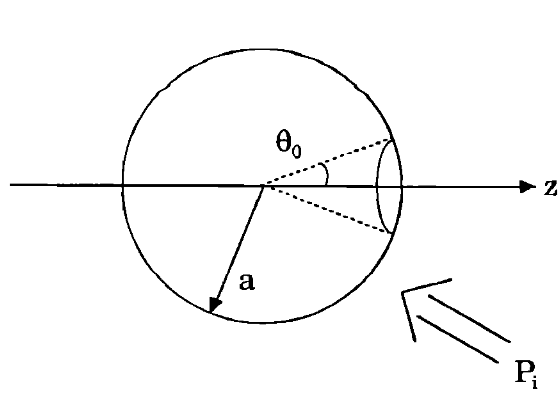

FIG. 1. The prototype Helmholtz resonator.

ance Z_h is the effective impedance of the hole, due either to viscosity or the presence of a porous material at the hole entrance.

The pressure fields inside and outside the sphere are determined by the wave equation and the boundary condition on the surface of the sphere,

$$\frac{\partial p}{\partial r} = i\rho\omega[v_s + (v_h - v_s)H(\theta_0 - \theta)]. \tag{3}$$

The interior and exterior fields can be written as

$$p_{\text{in}} = i\rho c \sum v_{lm} \left(\frac{j_l(kr)}{j_l'(ka)}\right) Y_{lm}(\Omega)$$

and

$$p_{\text{out}} = p_{\text{rig}} + i\rho c \sum v_{lm} \left(\frac{h_l(kr)}{h_l'(ka)}\right) Y_{lm}(\Omega). \tag{4}$$

The quantity p_{rig} is the total pressure field as if the entire sphere including the fluid at the hole location were rigid. This decomposition simply uses the superposition principle. It is convenient to define the total fluid impedance Z_f^l such that the net pressure difference across the sphere at $r = a$ is given by

$$p_{\text{out}} - p_{\text{in}} = p_{\text{rig}} + \sum Z_f^l v_{lm} Y_{lm}(\Omega). \tag{5}$$

I. THE MODIFIED RESONATOR PARAMETERS

Using Eq. (5) to eliminate the pressure difference from the equations of motion [Eqs. (1) and (2)], one obtains two algebraic equations for the fluid velocity at the hole and the shell velocity. The solution is most easily expressed in terms of various impedance combinations in addition to the shell and hole impedance; the stiffnesslike and masslike fluid impedance of the rigid resonator, Z_k and Z_m, the effective total shell impedance Z_t, and the $l = 0$ fluid impedance Z_f^0. In addition, define $\xi_0 = \Omega_h/4\pi$ to be the ratio of the hole area to the total surface area of the sphere; ξ_0 is typically a small parameter.

The stiffnesslike impedance, $Z_k = \xi_0 Z_f^0$ is the $l = 0$ component of the fluid impedance for the rigid resonator and contains the resistive radiation loading as well as a small piece of the mass loading. The total impedance to breathing motion of the shell,

$$Z_t = Z_s + Z_f^0 - Z_k - \xi_0(1 - \xi_0)^{-1} Z_m,$$

is reduced somewhat by the presence of the hole. For a small hole, we may neglect the term involving Z_m and the main effect of the hole is to reduce the fluid loading in a natural way by the amount $(1 - \xi_0)$.

The solution for the hole velocity is

$$Z_{\text{eff}} v_h = -p_{\text{eff}}, \tag{6}$$

where the effective impedance Z_{eff} is defined as

$$Z_{\text{eff}} = Z_h + Z_k(Z_s/Z_t) + Z_m[Z_s + Z_f^0/(1 - \xi_0)]/Z_t \tag{7}$$

and the effective driving pressure is

$$p_{\text{eff}} = p_{\text{rig}}^0 [Z_s + Z_m/(1 - \xi_0)]/ Z_t + \delta p(Z_s + Z_f^0)/Z_t. \tag{8}$$

The quantity δp is the contribution to the excitation due to modes with $l \geqslant 1$.

The resonator stiffness and the radiation resistance, both contained in Z_k, are reduced by the ratio (Z_s/Z_t). For a small hole, $Z_t \approx [Z_s + Z_f^0(1 - \xi_0)]$ and the stiffness of the internal fluid and the shell are intuitively combined as two mechanical elements in series. The radiation resistance is reduced because radiation from the hole and the accompanying motion of the shell tend to cancel.

The effective drive on the resonator is also reduced. For the case of small ka such that only the $l = 0$ term is important, the reduction is by the factor $[Z_s + Z_m/(1 - \xi_0)]/Z_t \underset{\xi_0 \to 0}{\to} Z_s/Z_t$; to leading order the same factor as for both the effective stiffness and the radiation resistance.

The effect of the elasticity on the effective resonator mass is in general more complicated. Let the mass due to the $l = 0$ mode be m_0, so that Z_k gives the contribution $-i\omega m_0$ and $Z_m = -i\omega(m - m_0)$. The effective mass is

$$m_{\text{eff}} = m_0(Z_s/Z_t) + (m - m_0)[Z_s + Z_f^0/(1 - \xi_0)]/Z_t. \tag{9}$$

The first term above, which tends to decrease the resonator mass, is due to the stiffness of the shell that opposes the motion of the fluid through the hole. The second term is due to the following mechanism: Flow out of the hole decreases the pressure in the cavity causing the shell to contract, but at the same time the exiting flow exerts a small reaction force driving the shell to contract further. This additional force, driving the shell to contract more than it otherwise would, mimics an inertial effect. For a small hole, $\xi_0 \leqslant 1$, and small ka we may make the above result explicit by using[4] $m_0/m \approx 9\pi\xi_0^{1/2}/80$,

$$m_{\text{eff}} \approx m[1 - Z_f^0/Z_t(9\pi\xi_0^{1/2}/80 - 2\xi_0) + O(\xi_0^{3/2})]. \tag{10}$$

For a very small hole the resonator mass is decreased, but for $\xi_0^{1/2} > 9\pi/160$, the effective inertia of the resonator is instead increased.

The resonance frequency is thus lowered by the compliance of the walls, to leading order by the factor $(Z_s/Z_t)^{1/2}$. The Q of the resonator is (again to leading order)

$$Q = \omega_0 m/(r_{\text{rad}} + r_i) \to \omega_0 m(Z_s/Z_t)^{1/2}/[r_{\text{rad}}(Z_s/Z_t) + r_i];$$

i.e., if radiation loading is the dominant source of damping the Q will be increased while if internal damping is dominant the Q is decreased. Above, r_{rad} and r_i are the radiation resistance and internal resistance respectively. How can one compensate for the compliance of the walls in the design of the resonator? The classic Helmholtz resonator result is[4]

$$\omega_0 \approx c(A_h^{1/2}/V)^{1/2},$$

where A_h is the area of the hole and V is the resonator volume. One may therefore either increase the area of the hole or decrease the volume, both increasing the stiffness of the cavity. An increase in hole area increases the radiation resistance relative to the mass and correspondingly decreases the Q.

II. SCATTERING

The total scattered field is produced by both flow through the hole and by the motion of the shell. The velocity of the shell is given by

$$Z_t v_s^0 = -p_{\rm rig}^0 (Z_h + Z_m)/Z_{\rm eff} + \delta p/(1-\xi_0)(Z_k + \xi_0 Z_h)/Z_{\rm eff}. \quad (11)$$

Assuming that $(Z_h + Z_m)$ is masslike and that the total shell impedance is stiffness like, one can see that the motion of the shell always opposes the motion of the fluid at the hole. The net volume outflow, $v^0 = v_s^0(1-\xi_0) + v_h^0 \xi_0$, is

$$v^0 = -p_{\rm rig}^0/Z_t + \xi_0 p_{\rm rig}^0/Z_{\rm eff}(Z_s/Z_t)^2 + O(\xi_0^2). \quad (12)$$

The first term is the contribution of the shell, almost independent of the presence of the hole. The second term yields the scattering from the Helmholtz resonance. Notice that this term has been reduced from the value expected from Eq. (7) by an additional factor of Z_s/Z_t; this is the result of a leading-order cancellation between the radiation from the hole and the radiation from the shell. It is interesting to note that if the shell impedance were masslike and substantial, the radiation of the shell could add in phase with the radiation from the hole and lead to amplification.

The far-field pressure near resonance (omitting the background field scattered by the shell) is

$$p_{\rm res} \approx ikA_h \frac{\exp(ikr)}{4\pi r} \frac{\rho c}{Z_{\rm eff}} \left(\frac{Z_s}{Z_t}\right)^2, \quad (13)$$

and the total cross section at resonance is thus reduced to,

$$\sigma_{\rm eff} = \sigma_0 \left(\frac{Z_s}{Z_t}\right)^4 = \frac{4\pi}{k^2} r_{\rm rad}(r_{\rm rad} + r_i)^{-1} \left(\frac{Z_s}{Z_t}\right)^4. \quad (14)$$

Evidently, one must have fairly stiff walls relative to the bulk stiffness of the internal fluid in order to observe the resonance in a scattering experiment.

ACKNOWLEDGMENT

I would like to thank Dr. M. Sevik for suggesting the idea of considering underwater Helmholtz resonators.

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).
[2] U. Ingard, "On the theory and design of acoustic resonators," J. Acoust. Soc. Am. **25**, 1037–1061 (1953).
[3] P. Morse and U. Ingard, *Theoretical Acoustics* (Princeton U.P., Princeton, NJ, 1968).
[4] U. Ingard, "The near field of a Helmholtz resonator exposed to a plane wave," J. Acoust. Soc. Am. **25**, 1062–1067 (1953).
[5] R. F. Lambert, "A study of the factors influencing the damping of an acoustical cavity resonator," J. Acoust. Soc. Am. **25**, 1068–1083 (1953).
[6] T. A. Henriquez and A. M. Young, "The Helmholtz resonator as a high power deep-submergence source for frequencies below 500 Hz," J. Acoust. Soc. Am. **67**, 1555–1558 (1980).
[7] D. Feit and M. Junger, *Sound, Structures, and Their Interaction* (MIT Cambridge, MA, 1972).