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GENERALIZATIONS OF NOETHER'S THEOREM IN CLASSICAL MECHANICS*

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Abstract. In this paper, a review is presented of various approaches to the generalization of the version of Noether's theorem, which is presented in most textbooks on classical mechanics. Its motivation is the controversy still persisting around the possible scope of a Noether-type theorem allowing for velocity-dependent transformations. Our analysis is centered around the one factor common to all known treatments, namely the structure of the related first integral. We first discuss the most general framework, in which a function of the above-mentioned structure constitutes a first integral of a given Lagrangian system, and show that one cannot really talk about an "interrelationship" between symmetries and first integrals there. We then compare different proposed generalizations of Noether's theorem, by describing the nature of the restrictions which characterize them, when they are situated within the broadest framework. We prove a seemingly new equivalence-result between the two main approaches: that of invariance of the action functional, and that of invariance of $d\theta$ (θ being the Cartan-form). A number of arguments are discussed in favor of this last version of a generalized Noether theorem.

Throughout the analysis we pay attention to practical considerations, such as the complexity of the Killing-type partial differential equations in each approach, which must be solved in order to identify "Noether-transformations".

1. Introduction. Since the publication of Emmy Noether's paper [42] on invariant variational problems more than half a century ago, there has been a never-ending stream of new contributions to the subject, aimed at establishing some generalization of the original theorem, or at clarifying certain methodological aspects. Let us mention some of the aspects which have frequently been discussed in the literature; it should be understood that the list of quoted references is not exhaustive, and that the papers in question usually contain much more than what is quoted here. In most treatments (in mechanics or in field theory), Noether-transformations are considered to be invariance transformations of an action functional (Lovelock and Rund [36], Logan [34], Hill [23]). Alternatively (but not equivalently), they can be regarded as invariance transformations of the Lagrangian density itself, up to gauge-terms (Palmieri and Vitale [45]). Still other treatments place a version of Noether's theorem, and corresponding generalizations, within the broader context of dynamical symmetries of the Euler–Lagrange equations (Katzin and Levine [27], [28], [29]). Finally, once the condition of some invariance of the action functional is no longer imposed, the second point of view allows further generalizations to be built up by allowing additional terms in the variation of the Lagrangian, which vanish along solutions of the motion equations (Candotti et al. [7], Rosen [49], [50]). In this last type of Noether-transformations, the bond with any invariance principle is completely lost, since these transformations do not even constitute dynamical symmetries of the equations of motion. Note, however, that in all cases considered above, the explicit formula for the computation of the related conservation law is the same. Obviously, if there is no agreement in the literature even about what, conceptually, should be called a Noether-transformation, questions about the existence and form of a converse to Noether's theorem (i.e., the determination of a Noether-transformation related to a given constant of the motion) must be somewhat controversial. So it is not surprising that this methodological aspect

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also has been under discussion in many publications (Fletcher [16], Dass [12], Steudel [56], Palmieri and Vitale [45], Saletan and Cromer [51], Candotti et al. [8], Crampin [11], Djukic and Vujanovic [14]).

We now focus our attention on the case of particle mechanics, more specifically, on systems described by second-order Euler–Lagrange equations. Before describing what type of generalizations are referred to in the title of the present paper, it is necessary that we first agree about the scope of the theorem to be generalized. Therefore, when talking about the *classical Noether theorem* in this paper, we will always be referring to the following statement.

Consider infinitesimal transformations of time and coordinates, whereby the first-order variations are assumed to be functionally dependent on time and coordinates only (i.e., not on velocity), then, to each such transformation, leaving the action functional invariant up to a constant (i.e., with gauge variance) corresponds a constant of the motion.

Keeping in mind a number of references quoted earlier, this is a very restrictive version of Noether's theorem. Even the original version by Noether [42], and particularly the way it is found in Bessel-Hagen's paper [5], are more general, and allow a dependence on velocities and derivatives of higher order, although the full consequences of such a dependence were not explored in any depth (see § 5 for more details). However, what we call the "classical Noether theorem" is the version which is mentioned in all the textbooks quoted earlier, and taking this as our starting point at least has the advantage that all treatments of it, although sometimes different in approach or in the complexity of the proof, are in full agreement. The disagreements start when generalizations of this theorem are presented, aimed at allowing the variations to depend on velocities. We refer here to papers by Lévy-Leblond [33], Djukic [13], Crampin [11] and Lutzky [38], for example.

A deeper analysis of these papers reveals conceptual differences, which are sometimes subtle but are nevertheless too fundamental to neglect. In other words, a generalization to velocity-dependent transformations, which one would expect to be a rather straightforward matter nowadays, still appears to create confusion. And such a generalization is needed, if only in order to establish an unambiguous inverse Noether theorem. It is interesting to note that a number of people quite recently have promoted the use of the so-called Lie-method of extended groups (which applies to general differential equations) in the case of Lagrangian systems (Prince and Eliezer [46], [47], Eliezer [15], Leach [32]). The use of Noether's theorem there, is criticized precisely because of that "troublesome" need for velocity-dependent transformations. By using, instead, the original Lie-method with velocity-independent transformations, the dimension of the Lie-algebra of infinitesimal generators is kept finite, which opens better perspectives for the determination of the complete algebra of symmetries and associated constants of the motion. We do not share this criticism of Noether's theorem, but will come back to this question later.

It is the purpose of the present paper to give a comparative survey of different approaches to the generalization of the classical Noether theorem for velocity-dependent transformations. The differences among previous treatments (or their equivalence) will be explained by situating them within the broadest possible framework. The nontrivial equivalence we will establish between the two main themes in the literature will be one of the arguments (among many others) in favor of what we feel should be called Noether's theorem. This will entail mild criticism of a too general concept of Noether-transformations, in which the elegant one-to-one correspondence between equivalence classes of symmetries and first integrals is completely lost. It is hoped that this contribution will help to resolve the confusion around Noether's theorem, although in

such an enterprise one always runs the risk of creating an opposite effect in the eyes of those who do not agree with one's ultimate conclusions.

In § 2, we briefly recall the classical Noether theorem, and prove an invariance property of the corresponding first integral, which seems to be largely unknown in the literature. As already mentioned, there is one common element in all treatments of generalizations of Noether's theorem: they all give rise to the same formula for the related first integral. We therefore will take this formula as the starting point for our analysis, which is the opposite of most other treatments. To be more precise, in § 4 we will determine necessary and sufficient conditions for the generator of an infinitesimal transformation to yield a first integral of a given Lagrangian system according to the accepted formula. This will set the stage, in a natural way, for the broadest possible version of a Noether theorem with an inverse. It will, however, also clearly show that there is too much freedom within this framework. We make use of the concise and powerful tools offered by the calculus on differentiable manifolds, but only in purely local considerations and in a way accessible to a large audience. Section 3, therefore, is devoted to a review of the way Lagrange's equations can be defined by a characteristic vector-field of the two-form $d\theta$, derived from the so-called Cartan-form [10]. It also recalls some basic results about curves on the tangent-bundle and symmetries of vectorfields and of the fundamental two-form $d\theta$.

The general scheme of the paper should now be clear. Realizing that too much freedom arises in the discussion presented in § 4, a more appropriate generalization of the classical Noether theorem must come from introducing supplementary restrictions. Therefore, the different extensions which are available in the literature, will be characterized by the nature of the supplementary restrictions they encompass.

In § 5, we distinguish between four possible restrictions. First, there is the method in which variations of the velocities are computed along arbitrary curves, leading to a "Noether-identity" which is required to hold for all t, q, $\dot{q}$, $\ddot{q}$, while an equality along integral curves of the given system would suffice to guarantee a similar invariant. This approach is, essentially, the original Noether theorem, and (apart from inevitable differences in details) can also be found in work by Djukic [13], Palmieri and Vitale [45] and Kobussen [30], [31].

A second type of restriction consists of requiring the term characterizing the gauge-variance to be independent of velocities. This has been advocated by Lutzky [38], and is also implicitly present in Lévy-Leblond's treatment [33]. Thirdly, one might think of imposing the condition that the generator of the infinitesimal transformation be a general dynamical symmetry of the vectorfield governing the given system. Finally, it might look advantageous to restrict the dynamical symmetry a bit further, by requiring the generator to be a $d\theta$-symmetry. In this last approach, a nice one-to-one correspondence between classes of symmetries and constants of the motion is most apparent (see, e.g., Crampin [11]).

Section 6 contains the key theorem of our analysis. It establishes the full equivalence between the first and the last alternative, which is an argument on its own for calling either of these the appropriate version of Noether's theorem. A number of other arguments are listed in the extensive discussion of § 7, in which, e.g., attention is paid to the connection with Hamiltonian mechanics and with the Lie-method of extended groups. Finally, the Poisson theorem in Lagrangian mechanics, which is most easily obtained within the context of $d\theta$-symmetries, is presented in an appendix.

2. The classical Noether theorem. Many excellent versions of the classical Noether theorem can be found in textbooks. We can refer e.g., to Saletan and Cromer

[51], Lovelock and Rund [36] or Logan [34]. For a rigorous modern treatment in continuum mechanics, see e.g., the paper by Trautman [58]. We therefore content ourselves here with a rather intuitive sketch, which is sufficient to provide us with formulae for later use. Consider the variational principle

$$\delta \int_{t_1}^{t_2} L(t, q(t), \dot{q}(t))\, dt = 0, \tag{1}$$

yielding the Euler–Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \qquad i = 1, \cdots, n. \tag{2}$$

The Lagrangian L is assumed to be regular, meaning that the Hessian $(\partial^2 L/\partial \dot{q}^i\, \partial \dot{q}^j)$ is invertible. Denoting the elements of the inverse matrix by $g^{jk}(t, q, \dot{q})$, we have (with summation convention)

$$\frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j} g^{jk} = \delta_i^k. \tag{3}$$

We denote the normal form of (2) by

$$\ddot{q}^i = \Lambda^i(t, q, \dot{q}), \tag{4}$$

where Λ^i is given by

$$\Lambda^i(t, q, \dot{q}) = g^{ij}\left(-\frac{\partial^2 L}{\partial \dot{q}^j\, \partial q^k}\dot{q}^k - \frac{\partial^2 L}{\partial \dot{q}^j\, \partial t} + \frac{\partial L}{\partial q^j}\right). \tag{5}$$

Consider now an infinitesimal transformation in the (t, q)-space, defined by

$$\bar{t} = t + \varepsilon\tau(t, q), \qquad \bar{q}^i = q^i + \varepsilon\xi^i(t, q), \tag{6}$$

where τ and ξ^i are functions of coordinates and time, but do not depend on velocities. By means of (6), each curve $t \to q(t)$, defined on an interval $[a, b]$, is transformed (for sufficiently small ε) into a (parameter-dependent) curve $\bar{t} \to \bar{q}(\bar{t})$ in the new variables (see Logan [34]). We then have, to first order in ε,

$$\frac{d\bar{q}^i}{d\bar{t}} = \frac{\dot{q}^i + \varepsilon\dot{\xi}^i}{1 + \varepsilon\dot{\tau}} = \dot{q}^i + \varepsilon(\dot{\xi}^i - \dot{q}^i\dot{\tau}). \tag{7}$$

The infinitesimal transformation (6) is said to leave the action integral invariant up to gauge terms, if a function $f(t, q)$ exists, such that for each differentiable curve $t \to q(t)$, we have

$$\begin{aligned}\int_{\bar{t}_1}^{\bar{t}_2} L\left(\bar{t}, \bar{q}(\bar{t}), \frac{d\bar{q}}{d\bar{t}}(\bar{t})\right) d\bar{t} &= \int_{t_1}^{t_2} L(t, q(t), \dot{q}(t))\, dt \\ &\quad + \varepsilon \int_{t_1}^{t_2} \frac{df}{dt}(t, q(t))\, dt + O(\varepsilon^2),\end{aligned} \tag{8}$$

where $[t_1, t_2]$ is any subinterval of the interval $[a, b]$ on which $q(t)$ is defined.

This will be the case if and only if

$$L\left(\bar{t}(t), \bar{q}(t), \frac{d\bar{q}}{d\bar{t}}(t)\right)\frac{d\bar{t}}{dt}(t) = L(t, q(t), \dot{q}(t)) + \varepsilon\frac{df}{dt}(t, q(t)) + O(\varepsilon^2).$$

Since this is required to hold for a whole family of curves $t \to q(t)$, we get the following identity in $t, q, \dot{q}$,

$$\frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial q^i}\xi^i + \frac{\partial L}{\partial \dot{q}^i}(\dot{\xi}^i - \dot{q}^i\dot{\tau}) + L\dot{\tau} = \frac{df}{dt}. \tag{9}$$

After some straightforward manipulations, this can be rewritten as

$$(\xi^i - \dot{q}^i\tau)\left[\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\right] + \frac{d}{dt}\left[L\tau + \frac{\partial L}{\partial \dot{q}^i}(\xi^i - \dot{q}^i\tau)\right] = \frac{df}{dt}. \tag{10}$$

An infinitesimal transformation (6), satisfying (10) for a given Lagrangian system and some f, will be called a *classical Noether-transformation* corresponding to L, and thus, we have the following theorem.

THEOREM 2.1. (Classical Noether theorem). *To each Noether-transformation* (6) *corresponds a constant of the motion* $F(t, q, \dot{q})$, *given by*

$$F(t, q, \dot{q}) = f(t, q) - \left[L\tau + \frac{\partial L}{\partial \dot{q}^i}(\xi^i - \dot{q}^i\tau)\right]. \tag{11}$$

This result follows trivially from (10).

Remark. While (10) was needed in order to recognize the explicit form of the related first integral, it is (9) which can be used as a partial differential equation to determine the components ξ^i, τ of a Noether-transformation, where of course, $\dot{\tau}$ and $\dot{\xi}^i$ must be interpreted as,

$$\dot{\tau} = \frac{\partial \tau}{\partial t}(t, q) + \dot{q}^i\frac{\partial \tau}{\partial q^i}(t, q), \qquad \dot{\xi}^i = \frac{\partial \xi^i}{\partial t}(t, q) + \dot{q}^j\frac{\partial \xi^i}{\partial q^j}(t, q). \tag{12}$$

We will refer to (9) as a *Killing-type equation.* The concept of a Killing-equation (Killing-vectorfield, etc.) is well known in Riemannian geometry. The use of this terminology in the present context can be motivated as follows. For a particle without external forces, and for a group of infinitesimal transformations (6) in which time is preserved, invariance of the action integral implies invariance of the metric tensor characterizing the kinetic energy of the particle, and therefore yields the original Killing-equation. The system of partial differential equations for ξ^i and τ, which results from (9) after substitution of a general Lagrangian which is polynomial in $\dot{q}$, have been called generalized Killing-equations by various authors (see e.g., Vujanovic [59], Djukic [13] and Logan [34]). For the sake of having a common terminology for comparable equations in the various approaches discussed later on, we will go one step further. Specifically, we will talk about Killing-type equations, whenever we encounter the set of partial differential equations from which Noether-transformations have to be determined in each approach. These equations, of course, are no longer related to invariance of a metric tensor.

Returning to (6), we can introduce the so-called generator of this infinitesimal transformation, namely the differential operator

$$Y^{(0)} = \tau(t, q)\frac{\partial}{\partial t} + \xi^i(t, q)\frac{\partial}{\partial q^i}, \tag{13}$$

and its extension to $(t, q, \dot{q})$-space,

$$Y^{(1)} = \tau(t, q)\frac{\partial}{\partial t} + \xi^i(t, q)\frac{\partial}{\partial q^i} + \eta^i(t, q, \dot{q})\frac{\partial}{\partial \dot{q}^i}, \tag{14}$$

with

$$\eta^i = \dot{\xi}^i - \dot{q}^i \dot{\tau}, \tag{15}$$

$\dot{\xi}^i$ and $\dot{\tau}$ being defined by (12).

We then come to an interesting property of the Noether-invariant (11), which is almost never mentioned in the quoted literature, at least not in the context of Lagrangian mechanics. The only explicit proof of it of which we are aware has been given by Lutzky [37] for the case of a system with one degree of freedom. In anticipation of the rest of the paper, we can announce, however, that the meaning of this property as well as the proof of it are much simpler when reinterpreted in the Hamiltonian framework (see § 7).

PROPOSITION 2.2. *The Noether-invariant* (11) *is also an invariant of the generator* $Y^{(1)}$ *of the Noether-symmetry itself, i.e.,*

$$Y^{(1)}(F) = 0. \tag{16}$$

Proof. We have from (11), in view of the independence of f, ξ^i and τ on $\dot{q}$,

$$\begin{aligned} Y^{(1)}(F) = {} & Y^{(0)}(f) - Y^{(1)}(L)\tau - L Y^{(0)}(\tau) \\ & - \frac{\partial L}{\partial \dot{q}^i} Y^{(1)}(\xi^i - \dot{q}^i \tau) - \left(Y^{(1)}\left(\frac{\partial L}{\partial \dot{q}^i} \right) \right)(\xi^i - \dot{q}^i \tau). \end{aligned} \tag{17}$$

Using the fact that F is a constant of the motion of system (4), we get

$$Y^{(1)}(L) = \dot{f} - L\dot{\tau}. \tag{18}$$

In addition, we have the identity

$$Y^{(1)}\left(\frac{\partial L}{\partial \dot{q}^i} \right) = \frac{\partial}{\partial \dot{q}^i}(Y^{(1)}(L)) - \frac{\partial L}{\partial \dot{q}^k} \frac{\partial \eta^k}{\partial \dot{q}^i}. \tag{19}$$

Using (18) and (19), (17) becomes,

$$\begin{aligned} Y^{(1)}(F) = {} & \left[Y^{(0)}(f) - \dot{f}\tau - \frac{\partial f}{\partial q^i}(\xi^i - \dot{q}^i \tau) \right] \\ & + L\left[\tau\dot{\tau} - Y^{(0)}(\tau) + \frac{\partial \tau}{\partial q^i}(\xi^i - \dot{q}^i \tau) \right] \\ & - \frac{\partial L}{\partial \dot{q}^k}\left[Y^{(1)}(\xi^k - \dot{q}^k \tau) - \frac{\partial \eta^k}{\partial \dot{q}^l}(\xi^l - \dot{q}^l \tau) - \dot{\tau}(\xi^k - \dot{q}^k \tau) \right], \end{aligned} \tag{20}$$

and it is straightforward to verify that the three expressions between square brackets in (20) vanish identically.

Operators of the form (13) also play the role of generators of symmetries in the so-called Lie-method of extended groups, which is applicable to general ordinary or partial differential equations (see Ovsjannikov [44] and Bluman and Cole [6]). Only recently, attempts have been made to introduce also in that context the notion of constants of the motion implied by a symmetry group, precisely by requiring it to have the invariance property (16) (see, e.g., Lutzky [38], Prince and Eliezer [47] and Leach [32]). A similar idea, in the context of Lagrangian systems with one degree of freedom, was expressed by Sarlet [52], also for certain simple cases of discrete symmetries (such as time-inversion). The above proposition therefore is important, because it illustrates how this idea is consistent with the classical Noether theory.

3. Preliminaries on Lagrangian systems and calculus on manifolds. It is of course impossible to give a complete introduction to the most fundamental notions of the calculus on manifolds. We will, however, try to list here some basic operations and properties which will be frequently used. For general background reading one can refer, e.g., to Hermann [22], and to the appendix on tensors and forms in Lovelock and Rund [36]. As said before, our analysis will be purely local in character. We will closely follow, in this section, the exposé given by Crampin [11].

Let M be a differentiable manifold of dimension n, and TM its tangent bundle. Adding the time-axis R, we get the bundle $R \times TM$, on which we choose a set of natural coordinates denoted by $(t, q^i, \dot{q}^i)$, $i = 1, \cdots, n$. As is well known, vectorfields on a general manifold N can be regarded as differential operators on $\mathcal{F}(N)$, the set of C^∞-functions on the manifold. One-forms are $\mathcal{F}(N)$-linear functionals on the set of vectorfields. More generally, p-forms are alternating $\mathcal{F}(N)$-multilinear functionals acting on vectorfields. In terms of the above local coordinates, a vectorfield X, and a 1-form α on $R \times TM$ have the representation,

$$X = h\frac{\partial}{\partial t} + f^i\frac{\partial}{\partial q^i} + g^i\frac{\partial}{\partial \dot{q}^i}, \qquad \alpha = \nu\, dt + \lambda_i\, dq^i + \mu_i\, d\dot{q}^i,$$

where the components h, f^i, g^i, ν, λ_i, μ_i are real-valued C^∞-functions on $R \times TM$. Pairing between the dual elements X and α yields the function

$$\alpha(X) = \langle X, \alpha \rangle = h\nu + \lambda_i f^i + \mu_i g^i. \tag{21}$$

The components of a vectorfield determine locally a system of first-order differential equations, whose solution curves are called integral curves of the given vectorfield. They define locally a 1-parameter family of mappings on the manifold, which is called the "flow" of the given vectorfield. In order to represent second-order equations in q by vectorfields on $R \times TM$, we have to pass to the equivalent first-order system in q, $\dot{q}$. A system like (4), e.g., is governed by the vectorfield

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^i\frac{\partial}{\partial q^i} + \Lambda^i\frac{\partial}{\partial \dot{q}^i}. \tag{22}$$

Integral curves of a vectorfield of type (22) are liftings of curves on the base manifold M. In general, if for t in some open interval $I \subset R$, the mapping $t \to q^i(t)$ is the local representation of a curve on M, then its lifting to $R \times TM$ is defined by the mapping

$$t \to (t, q^i(t), \dot{q}^i(t)), \quad \text{with } \dot{q}^i = \frac{dq^i}{dt}.$$

After a change of parametrization, the $\partial/\partial t$ component of a vectorfield tangent to a lifted curve need not be 1. In order that integral curves of a vectorfield X on $R \times TM$ be lifted curves, it is necessary and sufficient that

$$\langle X, dq^i - \dot{q}^i\, dt \rangle = 0, \qquad i = 1, \cdots, n. \tag{23}$$

Apart from the exterior derivative of differential forms, we will also make use of the Lie-derivative, and the inner product of a vectorfield with a differential form. The Lie-derivative of a general tensorfield with respect to a vectorfield Y is the appropriate operator for characterizing the evolution of the tensorfield under the flow of Y. For functions f and vectorfields X, we have

$$L_Y f = Y(f), \qquad L_Y X = [Y, X] = YX - XY.$$

The inner product of a vectorfield Y with a p-form ω yields a $(p-1)$-form, denoted by $i_Y\omega$ and defined by

$$i_Y\omega(X_1, \cdots, X_{p-1}) = \omega(Y, X_1, \cdots, X_{p-1}).$$

A vectorfield Y satisfying

$$i_Y\omega = 0, \qquad i_Y d\omega = 0 \tag{24}$$

is called a *characteristic vectorfield* of the p-form ω. Further properties which will be frequently used are (f being a function and α a p-form)

$$L_Y d\alpha = dL_Y\alpha, \tag{25}$$

$$i_{fX}\alpha = fi_X\alpha, \tag{26}$$

$$L_Y\alpha = i_Y d\alpha + di_Y\alpha, \tag{27}$$

$$i_{[X,Y]}\alpha = i_X L_Y\alpha - L_Y i_X\alpha. \tag{28}$$

We also recall that every exact form is closed, i.e., $d^2 = 0$, and that every closed form is locally exact (Poincaré's lemma), i.e.,

$$d\alpha = 0 \Rightarrow \alpha = d\beta \tag{29}$$

(possibly in a smaller neighborhood).

As an illustration of the conciseness with which certain evolution or conservation properties can be expressed in terms of the Lie-derivative, let us mention that the flow of a vectorfield Y on $R \times TM$ maps every lifted curve into a lifted curve if and only if

$$L_Y(dq^i - \dot{q}^i\, dt) = \lambda^i_j(dq^j - \dot{q}^j\, dt), \qquad i = 1, \cdots, n, \tag{30}$$

where the λ^i_j are functions on $R \times TM$.

Now, let us develop a definition of Lagrangian systems on the manifold $R \times TM$. Let L be a function on $R \times TM$, which satisfies the regularity condition on the Hessian, as in § 2, and consider the 1-form

$$\theta = Ldt + \frac{\partial L}{\partial \dot{q}^i}(dq^i - \dot{q}^i\, dt). \tag{31}$$

θ is called the *Cartan-form*; it is the pullback, under the Legendre transform, of the fundamental 1-form $p_i\, dq^i - H\, dt$ in Hamiltonian mechanics. We have

$$\begin{aligned} d\theta = &\frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j}(dq^j - \dot{q}^j\, dt) \wedge (dq^i - \dot{q}^i\, dt) \\ &+ \frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j}(d\dot{q}^i - \Lambda^i\, dt) \wedge (dq^j - \dot{q}^j\, dt), \end{aligned} \tag{32}$$

where the functions Λ^i are defined by (5). In view of the regularity of L, it is easy to verify that $d\theta$ has rank $2n$, i.e., the closed two-form $d\theta$ defines a so-called contact structure on the $(2n+1)$-dimensional manifold $R \times TM$. The space of characteristic vectorfields of $d\theta$ is one-dimensional, and the unique characteristic vectorfield with time-component one, defines the Lagrangian system corresponding to L; i.e., we define the vectorfield Γ by

$$i_\Gamma d\theta = 0, \tag{33}$$

$$\langle \Gamma, dt \rangle = 1. \tag{34}$$

Using the explicit expression (32) of $d\theta$, one easily verifies that condition (33) is equivalent to

$$\langle \Gamma, dq^i - \dot{q}^i\, dt\rangle = 0, \tag{35}$$

and

$$\langle \Gamma, d\dot{q}^i - \Lambda^i\, dt\rangle = 0. \tag{36}$$

The relations (34), (35), (36) then show that Γ is indeed the vectorfield (22), with Λ^i determined by (5). For later use, it is worthwhile writing down explicitly the formula by which the above stated equivalence, e.g., can be checked. Let Y be an arbitrary vectorfield on $R \times TM$, with local representation

$$Y = \tau(t, q, \dot{q})\frac{\partial}{\partial t} + \xi^i(t, q, \dot{q})\frac{\partial}{\partial q^i} + \eta^i(t, q, \dot{q})\frac{\partial}{\partial \dot{q}^i}. \tag{37}$$

Then we have

$$\begin{aligned} i_Y d\theta = {} & \left(\frac{\partial^2 L}{\partial \dot{q}^i\, \partial q^j} - \frac{\partial^2 L}{\partial \dot{q}^j\, \partial q^i}\right)(\xi^j - \dot{q}^j\tau)(dq^i - \dot{q}^i\, dt) \\ & + \frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j}(\eta^i - \Lambda^i\tau)(dq^j - \dot{q}^j\, dt) - \frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j}(\xi^j - \dot{q}^j\tau)(d\dot{q}^i - \Lambda^i\, dt). \end{aligned} \tag{38}$$

Note that the set of 1-forms

$$\{dt, dq^i - \dot{q}^i\, dt, d\dot{q}^i - \Lambda^i\, dt\}, \tag{39}$$

forms a local basis for all 1-forms on $R \times TM$. Of course, for the purposes of introducing such a basis, the given second-order system need not be of Lagrangian type; i.e., the Λ^i need not satisfy (5). Let us introduce the mapping ρ, which assigns a 1-form to each vectorfield Y on $R \times TM$, according to the rule

$$\rho(Y) = i_Y\, d\theta. \tag{40}$$

Since the kernel of this mapping, i.e., the set of characteristic vectorfields of $d\theta$, is one-dimensional, we see from (38) that the image of ρ, (Im (ρ)), is precisely the set of 1-forms generated by the $2n$ forms $dq^i - \dot{q}^i\, dt$ and $d\dot{q}^i - \Lambda^i\, dt$. In view of (35), (36), it is then clear that for an arbitrary 1-form α we have,

$$\langle \Gamma, \alpha\rangle = 0 \Leftrightarrow \alpha \in \text{Im}\,(\rho). \tag{41}$$

In terms of the basis (39), we can write, for an arbitrary function $F \in \mathcal{F}(R \times TM)$,

$$dF = \Gamma(F)\, dt + \frac{\partial F}{\partial q^i}(dq^i - \dot{q}^i\, dt) + \frac{\partial F}{\partial \dot{q}^i}(d\dot{q}^i - \Lambda^i\, dt), \tag{42}$$

while for a vectorfield Y, as in (37), we have

$$Y(F) = \tau\Gamma(F) + (\xi^i - \dot{q}^i\tau)\frac{\partial F}{\partial q^i} + (\eta^i - \Lambda^i\tau)\frac{\partial F}{\partial \dot{q}^i}. \tag{43}$$

Furthermore, F is a constant of the motion of the dynamical system generated by Γ if and only if $\Gamma(F) = 0$. Hence, from (42),

$$\Gamma(F) = 0 \Leftrightarrow dF \in \text{Im}\,(\rho). \tag{44}$$

To end this section, we need some notions about the concept of *symmetry*. In general, one can say that a vectorfield Y is a symmetry of a certain tensorfield, if that tensorfield

is invariant under the flow of Y. In this sense, Y is a symmetry of another vectorfield X if and only if

$$L_Y X = [Y, X] = 0, \tag{45}$$

and Y is a symmetry e.g., of the 2-form $d\theta$ if and only if

$$L_Y d\theta = 0. \tag{46}$$

However, when dealing with a system of differential equations, say, governed by a vectorfield Γ, the strict notion of symmetry incorporated in (45) is too restrictive. Indeed, all one is really interested in is that the flow of Y maps integral curves of Γ into integral curves. For that to be the case, the system of differential equations need not be strictly invariant, since one can allow the additional freedom of changing the parametrization along integral curves. This is reflected in the requirement that

$$L_Y \Gamma = [Y, \Gamma] = g\Gamma \tag{47}$$

for some function g. A vectorfield Y satisfying (47) will be called a *dynamical symmetry* of Γ. If Y and Γ are respectively given by (37) and (22), we have for their Lie-bracket

$$[Y, \Gamma] = (\eta^i - \Gamma(\xi^i)) \frac{\partial}{\partial q^i} + (Y(\Lambda^i) - \Gamma(\eta^i)) \frac{\partial}{\partial \dot{q}^i} - \Gamma(\tau) \frac{\partial}{\partial t}. \tag{48}$$

From this we can easily deduce the following result.

LEMMA 3.1. *Y is a dynamical symmetry of Γ if and only if*

$$\eta^i = \Gamma(\xi^i) - \dot{q}^i \Gamma(\tau), \tag{49}$$

$$\Gamma(\eta^i) - \Lambda^i \Gamma(\tau) - Y(\Lambda^i) = 0. \tag{50}$$

Proof. The proof follows immediately from the identification of (48) with $g\Gamma$. We obtain, in addition, that

$$g = -\Gamma(\tau). \tag{51}$$

Remark. For this result, the system governed by Γ need not be of Lagrangian type, in other words conditions (49), (50) are valid for a dynamical symmetry of an arbitrary second-order system (4), with Λ^i not necessarily satisfying (5). A term like $\Gamma(\tau)$ is, of course, nothing but the total time-derivative of $\tau(t, q, \dot{q})$ along solutions of the system (4). In this way we recover, with (49), (50), the conditions which, in the context of the generalization of the "Lie-method of extended groups" to velocity-dependent transformations, were derived, e.g., by Anderson and Davison [2] and Lutzky [38].

A special class of dynamical symmetries for Lagrangian systems is provided by the symmetries of the contact-form $d\theta$.

LEMMA 3.2. *A $d\theta$-symmetry is a dynamical symmetry of the Lagrangian vectorfield Γ.*

Proof. Using (28) we get

$$\begin{aligned} i_{[\Gamma, Y]} d\theta &= i_\Gamma L_Y d\theta - L_Y i_\Gamma d\theta \\ &= 0, \quad \text{in view of (46) and (33).} \end{aligned}$$

Since the set of characteristic vectorfields of $d\theta$ is one-dimensional, it follows that $[\Gamma, Y]$ must be proportional to Γ.

As a final remark, it is worthwhile giving a precise characterization of (49). The flow of a vectorfield Y satisfying (49) transforms integral curves of Γ into lifted curves,

as can be seen from the following argument. In view of (23), the transformed integral curves of Γ will be lifted curves, if and only if

$$\langle L_Y\Gamma, dq^i - \dot{q}^i\, dt\rangle = 0.$$

Taking the Lie-derivative with respect to Y of (35), we see that this is equivalent to

$$\langle \Gamma, L_Y(dq^i - \dot{q}^i\, dt)\rangle = 0,$$

or in view of (41), with

$$L_Y(dq^i - \dot{q}^i\, dt) \in \mathrm{Im}\,(\rho). \tag{52}$$

Now,

$$L_Y(dq^i - \dot{q}^i\, dt) = d\xi^i - \dot{q}^i\, d\tau - \eta^i\, dt.$$

Using (42) it is then clear that the dt-component (in the basis (39)) will vanish if and only if (49) holds.

The reason why we pay some attention to this characterization of (49) is that it originates from a very natural relaxation of the restrictions which arise if Noether's theorem is placed in its original context, that of the study of invariances of the action functional, and if, moreover, we want to generate the Noether-transformation by a vectorfield on $R \times TM$. Indeed, a functional like $\int L\, dt$ in the variational principle (1) clearly acts on a class of neighboring lifted curves. Hence, a particular one-parameter group of invariance transformations in that context will have to be sorted out from a class of mappings which allow the association of a new curve $\bar{t} \to \bar{q}(\bar{t})$ with each curve $t \to q(t)$. If such a mapping were generated by a vectorfield on $R \times TM$, we would end up with the requirement (30), and this would lead to severe restrictions on the velocity dependence of the functions τ and ξ^i, namely

$$\frac{\partial \xi^i}{\partial \dot{q}^j} - \dot{q}^i \frac{\partial \tau}{\partial \dot{q}^j} = 0,$$

which can most easily be seen from (7) (see also Crampin [11]). Actually, since this relation must hold for all i and j, one can easily deduce from it that ξ^i and τ must be independent of $\dot{q}$, except in the case of one degree of freedom. A similar result for the more general case of several independent variables was proved by Ovsjannikov, and reported in work by Ibragimov and Anderson [3], [24]. If the $\dot{q}$-dependence is not to be compromised from the beginning, and since our primary interest lies in the way integral curves of Γ transform, a natural relaxation is obtained when the admissible mappings at least transform integral curves of Γ (but not necessarily all lifted curves) into lifted curves.

4. Nonsymmetries and the Noether-invariant. As indicated in the introduction, we want to center our analysis around the explicit formula for the Noether-invariant, which in all treatments is the same. This explicit formula, for velocity-dependent transformations, has exactly the same structure as (11), but, of course, with f, τ and ξ^i functions of t, q and $\dot{q}$. In the terminology of the previous section we have

$$F = f - \langle Y, \theta\rangle. \tag{53}$$

So we can simply ask under what conditions for Y will F be an invariant for the system governed by Γ.

If F is a constant of the motion, we have

$$\Gamma(F) = \langle \Gamma, dF\rangle = 0.$$

On the other hand, since Γ is characteristic for $d\theta$,

$$0 = i_Y i_\Gamma \, d\theta = -\langle \Gamma, i_Y \, d\theta \rangle.$$

Combining these two results, we get

$$\langle \Gamma, i_Y \, d\theta - d(f - \langle Y, \theta \rangle) \rangle = 0,$$

from which, by (41), it follows that

$$i_Y \, d\theta = d(f - \langle Y, \theta \rangle) + \alpha \quad \text{with } \alpha \in \mathrm{Im}\,(\rho),$$

or, using property (27),

$$L_Y \theta = df + \alpha \quad \text{with } \alpha \in \mathrm{Im}\,(\rho). \tag{54}$$

In fact, in this context, no other requirements have to be imposed on Y. For a simple comparison with classical treatments, however, where the variation of $L\,dt$ is computed (or better the variation of its pullback $\gamma^*(L\,dt)$ under a lifted curve γ) and not the variation of θ, it is sufficient to add the mild restriction discussed in the previous section, namely, that Y should map integral curves of Γ into lifted curves. Indeed, in view of (52), (54) is then equivalent to,

$$L_Y(L\,dt) = df + \beta \quad \text{with } \beta \in \mathrm{Im}\,(\rho). \tag{55}$$

We reach, in this way, a framework in which, for velocity-dependent transformations, a Noether-type theorem with inverse can be formulated in the broadest possible way.

PROPOSITION 4.1. *Let Y be a vectorfield with property* (49), *and such that*

$$L_Y(L\,dt) = df + \beta$$

for some function f and some $\beta \in \mathrm{Im}\,(\rho)$. Then $F = f - \langle Y, \theta \rangle$ is a constant of the motion.

Proof. The proof consists in walking in the opposite order through the previous considerations, from (55) back to (53).

Conversely, we can state Proposition 4.2.

PROPOSITION 4.2. *To each constant of the motion F of Γ corresponds a vectorfield Y with property* (49), *such that* (55) *holds for some β and for*

$$f = F + \langle Y, \theta \rangle. \tag{56}$$

Proof. F a constant of Γ implies, according to (44), that $dF \in \mathrm{Im}\,(\rho)$. Hence there exists a vectorfield Y such that

$$i_Y \, d\theta = dF. \tag{57}$$

This implies

$$0 = d^2F = d i_Y \, d\theta = L_Y \, d\theta.$$

Hence Y is a $d\theta$-symmetry and has property (49) in view of Lemma 3.2. Moreover, (57), with the identity (27), yields (54), with $\alpha = 0$, and f given by (56), from which (55) follows.

The above propositions constitute, essentially, the type of general Noether theorem which (in classical field theory) was discussed, e.g., by Candotti et al. [7], and further generalized by Rosen [49], [50]. It looks attractive, and can certainly be useful in the search for constants of the motion. However, as a theoretical result, establishing a link between "Noether-transformations" and constants of the motion, it is a bit misleading, because there is too much freedom in the relationship, as illustrated by the following result.

PROPOSITION 4.3. *Every vectorfield Y with property* (49) *corresponds to every constant of the motion F via the rule* (55), *with f determined by* (56).

Proof. Let Y be an arbitrary vectorfield with property (49). Since (49) is the translation of (52), we have

$$\begin{aligned} L_Y(L\,dt) &= L_Y\theta + \alpha \\ &= d\langle Y, \theta\rangle + i_Y\,d\theta + \alpha \\ &= d(F + \langle Y, \theta\rangle) + \beta \end{aligned}$$

with $\alpha \in \mathrm{Im}\,(\rho)$ and with $\beta = -dF + i_Y\,d\theta + \alpha \in \mathrm{Im}\,(\rho)$, in view of the definition (40) of ρ and (44).

Note that Y here need not constitute any type of "symmetry". In our opinion, an elegant theoretical result must establish a sort of uniqueness in the relationship between Noether-transformations and constants of the motion, and, therefore, in the present framework must arise from some kind of supplementary restriction. For that reason we wish to test, in the next section, various versions of the Noether theorem in the literature, for the type of restrictions by which they are characterized when placed within the present general scheme. But first we derive Killing-type equations, i.e., partial differential equations for the components of a vectorfield Y, in this broadest possible framework. The following lemma will be useful for that; it is merely a paraphrase of the reasoning at the beginning of this section.

LEMMA 4.4. *F is a constant of the motion of* Γ; *i.e.*, $\Gamma(F) = 0$ *if and only if there exist functions $\mu_i(t, q, \dot{q})$ such that the relation*

$$\frac{dF}{dt} = \mu_i(\ddot{q}^i - \Lambda^i), \tag{58}$$

holds as an identity in $t, q, \dot{q}, \ddot{q}$.

Proof. If (58) holds for all $\ddot{q}$, we can take $\ddot{q}^i = \Lambda^i$, which yields $\Gamma(F) = 0$. In general we have the identity

$$\frac{dF}{dt} = \Gamma(F) + \frac{\partial F}{\partial \dot{q}^i}(\ddot{q}^i - \Lambda^i),$$

from which the converse follows with $\mu_i = \partial F/\partial \dot{q}^i$.

We now explicitly compute the identity (58) with F replaced by its expression (53). Since (58) is a linear relation in the $\ddot{q}^i$, requiring it to hold for all $t, q, \dot{q}, \ddot{q}$ will give rise to $n+1$ equations, which after straightforward calculations take the form

$$L\frac{\partial \tau}{\partial \dot{q}^j} + \frac{\partial L}{\partial \dot{q}^i}\left(\frac{\partial \xi^i}{\partial \dot{q}^j} - \dot{q}^i\frac{\partial \tau}{\partial \dot{q}^j}\right) = \frac{\partial f}{\partial \dot{q}^j} - (\xi^i - \dot{q}^i\tau)\frac{\partial^2 L}{\partial \dot{q}^i\,\partial \dot{q}^j} - \mu_j, \tag{59}$$

$$\begin{aligned} &\tau\frac{\partial L}{\partial t} + \xi^i\frac{\partial L}{\partial q^i} + L\left(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial q^i}\dot{q}^i\right) + \frac{\partial L}{\partial \dot{q}^i}\left[\frac{\partial \xi^i}{\partial t} + \frac{\partial \xi^i}{\partial q^j}\dot{q}^j - \dot{q}^i\left(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial q^j}\dot{q}^j\right)\right] \\ &\qquad = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^i}\dot{q}^i + (\xi^i - \dot{q}^i\tau)\frac{\partial^2 L}{\partial \dot{q}^i\,\partial \dot{q}^j}\Lambda^j + \mu_i\Lambda^i. \end{aligned} \tag{60}$$

How can we interpret the relations (59), (60)? We see that the μ_i in fact, are additional parameters, which we have at our disposal for finding solutions for τ and ξ^i. In all generality, the problem consists in finding functions τ and ξ^i such that (59), (60) hold identically for some functions f and μ_i. Note that the components η^i do not

occur in these equations, but once a solution for τ and ξ^i is found, we can again define the η^i-components by (49), following the same argument as for (54).

As a final remark, notice that the Killing-type equation (9), obtained for the classical Noether theorem, indeed constitutes a particular case of (59), (60). It is obtained when τ, ξ^i and f are required to be independent of the $\dot{q}$, and μ_i is chosen such that

$$\mu_i = -(\xi^j - \dot{q}^j\tau)\frac{\partial^2 L}{\partial \dot{q}^j\,\partial \dot{q}^i}. \tag{61}$$

5. Restricting the excessive freedom for Noether-transformations. We will distinguish among four possibilities for imposing supplementary restrictions on the too general Noether-type theorem, Proposition 4.1 of the previous section.

5.1. A strict interpretation of the Killing-type identity. Let us return here to the classical Noether theorem as treated in § 2. It is obvious that this treatment offers the possibility for a direct generalization to velocity-dependent transformations,

$$\bar{t} = t + \varepsilon\tau(t, q, \dot{q}), \qquad \bar{q}^i = q^i + \varepsilon\xi^i(t, q, \dot{q}). \tag{62}$$

Expressing invariance of the action integral as in (8), we end up with an equation of type (9), where f, of course, is allowed to depend on $\dot{q}$ and where $\dot{\xi}^i$ and $\dot{\tau}$ are computed along arbitrary curves $t \to q(t)$. Equation (9) in this case is an identity in t, q, $\dot{q}$ and $\ddot{q}$, which means that the coefficients of $\ddot{q}^i$ have to vanish separately, yielding the equations

$$L\frac{\partial\tau}{\partial\dot{q}^j} + \frac{\partial L}{\partial\dot{q}^i}\left(\frac{\partial\xi^i}{\partial\dot{q}^j} - \dot{q}^i\frac{\partial\tau}{\partial\dot{q}^j}\right) = \frac{\partial f}{\partial\dot{q}^j}, \tag{63}$$

$$\tau\frac{\partial L}{\partial t} + \xi^i\frac{\partial L}{\partial q^i} + L\left(\frac{\partial\tau}{\partial t} + \dot{q}^i\frac{\partial\tau}{\partial q^i}\right) + \frac{\partial L}{\partial\dot{q}^i}\left[\frac{\partial\xi^i}{\partial t} + \frac{\partial\xi^i}{\partial q^j}\dot{q}^j - \dot{q}^i\left(\frac{\partial\tau}{\partial t} + \dot{q}^j\frac{\partial\tau}{\partial q^j}\right)\right] = \frac{\partial f}{\partial t} + \dot{q}^i\frac{\partial f}{\partial q^i}. \tag{64}$$

The Killing-type equations (63), (64) were derived and studied in more detail by Djukic [13]. This version of Noether's theorem can also be found in work by Kobussen [30], [31]. In the first place, however, we should stress that Noether's original version [42] allowed for this $\dot{q}$-dependence in the transformations (62), and even for a dependence on higher-order derivatives. But Noether did not investigate to what extent such a dependence was needed, or important, or essential. We do not believe, e.g., that a $\ddot{q}$-dependence could substantially enlarge the picture here.

One might be tempted to introduce a dependence on higher-order derivatives, and then require that such transformations preserve the higher order tangency of curves. Just as with the preservation of lifted curves explained in § 3, however, such a requirement eventually would force the functions ξ^i and τ to depend on t and q only, unless one includes the derivatives of all orders up to infinity. Such questions, related to Lie tangent transformations and Lie-Bäcklund tangent transformations, are extensively discussed in references [3] and [24].

Let us now return to (62) and the resulting Killing-type equations (63), (64). Since this is really part of the original Noether theorem, one might wonder what the "restriction" is in this approach. That we indeed have a restriction, here, of the general setting in the previous section, is seen from the fact that (63), (64) follow from (59), (60) by fixing the μ_i according to (61). We can still better characterize the nature of

the present restriction as follows: (9) must express the constancy of a function F of the structure (11) (or (53)). For that purpose it is sufficient merely to compute derivatives like $\dot{\tau}$, $\dot{\xi}^i$ and $\dot{f}$ along integral curves of Γ. From this point of view therefore, the restriction under consideration arises from a strict interpretation of the Killing equation (9), where the $\ddot{q}^i$ are kept arbitrary, instead of being replaced by $\Lambda^i(t, q, \dot{q})$. Let us now translate (63), (64) into the terminology of vectorfields and differential forms. First, it should be noted that such a translation is not immediate. Indeed, (62), when completed by the transformation of the velocities

$$\frac{d\bar{q}^i}{d\bar{t}} = \dot{q}^i + \varepsilon(\dot{\xi}^i - \dot{q}^i\dot{\tau}), \tag{65}$$

as in (7), cannot represent the flow of a vectorfield on $R \times TM$, because of the $\ddot{q}$-dependence in (65). A posteriori, however, we can associate a vectorfield Y with a transformation like (62) by adding a prescription for the $\partial/\partial\dot{q}^i$-components η^i of Y, since after all the ultimate equations to be solved (63), (64) only involve τ and ξ^i. To be precise, to each $(n+1)$-tuple of functions (τ, ξ^i), satisfying (63), (64) for some f, we can unequivocally associate a vectorfield Y as in (37), with η^i defined by (49) (see also the discussion at the end of § 3). It is then straightforward to verify that the system (63), (64) can be rewritten in the form

$$\left\langle \frac{\partial}{\partial\dot{q}^j}, L_Y\theta - df \right\rangle = 0, \tag{66}$$

$$\langle \Gamma, d(f - \langle Y, \theta \rangle) \rangle = 0. \tag{67}$$

In other words, by this translation an infinitesimal transformation is said to be a Noether-transformation if it is generated by a vectorfield Y satisfying (66), (67) and (49), for some "gauge-function" f. The corresponding constant of the motion is as usual related to Y by the formula

$$F = f - \langle Y, \theta \rangle.$$

Concerning the freedom in this relationship, we can make the following observation. Let F be a constant of the motion corresponding to Y, then the same F also corresponds to the Noether-transformations $Y' = Y + h\Gamma$, where h is an arbitrary function of $t, q, \dot{q}$. Indeed, defining f' to be $f + hL$, we have

$$\begin{aligned} f' - \langle Y', \theta \rangle &= f + hL - \langle Y, \theta \rangle - h\langle \Gamma, \theta \rangle \\ &= f - \langle Y, \theta \rangle = F, \end{aligned} \tag{68}$$

and, using (33), (27) and (26),

$$\begin{aligned} L_{Y'}\theta - df' &= L_Y\theta + di_{h\Gamma}\theta - df - d(hL) \\ &= L_Y\theta - df, \end{aligned} \tag{69}$$

from which the result follows.

We will later see indirectly that there is an inverse Noether theorem in this framework, and that the above described freedom in the Y is the only freedom.

5.2. Restriction on the gauge-function f. As said before (disregarding the approach which led to (9)), we can consider that equation as expressing the constancy of a function F of the structure (11), in which case the total time-derivatives would be computed along integral curves of Γ. It is not so unreasonable, however, to compute

the left-hand side first, with $\dot{\xi}^i = \Gamma(\xi^i)$, $\dot{\tau} = \Gamma(\tau)$, and to require the resulting expression to be the total time-derivative of some function f. This was done e.g., in a paper by Lévy-Leblond [33]. Clearly, this procedure inevitably requires the gauge-function f to be independent of the velocities, which is also the restriction recently advocated by Lutzky [38]. The explicit form of the Killing-type equation in this approach is now clear,

$$\frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial q^i}\xi^i + \frac{\partial L}{\partial \dot{q}^i}(\Gamma(\xi^i) - \dot{q}^i\Gamma(\tau)) + L\Gamma(\tau) = \frac{d}{dt}f(t, q). \tag{70}$$

We can again consider this, although rather artificially, as a special case of (59), (60), corresponding to the choice

$$\mu_j = -\frac{\partial}{\partial \dot{q}^j}\left[L\tau + \frac{\partial L}{\partial \dot{q}^i}(\xi^i - \dot{q}^i\tau)\right] = -\frac{\partial}{\partial \dot{q}^j}\langle Y, \theta\rangle. \tag{71}$$

The questions of the possible existence of a converse to Noether's theorem, and of the freedom in the relationship between F and Y, were not treated by the above-quoted authors. It does not seem to be possible to give an elegant answer to these questions, but we can give the following description. In the present case, the relationship between Noether-transformations Y and constants of the motion F is completely determined by the conditions,

$$\Gamma(F) = \Gamma(f - \langle Y, \theta\rangle) = 0,$$

with f independent of $\dot{q}$.

So, for a given F, the problem consists in finding functions (τ, ξ^i), satisfying the system of partial differential equations

$$\frac{\partial F}{\partial \dot{q}^j} = -\frac{\partial}{\partial \dot{q}^j}\langle Y, \theta\rangle. \tag{72}$$

This gives rise to n partial differential equations for $n+1$ unknowns, so that solutions are likely to exist. Even this aspect, however, is not so clear, because even under the additional assumption that all coefficients would be real analytic, it does not seem to be possible to bring system (72) into the normal form for application of the Cauchy–Kovalevski existence theorem. Anyhow, if (τ_F, ξ^i_F) is a particular solution of (72), it is certainly not unique, and all solutions are of the form $(\tau_F + \tau_0, \xi^i_F + \xi^i_0)$, where (τ_0, ξ^i_0) is a solution of the system

$$\frac{\partial}{\partial \dot{q}^j}\langle Y_0, \theta\rangle = 0.$$

5.3. Restricting Y to be a dynamical symmetry. Recall that we started our analysis from the one common factor in all known treatments of Noether's theorem, namely, the structure of the formula (53) for a constant of the motion, so that, primarily, we wish to have

$$\Gamma(F) = \langle \Gamma, d(f - \langle Y, \theta\rangle)\rangle = 0.$$

In some sense one could say that the cases treated in §§ 5.1 and 5.2 arise from a "misreading" of that equation. Let us this time not change this equation, but instead discuss the quite natural additional restriction that Y be a dynamical symmetry of Γ, i.e., an invariance transformation of the first-order system in q and $\dot{q}$, equivalent to (4). Unfortunately, with this requirement we end up with a complicated system of

partial differential equations. Indeed, in view of (49), (50), we have to find solutions τ, ξ^i, η^i to the equations

$$\tau\frac{\partial L}{\partial t}+\xi^j\frac{\partial L}{\partial q^j}+\frac{\partial L}{\partial \dot{q}^j}(\Gamma(\xi^j)-\dot{q}^j\Gamma(\tau))+L\Gamma(\tau)=\Gamma(f),$$

(73)

$$\eta^i=\Gamma(\xi^i)-\dot{q}^i\Gamma(\tau),\qquad \Gamma(\eta^i)-\Lambda^i\Gamma(\tau)=\tau\frac{\partial \Lambda^i}{\partial t}+\xi^k\frac{\partial \Lambda^i}{\partial q^k}+\eta^k\frac{\partial \Lambda^i}{\partial \dot{q}^k}.$$

This approach certainly yields a converse to Noether's theorem, as can be seen from the proof of Proposition 4.2. Moreover, in view of (51), we have

$$[Y+h\Gamma,\Gamma]=[Y,\Gamma]-\Gamma(h)\Gamma=-(\Gamma(\tau+h))\Gamma, \tag{74}$$

so that the freedom discussed in § 5.1 also occurs here. But it is not the only freedom. An interesting additional freedom in the relationship between F and Y can be obtained as follows. Let F be a constant of the motion corresponding to the dynamical symmetry Y. Consider

$$Y'=\phi Y, \tag{75}$$

where ϕ is itself any first integral of Γ. Putting

$$f'=f+(\phi-1)\langle Y,\theta\rangle, \tag{76}$$

we have

$$f'-\langle Y',\theta\rangle=f-\langle Y,\theta\rangle=F.$$

Moreover,

$$[Y',\Gamma]=\phi Y\Gamma-\Gamma(\phi Y)=\phi[Y,\Gamma]-\Gamma(\phi)Y=-\phi\Gamma(\tau)\Gamma, \tag{77}$$

since $\Gamma(\phi)=0$. Hence, F also corresponds to any Y' of type (75).

There is one aspect of the classical Noether theorem which we have not yet discussed in the various generalizations so far considered, namely Proposition 2.2, which asserted that the constant of the motion F was also an invariant of the generator Y, itself. (Note that for τ and ξ^i independent of $\dot{q}$, the η^i of (15) coincide with the η^i of (49), so that $Y^{(1)}$ in (14) coincides with the vectorfield Y in the present context.) We could simply add this as a supplementary requirement, to the definition of the generator of a Noether-transformation. In the present context, e.g., the components of Y then should not only solve all equations (73) for some f, but should also satisfy the equation,

$$\langle Y,d(f-\langle Y,\theta\rangle)\rangle=0. \tag{78}$$

Unfortunately, this new restriction does not seem to simplify the problem of finding solutions for τ and ξ^i. Quite remarkably, however, the complexity of the equations significantly decreases if one restricts one's attention to a special class of dynamical symmetries, namely the $d\theta$-symmetries. But note, first, that the two simple degrees of freedom (74), (75), in the relation between F and Y still persist when the new restriction (78) is imposed.

5.4. Restricting Y to be a $d\theta$-symmetry. The study of the relation between $d\theta$-symmetries and constants of the motion stems from the work of Cartan [10]. It was recently discussed in full detail by Crampin [11], who, however, did not regard it as a generalization of the classical Noether theorem, but rather as a completely different approach, superior to Noether's theorem and showing its deficiencies.

Let Y be a $d\theta$-symmetry, i.e., satisfy the relation $L_Y d\theta = 0$. From (25) and the Poincaré lemma, this implies

$$L_Y\theta = df \quad \text{for some } f. \tag{79}$$

From property (27) of the Lie-derivative, (79) is seen to be equivalent with

$$i_Y d\theta = d(f - \langle Y, \theta\rangle). \tag{80}$$

If we take the inner product with Γ, (33) immediately implies

$$\langle \Gamma, d(f - \langle Y, \theta\rangle)\rangle = 0,$$

so that we get a constant of the motion of the by now familiar structure.

Conversely, in proving Proposition 4.2, we have already shown that to each constant of the motion F corresponds a $d\theta$-symmetry Y, via the relation

$$i_Y d\theta = dF. \tag{81}$$

Now, if two $d\theta$-symmetries Y_1 and Y_2 correspond to the same constant of the motion F, we have

$$i_{Y_1} d\theta = i_{Y_2} d\theta \quad \text{or} \quad i_{Y_1 - Y_2} d\theta = 0,$$

which, again from the fact that the set of characteristic vectorfields of $d\theta$ is one-dimensional, implies that

$$Y_1 - Y_2 = h\Gamma \quad \text{for some function } h. \tag{82}$$

On the other hand, if F_1 and F_2 are two constants of the motion corresponding to the same $d\theta$-symmetry Y, we see from (81) that they can only differ by a trivial constant.

Finally, taking the inner product of (80) or (81) with Y, we get

$$0 = i_Y i_Y d\theta = Y(F); \tag{83}$$

hence F automatically has the invariance property with respect to the generator Y.

Summarizing, we have the following results.

THEOREM 5.1. *Let Γ define a Lagrangian system according to* (33), (34). *Then*:

(i) *To each $d\theta$-symmetry Y corresponds a constant of the motion F of the form $F = f - \langle Y, \theta\rangle$, which is unique up to a trivial constant.*

(ii) *To each first integral F corresponds a $d\theta$-symmetry Y, which is unique up to a trivial dynamical symmetry $h\Gamma$.*

(iii) *F is in addition an invariant of the symmetry Y.*

A final question which is in order here is: What are the Killing-type equations within this framework? The function f playing the role of "gauge-function" appears in the relation (79), which is therefore the analogue of the Killing equation. Explicitly, (79) yields the partial differential equations

$$L\frac{\partial\tau}{\partial\dot{q}^i} + \frac{\partial L}{\partial\dot{q}^j}\left(\frac{\partial\xi^j}{\partial\dot{q}^i} - \dot{q}^j\frac{\partial\tau}{\partial\dot{q}^i}\right) = \frac{\partial f}{\partial\dot{q}^i}, \tag{84}$$

$$L\frac{\partial\tau}{\partial q^i} + \frac{\partial L}{\partial\dot{q}^j}\left(\frac{\partial\xi^j}{\partial q^i} - \dot{q}^j\frac{\partial\tau}{\partial q^i}\right) + \xi^j\frac{\partial^2 L}{\partial q^j\,\partial\dot{q}^i} + \eta^j\frac{\partial^2 L}{\partial\dot{q}^j\,\partial\dot{q}^i} + \tau\frac{\partial^2 L}{\partial t\,\partial\dot{q}^i} = \frac{\partial f}{\partial q^i}, \tag{85}$$

$$L\frac{\partial\tau}{\partial t}+\tau\frac{\partial L}{\partial t}+\xi^i\frac{\partial L}{\partial q^i}+\frac{\partial L}{\partial\dot q^i}\left(\frac{\partial\xi^i}{\partial t}-\dot q^i\frac{\partial\tau}{\partial t}\right)$$
$$-\dot q^i\left(\frac{\partial^2 L}{\partial\dot q^i\,\partial q^j}\xi^j+\frac{\partial^2 L}{\partial\dot q^i\,\partial\dot q^j}\eta^j+\frac{\partial^2 L}{\partial t\,\partial\dot q^i}\tau\right)=\frac{\partial f}{\partial t}. \tag{86}$$

Note that by Lemma 3.2 we are guaranteed that any solution (τ, ξ^i, η^i) of (84), (85), (86), will constitute the components of a dynamical symmetry, so that (by Lemma 3.1) use can be made of (49) in order to reduce the above relations to partial differential equations in τ and ξ^i only.

It is clear from the properties stated in Theorem 5.1 that the present approach offers a perfectly plausible candidate for being called Noether's theorem for velocity-dependent transformations; and we would certainly not be the first to do so. The predicate "Noether theorem" was assigned to $d\theta$-symmetries, e.g., by Gallissot [17], Marmo and Saletan [39] and in the context of Hamiltonian mechanics by Hermann [22] and Arnold [4]. A similar statement in the context of continuum mechanics was made, e.g., by Nôno and Mimura [43]. One of the principal purposes of this paper is to give more weight to this point of view, by establishing a rather unexpected equivalence between the approaches of §§ 5.1 and 5.4.

6. Equivalence between $d\theta$-symmetries and the Noether theorem of § 5.1. The four approaches presented in the previous section all have, in our opinion, a different origin, as reflected in the titles of the subsections.

Of course, § 5.4 is a particular case of § 5.3. Now, it is also easy to see that § 5.4 is at the same time a particular case of § 5.1. First of all, (84) is identical to (63). Second, multiplying (85) by $\dot q^i$, and adding (86) yields (64) exactly. Hence, for every solution (τ, ξ^i, η^i) of (84), (85), (86), the functions (τ, ξ^i) will satisfy (63), (64), while the η^i will be related to (τ, ξ^i) by (49). The converse, now, is not at all obvious, but is true. Recalling the equivalence between (63), (64) and (66), (67), under the additional prescription (49), we will indeed prove the following theorem.

THEOREM 6.1.

$$\text{(I)}\quad \left.\begin{aligned}\left\langle\frac{\partial}{\partial\dot q^i}, L_Y\theta-df\right\rangle&=0\\ \langle\Gamma, d(f-\langle Y,\theta\rangle)\rangle&=0\\ \eta^i=\Gamma(\xi^i)-\dot q^i\Gamma(\tau)&\end{aligned}\right\}\ \Leftrightarrow\ \text{(II)}\qquad L_Y\theta=df.$$

Proof. That (II) implies (I) is trivial and was explained above. Conversely, assume the three conditions (I). (I_1) implies (using property (27)),

$$\left\langle\frac{\partial}{\partial\dot q^i}, i_Y\,d\theta-dF\right\rangle=0, \tag{87}$$

where $F=f-\langle Y,\theta\rangle$ is a first integral in view of (I_2). Making use of the explicit formula (38), (87) immediately yields,

$$\xi^i-\dot q^i\tau=-g^{ij}\frac{\partial F}{\partial\dot q^j}. \tag{88}$$

On the other hand, we already know that to each constant of the motion corresponds a $d\theta$-symmetry. Suppose that F here corresponds to $\tilde Y$, with components $(\tilde\tau, \tilde\xi^i, \tilde\eta^i)$,

i.e., we have $i_{\tilde{Y}}\, d\theta = dF$, which obviously implies

$$\left\langle \frac{\partial}{\partial \dot{q}^j}, i_{\tilde{Y}}\, d\theta - dF \right\rangle = 0;$$

hence, as for (87),

$$\tilde{\xi}^i - \dot{q}^i \tilde{\tau} = -g^{ij} \frac{\partial F}{\partial \dot{q}^j}.$$

We therefore have

$$\xi^i - \dot{q}^i \tau = \tilde{\xi}^i - \dot{q}^i \tilde{\tau}. \tag{89}$$

If we define h by the relation $h = \tilde{\tau} - \tau$, we have $\tau = \tilde{\tau} - h$, and then from (89), $\xi^i = \tilde{\xi}^i - h\dot{q}^i$, and finally, from (I_3),

$$\begin{aligned} \eta^i &= \Gamma(\xi^i) - \dot{q}^i \Gamma(\tau) \\ &= \Gamma(\tilde{\xi}^i) - \dot{q}^i \Gamma(\tilde{\tau}) - \Gamma(h\dot{q}^i) + \dot{q}^i \Gamma(h) \\ &= \tilde{\eta}^i - h\Lambda^i. \end{aligned}$$

These three relations imply that $Y = \tilde{Y} - h\Gamma$, so that Y is itself a $d\theta$-symmetry corresponding to F. In other words we have $i_Y\, d\theta = dF$, from which (II) follows.

The equivalence established in the preceding theorem is after all quite remarkable. It asserts that once $\partial f/\partial \dot{q}^j$ is determined by the left-hand side of (63), and the sum of the $n+1$ terms $\partial f/\partial t + \dot{q}^i\, \partial f/\partial q^i$ is determined by (64), the individual terms $\partial f/\partial q^i$ and $\partial f/\partial t$ are given by (85), (86), with η^i defined by (49). More important, for practical purposes, is the observation that in order to find $d\theta$-symmetries for a given Lagrangian system, it suffices to look for "solution-triplets" (τ, ξ^i, f) of (63), (64) (the η^i afterwards, being immediately determined), and this problem indeed looks simpler than trying to solve the $(2n+1)$ equations (84), (85), (86). On the other hand, the treatment in § 5.4 was preferable on theoretical grounds, because it made it so easy to establish an inverse Noether theorem, and to describe the freedom in the relationship between Y and F. Through the above equivalence, these results are now also valid for the case treated in § 5.1.

As a further remark, note that for the classical Noether theorem 2.1, (63) or (I_1) is trivially satisfied because τ, ξ^i and f are all independent of $\dot{q}$. Nevertheless, (I_1) remains equivalent to (87), which provides us with the relation (88), which was crucial in proving the equivalence with a $d\theta$-symmetry. Hence, in the classical Noether theorem we are also dealing in some sense with a $d\theta$-symmetry, so that the invariance property (16), which needed a rather tedious proof in § 2, now merely appears as an immediate consequence of (83), which was trivial to prove.

The relations (88), obtained in the course of the proof of Theorem 6.1, deserve some special attention, because they allow an immediate computation of the $d\theta$-symmetries (which from now on are synonymous with Noether-symmetries for us), corresponding to a given constant of the motion. Because of their importance, we restate them separately as follows.

LEMMA 6.2. *Let F be an arbitrary constant of the motion of Γ. Then all $d\theta$-symmetries corresponding to F are determined by the relations*

$$\xi^i - \dot{q}^i \tau = -g^{ij} \frac{\partial F}{\partial \dot{q}^j},$$

and

$$\eta^i = \Gamma(\xi^i) - \dot{q}^i \Gamma(\tau) = \Gamma(\xi^i - \dot{q}^i \tau) + \Lambda^i \tau.$$

Since a Y corresponding to an F is only determined up to multiples of Γ, the choice of the time-component τ is completely free. Once such a choice is made, the other components ξ^i, η^i unequivocally follow from the above relations. These very simple relations do not seem to be widely known in the literature. We are not aware of any textbook mentioning them. As far as we know, similar relations were used for the first time by Palmieri and Vitale [45] and Candotti et al. [8]; they have also proven to be useful in generalizations of Noether's theorem to nonconservative systems, described by Lagrange-equations of the first type (Djukic and Vujanovic [14]).

There does not seem to exist a best device for fixing the Noether-transformation corresponding to a given constant in a unique way (i.e., fixing τ). A couple of more or less "natural" possibilities are presented below.

COROLLARY 6.3 (Possible restrictions for fixing τ).

(i) *The first possibility one can think of is, of course, that of taking* $\tau = 0$, *which means that one does not need to consider variations of the independent variable. This has been mentioned by Steudel* [55], *and can provide significant simplifications in constructive procedures for finding first integrals* (*see Kobussen* [31]). *The formulae for the determination of* Y *from* F *in this case simplify to*

$$\xi^i = -g^{ij}\frac{\partial F}{\partial \dot{q}^j}, \qquad \eta^i = \Gamma(\xi^i). \tag{90}$$

(ii) *The explicit formula for the first integral,*

$$F = f - \left[L\tau + \frac{\partial L}{\partial \dot{q}^i}(\xi^i - \dot{q}^i\tau)\right],$$

can in view of (88) *be written in the form*

$$f - L\tau = F - \frac{\partial L}{\partial \dot{q}^i} g^{ij}\frac{\partial F}{\partial \dot{q}^j}. \tag{91}$$

Hence, fixing τ *is equivalent to fixing the gauge-function* f. *In certain circumstances it might e.g., be interesting to take* $f = 0$, *which by* (91), *then immediately yields* τ.

Recently, Noether-symmetries have often been discussed as particular cases within applications of the Lie-method of extended groups (see e.g., Lutzky [38], Eliezer [15] and Leach [32]). From this point of view, one is merely interested in velocity-independent transformations, i.e., in applications of the classical Noether theorem of § 2. On the one hand, one, of course, loses universality by this restriction; i.e., not all constants of the motion can be related to a velocity-independent Noether-transformation. On the other hand, if such a velocity-independent transformation exists, it can certainly be advantageous to give it preference over other equivalent Noether-transformations, which illustrates that choosing $\tau = 0$ is not always the best choice. The above lemma enables us to give a simple characterization of this situation.

COROLLARY 6.4. *A constant of the motion* F *can be related to a classical* (*velocity-independent*) *Noether-transformation, if and only if for all* $i = 1, \cdots, n$, $g^{ij}\,\partial F/\partial \dot{q}^j$ *is a linear function of* $\dot{q}^i$, *does not depend on the other components of* $\dot{q}$, *and is such that the coefficient of* $\dot{q}^i$ *is the same for all* i. *Under these circumstances, the corresponding classical Noether-transformation is unique.*

Even if velocity-dependent transformations are allowed, it can still be advantageous, if only for the elegance of the end results, to take $\tau \neq 0$. Consider, e.g., the

Kepler problem, which reduced to its plane motion is described by

$$L=\frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2)+\frac{\mu}{r}, \qquad r=\sqrt{q_1^2+q_2^2}$$

(components are here labeled by subscripts for convenience). As a constant of the motion, let us take, e.g., the first component of the Runge–Lenz vector, namely

$$F_1=q_1\dot{q}_2^2-q_2\dot{q}_1\dot{q}_2-\mu\frac{q_1}{r}.$$

Then, from Lemma 6.2, all corresponding $d\theta$-symmetries must satisfy

$$\xi_1-\dot{q}_1\tau=q_2\dot{q}_2, \quad \xi_2-\dot{q}_2\tau=-2q_1\dot{q}_2+q_2\dot{q}_1.$$

Making the choice $\tau=q_1$, we thus get

$$\xi_1=q_1\dot{q}_1+q_2\dot{q}_2, \quad \xi_2=q_2\dot{q}_1-q_1\dot{q}_2,$$

while the η_i-components easily follow from (49).

Similarly, to the second component of the Runge–Lenz vector,

$$F_2=q_2\dot{q}_1^2-q_1\dot{q}_1\dot{q}_2-\mu\frac{q_2}{r},$$

corresponds the $d\theta$-symmetry

$$\tau=q_2, \qquad \xi_1=q_1\dot{q}_2-q_2\dot{q}_1, \quad \xi_2=q_1\dot{q}_1+q_2\dot{q}_2,$$

completed, of course, by the η_i from (49).

In the Appendix we discuss the Poisson theorem, because, first of all, it is rarely mentioned in Lagrangian mechanics; secondly it is typically related to $d\theta$-symmetries; thirdly its expression in local coordinates gives us again an opportunity to make use of the interesting formula (88).

7. Discussion. Summarizing the basic guidelines underlying our study, we can say that from a theoretical point of view it is inappropriate to define Noether-transformations in the broadest possible way, because of the excessive freedom which results in the correlation with first integrals (see Proposition 4.3). A similar criticism has been formulated recently by Martinez–Alonso [41]. The most attractive framework for a generalization of the classical Noether theorem to velocity-dependent transformations is offered by Theorem 5.1 which is about $d\theta$-symmetries, or the "equivalent" prescriptions of § 5.1.

A number of arguments in favor of this point of view are pointed out below.

(i) The equivalence result of Theorem 6.1.

(ii) The one-to-one correspondence between equivalence classes of symmetries and first integrals (of course, with respect to a given fixed Lagrangian).

(iii) The invariance property $Y(F)=0$, which fits in with a similar property for the classical Noether theorem.

(iv) The direct correspondence with the usual phase-space formulation of symmetries and conservation laws. Indeed, the classical theory of symmetries generated by infinitesimal canonical transformations and related constants of the motion in Hamiltonian mechanics is also a theory about $d\theta$-symmetries, with $\theta=p_i\,dq^i-H\,dt$. In fact, the $d\theta$-symmetry version of Noether's theorem is simply the translation of the canonical theory to Lagrangian coordinates, as is underscored by the Poisson theorem we prove in the Appendix. Note, by the way, that the invariance property $Y(F)=0$,

which was rather hard to prove in § 2, becomes quite obvious once this link with the phase-space formulation is made. Indeed, it simply follows from the fact that the Poisson bracket of a function with itself is identically zero. $d\theta$-symmetries for generalized Hamiltonian systems have been studied by Cantrijn and Sarlet [9].

(v) In the (too) general Noether-type theorems mentioned, e.g., by Candotti et al. [7] and Rosen [49], [50], and which have similar consequences as the case discussed in our § 4, the idea of invariance of some variational principle is completely lost, and this after all was the spirit of the original Noether theorem. With the formulation in the sense of $d\theta$-symmetries, one preserves the idea of invariance of a variational principle in the following sense. For a general vectorfield Γ, a relation like (33) expresses the fact that integral curves of Γ are extremals of the functional $\int_C \theta$, defined over a set of arbitrary curves on the tangent bundle (not necessarily "lifted") with fixed endpoints (see, e.g., Sternberg [54]). In view of $L_Y\theta = df$, it is then clear that $d\theta$-symmetries (to first order) preserve this functional up to a constant. This is not the invariance of the classical functional $\int L dt$, but it reduces to it in case where Y projects onto a vectorfield on the base manifold, and the variational principle is restricted to a class of lifted curves.

(vi) The theory of $d\theta$-symmetries and corresponding first integrals sets the stage for various interesting generalizations. Here we are not thinking of the type of generalizations in § 4, which merely discuss how to define Noether-transformations. Instead, we have intrinsic generalizations in mind, which significantly broaden the whole picture of symmetries and conservation laws.

It is in fact, more adequate to say that Noether's theorem can be deduced from more general theories under specific restrictions, and is then precisely recovered in the $d\theta$-symmetry version. A very simple generalization consists in introducing a "higher-order Noether theorem", in which "higher-order $d\theta$-symmetries" are linked to constants of the motion (see Sarlet and Cantrijn [53]). Another slight generalization was discussed by Losco [35] and Karaballi [25]. It essentially consists in relating a constant of the motion to general dynamical symmetries as in our § 5.3. Of course, when a dynamical symmetry is not a $d\theta$-symmetry, the computation of a first integral is not so straightforward and involves an integration procedure. As a result, it can happen that one has to leave the constant of the motion in an integral form (the so-called eleventh integral of the n-body problem is an example). A very general abstract framework, finally, is offered by the theory of "momentum mappings" (see e.g., Abraham and Marsden [1]), in which, roughly speaking, invariances are studied under the symplectic action of a Lie group.

In the above list of arguments, the emphasis lay on the $d\theta$-symmetries as they were discussed in § 5.4, because with this version of the Noether theorem we get the best insight into the relationship between generators and constants of the motion, while the various results are also most easily proven there. It was, however, not our intention to rule out, in this way, the (somehow) equivalent conceptions related to Noether's original version in § 5.1. As a matter of fact, for the practical determination of Noether-transformations, i.e., for finding solutions τ, ξ^i, η^i of Killing-type equations, the system (63), (64) has a marked advantage upon the system (84), (85), (86).

As a further remark, we want to stress that even the criticism we have formulated of a too general version of Noether's theorem is not meant to rule out that approach. That criticism is based on purely theoretical grounds, because a theoretician only talks about a link between symmetries and conservation laws if there is some kind of one-to-one correspondence between these concepts. In the practical search for constants of the motion, however, all attempts are worth trying. To be precise, let us go

back for a moment to the general equations (59), (60) of § 4. Although we do not know of any example, it should not be excluded that for a specific Lagrangian system, a clever choice of the functions μ_i might help to find a solution for τ and ξ^i.

To end this discussion, we would like to reply to recently formulated criticisms of the very idea of allowing velocity dependence in Noether's theorem, by some authors promoting the use of the Lie-method of extended groups [32], [47]. Let us first give a brief exposé of that method.

Consider an arbitrary second-order system (4) (Λ^i not necessarily satisfying (5)). Let the differential operator $Y^{(0)}$, as in (13), generate an infinitesimal transformation. The $Y^{(1)}$ of (14) is called its first extension, and $Y^{(0)}$ is said to generate a symmetry if the second-order system (4) is invariant under the "second extension" $Y^{(2)}$, which in the terminology of § 3 simply means that one is looking for dynamical symmetries of (4). In other words, the problem consists in finding vectorfields Y satisfying (49), (50), but where τ and ξ^i are independent of the velocities. Such symmetries constitute a finite subalgebra of the Lie-algebra of all dynamical symmetries. Hence it will often be possible to find explicit expressions for all elements of this subalgebra, by determining all solutions of the partial differential equations (49), (50). When applied to the special case of Lagrangian systems (i.e., for Λ^i satisfying (5)), it turns out that not all of these velocity-independent symmetries are classical Noether-transformations. In fact these Noether-transformations again form a subalgebra. Consequently, not all symmetries obtained will directly yield a first integral. In recent papers [38], [47], [32] a constant of the motion was said to correspond to a dynamical symmetry Y, if it has the invariance property $Y(F) = 0$. This is certainly an attractive definition, because it covers the case in which the symmetry is of Noether type. Moreover, such an invariance property has an elegant geometrical interpretation (see, e.g., Gonzáles-Gascón et al. [21]), and can be helpful in the reduction of the given system to a lower-dimensional one (see Gonzáles-Gascón and Moreno-Insertis [20], Marmo et al. [40]). It seems to us, however, that there is, as yet, no simple method available for the determination of such a constant in the case of non-Noether symmetries. Moreover, such an implied constant is not necessarily unique [19].

Now the criticism of Noether's theorem in that context stems from the following two arguments (see, e.g., Prince and Leach [48]). On the one hand, Noether's theorem (always keeping the symmetries velocity-independent) fails to produce certain interesting first integrals, which do correspond to non-Noether symmetries according to the above rule. On the other hand, a generalization of the classical Noether theorem to velocity-dependent transformations is rejected, because the algebra of symmetries obtained in this way becomes infinite, and consequently no systematic method exists for the determination of the complete algebra.

In our opinion, these arguments are not very relevant, at least if attention is focused on the determination of first integrals. First of all, not all constants of the motion of a given system can be related to a velocity-independent dynamical symmetry by the rule $Y(F) = 0$ (there even exist systems which simply do not have such symmetries [18]). Hence, if one wants to have, at least the theoretical possibility of finding all constants by the Lie-method, one has to allow velocity dependence too (this was done e.g., by Anderson and Davison [2] and Lutzky [38]). And once this step is taken, what can be more elegant than that the complete algebra of symmetries (as for $d\theta$-symmetries) is precisely homomorphic to the Lie-algebra of all constants of the motion under Poisson brackets (see the Poisson theorem)? By the way, this shows that in some sense there does exist a systematic method for the determination of the complete algebra of symmetries, namely it consists in solving the given system

of differential equations. Of course, this is not a very satisfactory counter-argument. However, also in the more realistic situation where the equations cannot be completely solved, it is possible to start searching, within the context of the Noether theorem, in a very systematic way for symmetries leading to all known first integrals. One example will illustrate this.

The Runge–Lenz vector in the Kepler problem cannot be obtained by the classical (i.e., velocity-independent) Noether theorem, whereas it can be related to one of the non-Noether dynamical symmetries [47], namely

$$Y = t\frac{\partial}{\partial t} + \frac{2}{3}q_1\frac{\partial}{\partial q_1} + \frac{2}{3}q_2\frac{\partial}{\partial q_2} - \frac{1}{3}\dot{q}_1\frac{\partial}{\partial \dot{q}_1} - \frac{1}{3}\dot{q}_2\frac{\partial}{\partial \dot{q}_2}. \tag{92}$$

As we have seen in § 6, however, these constants correspond to Noether-symmetries in which the quantities $\xi^i - \dot{q}^i\tau$ are linear in the velocity components. It is thus very natural to say that a systematic search for Noether-symmetries can consist of first determining all velocity-independent ones, then the ones which are linear in the velocities, followed by all kinds of special assumptions, depending on the problem at hand. Such a systematic application of Noether's theorem was very nicely presented by Kobussen [31].

In conclusion, we claim that all known integrals of Lagrangian systems can indeed be found by a systematic exploration via Noether's theorem. This should not be interpreted, however, as a complete rejection of the velocity-independent Lie-method. Indeed, because of certain theoretical considerations (we think, e.g., about quantization problems) the determination of first integrals can be subordinate to the determination of some finite algebra of symmetries, and the related structure constants.

As a final remark, it is interesting to note that exactly the same velocity-independent symmetry (92) (at least for a "rectilinear Kepler problem") was related, not to a Runge–Lenz-type constant, but to an "eleventh-integral-type constant", by Karaballi [25].

Appendix. The Poisson theorem in Lagrangian mechanics.

THEOREM. *If F_1 and F_2 are two constants of the motion for the system Γ, then*

$$F_{1,2} = i_{Y_1} i_{Y_2}\, d\theta,$$

is a new constant, where Y_1 and Y_2 are $d\theta$-symmetries corresponding respectively to F_1 and F_2.

Proof. Let Y_i be arbitrary $d\theta$-symmetries corresponding to F_i, $i = 1, 2$; i.e., we have

$$i_{Y_1}\, d\theta = dF_1, \qquad i_{Y_2}\, d\theta = dF_2.$$

Then, using (28), (27) and (46) we get

$$\begin{aligned} i_{[Y_1,Y_2]}\, d\theta &= -L_{Y_2} i_{Y_1}\, d\theta, \\ &= -d i_{Y_2} i_{Y_1}\, d\theta \\ &= d(i_{Y_1} i_{Y_2}\, d\theta), \end{aligned} \tag{A.1}$$

which shows that $[Y_1, Y_2]$ is a $d\theta$-symmetry corresponding to the constant of the motion

$$F_{1,2} = i_{Y_1} i_{Y_2}\, d\theta. \tag{A.2}$$

In view of (33) it is clear that $F_{1,2}$ does not depend on the particular $d\theta$-symmetries Y_1 and Y_2 we have chosen. For an explicit formula in local coordinates, we can go back to (38), with Y replaced by $Y_2(\tau_2, \xi_2^i, \eta_2^i)$, and again take the inner product with

$Y_1(\tau_1, \xi_1^i, \eta_1^i)$. In this way we get

$$i_{Y_1} i_{Y_2}\, d\theta = \left(\frac{\partial^2 L}{\partial \dot{q}^i\, \partial q^j} - \frac{\partial^2 L}{\partial \dot{q}^j\, \partial q^i}\right)(\xi_2^j - \dot{q}^j \tau_2)(\xi_1^i - \dot{q}^i \tau_1)$$
$$+ \frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j}(\eta_2^i - \Lambda^i \tau_2)(\xi_1^j - \dot{q}^j \tau_1)$$
$$- \frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j}(\xi_2^j - \dot{q}^j \tau_2)(\eta_1^i - \Lambda^i \tau_1).$$

From Lemma 6.2 we then obtain

$$F_{1,2} = \left(\frac{\partial^2 L}{\partial \dot{q}^i\, \partial q^j} - \frac{\partial^2 L}{\partial \dot{q}^j\, \partial q^i}\right) g^{ik} g^{jl} \frac{\partial F_1}{\partial \dot{q}^k} \frac{\partial F_2}{\partial \dot{q}^l}$$
$$+ \frac{\partial F_1}{\partial \dot{q}^i} \Gamma\left(g^{il} \frac{\partial F_2}{\partial \dot{q}^l}\right) - \frac{\partial F_2}{\partial \dot{q}^i} \Gamma\left(g^{il} \frac{\partial F_1}{\partial \dot{q}^l}\right). \tag{A.3}$$

Using the identities

$$\Gamma\left(\frac{\partial F}{\partial \dot{q}^l}\right) = \frac{\partial}{\partial \dot{q}^l}(\Gamma(F)) - \frac{\partial F}{\partial q^l} - \frac{\partial \Lambda^k}{\partial \dot{q}^l} \frac{\partial F}{\partial \dot{q}^k},$$
$$\Gamma\left(\frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j}\right) + \frac{\partial \Lambda^l}{\partial \dot{q}^i} \frac{\partial^2 L}{\partial \dot{q}^l\, \partial \dot{q}^j} = \frac{\partial^2 L}{\partial \dot{q}^i\, \partial q^j} - \frac{\partial^2 L}{\partial \dot{q}^j\, \partial q^i},$$

the expression (A.3) can be simplified to

$$F_{1,2} = -\left(\frac{\partial^2 L}{\partial \dot{q}^i\, \partial q^j} - \frac{\partial^2 L}{\partial \dot{q}^j\, \partial q^i}\right) g^{ik} g^{jl} \frac{\partial F_1}{\partial \dot{q}^k} \frac{\partial F_2}{\partial \dot{q}^l}$$
$$+ \frac{\partial F_1}{\partial q^i} g^{il} \frac{\partial F_2}{\partial \dot{q}^l} - \frac{\partial F_2}{\partial q^i} g^{il} \frac{\partial F_1}{\partial \dot{q}^l}. \tag{A.4}$$

We repeat that, if F_1 and F_2 are constants of the motion of the given Lagrangian system, then $F_{1,2}$ computed via (A.3) or (A.4) yields a new (not necessarily independent) constant of the motion. The simpler expression (A.4) coincides, as expected, with the formula for the Poisson bracket of F_1 and F_2 in Lagrangian coordinates given by Sudarshan and Mukunda [57].

Remark. Obviously, we have

$$i_{Y_1} i_{Y_2}\, d\theta = i_{Y_1}\, dF_2 = -i_{Y_2}\, dF_1 = Y_1(F_2) = -Y_2(F_1).$$

Hence, the $d\theta$-symmetry Y_1, acting on the constant of the motion F_2, yields a new constant of the motion. This property remains true if Y_1 is a general dynamical symmetry. Indeed, from

$$\Gamma(F) = 0, \quad \text{and} \quad [Y, \Gamma] = g\Gamma$$

one immediately obtains

$$0 = [Y, \Gamma](F) = -\Gamma(Y(F)),$$

so that $Y(F)$ is another constant of the motion. This is a simple way to recognize that the Poisson theorem is a special case of what was called the "related integral theorem" by Katzin and Levine [26], [27].

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REFERENCES

[1] R. ABRAHAM AND J. E. MARSDEN, *Foundations of Mechanics*, 2nd ed., Benjamin/Cummings, Reading, MA, 1978.

[2] R. L. ANDERSON AND S. M. DAVISON, *A generalization of Lie's "counting" theorem for second-order ordinary differential equations*, J. Math. Anal. Appl., 48 (1974), pp. 301–315.

[3] R. L. ANDERSON AND N. H. IBRAGIMOV, *Lie-Bäcklund Transformations in Applications*, SIAM Studies 1, Society of Industrial and Applied Mathematics, Philadelphia, 1979.

[4] V. I. ARNOLD, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1978.

[5] E. BESSEL-HAGEN, *Über die Erhaltungssätze der Elektrodynamik*, Math. Ann., 84 (1921), pp. 258–276.

[6] G. W. BLUMAN AND J. D. COLE, *Similarity Methods for Differential Equations*, Springer-Verlag, New York, 1974.

[7] E. CANDOTTI, C. PALMIERI AND B. VITALE, *On the inversion of Noether's theorem in the Lagrangian formalism. II. Classical field theory*, Nuovo Cimento, 70A (1970), pp. 233–246.

[8] ———, *On the inversion of Noether's theorem in classical dynamical systems*, Amer. J. Phys., 40 (1972), pp. 424–429.

[9] F. CANTRIJN AND W. SARLET, *Symmetries and conservation laws for generalized Hamiltonian systems*, Internat. J. Theor. Phys., to appear.

[10] E. CARTAN, *Leçons sur les invariants intégraux*, Hermann, Paris, 1922.

[11] M. CRAMPIN, *Constants of the motion in Lagrangian mechanics*, Internat. J. Theor. Phys., 16 (1977), pp. 741–754.

[12] T. DASS, *Conservation laws and symmetries*: II, Phys. Rev., 150 (1966), pp. 1251–1255.

[13] DJ. S. DJUKIC, *A procedure for finding first integrals of mechanical systems with gauge-variant Lagrangians*, Internat. J. Non-Linear Mech., 8 (1973), pp. 479–488.

[14] DJ. S. DJUKIC AND B. D. VUJANOVIC, *Noether's theory in classical nonconservative mechanics*, Acta Mech., 23 (1975), pp. 17–27.

[15] C. J. ELIEZER, *The symmetries and first integrals of some differential equations of dynamics*, Hadronic J., 2 (1979), pp. 1067–1109.

[16] J. G. FLETCHER, *Local conservation laws in generally covariant theories*, Rev. Modern Phys., 32 (1960), pp. 65–87.

[17] F. GALLISSOT, *Les formes extérieures en mécanique*, Ann. Inst. Fourier, 4 (1952), pp. 145–297.

[18] F. GONZÁLEZ-GASCÓN, *Notes on the symmetries of systems of differential equations*, J. Math. Phys., 18 (1977), pp. 1763–1767.

[19] ———, *Geometrical obtention and generalization of recent results concerning symmetries and first integrals*, Lett. Nuovo Cimento, 26 (1979), pp. 385–387.

[20] F. GONZÁLEZ-GASCÓN AND F. MORENO-INSERTIS, *Symmetries, first integrals and splittability of dynamical systems*, Lett. Nuovo Cimento, 21 (1978), pp. 253–257.

[21] F. GONZÁLEZ-GASCÓN, F. MORENO-INSERTIS AND E. RODRIGUEZ-CAMINO, *Geometrical foundations and results on a problem suggested in a paper by Anderson et al.*, Lett. Nuovo Cimento, 21 (1978), pp. 595–599.

[22] R. HERMANN, *Differential Geometry and the Calculus of Variations*, Academic Press, New York, 1968.

[23] E. L. HILL, *Hamilton's principle and the conservation theorems of mathematical physics*, Rev. Modern Phys., 23 (1951), pp. 253–260.

[24] N. H. IBRAGIMOV AND R. L. ANDERSON, *Lie-Bäcklund tangent transformations*, J. Math. Anal. Appl., 59 (1977), pp. 145–162.

[25] A. KARABALLI, *Transformations infinitésimales et stabilité des systèmes Hamiltoniens*, Thèse, Université de Franche-Comté, Besançon, 1979.

[26] G. H. KATZIN, *The related integral theorem in phase space: A generalization of Poisson's theorem on constants of the motion*, Lett. Nuovo Cimento, 7 (1973), pp. 213–216.

[27] G. H. KATZIN AND J. LEVINE, *Dynamical symmetries and constants of the motion for classical particle systems*, J. Math. Phys., 15 (1974), pp. 1460–1470.

[28] ———, *A gauge invariant formulation of time-dependent dynamical symmetry mappings and associated constants of motion for Lagrangian particle mechanics.* I, J. Math. Phys., 17 (1976), pp. 1345–1350.
[29] ———, *Time-dependent dynamical symmetries and constants of motion.* II: *Time-dependent harmonic oscillator*, J. Math. Phys., 18 (1977), pp. 1267–1274.
[30] J. A. KOBUSSEN, *Some methods of classical mechanics applied to continuous systems*, Thesis, University of Technology, Eindhoven, the Netherlands, 1973.
[31] ———, *On a systematic search for integrals of the motion*, Helv. Phys. Acta., 53 (1980), pp. 183–200.
[32] P. G. L. LEACH, *An exact invariant for a class of time-dependent anharmonic oscillators with cubic anharmonicity*, Research Rep. AM-79:10, La Trobe University, Bundoora, 1979.
[33] J. M. LÉVY-LEBLOND, *Conservation laws for gauge-invariant Lagrangians in classical mechanics*, Amer. J. Phys., 39 (1971), pp. 502–506.
[34] J. D. LOGAN, *Invariant Variational Principles*, Academic Press, New York, 1977.
[35] L. LOSCO, *Le problème des n corps et les invariants intégraux*, Celest. Mech., 15 (1977), pp. 477–488.
[36] D. LOVELOCK AND H. RUND, *Tensors, Differential Forms, and Variational Principles*, John Wiley, New York, 1975.
[37] M. LUTZKY, *Symmetry groups and conserved quantities for the harmonic oscillator*, J. Phys. A, 11 (1978), pp. 249–258.
[38] ———, *Dynamical symmetries and conserved quantities*, J. Phys. A, 12 (1979), pp. 973–981.
[39] G. MARMO AND E. J. SALETAN, *Ambiguities in the Lagrangian and Hamiltonian formalism: transformation properties*, Nuovo Cimento, 40B (1977), pp. 67–89.
[40] G. MARMO, E. J. SALETAN AND A. SIMONI, *A general setting for reduction of dynamical systems*, J. Math. Phys., 20 (1979), pp. 856–860.
[41] L. MARTINEZ-ALONSO, *On the Noether map*, Lett. Math. Phys., 3 (1979), pp. 419–424.
[42] E. NOETHER, *Invariante Variationsprobleme*, Nachr. kgl. Ges. Wiss. Göttingen, Math. Phys. Kl. (1918), pp. 235–257.
[43] T. NÔNO AND F. MIMURA, *Dynamical symmetries I.*, Bull. Fukuoka Univ. of Education, 25 Part III, 1975, pp. 9–26.
[44] L. V. OVSJANNIKOV, *Group Analysis of Differential Equations*, Nauka, Moscow, 1978. (In Russian.)
[45] C. PALMIERI AND B. VITALE, *On the inversion of Noether's theorem in the Lagrangian formalism*, Nuovo Cimento, 66A (1970), pp. 299–309.
[46] G. E. PRINCE AND C. J. ELIEZER, *Symmetries of the time-dependent N-dimensional oscillator*, J. Phys. A, 13 (1980), pp. 815–823.
[47] ———, *On the Lie symmetries of the classical Kepler problem*, J. Phys. A, 14 (1981), pp. 587–596.
[48] G. E. PRINCE AND P. G. L. LEACH, *The Lie theory of extended groups in Hamiltonian mechanics*, Hadronic J., 3 (1980), pp. 941–961.
[49] J. ROSEN, *Noether's theorem in classical field theory*, Ann. Phys., 69 (1972), pp. 349–363.
[50] ———, *Generalized Noether's theorem I: Theory*, Ann. Phys., 82 (1974), pp. 54–69.
[51] E. J. SALETAN AND A. H. CROMER, *Theoretical Mechanics*, John Wiley, New York, 1971.
[52] W. SARLET, *Invariance and conservation laws for Lagrangian systems with one degree of freedom*, J. Math. Phys., 19 (1978), pp. 1049–1054. Addendum, J. Math. Phys., 20 (1979), p. 1295.
[53] W. SARLET AND F. CANTRIJN, *Higher-order Noether symmetries and constants of the motion*, J. Phys. A, 14 (1981), pp. 479–492.
[54] S. STERNBERG, *Lectures on Differential Geometry*, Prentice-Hall Inc., Englewood Cliffs, NJ, 1964.
[55] H. STEUDEL, *Die Struktur der Invariantzgruppe für lineare Feldtheorien*, Z. Naturforschg., 21a (1966), pp. 1826–1828.
[56] ———, *Zur Umkehrung des Noeterschen Satzes*, Ann. der Physik, 20 (1967), pp. 110–112.
[57] E. C. G. SUDARSHAN AND N. MUKUNDA, *Classical Dynamics: A Modern Perspective*, John Wiley, New York, 1974.
[58] A. TRAUTMAN, *Noether equations and conservation laws*, Comm. Math. Phys., 6 (1967), pp. 248–261.
[59] B. VUJANOVIC, *A group-variational procedure for finding first integrals of dynamical systems*, Int. J. Non-Linear Mech., 5 (1970), pp. 269–278.