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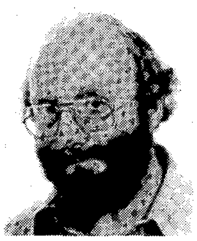

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Synthesis of Lumped-Distributed Cascades with Lossy Transmission Lines

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***Abstract*—In the analysis of large systems such as high-speed digital computing networks and circuits on an LSI or VLSI silicon chip, lossy lumped-distributed networks have been used to model their interconnections. A solution of the synthesis problem for these networks will aid in the *design* of these circuits.**

This paper establishes single-variable realizability conditions and synthesis procedures for the class of lossy lumped-distributed cascade networks which contain lossy transmission lines and are described by a driving point impedance expression of the form

$$Z_0 = \frac{\sum_{i=0}^{n} a_i(s, z_0)\, e^{(2i-n)T_0\gamma(s)}}{\sum_{i=0}^{n} b_i(s, z_0)\, e^{(2i-n)T_0\gamma(s)}}$$

where $a_i(s, z_0)$, $b_i(s, z_0)$ are two-variable, real polynomials in s and z_0, with z_0 the characteristic impedance, $\gamma(s)$ the progagation constant, and T_0 the total "electrical length" characterizing each of the lossy lines.

The cascade networks consist of commensurate, uniform and/or tapered, lossy (except distortionless [3], [4]) transmission lines interconnected by passive, lumped (lossless and/or lossy) two-ports and terminated in a passive load. This class includes general lines, leakage-free lines, RC-lines and acoustic filters. The results also apply to cascades with noncommensurate lines and to cascades of mixed transmission-line types.

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TABLE I

type of commensurate line	$z_o(s) = \frac{n_o}{d_o}$	$T_o \gamma(s)$
general	$k_i\sqrt{\frac{sT_1+1}{sT_2+1}}$	$T_o\sqrt{(sT_1+1)(sT_2+1)}$
diffusive or RC-line L = 0, G = 0	$\frac{k_i}{\sqrt{s}}$	$T_o\sqrt{s}$
R = 0, C = 0	$k_i\sqrt{s}$	$T_o\sqrt{s}$
L = 0	$\frac{k_i}{\sqrt{sT+1}}$	$T_o\sqrt{sT+1}$
C = 0	$k_i\sqrt{sT+1}$	$T_o\sqrt{sT+1}$
R = 0	$k_i\sqrt{\frac{s}{sT+1}}$	$T_o\sqrt{s(sT+1)}$
G = 0	$k_i\sqrt{\frac{sT+1}{s}}$	$T_o\sqrt{s(sT+1)}$
acoustic [5] (viscous medium)	$k_i\, s\sqrt{sT+1}$	$\frac{T_o\, s}{\sqrt{sT+1}}$

Introduction

PRESENT day technology requires that the analysis and synthesis of mixed lumped-distributed networks, that is, networks containing both lumped and distributed-parameter elements, be treated. Many design and simulation problems where loss is unavoidable or non-negligible, as in the modeling of the wiring on a silicon chip for large-scale integration (LSI) and very large-scale integration (VLSI) [1], demand a synthesis theory that realizes lossly mixed lumped-distributed coupling networks terminated in an arbitrary load. Other applications [2] of the lumped-distributed network model are to tunnel diodes, to the interconnecting wires in high-speed computing networks to cascaded transmission lines with differing characteristic impedances and accompanying parasitic lumped elements, and, perhaps most importantly, to the approximation problem of more powerful filters.

In the first of our two previous papers [3] we treated the synthesis of lossless lumped-distributed cascade networks and then extended the results in the second paper [4] to cascade networks containing lossless and/or lossy lumped two-ports and lossless transmission lines. This paper, the third in a series of four presenting a unified theory for the synthesis of mixed lumped-distributed networks, treats coupling networks containing lossless and/or lossy lumped networks and lossy transmission lines connected in a cascade configuration. Included in this class are lumped-distributed cascade networks with commensurate, uniform or tapered, and lossy transmission lines such as *RC*-lines, general lines, leakage-free lines, and even acoustic filters [5].

The analysis in [3], [4] establishing the expansion conditions for a driving-point impedance expression of the prescribed form deals with a completely general lumped-distributed cascade and hence is easily adapted to chains containing lossy transmission lines. Thus the realizability conditions first guarantee a cascade representation for any input-impedance expression satisfying them and secondly, ensure the positive realness of the lumped networks and the transmission lines in the cascade representation of the input impedance Z_0.

After stating the results in the form of a theorem and immediately illustrating them with a simple example, the incorporation of the characteristic impedances of the $(i-1)$th and ith transmission lines into the chain-matrix description of the ith lumped network is investigated and then used to formulate the realizability conditions. Specific examples to illustrate this process and the synthesis procedure are then given. Within the examples section we also give a realizability lemma which represents a unified synthesis theory for distributed cascade networks consisting entirely of transmission lines.

Results

The contributions of this paper are stated in the following theorem:

Theorem :

Any irreducible

$$Z_0 = \frac{\sum_{i=0}^{n} a_i(s, z_0)\, e^{(2i-n)T_0\gamma(s)}}{\sum_{i=0}^{n} b_i(s, z_0)\, e^{(2i-n)T_0\gamma(s)}}$$

where z_0, $\gamma(s)$, T_0, and their various permitted combinations are as specified in Table I, and a_i, b_i are single-variable algebraic expressions involving s^j with $j=0,1,2,\cdots m$ and irrational functions of the type $\sqrt{s}$, $\sqrt{sT+1}$, or equivalently a_i, b_i are two-variable real polynomials in s, z_0, is realizable as a cascade of lossless and/or lossy lumped two-ports interconnecting n lossy, uniform or tapered, commensurate transmission lines characterized by z_0, their characteristic impedance, $\gamma(s)$ their propagation constant, T_0 their electric length, and which is terminated in a passive load, Table I, if and only if:

a) The $2i \times 2i$ determinants, formed from the algebraic-expression coefficients of Z_0 and defined as

$$C_i \triangleq \left| \begin{array}{ccccc|ccccc} a_n & & & & & b_n & & & & \\ a_{n-1} & a_n & & & & b_{n-1} & b_n & & & \\ \vdots & \vdots & \ddots & & & \vdots & \vdots & \ddots & & \\ a_{n-i+2} & a_{n-i+3} & \cdots & a_n & & b_{n-i+2} & b_{n-i+3} & \cdots & b_n & \\ a_1 & a_2 & \cdots & & a_i & b_1 & b_2 & \cdots & & b_i \\ \hline a_0 & a_1 & \cdots & & a_{i-1} & b_0 & b_1 & \cdots & & b_{i-1} \\ & a_0 & \cdots & & a_{i-2} & & b_0 & \cdots & & b_{i-2} \\ & & \ddots & & \vdots & & & \ddots & & \vdots \\ & & & & a_0 & & & & & b_0 \end{array} \right|, \qquad i=1,2,\cdots,n-1$$

$$D_i \triangleq \left| \begin{array}{cccc|cccc} a_n & & & & b_n & & & \\ a_{n-1} & a_n & & & b_{n-1} & b_n & & \\ \vdots & & \ddots & & \vdots & & \ddots & \\ a_{n-i+1} & a_{n-i+2} & \cdots & a_n & b_{n-i+1} & b_{n-i+2} & \cdots & b_n \\ \hline a_0 & a_1 & \cdots & a_{i-1} & b_0 & b_1 & \cdots & b_{i-1} \\ & a_0 & \cdots & a_{i-2} & & b_0 & \cdots & b_{i-2} \\ & & \ddots & \vdots & & & \ddots & \vdots \\ & & & a_0 & & & & b_0 \end{array} \right|, \qquad i=1,2,\cdots,n$$

and

$$E_i \triangleq \left| \begin{array}{cccc|cccc} a_n & & & & b_n & & & \\ a_{n-1} & a_n & & & b_{n-1} & b_n & & \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & \\ a_{n-i+1} & a_{n-i+2} & \cdots & a_n & b_{n-i+1} & b_{n-i+2} & \cdots & b_n \\ \hline a_{n-i} & a_{n-i+1} & \cdots & a_{n-1} & b_{n-i} & b_{n-i+1} & \cdots & b_{n-1} \\ & a_0 & \cdots & a_{i-2} & & b_0 & \cdots & b_{i-2} \\ & & \ddots & \vdots & & & \ddots & \vdots \\ & & & a_0 & & & & b_0 \end{array} \right|, \qquad i=1,2,\cdots,n-1$$

with the nonspecified values of the above determinants with zeros;

b) The algebraic expressions Q_i, P_i, K_i, L_i defined by the irreducible ratios of the algebraic expressions for all $i=1,2,\cdots,n+1$:

$$\frac{P_i}{Q_i} \triangleq \frac{(a_0-b_0)E_{i-1}\prod_{k=0}^{i-1} Q_k}{(a_n+b_n)D_{i-1}\prod_{k=0}^{i-1} L_k}$$

$$\frac{K_i}{L_i} \triangleq \frac{(a_n+b_n)C_{i-1}\prod_{k=0}^{i-1} L_k}{(a_0-b_0)D_{i-1}\prod_{k=0}^{i-1} Q_K}$$

with

$$\frac{E_0}{D_0} \triangleq \frac{a_n-b_n}{a_0-b_0}; \quad \frac{C_0}{D_0} \triangleq \frac{a_0+b_0}{a_n+b_n}$$

$$\frac{E_n}{D_n} = \frac{C_n}{D_n} = Q_0 = L_0 \triangleq 1$$

and

c) The matrix $[T_i]$ for $z_{0i} = n_{0i}/d_{0i}$, $i=1,2,\cdots,n$

$$[T_i] \triangleq \frac{1}{4G_iM_i} \begin{bmatrix} n_{0i-1}d_{0i}[(Q_i+e_{i-1}P_i)+e_i(K_i+e_{i-1}L_i)] & n_{0i-1}n_{0i}[(Q_i+e_{i-1}P_i)-e_i(K_i+e_{i-1}L_i)] \\ d_{0i}d_{0i-1}[(Q_i-e_{i-1}P_i)+e_i(K_i-e_{i-1}L_i)] & n_{0i-1}d_{0i}[(Q_i-e_{i-1}P_i)-e_i(K_i-e_{i-1}L_i)] \end{bmatrix}$$

with the common factors (resultants) of the matrix elements

$$M_i=\{n_{0i-1}d_{0i}[(Q_i+e_{i-1}P_i)+e_i(K_i+e_{i-1}L_i)];\ n_{0i-1}n_{0i}[(Q_i+e_{i-1}P_i)-e_i(K_i+e_{i-1}L_i)];\ d_{0i-1}d_{0i}[(Q_i-e_{i-1}P_i)+e_i(K_i-e_{i-1}L_i)];\ n_{0i-1}d_{0i}[(Q_i-e_{i-1}P_i)-e_i(K_i-e_{i-1}L_i)]\}$$

$e_0=n_{00}=d_{00}\triangleq 1$; $e_i=\pm 1$; n_{0i}, d_{0i} and the real polynomials $2G_i, 2F_i$ of the definition

$$2G_i2F_iM_i^2\triangleq(e_{i-1}e_in_{0i-1}n_{0i}d_{0i-1}d_{0i})(Q_iL_i-P_iK_i)$$

as specified below are such that

1) $D_i\not\equiv 0$, for $i=1,2,\cdots,n$;

2) $[T_i]$ is a real rational, positive real (PR) chain matrix [4] for all $i=1,2,\cdots,n$ with $e_i=+1$ and/or $e_i=-1$, n_{0i}, d_{0i} either explicitly specified by prescribing the transmission-line type of the cascade or determined to be compatible with $\gamma(s)$ as in Table I, $n_{00}=d_{00}\triangleq 1$ and appropriate real polynomials $2G_i, 2F_i$ chosen from the definition

$$2G_i2F_iM_i^2\triangleq e_{i-1}e_in_{0i-1}n_{0i}d_{0i-1}d_{0i}(Q_iL_i-P_iK_i)$$

and

3) The function $Z_L=n_{0n}(Q_{n+1}+e_nP_{n+1})/d_{0n}(Q_{n+1}-e_nP_{n+1})$ is real rational and positive real (pr).

Furthermore, $[T_i]$ represents the chain-matrix description of the ith lumped two-port connecting the $(i-1)$th and ith transmission lines characterized by the propagation constant $\gamma(s)$ and the characteristic impedance z_{0i} in the lumped-distributed cascade realization of Z_0, while Z_L is its termination.

We illustrate the use of the theorem at the outset to demonstrate the ease of the realization technique and to help in the proof of the theorem: Can the following given Z_0 be realized with commensurate uniform RC-lines ($z_{0i}=z_0(s)=1/\sqrt{s}$, $\gamma(s)=\sqrt{s}$ and $T_0=$ real constant)? And if yes, what is its network realization?

$$Z_0=\frac{\overbrace{(1+s\sqrt{s})(2+2\sqrt{s}+s)(2\sqrt{s}+1)}^{a_2}e^{2\sqrt{s}}+\overbrace{2s\sqrt{s}(2-s)}^{a_1}+\overbrace{(1-s\sqrt{s})(2-2\sqrt{s}+s)(2\sqrt{s}-1)}^{a_0}e^{-2\sqrt{s}}}{\underbrace{\sqrt{s}(2+2\sqrt{s}+s)(2\sqrt{s}+1)}_{b_2}e^{2\sqrt{s}}\underbrace{-2s\sqrt{s}}_{b_1}\underbrace{-\sqrt{s}(2-2\sqrt{s}+s)(2\sqrt{s}-1)}_{b_0}e^{-2\sqrt{s}}}\quad(n=2).$$

This Z_0 is of the specified form for an RC-line cascade and has an initial lumped two-port characterized by

$$\frac{P_1}{Q_1}\triangleq\frac{a_0-b_0}{a_n+b_n}\cdot\frac{a_n-b_n}{a_0-b_0}=\frac{(1+\sqrt{s}-s\sqrt{s})(2-2\sqrt{s}+s)(2\sqrt{s}-1)}{(1+\sqrt{s}+s\sqrt{s})(2+2\sqrt{s}+s)(2\sqrt{s}+1)}\,\frac{(1+s\sqrt{s}-\sqrt{s})(2+2\sqrt{s}+s)(1+2\sqrt{s})}{(1+\sqrt{s}-s\sqrt{s})(2-2\sqrt{s}+s)(2\sqrt{s}-1)}=\frac{1+s\sqrt{s}-\sqrt{s}}{1+s\sqrt{s}+\sqrt{s}}$$

$$\frac{K_1}{L_1}\triangleq\frac{a_n+b_n}{a_0-b_0}\cdot\frac{a_0+b_0}{a_n+b_n}=\frac{1-s\sqrt{s}-\sqrt{s}}{1-s\sqrt{s}+\sqrt{s}}$$

or a chain-matrix description for $z_0=n_0/d_0=1\sqrt{s}$ (RC- lines prescribed) of

$$[T_1]\triangleq\frac{1}{4G_1M_1}\begin{bmatrix}d_0[(Q_1+P_1)+e_1(K_1+L_1)] & n_0[(Q_1+P_1)-e_1(K_1+L_1)]\\ d_0[(Q_1-P_1)+e_1(K_1-L_1)] & n_0[(Q_1-P_1)-e_1(K_1-L_1)]\end{bmatrix}=\frac{1}{4G_1M_1}\begin{bmatrix}2[(1+s\sqrt{s})+e_1(1-s\sqrt{s})]\sqrt{s} & 2[(1+s\sqrt{s})-e_1(1-s\sqrt{s})]\\ [2\sqrt{s}+e_1(-2\sqrt{s})]\sqrt{s} & [2\sqrt{s}-e_1(-2\sqrt{s})]\end{bmatrix}$$

which for $e_1=+1$ becomes

$$[T_1]_{e_1=+1}=\begin{bmatrix}1 & s\\ 0 & 1\end{bmatrix},\qquad\text{for } 4G_1M_1=4\sqrt{s}$$

and for $e_1=-1$

$$[T_1]_{e_1=-1}=\begin{bmatrix}s^2 & 1\\ s & 0\end{bmatrix},\qquad\text{for } 4G_1M_1=4.$$

Note only $[T_1]_{e_1=+1}$ is realizable. The second lumped network is obtained from

$$\frac{P_2}{Q_2} \triangleq \frac{a_0-b_0}{a_n+b_n}\cdot\frac{E_1}{D_1}\cdot\frac{Q_1}{L_1}$$
$$=\frac{(1-s\sqrt{s}+\sqrt{s})(2-2\sqrt{s}+s)(2\sqrt{s}-1)}{(1+s\sqrt{s}+\sqrt{s})(2+2\sqrt{s}+s)(2\sqrt{s}+1)}$$
$$\cdot\frac{\begin{vmatrix}(1+s\sqrt{s})(2+2\sqrt{s}+s)(2\sqrt{s}+1) & \sqrt{s}(2+2\sqrt{s}+s)(2\sqrt{s}+1)\\ 2s\sqrt{s}(2-s) & -2s\sqrt{s}\end{vmatrix}}{\begin{vmatrix}(1+s\sqrt{s})(2+2\sqrt{s}+s)(2\sqrt{s}+1) & \sqrt{s}(2+2\sqrt{s}+s)(2\sqrt{s}+1)\\ (1-s\sqrt{s})(2-2\sqrt{s}+s)(2\sqrt{s}-1) & -\sqrt{s}(2-2\sqrt{s}+s)(2\sqrt{s}-1)\end{vmatrix}}\cdot\frac{(1+s\sqrt{s}+\sqrt{s})}{(1-s\sqrt{s}+\sqrt{s})}$$

$$\frac{P_2}{Q_2}=\frac{s}{2+2\sqrt{s}+s}$$

and

$$\frac{K_2}{L_2} \triangleq \frac{a_n+b_n}{a_0-b_0}\cdot\frac{C_1}{D_1}\cdot\frac{L_1}{Q_1}$$
$$=\frac{(1+s\sqrt{s}+\sqrt{s})(2+2\sqrt{s}+s)(2\sqrt{s}+1)}{(1-s\sqrt{s}+\sqrt{s})(2-2\sqrt{s}+s)(2\sqrt{s}-1)}$$
$$\cdot\frac{\begin{vmatrix}2s\sqrt{s}(2-s) & -2s\sqrt{s}\\ (1-s\sqrt{s})(2-2\sqrt{s}+s)(2\sqrt{s}-1) & -\sqrt{s}(2-2\sqrt{s}+s)(2\sqrt{s}-1)\end{vmatrix}}{\begin{vmatrix}(1+s\sqrt{s})(2+2\sqrt{s}+s)(2\sqrt{s}+1) & \sqrt{s}(2+2\sqrt{s}+s)(2\sqrt{s}+1)\\ (1-s\sqrt{s})(2-2\sqrt{s}+s)(2\sqrt{s}-1) & -\sqrt{s}(2-2\sqrt{s}+s)(2\sqrt{s}-1)\end{vmatrix}}$$
$$\cdot\frac{(1-s\sqrt{s}+\sqrt{s})}{(1+s\sqrt{s}+\sqrt{s})}$$

$$\frac{K_2}{L_2}=\frac{s}{2-2\sqrt{s}+s}$$

from which for $z_{0i-1}=z_{0i}=z_0=n_0/d_0=k_i/\sqrt{s}$ and

$$[T_2] \triangleq \frac{1}{4G_2M_2}\begin{bmatrix}n_0d_0[(Q_2+e_1P_2)+e_2(K_2+e_1L_2)] & n_0^2[(Q_2+e_1P_2)-e_2(K_2+e_1L_2)]\\ d_0^2[(Q_2-e_1P_2)+e_2(K_2-e_1L_2)] & n_0d_0[(Q_2-e_1P_2)-e_2(K_2-e_1L_2)]\end{bmatrix}$$

$$[T_2]_{e_1=e_2=+1}=\begin{bmatrix}1+s & 1\\ s & 1\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 0\\ s & 1\end{bmatrix}\quad \text{with } 4G_2M_2=4\sqrt{s}$$

and

$$[T_2]_{e_1=+1,e_2=-1}=\frac{1}{4G_2'}\begin{bmatrix}s & 1+s\\ s & s\end{bmatrix}\qquad \text{(not PR)}$$

are obtained. There is no need to find $[T_2]_{e_1=-1,e_2=+1}$ and $[T_2]_{e_1=e_2=-1}$ since $[T_1]_{e_1=-1}$ is not PR.

Finally, for the termination ($n=2$)

$$\frac{P_3}{Q_3}=\frac{P_{n+1}}{Q_{n+1}}=\frac{L_{n+1}}{K_{n+1}}$$
$$\triangleq \frac{a_0-b_0}{a_n+b_n}\cdot(1)\cdot\frac{Q_1Q_2}{L_1L_2}=\frac{2\sqrt{s}-1}{2\sqrt{s}+1}$$

or

$$Z_L=\frac{n_0(Q_{n+1}+e_nP_{n+1})}{d_0(Q_{n+1}-e_nP_{n+1})}$$

with

$$Z_L|_{e_n=+1}=2.$$

Again

$$Z_L|_{e_n=-1}$$

need not be determined since $[T_2]_{e_1=+1,\,e_2=-1}$ is not PR. Hence, one possible cascade representation for the given

Z_0 is

$$\begin{bmatrix}1 & s\\0 & 1\end{bmatrix}\begin{bmatrix}\cosh\sqrt{s} & \frac{1}{\sqrt{s}}\sinh\sqrt{s}\\ \sqrt{s}\sinh\sqrt{s} & \cosh\sqrt{s}\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\s & 1\end{bmatrix}\cdot\begin{bmatrix}\cosh\sqrt{s} & \frac{1}{\sqrt{s}}\sinh\sqrt{s}\\ \sqrt{s}\sinh\sqrt{s} & \cosh\sqrt{s}\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}$$

and thus Z_0 is realizable as a mixed lumped-distributed cascade network containing RC-lines. Note that there is no unique mixed lumped-distributed cascade representation but rather there exist 2^n distinct (not all necessarily PR) representations for any given Z_0 of the theorem if the types of lines are prescribed.

Necessity

The necessity of the realizability conditions is established by an analysis of the general cascade specified in the theorem. A large part of this analysis and thus of the necessity proof has already been carried out by the authors in two recent papers [3], [4] for lossless transmission-line cascade networks. The results of this analysis are easily extended to cascade networks containing lossy transmission lines. The identical analysis becomes valid for lossy transmission-line cascades upon a redefinition of the chain matrix

$$\frac{1}{2g_i}\begin{bmatrix}x_i+u_i & x_i-u_i\\ y_i+v_i & y_i-v_i\end{bmatrix}$$

which for lossless line cascades characterizes the ith lumped two-port network interconnecting the $(i-1)$th and ith transmission lines. For lossy-line cascades (distortionless lines excepted) this matrix is redefined by incorporating the characteristic impedances of the $(i-1)$th and ith lossy transmission lines into the chain-matrix description of the lumped two-port network

$$[T_i]=\frac{1}{2g_i'}\begin{bmatrix}\alpha_i & \beta_i\\ \gamma_i & \delta_i\end{bmatrix}.$$

In particular,

$$\begin{bmatrix}\cosh\gamma(s)T_0 & z_{0i-1}\sinh\gamma(s)T_0\\ (1/z_{0i-1})\sinh\gamma(s)T_0 & \cosh\gamma(s)T_0\end{bmatrix}\frac{1}{2g_i'}\begin{bmatrix}\alpha_i & \beta_i\\ \gamma_i & \delta_i\end{bmatrix}\cdot\begin{bmatrix}\cosh\gamma(s)T_0 & z_{0i}\sinh\gamma(s)T_0\\ (1/z_{0i})\sinh\gamma(s)T_0 & \cosh\gamma(s)T_0\end{bmatrix}$$

can be rewritten as

and

$$\begin{bmatrix}n_L\\ d_L\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & z_{0n}\end{bmatrix}\begin{bmatrix}N_L\\ D_L\end{bmatrix}$$

for a termination $Z_L=N_L/D_L$ and where for $z_{0i}=n_{0i}/d_{0i}$ and $z_{00}=1$

$$x_i=\frac{1}{2}d_{0i-1}(n_{0i}\alpha_i+d_{0i}\beta_i)\qquad u_i=\frac{1}{2}d_{0i-1}(n_{0i}\alpha_i-d_{0i}\beta_i)$$

$$y_i=\frac{1}{2}n_{0i-1}(n_{0i}\gamma_i+d_{0i}\delta_i)\qquad v_i=\frac{1}{2}n_{0i-1}(n_{0i}\gamma_i-d_{0i}\delta_i).$$

For these definitions the analysis in the necessity proof of [3], [4] for lossless-line cascades becomes directly applicable to lossy-line cascades. Thus if the ith section of the proposed lumped-distributed cascade is described by the chain-matrix relation:

$$\begin{bmatrix}E_i\\ I_i\end{bmatrix}=\frac{1}{2g_i}\begin{bmatrix}x_i+u_i & x_i-u_i\\ y_i+v_i & y_i-v_i\end{bmatrix}\cdot\begin{bmatrix}\cosh\gamma(s)T_0 & \sinh\gamma(s)T_0\\ \sinh\gamma(s)T_0 & \cosh\gamma(s)T_0\end{bmatrix}\begin{bmatrix}E_{i+1}\\ I_{i+1}\end{bmatrix}$$

then the input impedance of a general lumped-distributed cascade from the ith section through to and including the modified load

$$E_{n+1}/I_{n+1}=a_{n+1,0}/b_{n+1,0}\triangleq n_L/d_L=d_{0n}N_L/n_{0n}D_L$$

is given by

$$\frac{E_i}{I_i}=Z_{0,i}=\frac{\sum_{k=0}^{n-i+1}a_{i,k}e^{[2k-n+i-1]\gamma(s)T_0}}{\sum_{k=0}^{n-i+1}b_{i,k}e^{[2k-n+i-1]\gamma(s)T_0}},\qquad n+1\geqslant i\geqslant 1$$

with the two-variable, real polynomials in s and z_{0i}

$$\begin{aligned}a_{i,k}&=x_ir_{i,k}+u_it_{i,k}\\ b_{i,k}&=y_ir_{i,k}+v_it_{i,k},\qquad n+1\geqslant i\geqslant 1;\ 0\leqslant k\leqslant n-i+1\\ r_{i-1,k+1}&=a_{i,k}+b_{i,k}\\ t_{i-1,k}&=a_{i,k}-b_{i,k}\end{aligned}$$

$$\begin{bmatrix}1 & 0\\ 0 & \frac{1}{z_{0i-1}}\end{bmatrix}\begin{bmatrix}\cosh\gamma(s)T_0 & \sinh\gamma(s)T_0\\ \sinh\gamma(s)T_0 & \cosh\gamma(s)T_0\end{bmatrix}\frac{1}{2g_i'}\begin{bmatrix}1 & 0\\ 0 & z_{0i-1}\end{bmatrix}\begin{bmatrix}\alpha_i & \beta_i\\ \gamma_i & \delta_i\end{bmatrix}\cdot\begin{bmatrix}1 & 0\\ 0 & \frac{1}{z_{0i-1}}\end{bmatrix}\begin{bmatrix}\cosh\gamma(s)T_0 & \sinh\gamma(s)T_0\\ \sinh\gamma(s)T_0 & \cosh\gamma(s)T_0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & z_{0i}\end{bmatrix}$$

Thus we may define for lossy lines the matrices in [3], [4] as

$$\frac{1}{2g_i}\begin{bmatrix}x_i+u_i & x_i-u_i\\ y_i+v_i & y_i-v_i\end{bmatrix}=\frac{1}{2g_i'}\begin{bmatrix}1 & 0\\ 0 & z_{0i-1}\end{bmatrix}\begin{bmatrix}\alpha_i & \beta_i\\ \gamma_i & \delta_i\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1/z_{0i}\end{bmatrix}$$

and, in particular,

$$r_{i-1,0} \equiv 0$$
$$r_{i-1,1} = a_{i,0} + b_{i,0} = (u_i + v_i)t_{i,0} = (u_i + v_i)\prod_{k=i+1}^{n}(u_k - v_k)(n_L - d_L)$$
$$t_{i-1,0} = a_{i,0} - b_{i,0} = (u_i - v_i)t_{i,0} = (u_i - v_i)\prod_{k=i+1}^{n}(u_k - v_k)(n_L - d_L)$$
$$t_{i-1,n-i+2} \equiv 0$$
$$r_{i-1,n-i+2} = a_{i,n-i+1} + b_{i,n-i+1} = (x_i + y_i)r_{i,n-i+1} = (x_i + y_i)\prod_{k=i+1}^{n}(x_k + y_k)(n_L + d_L)$$
$$t_{i-1,n-i+1} = a_{i,n-i+1} - b_{i,n-i+1} = (x_i - y_i)r_{i,n-i+1} = (x_i - y_i)\prod_{k=i+1}^{n}(x_k + y_k)(n_L + d_L)$$
$$(a_0 - b_0) = (a_{1,0} - b_{1,0}) = \prod_{k=1}^{n}(u_k - v_k)(n_L - d_L)$$
$$(a_n + b_n) = (a_{1,0} - b_{1,0}) = \prod_{k=1}^{n}(x_k + y_k)(n_L + d_L).$$

Also derived in [3] is the evaluation of the determinants D_i and E_i of the theorem in terms of the chain-matrix elements of the lumped networks in a lumped-distributed cascade. The identical column manipulation procedure used to evaluate D_i and E_i is valid for obtaining the evaluation of C_i. The expressions for C_i, D_i, and E_i, which also apply to lossy-line cascades, are as follows:

$$C_i = (-1)^i(2)^{i(i-2)/2}\prod_{k=1}^{i}\left\{(-v_kx_k + u_ky_k)^{i-k-1}r_{k,n-k+1}t_{k,0}\right\}r_{i,1}/r_{i,n-i+1}$$
$$D_i = (-1)^i(2)^{i(i-2)/2}\prod_{k=1}^{i}\left\{(-v_kx_k + u_ky_k)^{i-k+1}r_{k,n-k+1}t_{k,0}\right\}$$
$$E_i = (-1)^i(2)^{i(i-2)/2}\prod_{k=1}^{i}\left\{(-v_kx_k + u_ky_k)^{i-k+1}r_{k,n-k+1}t_{k,0}\right\}t_{i,m-1}/t_{i,0}$$

valid for $i = 1, 2, \cdots, n$.

It is therefore possible to express the irreducible algebraic-expression ratios P_i/Q_i and K_i/L_i of the theorem in terms of modified lumped-circuit parameters of the general cascade by substituting the various relations given above into these algebraic expressions to yield

$$\frac{P_i}{Q_i} \triangleq \frac{(a_0 - b_0)}{(a_n + b_n)}\cdot\frac{E_{i-1}}{D_{i-1}}\cdot\frac{\prod_{k=0}^{i-1}Q_k}{\prod_{k=0}^{i-1}L_k} = \frac{(x_i - y_i)}{(x_i + y_i)}\cdot\frac{\prod_{k=0}^{i-1}(u_k - v_k)Q_k}{\prod_{k=0}^{i-1}(x_k + y_k)L_k}, \qquad 1 \leqslant i \leqslant n+1$$

$$\frac{K_i}{L_i} \triangleq \frac{(a_n + b_n)}{(a_0 - b_0)}\cdot\frac{C_{i-1}}{D_{i-1}}\cdot\frac{\prod_{k=0}^{i-1}L_k}{\prod_{k=0}^{i-1}Q_k} = \frac{(u_i + v_i)}{(u_i - v_i)}\cdot\frac{\prod_{k=0}^{i-1}(x_k + y_k)L_k}{\prod_{k=0}^{i-1}(u_k - v_k)Q_k} \qquad 1 \leqslant i \leqslant n+1$$

where

$$(u_0 - v_0) = (x_0 + y_0) \triangleq 1$$

and

$$x_{n+1}/y_{n+1} = u_{n+1}/v_{n+1} = n_L/d_L.$$

These new expressions for the irreducible ratios of the theorem can be refined further by induction to read

$$\frac{k_{i-1}(x_i - y_i)}{h_{i-1}(x_i + y_i)} = \frac{h_iP_i}{h_iQ_i}, \qquad \frac{h_{i-1}(u_i + v_i)}{k_{i-1}(u_i - v_i)} = \frac{k_iK_i}{k_iL_i}, \qquad 1 \leqslant i \leqslant n+1$$

if $k_0 = h_0 \triangleq 1$ and the common factors (or resultants) are defined by

$$k_i \triangleq \{[k_{i-1}(u_i + v_i)], [h_{i-1}(u_i - v_i)]\}$$
$$h_i \triangleq \{[k_{i-1}(x_i - y_i)], [h_{i-1}(x_i + y_i)]\}$$

with $\{k_i, h_i\} = 1$.

However, substitution of the definitions for x_i, y_i, u_i, and v_i in terms of the lumped-parameter, chain-matrix elements and the characteristic impedance $z_{0i} = n_{0i}/d_{0i}$ in the common-factor expressions:

$$k_i = \{\langle k_{i-1}[d_{0i-1}(n_{0i}\alpha_i - d_0\beta_i) + n_{0i-1}(n_{0i}\gamma_i - d_{0i}\delta_i)]\rangle; \times\langle h_{i-1}[d_{0i-1}(n_{0i}\alpha_i - d_{0i}\beta_i) - n_{0i-1}(n_{0i}\gamma_i - d_{0i}\delta_i)]\rangle\}$$
$$h_i = \{\langle k_{i-1}[d_{0i-1}(n_0\alpha_i + d_{0i}\beta_i) - n_{0i-1}(n_{0i}\gamma_i + d_{0i}\delta_i)]\rangle; \times\langle h_{i-1}[d_{0i-1}(n_{0i}\alpha_i + d_{0i}\beta_i) + n_{0i-1}(n_{0i}\gamma_i - d_{0i}\delta_i)]\rangle\}$$

demonstrates that at most $k_i = e_i \triangleq \pm 1$ and $h_i = +1$ for the modified lumped two-port description. Thus no common factors are present in the various components of the k_i, h_i expressions above or

$$\{k_{i-1}, h_{i-1}\} = 1$$
$$\{k_{i-1}, (u_i - v_i)\} = 1, \quad \{h_{i-1}, (u_i + v_i)\} = 1$$
$$\{k_{i-1}, (x_i + y_i)\} = 1, \quad \{h_{i-1}, (x_i + y_i)\} = 1$$
$$\{(u_i + v_i), (u_i - v_i)\} = 1 \quad \{(x_i - y_i), (x_i + y_i)\} = 1$$

even if $n_{0i-1} = n_{0i} = n_0$ and $d_{0i-1} = d_{0i} = d_0$. For otherwise $\{k_i, h_i\} \neq 1$ and a common factor in x_i, y_i, u_i, v_i would yield a Z_0 not in an irreducible form since $\{k_{i-1}, (u_i - v_i)\} = n_0^m d_0^j f(s)$ or $\{h_{i-1}, (u_i + v_i)\} = n_0^m d_0^j f(s)$ implies $\{k_{i-1}, (x_i + y_i)\} = n_0^m d_0^j f(s)$ or $\{h_{i-1}, (x_i - y_i)\} = n_0^m d_0^j f(s)$, respectively, and similarly $\{(u_i + v_i), (u_i - v_i)\} = n_0^m d_0^j f(s)$ implies $\{(x_i - y_i), (x_i + y_i)\} = n_0^m d_0^j f(s)$ for n_0 and/or d_0 containing irrational functions of s and α_i, β_i, γ_i, δ_i real polynomials.

Hence,

$$\frac{k_{i-1}}{h_{i-1}} \cdot \frac{x_i - y_i}{x_i + y_i} = \frac{h_i P_i}{h_i Q_i}, \qquad \frac{h_{i-1}}{k_{i-1}} \cdot \frac{u_i + v_i}{u_i - v_i} = \frac{k_i K_i}{k_i L_i}$$

become for $k_i = e_i = \pm 1$ and $h_i = +1$ (all terms in Q_i must have a plus sign)

$$\frac{x_i - y_i}{x_i + y_i} = \frac{e_{i-1} P_i}{Q_i}, \qquad \frac{u_i - v_i}{u_i + v_i} = \frac{e_i K_i}{e_{i-1} e_i L_i}$$

and we may write

$$\frac{1}{2g_i}\begin{bmatrix} x_i + u_i & x_i - u_i \\ y_i + v_i & y_i - v_i \end{bmatrix} = \frac{1}{4G_i}\begin{bmatrix} (Q_i + e_{i-1}P_i) + e_i(K_i + e_{i-1}L_i) & (Q_i + e_{i-1}P_i) - e_i(K_i + e_{i-1}L_i) \\ (Q_i - e_{i-1}P_i) + e_i(K_i - e_{i-1}L_i) & (Q_i - e_{i-1}P_i) - e_i(K_i - e_{i-1}L_i) \end{bmatrix}.$$

Thus the original matrix

$$[T_i] = \frac{1}{2g_i'}\begin{bmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{bmatrix}$$

is recovered from the modified lumped two-port description

$$\frac{1}{2g_i}\begin{bmatrix} x_i + u_i & x_i - u_i \\ y_i + v_i & y_i - v_i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & z_{0i-1} \end{bmatrix} \frac{1}{2g_i'}\begin{bmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{z_{0i}} \end{bmatrix}$$

as follows:

$$[T_i] = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{z_{0i-1}} \end{bmatrix} \frac{1}{2g_i}\begin{bmatrix} x_i + u_i & x_i - u_i \\ y_i + v_i & y_i - v_i \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & z_{0i} \end{bmatrix}$$

or in terms of the algebraic-expression ratios

$$[T_i] = \frac{1}{4G_iM_i}\begin{bmatrix} n_{0i-1}d_{0i}[(Q_i + e_{i-1}P_i) + e_i(K_i + e_{i-1}L_i)] & n_{0i-1}n_{0i}[(Q_i + e_{i-1}P_i) - e_i(K_i + e_{i-1}L_i)] \\ d_{0i-1}d_{0i}[(Q_i - e_{i-1}P_i) + e_i(K_i - e_{i-1}L_i)] & d_{0i-1}n_{0i}[(Q_i - e_{i-1}P_i) - e_i(K_i - e_{i-1}L_i)] \end{bmatrix}$$

where M_i represents any common matrix-element factor and $e_0 = n_{00} = d_{00} \triangleq 1$, while the termination becomes

$$\frac{N_L}{D_L} = \frac{n_{0n}}{d_{0n}} \cdot \frac{n_L}{d_L} \triangleq \frac{n_{0n}}{d_{0n}} \cdot \frac{x_{n+1}}{y_{n+1}} = \frac{n_{0n}}{d_{0n}} \cdot \frac{(Q_{n+1} + e_n P_{n+1})}{(Q_{n+1} - e_n P_{n+1})}.$$

Because of the presence of e_{i-1}, e_i in these last relations, there exist a total of 2^n distinct lumped-distributed cascade representations for any Z_0 with the prescribed form and Z_0 for which $D_i \neq 0$ $i = 1, 2, \cdots, n$. For n_{0i-1}, n_{0i} unspecified there are at least $(2^n)^2$ distinct representations. However, none of these representations need be physically realizable.

However, if a given Z_0 is to represent the physically realizable lossy-line cascade:

$$[T_1]\begin{bmatrix} \cosh\gamma(s)T_0 & z_0\sinh\gamma(s)T_0 \\ \frac{1}{z_0}\sinh\gamma(s)T_0 & \cosh\gamma(s)T_0 \end{bmatrix}\cdots$$

$$[T_i]\begin{bmatrix} \cosh\gamma(s)T_0 & z_0\sinh\gamma(s)T_0 \\ \frac{1}{z_0}\sinh\gamma(s)T_0 & \cosh\gamma(s)T_0 \end{bmatrix}\cdots\begin{bmatrix} N_L \\ D_L \end{bmatrix}$$

then

$$[T_i], \; i = 1, 2, \cdots, n$$

must be PR chain matrices [4], [6] and N_L/D_L pr while z_0, γ, and T_0 must be such that the transcendental chain matrices characterize a particular transmission line type (e.g., tapered RC-line). These conditions, specified in the theorem, are thus necessary for a Z_0 which is to represent any of the prescribed cascade networks of the theorem.

Sufficiency

Sufficiency of the theorem's conditions is established by showing that any Z_0 satisfying these conditions has a realizable lumped-distributed cascade representation of the types described in the theorem.

In their recent paper [3] the authors showed that any Z_0 of a specified form similar to the one of this paper and for which all the determinants $D_i \neq 0$ $(i = 1, 2, \cdots, n)$ has a (not necessarily realizable) lumped-distributed cascade representation. This contention was established by using a generalization of the Kinariwala-synthesis procedure [7] which yielded an ith expansion section described by the chain matrix:

$$[{}_KT_i] = \frac{1}{2G_i'}\begin{bmatrix} X_i+U_i & X_i-U_i \\ Y_i+V_i & Y_i-V_i \end{bmatrix}\begin{bmatrix} \cosh sT & \sinh sT \\ \sinh sT & \cosh sT \end{bmatrix}$$

where X_i, Y_i, U_i, V_i, G_i' were the real polynomials in s that collectively represented the ith lumped expansion network and the hyperbolic matrix represented the uniform, lossless transmission line of delay T and unity characteristic impedance.

The identical expansion process may be applied to any Z_0 of the forms specified in this paper since the D_i for both lossless-line cascades and lossy-line cascades involve only the coefficients of their Z_0 expressions and not the Z_0 exponentials. For lossy-line cascades, however, the ith expansion section is described by

$$[{}^L_KT_i] = \frac{1}{2G_i'}\begin{bmatrix} X_i+U_i & X_i-U_i \\ Y_i+V_i & Y_i-V_i \end{bmatrix} \cdot \begin{bmatrix} \cosh\gamma(s)T_0 & \sinh\gamma(s)T_0 \\ \sinh\gamma(s)T_0 & \cosh\gamma(s)T_0 \end{bmatrix}$$

where X_i, Y_i, U_i, V_i, G_i' are algebraic expressions (or two-variable, real polynomials in s and z_0) and γ, z_0, and T_0 are to specify the transmission line type. It remains to be demonstrated that this expansion section represents a physically realizable network of a lumped two-port and a specified transmission line.

The defining two-variable polynomials of the lossy-line expansion section are obtained in a manner identical to the method that yielded the lossless-line section parameters in [3], [4]. They are determined from the irreducible ratios of algebraic expressions

$$\frac{X_i'}{Y_i'} = \frac{A_{i,n-i+1}}{B_{i,n-i+1}}, \quad \frac{U_i'}{V_i'} = \frac{A_{i,0}}{B_{i,0}}, \qquad i=1,2,\cdots,n+1$$

with $X_i = X_i'$, $Y_i = Y_i'$, $U_i = U_i'$, $V_i = V_i'$, except when the s^j coefficients $(j=0,1,\cdots,n)$ of X_i' and Y_i' or U_i' and V_i' are either both odd or even real constants (zero of course considered as even), for which case $X_i = \frac{1}{2}X_i'$ and $Y_i = \frac{1}{2}Y_i'$ or $U_i = \frac{1}{2}U_i'$ and $V_i = \frac{1}{2}V_i'$, respectively; while for $A_{i,n-i+1} \equiv 0$, $X_i \equiv 0$ and $Y_i = 1$; for $B_{i,n-i+1} \equiv 0$, $X_i = 1$ and $Y_i \equiv 0$; for $A_{i,0} \equiv 0$, $U_i \equiv 0$ and $V_i = 1$; and for $B_{i,0} \equiv 0$, $U_i = 1$, and $V_i \equiv 0$. The irreducible ratios themselves are obtained from the remainder function after the $(i-1)$th expansion step:

$${}_{i-1}Z_0 = \frac{\sum_{k=0}^{n-i+1} A_{i,k} e^{[2k-(n-i+1)]\gamma(s)T_0}}{\sum_{k=0}^{n-i+1} B_{i,k} e^{[2k-(n-i+1)]\gamma(s)T_0}}$$

where the algebraic-expression coefficients of ${}_{i-1}Z_0$ are defined in terms of the $(i-1)$th expansion parameters:

$$\begin{aligned} 2A_{i,k} &= (R_{i-1,k} + T_{i-1,k}) \\ 2B_{i,k} &= (R_{i-1,k} - T_{i-1,k}) \\ R_{i-1,k} &= -V_{i-1}A_{i-1,k+1} + U_{i-1}B_{i-1,k+1}, \qquad 0 \leqslant k \leqslant n-i+1 \\ T_{i-1,k} &= Y_{i-1}A_{i-1,k} - X_{i-1}B_{i-1,k}, \qquad 2 \leqslant i \leqslant n+1 \end{aligned}$$

and specifically,

$$\begin{aligned} R_{i-1,n-i+1} &= (A_{i,n-i+1} + B_{i,n-i+1}) = (X_i + Y_i)p_{i,n-i+1} \\ R_{i-1,0} &= (A_{i,0} + B_{i,0}) = (U_i + V_i)q_{i,0} \end{aligned}$$

with

$$\begin{aligned} R_{n,0} &= (A_{n+1,0} + B_{n+1,0}) = (X_{n+1} + Y_{n+1})p_{n+1,0} \\ &= (U_{n+1} + V_{n+1})q_{n+1,0}; \\ T_{i-1,n-i+1} &= (A_{i,n-i+1} - B_{i,n-i+1}) = (X_i - Y_i)p_{i,n-i+1} \\ T_{i-1,0} &= (A_{i,0} - B_{i,0}) = (U_i - V_i)q_{i,0} \end{aligned}$$

with

$$\begin{aligned} T_{n,0} &= (A_{n+1,0} - B_{n+1,0}) = (X_{n+1} - Y_{n+1})p_{n+1,0} \\ &= (U_{n+1} - V_{n+1})q_{n+1,0} \end{aligned}$$

and

$$\prod_{k=1}^{i} (X_k + Y_k)p_{i,n-i+1} = (a_n + b_n)\prod_{k=1}^{i-1}(-V_kX_k + U_kY_k)$$

$$\prod_{k=1}^{i} (U_k - V_k)q_{i,0} = (a_0 - b_0)\prod_{k=1}^{i-1}(-V_kX_k + U_kY_k)$$

with the common factors $p_{i,n-i+1} = A_{i,n-i+1}$; $B_{i,n-i+1}$, $q_{i,0} = A_{i,0}$; $B_{i,0}$; and $T_{i-1,n-i+2} = R_{i-1,-1} \equiv 0$; $A_{1,k} = a_k$, $B_{1,k} \equiv b_k$; $p_{n+1,0} = q_{n+1,0}$ and $X_{n+1} = U_{n+1}$; $Y_{n+1} = V_{n+1}$.

Notice that no relationship is assumed to exist between the expansion polynomials X_i, Y_i, U_i, V_i, G_i' (or $A_{i,j}$; $B_{i,j}$; $R_{i,j}$; $T_{i,j}$) of the sufficiency proof and the x_i, y_i, u_i, v_i, g_i (or $a_{i,j}$; $b_{i,j}$; $r_{i,j}$; $t_{i,j}$) used in the necessity proof.

Also derived in [3] are the evaluations of the determinants E_i, D_i in terms of the generalized synthesis parameters. The identical evaluation process used to determine E_i, D_i can also be applied to C_i for its evaluation and is valid for lossy-line cascades with the result that

$$E_i = (2)^{i(i-1)/2}\prod_{k=1}^{i}\left[R_{k,n-k}\left(-T_{k,0}\Big/\prod_{m=1}^{k}H_m\right)\right]T_{i,n-i}/T_{i,0}$$

$$D_i = (2)^{i(i-1)/2}\prod_{k=1}^{i}\left[R_{k,n-k}\left(-T_{k,0}\Big/\prod_{m=1}^{k}H_m\right)\right]$$

$$C_i = (2)^{i(i-1)/2}\prod_{k=1}^{i}\left[R_{k,n-k}\left(-T_{k,0}\Big/\prod_{m=1}^{k}H_m\right)\right]R_{i,0}/R_{i,n-i}$$

where $H_m \triangleq -V_mX_m + U_mY_m$.

These results allow us to express the irreducible ratios of algebraic expressions of the theorem in terms of the generalized synthesis parameters:

$$\frac{P_i}{Q_i} \triangleq \frac{a_0-b_0}{a_n+b_n}\cdot\frac{E_{i-1}}{D_{i-1}}\cdot\frac{\prod_{k=0}^{i-1} Q_k}{\prod_{k=0}^{i-1} L_k} = \frac{a_0-b_0}{a_n+b_n}\cdot\frac{T_{i-1,n-i+1}}{T_{i-1,0}}\cdot\frac{\prod_{k=0}^{i-1} Q_k}{\prod_{k=0}^{i-1} L_k}, \qquad 1 \leqslant i \leqslant n+1$$

$$\frac{K_i}{L_i} \triangleq \frac{a_n+b_n}{a_0-b_0}\cdot\frac{C_{i-1}}{D_{i-1}}\cdot\frac{\prod_{k=0}^{i-1} L_k}{\prod_{k=0}^{i-1} Q_k} = \frac{a_n+b_n}{a_0-b_0}\cdot\frac{R_{i-1,0}}{R_{i-1,n-i+1}}\cdot\frac{\prod_{k=0}^{i-1} L_k}{\prod_{k=0}^{i-1} Q}$$

with $E_n/D_n = C_n/D_n \triangleq 1$ or

$$\frac{P_i}{Q_i} = \frac{(X_i-Y_i)}{(X_i+Y_i)}\cdot\frac{\prod_{k=0}^{i-1}(U_k-V_k)Q_k}{\prod_{k=0}^{i-1}(X_k+Y_k)L_k}$$

$$\frac{P_{n+1}}{Q_{n+1}} = \frac{(U_{n+1}-V_{n+1})}{(X_{n+1}+Y_{n+1})}\cdot\frac{\prod_{k=0}^{n}(U_k-V_k)Q_k}{\prod_{k=0}^{n}(X_k+Y_k)L_k}$$

$$\frac{K_i}{L_i} = \frac{U_i+V_i}{U_i-V_i}\cdot\frac{\prod_{k=0}^{i-1}(X_k+Y_k)L_k}{\prod_{k=0}^{i-1}(U_k-V_k)Q_k}$$

$$\frac{K_{n+1}}{L_{n+1}} = \frac{X_{n+1}+Y_{n+1}}{U_{n+1}-V_{n+1}}\cdot\frac{\prod_{k=0}^{n}(X_k+Y_k)L_k}{\prod_{k=0}^{n}(U_k-V_k)Q_k}$$

with $(U_0-V_0)=(X_0+Y_0)=Q_0=L_0\triangleq 1$.

The specification of the generalized expansion parameters X_i, Y_i and U_i, V_i as irreducible or their special definitions ensures that the algebraic expressions (X_i-Y_i), (X_i+Y_i) and (U_i+V_i), (U_i-V_i) are also irreducible.

Hence the irreducible ratios of the theorem can be expressed in terms of the generalized synthesis parameters as

$$\frac{P_i}{Q_i} = \frac{X_i-Y_i}{X_i+Y_i}, \qquad \frac{K_i}{L_i} = \frac{U_i+V_i}{U_i-V_i}, \qquad 1 \leqslant i \leqslant n+1$$

with $X_{n+1}=U_{n+1}$, $Y_{n+1}=V_{n+1}$ since $Q_i=(X_i+Y_i)$ and $L_i=(U_i-V_i)$ by induction and

$$[T_i'] = \frac{1}{2G_i'}\begin{bmatrix} X_i+U_i & X_i-U_i \\ Y_i+V_i & Y_i-V_i \end{bmatrix} = \frac{1}{4G_i}\begin{bmatrix} [(Q_i+P_i)+(K_i+L_i)] & [(Q_i+P_i)-(K_i+L_i)] \\ [(Q_i-P_i)+(K_i-L_i)] & [(Q_i-P_i)-(K_i-L_i)] \end{bmatrix}$$

with ${}_nZ_0 = X_{n+1}/Y_{n+1} = (Q_{n+1}+P_{n+1})/(Q_{n+1}-P_{n+1})$.

However, no unique determination of the ith lumped expansion network from the given Z_0 in terms of the expansion parameters is possible in view of the following identities:

$$\begin{bmatrix} (X_{i-1}+U_{i-1}) & (X_{i-1}-U_{i-1}) \\ (Y_{i-1}+V_{i-1}) & (Y_{i-1}-V_{i-1}) \end{bmatrix}[\times]\begin{bmatrix} (X_i+U_i) & (X_i-U_i) \\ (Y_i+V_i) & (Y_i-V_i) \end{bmatrix}[\times]\begin{bmatrix} (X_{i+1}+U_{i+1}) & (X_{i+1}-U_{i+1}) \\ (Y_{i+1}+V_{i+1}) & (Y_{i+1}-V_{i+1}) \end{bmatrix}$$

$$= \begin{bmatrix} (X_{i-1}-U_{i-1}) & (X_{i-1}+U_{i-1}) \\ (Y_{i-1}-V_{i-1}) & (Y_{i-1}+V_{i-1}) \end{bmatrix}[\times]\begin{bmatrix} (Y_i+V_i) & (Y_i-V_i) \\ (X_i+U_i) & (X_i-U_i) \end{bmatrix}[\times]\begin{bmatrix} (X_{i+1}+U_{i+1}) & (X_{i+1}-U_{i+1}) \\ (Y_{i+1}+V_{i+1}) & (Y_{i+1}-V_{i+1}) \end{bmatrix}$$

$$= \begin{bmatrix} (X_{i-1}-U_{i-1}) & (X_{i-1}+U_{i-1}) \\ (Y_{i-1}-V_{i-1}) & (Y_{i-1}+V_{i-1}) \end{bmatrix}[\times]\begin{bmatrix} (Y_i-V_i) & (Y_i+V_i) \\ (X_i-U_i) & (X_i+U_i) \end{bmatrix}[\times]\begin{bmatrix} (Y_{i+1}+V_{i+1}) & (Y_{i+1}-V_{i+1}) \\ (X_{i+1}+U_{i+1}) & (X_{i+1}-U_{i+1}) \end{bmatrix}$$

where

$$[\times] \triangleq \begin{bmatrix} \cosh\gamma(s)T_0 & \sinh\gamma(s)T_0 \\ \sinh\gamma(s)T_0 & \cosh\gamma(s)T_0 \end{bmatrix}.$$

The above matrix relations are specific cases of the general matrix product

$$\begin{bmatrix} (Q_{i-1}+e_{i-2}P_{i-1})+e_{i-1}(K_{i-1}+e_{i-2}P_{i-1}) & (Q_{i-1}+e_{i-2}P_{i-1})-e_{i-1}(K_{i-1}+e_{i-2}P_{i-1}) \\ (Q_{i-1}-e_{i-2}P_{i-1})+e_{i-1}(K_{i-1}-e_{i-2}P_{i-1}) & (Q_{i-1}-e_{i-2}P_{i-1})-e_{i-1}(K_{i-1}-e_{i-2}P_{i-1}) \end{bmatrix}[\times]$$

$$\cdot\begin{bmatrix} (Q_i+e_{i-1}P_i)+e_i(K_i+e_{i-1}L_i) & (Q_i+e_{i-1}P_i)-e_i(K_i+e_{i-1}L_i) \\ (Q_i-e_{i-1}P_i)+e_i(K_i-e_{i-1}L_i) & (Q_i-e_{i-1}P_i)-e_i(K_i-e_{i-1}L_i) \end{bmatrix}[\times]$$

$$\cdot\begin{bmatrix} (Q_{i+1}+e_iP_{i+1})+e_{i+1}(K_{i+1}+e_iL_{i+1}) & (Q_{i+1}+e_iP_{i+1})-e_{i+1}(K_{i+1}+e_iL_{i+1}) \\ (Q_{i+1}-e_iP_{i+1})+e_{i+1}(K_{i+1}-e_iL_{i+1}) & (Q_{i+1}-e_iP_{i+1})-e_{i+1}(K_{i+1}-e_iL_{i+1}) \end{bmatrix}$$

for

$$e_{i-2}=+1,\quad e_{i-1}=+1,\quad e_i=+1,\quad e_{i+1}=+1$$
$$e_{i-2}=+1,\quad e_{i-1}=-1,\quad e_i=+1,\quad e_{i+1}=+1$$
$$e_{i-2}=+1,\quad e_{i-1}=-1,\quad e_i=-1,\quad e_{i+1}=+1$$

respectively, and illustrate the fact that for $e_i=\pm1$ there are 2^n distinct cascade representations in terms of the expansion parameters for any Z_0 of the prescribed form and for which $D_i \not\equiv 0$, $i=1,2,\cdots,n$.

Thus the general expansion cascade

$$[T_1''][\times]\cdots[T_i''][\times]\cdots[\times]\begin{bmatrix}X_{n+1}\\ Y_{n+1}\end{bmatrix}$$

with

$$[T_i'']=\frac{1}{4G_i''}\begin{bmatrix}(Q_i+e_{i-1}P_i)+e_i(K_i+e_{i-1}L_i) & (Q_i+e_{i-1}P_i)-e_i(K_i+e_{i-1}L_i)\\ (Q_i-e_{i-1}P_i)+e_i(K_i-e_{i-1}L_i) & (Q_i-e_{i-1}P_i)-e_i(K_i-e_{i-1}L_i)\end{bmatrix}$$

$$[\times]=\begin{bmatrix}\cosh\gamma(s)T_0 & \sinh\gamma(s)T_0\\ \sinh\gamma(s)T_0 & \cosh\gamma(s)T_0\end{bmatrix}$$

and

$$X_{n+1}/Y_{n+1}=(Q_{n+1}+e_nP_{n+1})/(Q_{n+1}-e_nP_{n+1})$$

and $e_0=1$, may be rewritten as

$$[T_1'']\begin{bmatrix}1&0\\0&z_{01}\end{bmatrix}\begin{bmatrix}1&0\\0&1/z_{01}\end{bmatrix}[\times]\begin{bmatrix}1&0\\0&z_{01}\end{bmatrix}\begin{bmatrix}1&0\\0&1/z_{01}\end{bmatrix}\cdots$$
$$\begin{bmatrix}1&0\\0&z_{0i-1}\end{bmatrix}\begin{bmatrix}1&0\\0&1/z_{0i-1}\end{bmatrix}[T_i'']\begin{bmatrix}1&0\\0&z_{0i}\end{bmatrix}\begin{bmatrix}1&0\\0&1/z_{0i}\end{bmatrix}[\times]\begin{bmatrix}1&0\\0&z_{0i}\end{bmatrix}\begin{bmatrix}1&0\\0&1/z_{0i}\end{bmatrix}\cdots$$
$$\begin{bmatrix}1&0\\0&z_{0n}\end{bmatrix}\begin{bmatrix}1&0\\0&1/z_{0n}\end{bmatrix}[\times]\begin{bmatrix}1&0\\0&z_{0n}\end{bmatrix}\begin{bmatrix}1&0\\0&1/z_{0n}\end{bmatrix}\begin{bmatrix}X_{n+1}\\ Y_{n+1}\end{bmatrix}$$

which for the following definitions:

$$[T_i]\triangleq\begin{bmatrix}1&0\\0&1/z_{0i-1}\end{bmatrix}[T_i'']\begin{bmatrix}1&0\\0&z_{0i}\end{bmatrix}$$
$$=\frac{1}{4G_iM_i}\begin{bmatrix}n_{0i-1}d_{0i}[(Q_i+e_{i-1}P_i)+e_i(K_i+e_{i-1}L_i)] & n_{0i-1}n_{0i}[(Q_i+e_{i-1}P_i)-e_i(K_i+e_{i-1}L_i)]\\ d_{0i-1}d_{0i}[(Q_i-e_{i-1}P_i)+e_i(K_i-e_{i-1}L_i)] & d_{0i-1}n_{0i}[(Q_i-e_{i-1}P_i)-e_i(K_i-e_{i-1}L_i)]\end{bmatrix}$$

$$\begin{bmatrix}N_L\\ D_l\end{bmatrix}\triangleq\begin{bmatrix}1&0\\0&1/z_{0n}\end{bmatrix}\begin{bmatrix}X_{n+1}\\ Y_{n+1}\end{bmatrix}\triangleq\begin{bmatrix}n_{0n}(Q_{n+1}+e_nP_{n+1})\\ d_{0n}(Q_{n+1}-e_nP_{n+1})\end{bmatrix}$$

with M_i equated to any common matrix-element factors and $2G_i, 2F_i$ chosen from the definition

$$2G_i2F_iM_i^2\triangleq e_{i-1}e_in_{0i-1}n_{0i}d_{0i-1}d_{0i}(Q_iL_i-P_iK_i)$$

such that $[T_i]$ is PR (if possible), becomes

$$[T_1]\begin{bmatrix}\cosh\gamma(s)T_0 & z_{01}\sinh\gamma(s)T_0\\ \dfrac{1}{z_{01}}\sinh\gamma(s)T_0 & \cosh\gamma(s)T_0\end{bmatrix}\cdots$$
$$\cdot[T_i]\begin{bmatrix}\cosh\gamma(s)T_0 & z_{0i}\sinh\gamma(s)T_0\\ \dfrac{1}{z_{0i}}\sinh\gamma(s)T_0 & \cosh\gamma(s)T_0\end{bmatrix}\cdots\begin{bmatrix}N_L\\ D_L\end{bmatrix}.$$

Therefore, the conditions of the theorem are sufficient to guarantee any Z_0 satisfying them to be the input impedance of a realizable lumped-distributed cascade containing lossy transmission lines.

Examples

We illustrate the realization techniques of this paper with two additional examples: the first specializes the results to cascades consisting of lossy or lossless transmission lines (no lumped two-ports) and the second demonstrates the applicability of the cascade networks for which z_{0i} is not specifically prescribed for a given $\gamma_i(s)$ or to cascades consisting of a limited mix of transmission line types (e.g., for $\gamma_i(s)=\sqrt{s}$ RC lines and/or LG lines). The second example serves as an introduction to a forthcoming paper on noncommensurate lines and mixed-line cascades.

Before the first example we state a realization lemma for a special class of distributed cascades which the results of this paper and those of [3], [4] prove. These distributed cascades are composed exclusively of transmission lines of the same type whether the type be lossy or lossless (e.g., all lossless, all distortionless, all RC, all general, etc.).

Lemma: Any Z_0 of the forms prescribed in the theorems of this paper and [3] is realizable as a distributed cascade network composed exclusively of n commensurate, uniform or tapered, lossless or lossy transmission lines of the same or similar type (e.g., all RC or a mix of RC and LG lines, etc.) and terminated in a passive load if and only if for the definitions a) and b) of the two theorems and

$i=1,2,\cdots,\ n+1$

1) $\dfrac{P_i}{Q_i}$ exists (or $D_i \neq 0$),

2) $\dfrac{P_i}{Q_i}=\dfrac{K_i}{L_i}$; and

3) $z_{0i}=z_{0i-1}\cdot\dfrac{Q_i+P_i}{Q_i-P_i}=\dfrac{\prod_{k=0}^{i}(Q_k+P_k)}{\prod_{k=0}^{i}(Q_k-P_k)}$

with $z_{00}\triangleq 1$ is such that

a) for a prescribed line-type cascade

$$z_{0i}=k_i z_0$$

with z_0 the characteristic impedance of the prescribed lines and k_i a positive real constant;

b) for a limited mixed line-type cascade

$$z_{0i}=k_i n_0/d_0$$

with z_{0i} compatible with $\gamma(s)=T_0 n_0 d_0$ (e.g., for $\gamma(s)=T_0\sqrt{s}$, $z_{0i}=k_i\sqrt{s}$ or $k_i/\sqrt{s}$; for $\gamma(s)=T_0\sqrt{(sT_1+1)(sT_2+1)}$, $z_{0i}=k_1\sqrt{(sT_1+1)/(sT_2+1)}$, or $k_i\sqrt{(sT_2+1)/(sT_1+1)}$; etc.) and k_i a real positive constant; and

c) z_{0n+1} is a pr rational function of s.

Furthermore, the ith transmission line or ith expansion section in the distributed-cascade synthesis of Z_0 is characterized by the propagation constant $\gamma(s)$ with the total "electric length" T_0 and the characteristic impedance z_{0i}, while the cascade network termination is given by z_{0n+1}.

Now we consider the given impedance

$$Z_0=\frac{3(2\sqrt{s}+3)e^{3.3\sqrt{s}}+6\sqrt{s}\,e^{1.1\sqrt{s}}+6\sqrt{s}\,e^{-1.1\sqrt{s}}+3(2\sqrt{s}-3)e^{-3.3\sqrt{s}}}{2\sqrt{s}\,3(2\sqrt{s}+3)e^{3.3\sqrt{s}}+(2\sqrt{s}-6)e^{1.1\sqrt{s}}-(2\sqrt{s}+6)e^{-1.1\sqrt{s}}-3(2\sqrt{s}-3)e^{-3.3\sqrt{s}}}$$

and determine whether this Z_0 possesses an RC line cascade realization. An examination of the form of Z_0 shows that it could represent a realizable, commensurate, uniform or tapered, RC-line cascade since $\gamma(s)=T_0\sqrt{s}$ with $T_0=1.1$ and a_i, b_i are real polynomials of $\sqrt{s}$.

The initial network is obtained from

$$\frac{P_1}{Q_1}\triangleq\frac{a_n-b_n}{a_n+b_n}=\frac{(3-6\sqrt{s})(2\sqrt{s}+3)}{(3+6\sqrt{s})(2\sqrt{s}+3)}=\frac{1-2\sqrt{s}}{1+2\sqrt{s}}$$

$$\frac{K_1}{L_1}\triangleq\frac{a_0+b_0}{a_0-b_0}=\frac{(3-6\sqrt{s})(2\sqrt{s}-3)}{(3+6\sqrt{s})(2\sqrt{s}-3)}=\frac{1-2\sqrt{s}}{1+2\sqrt{s}}.$$

Note that P_1/Q_1 exists, $P_1/Q_1=K_1/L_1$ and $z_{01}=(Q_1+P_1)/(Q_1-P_1)=1/2\sqrt{s}$. Hence, by the lemma the first expansion section is the RC-line:

$$\begin{bmatrix}\cosh(1.1\sqrt{s}) & 1/2\sqrt{s}\,\sinh(1.1\sqrt{s})\\ 2\sqrt{s}\,\sinh(1.1\sqrt{s}) & \cosh(1.1\sqrt{s})\end{bmatrix}$$

while according to the theorem the first lumped-network description is

$$[T_1]=\frac{1}{4G_1M_1}\begin{bmatrix}d_{01}[(Q_1+P_1)+e_1(K_1+L_1)] & n_{01}[(Q_1+P_1)-e_1(K_1+L_1)]\\ d_{01}[(Q_1-P_1)+e_1(K_1-L_1)] & n_{01}[(Q_1-P_1)-e_1(K_1-L_1)]\end{bmatrix}$$
$$=\frac{1}{4G_1M_1}\begin{bmatrix}4d_{01} & 0\\ 0 & 8\sqrt{s}\,n_{01}\end{bmatrix}$$

where $e_1=+1$ is chosen to eventually achieve a unit matrix for $[T_1]$, which is required if the realization cascade is to consist entirely of RC transmission lines. Thus for $n_{01}=1$ and $d_{01}=2\sqrt{s}$ or $z_{01}=1/2\sqrt{s}$ with $2G_1M_1=8\sqrt{s}$, $[T_1]$ becomes the unit matrix and the first expansion-section description from the theorem is identical to the one obtained by application of the lemma.

$$\frac{P_2}{Q_2}\triangleq\frac{a_0-b_0}{a_n+b_n}\cdot\frac{E_1}{D_1}\cdot\frac{Q_1}{L_1}=\frac{(3+6\sqrt{s})(2\sqrt{s}-3)}{(3+6\sqrt{s})(2\sqrt{s}+3)}\cdot\frac{\begin{vmatrix}3(2\sqrt{s}+3) & 2\sqrt{s}(3)(2\sqrt{s}+3)\\ 6\sqrt{s} & 2\sqrt{s}(2\sqrt{s}-6)\end{vmatrix}}{\begin{vmatrix}3(2\sqrt{s}+3) & 2\sqrt{s}(3)(2\sqrt{s}+3)\\ 3(2\sqrt{s}-3) & -2\sqrt{s}(3)(2\sqrt{s}-3)\end{vmatrix}}\cdot\frac{1+2\sqrt{s}}{1+2\sqrt{s}}$$

which yields

$$\frac{P_2}{Q_2}=\frac{1}{3}$$

$$\frac{K_2}{L_2}\triangleq\frac{a_n+b_n}{(a_0-b_0)}\cdot\frac{C_1}{D_1}\cdot\frac{L_1}{Q_1}=\frac{(3+6\sqrt{s})(2\sqrt{s}+3)}{(3+6\sqrt{s})(2\sqrt{s}-3)}\cdot\frac{\begin{vmatrix}6\sqrt{s} & -2\sqrt{s}(2\sqrt{s}+6)\\ 3(2\sqrt{s}-3) & -2\sqrt{s}(3)(2\sqrt{s}-3)\end{vmatrix}}{\begin{vmatrix}3(2\sqrt{s}+3) & 2\sqrt{s}(3)(2\sqrt{s}+3)\\ 3(2\sqrt{s}-3) & -2\sqrt{s}(3)(2\sqrt{s}-3)\end{vmatrix}}\cdot\frac{1+2\sqrt{s}}{1+2\sqrt{s}}$$

so that we get

$$\frac{K_2}{L_2} = \frac{1}{3}.$$

Note that all terms in the algebraic expression Q_i must have the plus sign, otherwise the section in question will not be realizable. Using the lemma

$$z_{02} = z_{01} \cdot \frac{Q_2 + P_2}{Q_2 - P_2} = \frac{1}{2\sqrt{s}} \cdot \frac{4}{2} = \frac{1}{\sqrt{s}}$$

and the second section is the RC transmission line described by

$$\begin{bmatrix} \cosh(1.1\sqrt{s}) & 1/\sqrt{s}\,\sinh(1.1\sqrt{s}) \\ \sqrt{s}\,\sinh(1.1\sqrt{s}) & \cosh(1.1\sqrt{s}) \end{bmatrix}.$$

Similarly

$$\frac{P_3}{Q_3} = \frac{1}{2} \quad \text{and} \quad \frac{K_3}{L_3} = \frac{1}{2}$$

or

$$z_{03} = \frac{3}{\sqrt{s}} \quad \text{while for}$$

$$\frac{P_4}{Q_4} = \frac{P_{n+1}}{Q_{n+1}} = \frac{2\sqrt{s} - 3}{2\sqrt{s} + 3}$$

$$Z_L = \frac{3}{\sqrt{s}} \cdot \frac{4\sqrt{s}}{6} = 2.$$

Hence the given Z_0 has an RC-line cascade representation of

$$\begin{bmatrix} \cosh(1.1\sqrt{s}) & 1/2\sqrt{s}\,\sinh(1.1\sqrt{s}) \\ 2\sqrt{s}\,\sinh(1.1\sqrt{s}) & \cosh(1.1\sqrt{s}) \end{bmatrix} \cdot \begin{bmatrix} \cosh(1.1\sqrt{s}) & 1/\sqrt{s}\,\sinh(1.1\sqrt{s}) \\ \sqrt{s}\,\sinh(1.1\sqrt{s}) & \cosh(1.1\sqrt{s}) \end{bmatrix}$$
$$\cdot \begin{bmatrix} \cosh(1.1\sqrt{s}) & 3/\sqrt{s}\,\sinh(1.1\sqrt{s}) \\ \sqrt{s}/3\,\sinh(1.1\sqrt{s}) & \cosh(1.1\sqrt{s}) \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and is pr.

It must be emphasized that the lemma applied to all distributed cascades even to those containing a variety of transmission line types. In fact, $z_{0i} = z_{0i-1}(Q_i + P_i)/(Q_i - P_i)$ whenever $P_i/Q_i \equiv K_i/L_i$ even in a lumped-distributed cascade realization of a given Z_0, and $P_i/Q_i = K_i/L_i \equiv 0$ implies that the ith line is identical to the $(i-1)$th or a two-unit length line ($2T_0$) can be extracted from Z_0 as the $(i-1)$th line.

As a second example we examine

$$Z_0 = \frac{\sqrt{s+1}\,\{[s\sqrt{s+1} + \sqrt{2s+1}][2\sqrt{2s+1} + \sqrt{s+1}]e^{3\sqrt{(s+1)(2s+1)}} + s\sqrt{s+1}[2\sqrt{2s+1} - \sqrt{s+1}]e^{\sqrt{(s+1)(2s+1)}}}{\sqrt{2s+1}\,\{[s\sqrt{s+1} + \sqrt{2s+1}][2\sqrt{2s+1} + \sqrt{s+1}]e^{3\sqrt{(s+1)(2s+1)}} + s\sqrt{s+1}[2\sqrt{2s+1} - \sqrt{s+1}]e^{\sqrt{(s+1)(2s+1)}}}$$
$$\cdot \frac{- s\sqrt{s+1}[2\sqrt{2s+1} + \sqrt{s+1}]e^{-\sqrt{(s+1)(2s+1)}} + [\sqrt{2s+1} - s\sqrt{s+1}][2\sqrt{2s+1} - \sqrt{s+1}]e^{-3\sqrt{(s+1)(2s+1)}}\}}{+ s\sqrt{s+1}[2\sqrt{2s+1} + \sqrt{s+1}]e^{-\sqrt{(s+1)(2s+1)}} - [\sqrt{2s+1} - s\sqrt{s+1}][2\sqrt{2s+1} - \sqrt{s+1}]e^{-3\sqrt{(s+1)(2s+1)}}\}}$$

and determine whether it is realizable as a lumped-distributed cascade. Note we are not specifying the lines in this cascade expansion of Z_0. However, $T_0\gamma(s) = \sqrt{(s+1)(2s+1)}$ indicates that the only permitted z_{0i} are $z_{0i} = k_i\sqrt{(s+1)/(2s+1)}$ and $k_i\sqrt{(2s+1)/(s+1)}$ (Assume $k_i = 1$ to simplify matters.)

Calculating the various P_i/Q_i and K_i/L_i yields

$$\frac{P_1}{Q_1} = \frac{\sqrt{s+1} - \sqrt{2s+1}}{\sqrt{s+1} + \sqrt{2s+1}} \qquad \frac{K_1}{L_1} = \frac{\sqrt{s+1} - \sqrt{2s+1}}{\sqrt{s+1} + \sqrt{2s+1}}$$

$$\frac{P_2}{Q_2} = \frac{0}{1} \qquad \frac{K_2}{L_2} = \frac{0}{1}$$

$$\frac{P_3}{Q_3} = \frac{- s\sqrt{s+1}}{s\sqrt{s+1} + \sqrt{2s+1}} \qquad \frac{K_3}{L_3} = \frac{s\sqrt{s+1}}{\sqrt{2s+1} - 2\sqrt{2s+1}}$$

$$\frac{P_4}{Q_4} = \frac{\sqrt{s+1} - 2\sqrt{2s+1}}{\sqrt{s+1} + 2\sqrt{2s+1}} \qquad \frac{K_4}{L_4} = \frac{\sqrt{s+1} + 2\sqrt{2s+1}}{\sqrt{s+1} - 2\sqrt{2s+1}}.$$

Because $P_1/Q_1 = K_1/L_1$ and $P_2/Q_2 = K_2/L_2$ we can immediately write

$$z_{01} = \frac{Q_1 + P_1}{Q_1 - P_1} = \sqrt{\frac{s+1}{2s+1}} \qquad z_{02} = z_{01} \cdot \frac{Q_2 + P_2}{Q_2 - P_2} = \sqrt{\frac{s+1}{2s+1}}.$$

These calculations assume the first two lumped networks desired are described by the unit chain matrix.

But $e_1 = +1$ and $z_{0i} = z_1 = \sqrt{(s+1)/(2s+1)}$ or $Z_{0i} = z_2 = \sqrt{(2s+1)/(s+1)}$ implies that there are four possible chain-matrix descriptions for the initial lumped network or

$$[T_1]^{e_1=1}_{z_1} = [T_{11}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[T_1]^{e_1=1}_{z_2} = [T_{12}] = \begin{bmatrix} s+1 & 0 \\ 0 & 2s+1 \end{bmatrix}$$

$$[T_1]^{e_1=-1}_{z_1} = [T_{13}] = \begin{bmatrix} 0 & s+1 \\ 2s+1 & 0 \end{bmatrix}$$

$$[T_1]^{e_1=-1}_{z_2} = [T_{14}] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

of which only $[T_{11}]$ and $[T_{14}]$ are PR. The second lumped two-port will have four distinct chain-matrix representations ($e_2 = \pm 1$, $z_{02} = z_1$ or z_2) for each of the four initial lumped-network descriptions or a total of 16 matrices. However, there is no need to find the 8 chain matrices generated by $[T_{12}]$ and $[T_{13}]$ since the latter are not PR. Similarly, not all of the remaining 8 chain-matrix description of the second lumped network need be PR. In fact, only four

$$[T_{21}]_{z_1}^{e_2=\pm 1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad [T_{21}]_{z_2}^{e_2=-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$[T_{24}]_{z_1}^{e_2=+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad [T_{24}]_{z_2}^{e_2=-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

are realizable. Carrying the expansion process to a conclusion yields the following 16 realizable lumped-distributed cascade network representations for the given Z_0:

Conclusions

With the results of this paper and of their previous work [3], [4] the authors have established a complete and unified synthesis theory for commensurate, lumped-distributed cascade networks: the first paper [3] treats the class of lossless lumped two-ports and lossless transmission lines and the class of lossless lumped two-ports and distortionless lines. (Although this class is not specifically discussed, the theorems in [3], [4] hold for distortionless line cascades if sT is replaced by $sT + \beta$.) The second paper [4] treats lossless/lossy lumped networks and lossless or distortionless lines; while this third paper treats lossless/lossy lumped two-ports and lossy (except distortionless) lines. The third paper also gives a single set of realizability conditions for

$$\left[\begin{matrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} X \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} X \\ 1 \end{bmatrix} \\ \text{or} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} X \\ 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} X \\ 1 \end{bmatrix} \end{matrix}\right] \cdot \left[\begin{matrix} \begin{bmatrix} 1 & 0 \\ 2s & 1 \end{bmatrix}\begin{bmatrix} X \\ 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ \text{or} \\ \begin{bmatrix} s+1 & 0 \\ 2s(s+1) & (2s+1) \end{bmatrix}\begin{bmatrix} X \\ 2 \end{bmatrix}\begin{bmatrix} 2(2s+1) \\ (s+1) \end{bmatrix} \\ \text{or} \\ \begin{bmatrix} 0 & (s+1) \\ (2s+1) & 2s(s+1) \end{bmatrix}\begin{bmatrix} X \\ 1 \end{bmatrix}\begin{bmatrix} (s+1) \\ 2(2s+1) \end{bmatrix} \\ \text{or} \\ \begin{bmatrix} 0 & 1 \\ 1 & 2s \end{bmatrix}\begin{bmatrix} X \\ 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{matrix}\right]$$

$$\left[\begin{matrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} X \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} X \\ 2 \end{bmatrix} \\ \text{or} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} X \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} X \\ 2 \end{bmatrix} \end{matrix}\right] \cdot \left[\begin{matrix} \begin{bmatrix} 2s & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} X \\ 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ \text{or} \\ \begin{bmatrix} 2s(s+1) & (2s+1) \\ s+1 & 0 \end{bmatrix}\begin{bmatrix} X \\ 2 \end{bmatrix}\begin{bmatrix} 2(2s+1) \\ (s+1) \end{bmatrix} \\ \text{or} \\ \begin{bmatrix} (2s+1) & 2s(s+1) \\ 0 & (s+1) \end{bmatrix}\begin{bmatrix} X \\ 1 \end{bmatrix}\begin{bmatrix} (s+1) \\ 2(2s+1) \end{bmatrix} \\ \text{or} \\ \begin{bmatrix} 1 & 2s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} X \\ 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{matrix}\right]$$

where

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} \cosh\sqrt{(s+1)(2s+1)} & \sqrt{(s+1)/(2s+1)}\,\sinh\sqrt{(s+1)(2s+1)} \\ \sqrt{(2s+1)/(s+1)}\,\sinh\sqrt{(s+1)(2s+1)} & \cosh\sqrt{(s+1)(2s+1)} \end{bmatrix}$$

and

$$\begin{bmatrix} X \\ 2 \end{bmatrix} = \begin{bmatrix} \cosh\sqrt{(s+1)(2s+1)} & \sqrt{(2s+1)/(s+1)}\,\sinh\sqrt{(s+1)(2s+1)} \\ \sqrt{(s+1)/(2s+1)}\,\sinh\sqrt{(s+1)(2s+1)} & \cosh\sqrt{(s+1)(2s+1)} \end{bmatrix}.$$

The easiest to realize would be

$$\begin{bmatrix} \cosh 2T_0\gamma(s) & \sqrt{(s+1)/(2s+1)}\,\sinh 2T_0\gamma(s) \\ \sqrt{(2s+1)/(s+1)}\,\sinh 2T_0\gamma(s) & \cosh 2T_0\gamma(s) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2s & 1 \end{bmatrix}$$

$$\cdot\begin{bmatrix} \cosh T_0\gamma(s) & z_1 \sinh T_0\gamma(s) \\ 1/z_1 \sinh T_0\gamma(s) & \cosh T_0\gamma(s) \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

the total class of distributed cascade networks not containing lumped-element networks. A forthcoming paper, completing this series, will treat lumped-distributed cascades consisting of noncommensurate lines or transmission lines of different types.

The theory is a unified synthesis theory because it is formulated in terms of definitions which are identical for all classes of commensurate lumped-distributed cascade networks. In fact, the same definitions are also used in the formulation of the synthesis theory for non-commensurate lines.

Note that no transformations of variables are used to formulate the various realizability conditions, and that the definitions and synthesis procedures are given in terms of the algebraic-expression or real-polynomial coefficients of the prescribed Z_0. Hence, the synthesis theory is given entirely as a single-variable formulation.

The evaluation of the various definitions or determinants which yield the cascade-expansion sections can easily be obtained through a column manipulation process [3] for one determinant, D_n, since C_i, D_i, E_i are obtained from D_n by appropriate column and row deletions.

Realizability conditions for more specialized cascade configurations are easily written simply by adding appropriate conditions to those of the stated theorems. Thus if all the lumped networks in the cascade realization of a Z_0 of this paper are to be reciprocal two-ports, then the additional condition becomes

$$e_{i-1}e_i n_{0i-1}d_{0i-1}n_{0i}d_{0i}(Q_iL_i - P_iK_i) = 4M_i^2G_i^2,$$

while the additional realizability condition which permits the transfer of all loss from the lumped networks to the transmission lines in a cascade consisting of lossy lumped networks and lossless transmission lines to yield a cascade of lossless lumped networks and distortionless lines for a

$$Z_0 = \frac{\sum_{i=0}^{n} a_i e^{(2i-n)sT}}{\sum_{i=0}^{n} b_i e^{(2i-n)sT}} \qquad a_i, b_i \text{ real polynomials in } s$$

requires that

$$Z_0' = \frac{\sum_{i=0}^{n} a_i' e^{(2i-n)sT}}{\sum_{i=0}^{n} b_i' e^{(2i-n)sT}}$$

with $a_i' = e^{-(2i-n)\beta}a_i$, $b_i' = e^{-(2i-n)\beta}b_i$ satisfy the realizability conditions of the theorem in [3]. To obtain the network realization of the given Z_0 in terms of the lossless lumped networks and the distortionless lines we simply replace the lossless transmission lines in the realization of Z_0' with distortionless lines possessing a propagation constant $(sT + \beta)$.

In addition, the synthesis procedures of these papers extract the entire ith lumped two-port and its associated transmission line as an entity at each synthesis step. Hence, at most $(n+1)$ expansion steps are required to obtain a cascade representation of any given Z_0. Should any of the $C_i \equiv E_i \equiv 0$ then the corresponding lumped network is simply a connection made by a pair of wires and the corresponding transmission lines combine to a two-unit length line of the same characteristic impedance.

The commensurate specification for the n transmission lines in the cascade realization of Z_0 requires that their time constants (if they exist) be related as follows:

$$\frac{L_1}{R_1} = \cdots = \frac{L_i}{R_i} = \cdots = \frac{L_n}{R_n} \triangleq T_1$$

and/or

$$\frac{C_1}{G_1} = \cdots = \frac{C_i}{G_i} = \cdots = \frac{C_n}{G_n} \triangleq T_2$$

and that their electric lengths are given by either

$$x_i\sqrt{R_iG_i} \triangleq T_0, \quad x_i\sqrt{R_iC_i} \triangleq T_0, \quad \text{or} \quad x_i\sqrt{L_iG_i} \triangleq T_0$$

where R_i, L_i, G_i, L_i, and x_i are either the uniform or tapered (per unit length) [5] line parameters of the ith line with x_i its total length.

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Transactions Briefs

CAD Modeling of Three-Terminal Piecewise-Linear Device Characteristics

SAMIR S. ROFAIL

Abstract—**The purpose of this work is to construct a circuit analysis model synthesizing piecewise-linear characteristics of a three terminal device, using only *two-terminal* piecewise-linear resistors together with *linear* controlled sources. The model is based on representing the function, to be modeled, by a finite amount of tabulated data.**

I. Introduction

The piecewise-linear technique is one of the methods that can be applied to handle nonlinear networks [1]–[3]. In applying such technique, one could first obtain the circuit model that realizes the network characteristics then piecewise linearize its elements [2]. Another approach is to express the terminal characteristics in a piecewise-linear spatial form [4], then realize the resulting surface using the appropriate piecewise-linear elements together with controlled sources. Van Eijndhoven and Jess [6] demonstrated that piecewise-linear systems can be decomposed into subsystems which, in turn, can be stored using a hierarchical tree structure. In their work, however, the $(V-I)$ characteristics and dimensions of the *basic piecewise-linear network elements* forming a subsystem have not been investigated. This paper proposes a modeling scheme based on the second approach to realize the three-terminal piecewise-linear characteristic using *only* two-terminal piecewise-linear elements together with *linear* controlled sources. Investigating such an equivalent is conceptually important to show whether multidimensional piecewise-linear elements extend the existing range of mathematical network elements.

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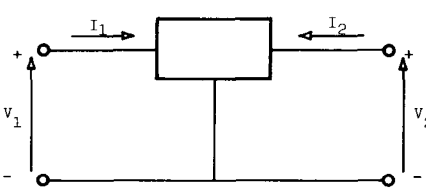

Fig. 1. Three-terminal device.

II. The Circuit Realization of a Three-Terminal PWL Characteristics

Consider the three-terminal PWL device, Fig. 1, whose characteristics are represented by the following input and output descriptions, Fig. 2(a), (b):

$$I_1 = G_1(V_1, V_2)$$
$$I_2 = G_2(V_1, V_2) \qquad (1)$$

where (V_1, V_2), (I_1, I_2) are the independent and dependent sets of variables respectively; G_1, G_2 are piecewise-linear functions and the partitioning of V_1 and V_2 into linear regions, Fig. 3, is

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