Numerical Solution for Sound Velocity and Absorption in Cylindrical Tubes

F. DOUGLAS SHIELDS, K. P. LEE, AND W. J. WILEY

University of Mississippi, University, Mississippi 38677 (Received 12 October 1964)

A numerical solution of the Kirchhoff equation for the propagation constant of longitudinal sound waves in infinitely long cylindrical tubes has been obtained. The solution, which avoids the wide-tube approximations, shows that the percentage errors in the von Helmholtz-Kirchhoff tube velocity correction and tube absorption are both roughly equal to the percentage the velocity correction is of the free-space velocity. The error in the von Helmholtz-Kirchhoff equations can be plotted as a function of fD/a, $pD/\eta a$, and γ . (f is the sound frequency, D the tube diameter, a the free-space velocity, p the gas pressure, η the viscosity, and 2 the ratio of specific heats.) Recent absorption measurements in Ar are in agreement with values calculated numerically, but measured velocities indicate the need for considering molecular slip at the tube wall. Thermal relaxation is introduced into Kirchhoff's basic equation by using the Eucken relation $k/c_{v\eta}$ $-(9_{2}-5)/4$ and considering γ to be the ratio of complex relaxing specific heats. Viscothermal and relaxation effects are found to be additive only if the frequency is near the cutoff frequency for the first unsymmetric mode and the f/p values do not extend to the megacycle/second atmospheres range.

INTRODUCTION

THE classical equations for sound absorption and velocity developed by Kirchhoff¹ in 1868 have often been used without regard for either the boundary conditions or the mathematical approximations that limit their applicability and accuracy. For this reason, many hours have been needlessly spent looking for explanations of discrepancies between experimental results and the equations. This paper presents the results of a numerical solution of the exact equation for the propagation constant. Accurate values of the absorption and velocity of sound have been calculated for radially symmetric waves propagated through gases enclosed in rigid-walled cylindrical tubes. The results cover wide ranges of frequency, pressure, and tube radius. The effect of thermal relaxation is incorporated directly into the solution for the first time. Results of the numerical solution are then compared with the approximate solutions of Kirchhoff¹ and with some recent experimental measurements.

I. KIRCHHOFF EQUATION

Beginning with the conservation equations for mass, energy, and momentum (the last two containing the effects of thermal conductivity and viscosity), Kirchhoff obtained an algebraic equation for the propagation constant *m* for radially symmetric waves. He did this by assuming the boundary conditions of zero particle velocity and sound-temperature at the tube wall. This important equation is reproduced by both Ravleigh² and Weston.4 For the calculation presented here, it can be put in the form

$$m^{2} \left[h \left(m^{2} - \frac{h}{\mu} \right)^{-\frac{1}{2}} \left(\frac{1}{\lambda_{1}} - \frac{1}{\lambda_{2}} \right) \frac{J_{1}}{J_{0}} \right] + (m^{2} - \lambda_{1}) \left[\left(\nu - \frac{h}{\lambda_{1}} \right) (m^{2} - \lambda_{1})^{-\frac{1}{2}} \frac{J_{1}'}{J_{0}'} \right] - \left(\frac{h}{\lambda_{2}} - \nu \right) (m^{2} - \lambda_{2})^{\frac{1}{2}} \frac{J_{1}''}{J_{0}''} = 0, \quad (1)$$

where λ_1 and λ_2 are the small and large roots of the equation

$$\begin{array}{c} h^{2} - \left[a^{2} + h(\mu + \mu' + \nu)\right]\lambda \\ + (\nu/h) \left[b^{2} + h(\mu + \mu')\right]\lambda^{2} = 0, \quad (2) \end{array}$$

¹ G. Kirchhoff, Ann. Phys. Leipzig 134, 177-193 (1868).

² Lord Rayleigh, *Theory of Sound* (Dover Publications, Inc., New York, 1945), 2nd ed., Vols. 1 and 2, Topic 350, p. 324. ³ D. E. Weston, Proc. Phys. Soc. (London) **B66**, 695-709 (1953).

and the J_1 's and J_0 's are Bessel functions with arguments of $r(m^2 - h/\mu)^{\frac{1}{2}}$, $r(m^2 - \lambda_1)^{\frac{1}{2}}$, and $r(m^2 - \lambda_2)^{\frac{1}{2}}$ for the unprimed, primed, and doubled-primed ratios, respectively. r is the tube radius; a is the adiabatic sound velocity, which is $(RT\gamma/M)^{\frac{1}{2}}$ for an ideal gas; b, the isothermal sound velocity, $-a/\gamma^{\frac{1}{2}}$; $h=i\omega$; μ is viscosity/density; $\nu = k/c_r\rho = \mu(9\gamma - 5)/4$ (according to the Eucken equation); k is thermal conductivity; c_r is the specific heat at constant volume; ρ is the density; and $\mu' = \mu/3$ (according to the Stokes hypothesis).

The only approximation involved in Eq. (1) is that of small amplitude waves. An accurate numerical solution of Eq. (1), therefore, should give values of absorption and velocity that are accurate even at extreme conditions of pressure, frequency, and tube radius where earlier approximate solutions fail. Furthermore, a comparison of the numerical solution with experimental values should indicate where such physical phenomena as slip at the tube wall become important.

II. NUMERICAL SOLUTION FOR GASES WITHOUT RELAXATION

Kirchhoff's approximate solution of Eq. (1) (the wide-tube case) was obtained by including only the first term in the ratio of the Bessel functions and by approximating λ_1 , as h^2/a^2 and λ_2 as ha^2/vb^2 . Thus, λ_1 is real and λ_2 is imaginary in this approximation. His familiar expression for the amplitude absorption coefficient (tube absorption) and sound velocity inside the tube (tube velocity) are given by

 $\alpha_{1} - \gamma'(\pi f)^{\frac{1}{2}} ar \tag{3}$

$$v_{t} = a\{1 - \gamma'_{/}[(4\pi f)^{\frac{1}{2}}r]\}, \qquad (4)$$

where $\gamma' = \sqrt{\mu} + (a \ b - b/a)\sqrt{\nu}$ and f is the frequency. Because we will need to use it later, we define Δv_1 as the fractional decrease in velocity predicted by this equation. Thus,

$$\Delta v_{\rm t} = (a - v_{\rm t})/a \quad \gamma' / [(4\pi f)^{\frac{1}{2}}r]. \tag{5}$$

For want of a better expression, the total soundabsorption coefficient is generally taken to be the sum of α_t and the classical free-space absorption α_{clas} given by

$$\alpha_{\rm elas} = \frac{2\pi^2 f^2}{a^3} \bigg[\frac{4}{3} + \bigg(1 + \frac{b^2}{a^2} \bigg) v \bigg]. \tag{6}$$

In Ar at 300°K, for example, with f in kilocycles ' second, p in millimeters of Hg, and r in centimeters, these equations become

$$\alpha_{t} = (0.2842/r)(f/p)^{1},$$

$$\Delta r_{t} = (0.1680/r)(fp)^{-1},$$

$$\alpha_{t-1:ts} = (0.00128)f^{2}/p.$$
(7)

In what follows, we take the first approximation of the



FIG. 1. Fractional error in the Kirchhoff tube velocity correction for an ideal gas with $C_{\nu} = 1.5R$. The abscissa is the tube correction to the velocity divided by the free-space velocity [Eq. (5)]. The ordinate is the velocity calculated numerically minus the tube velocity divided by the tube correction to the velocity [Eq. (8)].

Constant frequency times tube diameter divided by free space velocity. -- Constant pressure times tube diameter divided by viscosity times free-space velocity.

absorption as $\alpha_{\text{telas}} - \alpha_t + \alpha_{\text{clas}}$ and the first approximation of the velocity to be r_t .

Weston³ has obtained more accurate expressions for the absorption and velocity by including more terms in the expansion of the Bessel functions. He obtained several equations for the velocity and absorption, each of which is applicable over a limited range of frequencies and tube radii. However, his approximations also are inaccurate at low pressure.

In the numerical solution that we have performed, both the real and imaginary parts of λ_1 and λ_2 obtained from the solution of Eq. (2) are used. In this way, it is possible to include relaxation effects (see below). Equation (1) is then solved for m^2 by an iteration process. In the first approximation, the real part of mis set equal to α_{telas} and the imaginary part to ω/v_t . The value of m^2 thus obtained is substituted into the argument of the Bessel function and in the radicals of Eq. (1). The equation is then solved for a new value of m. In this process, the Bessel functions are evaluated to a minimum accuracy of 1 part in 10⁴ by the computer program. It should be noted in evaluating the Bessel functions that the desired square root is in the second quadrant of the complex plane for J and J'' but in the first quadrant for J'. The new value of m thus obtained is substituted back into the equation and the iteration continued until the error in the absorption is less than 0.5% and in the velocity is less than 0.01%. Iteration is not continued beyond this accuracy since this is about the best that can be expected from experimental measurements at the present time. The velocity and absorption coefficients thus obtained are labeled r and α in what follows. It is also useful to define dimensionless



Fig. 2. The fractional error in the Kirchhoff absorption coefficient α_{telas} for a gas with $C_v = 1.5R$. The ordinate is the absorption calculated numerically minus α_{telas} divided by α_{telas} [Eq. (9)]. The other symbols are the same as in Fig. 1.

2nd-order velocity and absorption corrections by the following equations:

$$\Delta v' = (v - v_t) / (a - v_t) \tag{8}$$

$$\Delta \alpha' = (\alpha - \alpha_{\text{telas}}) / \alpha_{\text{telas}} \tag{9}$$

Thus, $\Delta v'$ is the fractional error in the Kirchhoff velocity correction and $\Delta \alpha'$ is the fractional error in the absorption $\alpha_{clas} + \alpha_t$.

As is expected from a dimensional analysis of Eq. (1), $\Delta \alpha'$ and $\Delta v'$ are found to be functions of the quantities fD/a, $pD/\eta a$, and γ (D is the tube diameter and η is the gas viscosity). This is shown in Figs. 1 and 2. These Figures are for any gas with a specific heat at constant volume equal to 1.5R. Note that, in evaluating Eq. (1), $k/c_{\nu\eta}$ has been set equal to $(9\gamma - 5)/4$. The family of solid curves in Figs. 1 and 2 is for values of constant fD/a, and the family of dotted curves is for values of constant $pD/\eta a$. The highest value of fD/a considered was 0.55 since the value corresponding to the cutoff frequency of the first unsymmetric mode is 0.586. The curves are not extended to values of frequency or pressure where the unsymmetric modes might be present. Most experimental measurements at present are for pressures and tube diameters such that $pD/\eta a$ is greater than 200. Equation (1) should hold to much lower pressures and tube diameters, however, and the correction factors have therefore been calculated and recorded for the extreme cases. The calculations were made using an IBM-1620 computer.

In cases where $pD/\eta a$ is greater than 200, the Kirchhoff velocity correction is seen to be in error by about the same percentage that this correction is of the total velocity. That is, $\Delta v' = \Delta v_t$. In this same region, the error in the absorption when figured as $\alpha_t + \alpha_{clas}$ is from $\frac{1}{2}$ to 2 times Δr_t . Note that Kirchhoff's velocity correction is always too large (giving velocities too small) but the absorption calculated as $\alpha_t + \alpha_{clas}$ is always too small.

Values of the absorption and velocity have also been calculated for gases with $C_v/R=2.5$, 3.5, and 5.5. The results are given in Table I, along with those for $C_{p} = 1.5R$. To use this Table, one first calculates the absorption and velocity corrections using the Kirchhoff approximate equations [Eqs. (3), (6), and (5)]. The error in these values can then be interpolated from the Table for the particular gas, frequency, and tube diameter desired. For example, for Ar at 300°K the ideal gas velocity is 32 260 cm/sec. Therefore, for a 1-cm-diam tube with a sound frequency of 10 kc/sec and a gas pressure of 10 mm, fD/a=0.310 and $v_t = 3.63\%$ [Eq. (7)]. From Table I or Fig. 1, the velocity calculated by the Kirchhoff approximation would be too small by approximately 0.1% or 32 cm/sec. The absorption figured as $\alpha_t + \alpha_{clas}$ would be too low by approximately 3.4%.

III. COMPARISON WITH EXPERIMENTAL RESULTS

Equation (1) assumes that the particle-velocity profile across the tube is independent of axial distance down the tube. Therefore, neither the Kirchhoff tube absorption and velocity equations nor the numerical analysis that is reported here would be expected to apply to standing waves in short tubes or resonant cavities. It has been common practice, however, to try to apply Kirchhoff's equations to such situations by multiplying the right-hand side of Eqs. (3) and (5) by an empirical correction factor. On the other hand, experiments that avoid the end effects and relaxation effects now find general agreement with the Kirchhoff tube velocity and absorption equations over the range of frequencies, pressures, and tube radii where they should apply $^{4-9}$ Chandler¹⁰ accounts for about $\frac{1}{2}$ of the 4% discrepancy in absorption values reported by Angona⁹ by using more accurate values of physical constants in evaluating the Kirchhoff equations. This produces substantial agreement between the results of Angona and Shields and Lagemann,⁴ both of whom used a method that avoids standing waves. The remaining small excess absorption reported in these two papers would be expected from the more accurate numerical analysis reported here.

In the past, the effect of departures from the Kirchhoff wide-tube approximations have been tested by making measurements at high frequencies in capillary tubes. The results generally are of the proper magnitude and sign to confirm Weston's³ more accurate

- ⁶ G. A. Norton, J. Acoust. Soc. Am. 7, 16-26 (1935).
 ⁸ W. P. Mason, Phys. Rev. 31, 283-295 (1928).
 ⁹ F. A. Angona, J. Acoust. Soc. Am. 25, 1111-1116 (1953).
 ¹⁰ D. E. Chandler, Univ. Calif., Los Angeles, Phys. Dept. Tech. ev. 11 ONP contract Norm 233(49, 14052). Rept. 11 ONR contract Nonr 233(48) (1958).

^{*} F. D. Shields and R. T. Lagemann, J. Acoust. Soc. Am. 29, 470-475 (1957).

⁵ F. A. Angona, J. Acoust. Soc. Am. **25**, 336(L) (1953). ⁶ R. D. Fay, J. Acoust. Soc. Am. **12**, 62–67 (1940).

$\frac{D}{\Delta v_t} = 0.0$ $\frac{\Delta v_t}{a} = 0.0$ $\frac{\Delta v_t}{a} = 0.0$ I. $C_y = 1.5R$ $0.05 = 0.04$ $0.10 = 0.09$ $0.15 = 0.13$ $0.20 = 0.18$ $0.25 = 0.22$ $0.30 = 0.25$ $0.35 = 0.28$ $0.40 = 0.31$ II. $C_y = 2.5R$ $0.05 = 0.044$ $0.10 = 0.09$ $0.15 = 0.14$ $0.20 = 0.19$ $0.25 = 0.22$	05 0.15 487 0.0490 949 0.0961 386 0.1414 804 0.1856 214 0.2298 584 0.2709 884 0.3059 119 0.2252	0.25 0.0497 0.0990 0.1485 0.1993 0.2522 0.3049	0.35 0.0508 0.1047 0.1630	0.45 0.0529 0.1147	0.55	0.05	0.15	0.25	0.35	0.45	0.55
Δv_t I. $C_v = 1.5R$ 0.05 0.04 0.10 0.09 0.15 0.13 0.20 0.15 0.22 0.30 0.25 0.35 0.28 0.40 0.31 II. $C_v = 2.5R$ 0.05 0.04 0.10 0.09 0.15 0.14 0.20 0.25 0.25 0.25 0.24 0.10 0.09 0.15 0.14 0.20 0.19 0.25 0.25 0.25 0.25 0.25 0.25 0.25 0.25	487 0.0490 949 0.0961 386 0.1414 804 0.1856 214 0.2298 584 0.2709 884 0.3059	$\begin{array}{c} 0.0497 \\ 0.0990 \\ 0.1485 \\ 0.1993 \\ 0.2522 \\ 0.3049 \end{array}$	0.0508 0.1047 0.1630	0.0529 0.1147	0.0562	0.0316	0.02/5				
I. $C_v = 1.5R$ 0.05 0.04 0.10 0.09 0.15 0.13 0.20 0.18 0.25 0.22 0.30 0.25 0.35 0.28 0.40 0.31 II. $C_v = 2.5R$ 0.05 0.04 0.10 0.09 0.15 0.14 0.20 0.19 0.25 0.22	487 0.0490 949 0.0961 386 0.1414 804 0.1856 214 0.2298 584 0.2709 884 0.3059 114 0.2252	$\begin{array}{c} 0.0497 \\ 0.0990 \\ 0.1485 \\ 0.1993 \\ 0.2522 \\ 0.3049 \end{array}$	0.0508 0.1047 0.1630	0.0529 0.1147	0.0562	0.0316	0.01/5				
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{rrrrr} 487 & 0.0490\\ 949 & 0.0961\\ 386 & 0.1414\\ 804 & 0.1856\\ 214 & 0.2298\\ 584 & 0.2709\\ 884 & 0.3059\\ 118 & 0.275\\ \end{array}$	$\begin{array}{c} 0.0497 \\ 0.0990 \\ 0.1485 \\ 0.1993 \\ 0.2522 \\ 0.3049 \end{array}$	$0.0508 \\ 0.1047 \\ 0.1630$	$0.0529 \\ 0.1147$	0.0562	0.0316	0.02/7				
$\begin{array}{c} \hline \textbf{II.} \ C_v = 2.5R \\ 0.05 & 0.04 \\ 0.10 & 0.09 \\ 0.15 & 0.14 \\ 0.20 & 0.19 \\ 0.25 & 0.24 \\ 0.25 & 0.25 \\ 0.25 & 0.24 \\ 0.25 & 0.25 \\$	118 0.3353	$0.3547 \\ 0.4024$	0.2280 0.3000 0.3787 0.4609 0.5476	0.1891 0.2798		$\begin{array}{c} 0.0654\\ 0.1017\\ 0.1402\\ 0.1843\\ 0.2376\\ 0.2974\\ 0.3592 \end{array}$	$\begin{array}{c} 0.0305\\ 0.0753\\ 0.1164\\ 0.1595\\ 0.2079\\ 0.2646\\ 0.3267\\ 0.3900 \end{array}$	$\begin{array}{c} 0.0462 \\ 0.0944 \\ 0.1442 \\ 0.1950 \\ 0.2492 \\ 0.3086 \\ 0.3691 \\ 0.4262 \end{array}$	$\begin{array}{c} 0.0604\\ 0.1214\\ 0.1821\\ 0.2406\\ 0.2967\\ 0.3493\\ 0.3915\\ 0.4181\end{array}$	$\begin{array}{c} 0.0787 \\ 0.1549 \\ 0.2258 \\ 0.2869 \end{array}$	0.1005
$\begin{array}{cccc} 0.05 & 0.04 \\ 0.10 & 0.09 \\ 0.15 & 0.14 \\ 0.20 & 0.19 \\ 0.25 & 0.21 \end{array}$											
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	493 0.0498 970 0.0992 433 0.1483 904 0.1995 366 0.2512 716 0.2933 935 0.3239 061 0.3464	$\begin{array}{c} 0.0510\\ 0.1043\\ 0.1605\\ 0.2227\\ 0.2894\\ 0.3516\\ 0.4077\\ 0.4604 \end{array}$	$\begin{array}{c} 0.0530\\ 0.1137\\ 0.1844\\ 0.2691\\ 0.3684\\ 0.4748\\ 0.5847\\ 0.6998 \end{array}$	0.0563 0.1297 0.2263 0.3534	0.0614	$\begin{array}{c} 0.0528\\ 0.1091\\ 0.1687\\ 0.2328\\ 0.3128\\ 0.4086\\ 0.5095\\ 0.6056\end{array}$	$\begin{array}{c} 0.0578\\ 0.1189\\ 0.1833\\ 0.2517\\ 0.3348\\ 0.4319\\ 0.5327\\ 0.6282\\ \end{array}$	$\begin{array}{c} 0.0675\\ 0.1377\\ 0.2102\\ 0.2848\\ 0.3701\\ 0.4633\\ 0.5534\\ 0.6314 \end{array}$	$\begin{array}{c} 0.0815\\ 0.1639\\ 0.2454\\ 0.3233\\ 0.4010\\ 0.4719\\ 0.5231\\ 0.5460 \end{array}$	0.0993 0.1953 0.2830 0.3545	0.1204
III. $C_v = 3.5 R$				-							
$\begin{array}{ccccccc} 0.05 & 0.04'\\ 0.10 & 0.09\\ 0.15 & 0.14\\ 0.20 & 0.19\\ 0.25 & 0.24\\ 0.30 & 0.27\\ 0.35 & 0.29\\ 0.40 & 0.29\\ \end{array}$	497 0.0504 988 0.1016 472 0.1537 993 0.2110 468 0.2656 770 0.3049 909 0.3296 953 0.3460	$\begin{array}{c} 0.0519\\ 0.1080\\ 0.1692\\ 0.2404\\ 0.3144\\ 0.3796\\ 0.4365\\ 0.4900\\ \end{array}$	$\begin{array}{c} 0.0545\\ 0.1199\\ 0.1993\\ 0.2991\\ 0.4149\\ 0.5360\\ 0.6605\\ 0.7891 \end{array}$	0.0586 0.1398 0.2514 0.4045	0.0647	$\begin{array}{c} 0.0649\\ 0.1342\\ 0.2076\\ 0.2893\\ 0.3953\\ 0.5187\\ 0.6415\\ 0.7533\end{array}$	$\begin{array}{c} 0.0698\\ 0.1439\\ 0.2218\\ 0.3073\\ 0.4152\\ 0.5381\\ 0.6591\\ 0.7688\end{array}$	$\begin{array}{c} 0.0795\\ 0.1624\\ 0.2477\\ 0.3380\\ 0.4446\\ 0.5585\\ 0.6624\\ 0.7465\end{array}$	$\begin{array}{c} 0.0933\\ 0.1878\\ 0.2806\\ 0.3707\\ 0.4629\\ 0.5432\\ 0.5945\\ 0.6100 \end{array}$	0.1109 0.2176 0.3137 0.3907	0.1314
IV. $C_v = 5.5R$											
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	503 0.0512 011 0.1046 529 0.1610 110 0.2257 570 0.2807 786 0.3136 822 0.3030	$\begin{array}{c} 0.0530\\ 0.1126\\ 0.1802\\ 0.2625\\ 0.3419\\ 0.4074\\ 0.4633\\ 0.5169\end{array}$	$\begin{array}{c} 0.0562 \\ 0.1272 \\ 0.2175 \\ 0.3355 \\ 0.4678 \\ 0.6024 \\ 0.7405 \\ 0.8834 \end{array}$	$\begin{array}{c} 0.0612 \\ 0.1515 \\ 0.2815 \\ 0.4648 \end{array}$	0,0686	0.0778 0.1613 0.2497 0.3545 0.4923 0.6438 0.7865	0.0827 0.1708 0.2633 0.3710 0.5086 0.6573 0.7961	$\begin{array}{c} 0.0922\\ 0.1887\\ 0.2876\\ 0.3977\\ 0.5293\\ 0.6624\\ 0.7762\\ 0.762\end{array}$	$\begin{array}{c} 0.1058\\ 0.2129\\ 0.3173\\ 0.4223\\ 0.5301\\ 0.6168\\ 0.6634\\ 0.6654\end{array}$	$\begin{array}{c} 0.1229 \\ 0.2407 \\ 0.3446 \\ 0.4281 \end{array}$	0.1427

TABLE I. Fractional error in the Kirchhoff expression for the tube velocity and absorption of sound.

equations, which include terms so as to apply to this case.^{11–13} Very few measurements have been made at low pressures where the corrections given in Table I become appreciable. We therefore have performed some measurements for this purpose. The method was essentially the same as that previously described⁴ but employs a tube about one-half as big (0.785 cm diam). As before, reflections within the sound tube are kept to a minimum so as to avoid standing waves. The results of the absorption and velocity measurements in Ar are shown in Figs. 3 and 4, respectively. For these measurements, the frequency was maintained at 16.9 kc/sec and the pressure varied from 603 to 3.5 mm Hg. The circles in the Figures are the experimental points.

Shown also in Fig. 3 are the values of α_t , α_{telas} , and α . It is seen that the numerical calculation does account for the departure of the measured points from the $\alpha_t + \alpha_{clas}$ line. The measured velocities are seen in Fig. 4 to be considerably larger than the tube velocity curve at low pressures. The numerical analysis gives values that are somewhat better but still too low. The remaining discrepancy can be accounted for by assuming a molecular slip at the tube wall. Henry¹⁴ has considered the effects of molecular slip on Kirchhoff's equation and finds that it produces a negligible effect on the absorption but adds a small term to the velocity. The velocityslip correction that he develops is given by

$$\Delta v_{s} = \pi^{\frac{1}{2}} a^{2} \eta \left(\frac{2-g}{g}\right) / (2\gamma)^{\frac{1}{2}} r \left[p + \left(\frac{2-g}{g}\right) (2\pi^{2} p j \eta)^{\frac{1}{2}} \right], (10)$$

¹¹ P. S. H. Henry, Proc. Phys. Soc. (London) 43, 340-363 (1931).

¹¹ D. E. Weston and I. D. Campbell, Proc. Phys. Soc. (London) B66, 769-774 (1953).

 ¹⁹ L. E. Lawley, Proc. Phys. Soc. (London) B65, 181–188 (1952).
 ¹³ G. T. Kemp and A. W. Nolle, J. Acoust. Soc. Am. 25, 1083 (1953).



FIG. 3. Sound absorption in Ar(25.6°C). The measurements shown by the open circles were made at 16.9 kc/sec inside a tube 0.785 cm in diameter. -- Tube absorption α_t [Eq. (3)]. --- Tube absorption plus the classical absorption [Eq. (6)]. --- Results of the numerical calculation.

where g is the fraction of molecules diffusely reflected from the surface and will be taken to be unity. When this term is added to the velocity, v, calculated by the numerical analysis, an acceptable fit to the experimental points is obtained as seen in Fig. 4.

IV. INTRODUCTION OF THERMAL RELAXATION

Experimentalists have generally assumed, without proof, that the viscothermal and relaxation effects on the absorption and velocity are additive. Thus, it is common practice to obtain the relaxation absorption $\alpha_{\rm r}$ by subtracting $\alpha_{\rm telas}$ from the measured absorption, and to obtain the free-space relaxation velocity by adding the tube correction to the measured velocity.¹³ It is possible to check this assumption with the numerical solution discussed above if we can incorporate the relaxation phenomena into Eq. (1). This we have done by first replacing $k/c_{v\eta}$ with the Eucken expression $(9\gamma-5)/4$ and then using the complex, relaxing specific heat to obtain γ . This complex specific heat is obtained from the usual equation,

$$C_{\nu} = C_{\omega} + C_i / (1 + i\omega\tau). \tag{11}$$

 C_{∞} and C_i are, respectively, the nonlagging and lagging parts of the specific heat.

In Eq. (1), ν now becomes a complex quantity with an appreciable imaginary part. This augments the imaginary part of λ_1 , and λ_2 and results in an increased absorption and dispersion.

The results of such numerical calculations using CO₂ as an example illustrate the magnitude of the error to be expected when the absorption is calculated as $\alpha_{tclas} + \alpha_r$. For a 1-cm-diam tube, a frequency of 8 kc/sec (fD/a=0.3), and assuming the maximum relaxation absorption to fall at 30 kc/sec/atm, the numerical solution gives values that are less than $\alpha_{telas} + \alpha_r$ by only 0.5%, 0.9%, and 0.6% for f/p values of 10, 30, and 100 kc/sec/atm, respectively. If the absorption peak is

shifted to 300 kc/sec/atm, the corresponding errors at 100, 300, and 1000 kc/sec/atm are 0.6%, 1.2%, and 1.1%. If the peak were at 3 Mc/sec/atm, it would be necessary to make the measurements at pressures of 6.2 and 0.6 mm Hg to span the absorption peak as before, and would therefore push the tube method to its present limit. In this case, $\alpha_{\text{telas}} + \alpha_{\text{r}}$ would again be too big by 1.1%, 1.9%, and 9.9%.

If we look at the corresponding differences between the velocity obtained from the numerical solution and that obtained by adding the tube and relaxation dispersion, we find the numerical values of the velocity larger by 0.02%, 0.05%, and 0.01% for the case where (f/p) max is 30 kc/sec/atm, 0.02%, 0.04%, and 0.06% for (f/p) max=300 kc/sec/atm, and 0.06%, 0.19%, 0.91% for (f/p) max=3 Mc/sec/atm.

Thus, for a value of (fD/a)=0.3, the numerical values differ only slightly from values obtained in the usual way by adding viscothermal and relaxation effects. Only at low pressures is the difference much greater than the usual experimental error. Since the addition method also has been found to give values in agreement with experiment, we can conclude that the method used to incorporate relaxation effects into Eq. (1) is legitimate and that $k/c_s\eta$ can be set equal to $(9\gamma-5)/4$, even in the case of a relaxing gas if the complex γ is used.

The error in the absorption when calculated as $\alpha_{\text{telss}} + \alpha_{\text{r}}$ is highly frequency-dependent. As far as the relaxation effects are concerned, the absorption and velocity should be the same at low frequencies and low pressures as at high frequencies and high pressures, since the relaxation phenomena is a function of f/p. However, the numerical calculation shows significant departures of α from $\alpha_{\text{telss}} + \alpha_{\text{r}}$ as the frequency and



FIG. 4. Velocity of sound in Ar(25.6° C). The measurements, as in Fig. 3, are shown by open circles. --- Kirchhoff tube velocity [Eq. (4)]. --- Result of the numerical calculation. --- Numerical solution plus the slip correction [Eq. (10)].

¹⁶ F. D. Shields, J. Acoust. Soc. Am. 29, 450-454 (1957).

pressure are lowered keeping f/p constant. Starting near the cutoff frequency for the unsymmetric mode, α is only slightly smaller than $\alpha_{\text{telas}}+\alpha_{\text{r}}$ but it gets significantly larger than $\alpha_{\text{telas}}+\alpha_{\text{r}}$ as the frequency is lowered. For a 1-cm-diam tube and a gas like CO₂, for example, with an absorption peak at 300 kc/sec/atm, the percentage differences between α and $\alpha_{\text{telas}}+\alpha_{\text{r}}$ are -0.8, -1.2, -0.7, +1.7, +8.3, and +26 as the frequency takes the values 16, 8, 4, 2, 1, and 0.5 kc/sec while maintaining f/p at the peak value of 300 kc/sec atm.

A similar frequency dependence is found for the velocity error. At the same frequencies as in the above example, τ is greater than v_t by 0.05%, 0.05%, 0.08%, 0.03%, 1.5%, and 7.4%.

V. CONCLUSIONS

Kirchhoff's tube absorption and velocity equations have been found to be in error by a percentage roughly equal to the percentage the tube correction is of the velocity. A numerical solution to Kirchhoff's basic equation for the propagation constant shows the corrections to his approximate equations to be functions of fD/a, $pD/\eta a$, and γ . Recent absorption measurements in Ar, using a tube 0.785 cm in diameter are in agreement with the values obtained by the numerical analysis. Measured velocities are greater than those predicted by either the numerical analysis or the Kirchhoff equation. The results can be explained by assuming a molecular slip at the tube walls.

The effects of thermal relaxation can be incorporated into the numerical solution by replacing $k/c_{\nu\eta}$ by $(9\gamma-5)/4$, where γ is the ratio of complex, relaxing specific heats. Viscothermal and relaxation effects are additive only if the frequency is close to the cutoff frequency of the first unsymmetric mode and f/p values do not extend into the megacycle/second/atmospheres range.

ACKNOWLEDGMENT

We wish to thank J. W. L. Lewis for suggesting molecular slip at the tube wall as a possible explanation for the observed velocity discrepancy. This work has been supported by the U. S. Office of Naval Research.