Sound Transmission through Thin Cylindrical Shells*

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(Received February 4, 1957)

An analysis is presented of the impedance presented by a thin cylindrical elastic shell to a pressure or normal stress as a function of the axial wavelength and the angular dependence of the forces. Results of computation are presented graphically. This information is then used to compute a measure of the sound transmitted through the shell immersed in air for various particular cases. The theory of the scattering and absorption of waves incident upon a cylinder at angles other than normal is developed for this purpose. The results and their implications are discussed in detail.

LIST OF MAJOR SYMBOLS

a	radius of shell
c	phase velocity
h	half-thickness of shell
j	$(-1)^{\frac{1}{2}}$
k	phase constant: ω/c
m	order number of mode
p	acoustic pressure
P	acoustic pressure amplitude
Q^a	absorption coefficient
r	specific acoustic resistance ratio; radial cylindrical coordinate
u, v, w	axial, circumferential, and radial components of displacements of shell
U, V, W	time—and space—independent amplitudes of u, v, w
x	dimensionless ratio: $x=ka=\omega a/c$
z	dimensionless specific acoustic impedance ratio; axial cylindrical coordinate
Z	specific acoustic impedance
ϵ_m	Neumann factor: $\epsilon_0=1$; $\epsilon_m=2$, $m>0$.
θ	angle of incidence of plane wave upon cylinder
ν	Poisson's ratio
Π	acoustic power, per unit length
ρ	density
ϕ	angular cylindrical coordinate

Superscripts

a	absorbed
c	contents of shell
i	incident wave
s	scattered wave
sh	shell

INTRODUCTION

THIS paper is a theoretical study of the transmission of sound energy through a thin, elastic cylindrical shell immersed in a fluid medium. Particular attention is focused on evaluating a "transmission loss" for the shell, considering it to be a structure isolating the interior region from sound waves incident upon the exterior. In order to simplify the analysis, the shell is assumed to be uniform and infinitely long.

The incident sound wave is taken as an infinite plane wave incident upon the shell at an angle θ relative to the normal to the shell axis. (See Fig. 1.) The wave will be considered as approaching in the radial plane defined by the equation $\phi=\pi$. Then the incident wave can be expressed as a sum of partial waves each characterized by a dependence upon the ϕ coordinate of the form $\cos(m\phi)$, where m is an integer. All the partial waves have the same dependence upon the cylindrical axial coordinate z, namely that of a wave traveling in the z direction with a velocity $c_z=\omega/k_z=c/\sin\theta$; this dependence has the form: $\exp j(\omega t-k_z z)$. These partial waves constitute the forcing functions acting upon the shell. Under certain conditions (which are assumed in this analysis and will be discussed later) the total response of the shell and its contents is the sum of the responses to the individual partial waves of the incident sound. The analysis can then be carried through for each partial wave independently.

In this paper the sound energy transmitted through the shell is determined by evaluating the absorption cross section of the shell by an extention of methods developed for studies of the scattering and absorption of normally incident sound.[1] An essential part of this method is the evaluation of the acoustic impedance of the cylinder for a given partial wave. Therefore a large

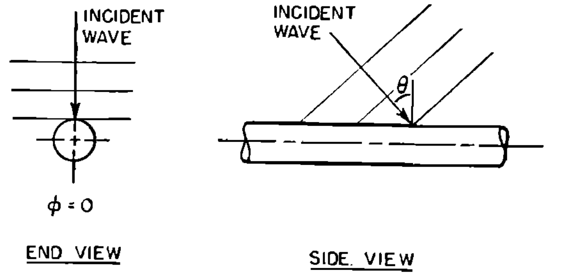

FIG. 1. Geometrical relationships.

* This research has been aided in part by funds made available under a contract with the Office of Naval Research. A summary of this work was presented orally by the author at the 49th meeting of The Acoustical Society of America, in July, 1955. [See, J. Acoust. Soc. Am. 27, 1011 (1955).]

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[1] B. Lax and H. Feshbach J. Acoust. Soc. Am. 20, 108–124 (1948).

part of this paper is devoted to the evaluation of the impedance to normal forces presented by a thin shell, as a function of the axial wavelength and angular dependence of the forces. In this analysis a bending theory of shell vibration[2] is used which is valid so long as the wavelengths of shell motion are large compared with the shell thickness. In other words, the analysis cannot be extended to the very high frequencies where the shell motion is like a Rayleigh wave. These impedance functions, and the use of an impedance concept, can be very useful in the consideration of other problems involving the interaction of an elastic shell with fluid media.

SCATTERING OF WAVES NOT NORMALLY INCIDENT

Consider an infinite plane acoustic wave in a fluid incident upon a cylindrical surface at an angle θ relative to the normal to the cylindrical axis. The wave is assumed to be *approaching* that axis in a direction lying in the radial plane $\phi=\pi$. It is readily verified that the expression for this incident pressure wave in cylindrical coordinates is a series of the form

$$p^i=P_0e^{-jk_zz}e^{j\omega t}\Sigma_{m=0}\epsilon_m(-j)^mJ_m(k_rr)\cos m\phi, \tag{1}$$

where $k_z=k\sin\theta$ and $k_r=k\cos\theta$. The acoustic disturbance from this wave which is caused by the presence of the shell—the scattered wave—is a solution of the wave equation representing an outgoing wave. If the assumption is made that the scattered pressure wave has the same z dependence as the incident wave,[3] the general solution can be represented in a series of the form

$$p^s=e^{-jk_zz}e^{j\omega t}\Sigma A_mH_m^{(2)}(k_rr)\cos m\phi. \tag{2}$$

The sum of the two series is the general solution in the presence of the shell. The complex amplitude factors A_m remain to be determined.

The total radial outward particle velocity $\dot{w}$ in the fluid medium outside the cylinder which corresponds to this total acoustic pressure $p=(p^i+p^s)$ can be determined from the acoustic force equation

$$\partial p/\partial r=-\rho(\partial^2w/\partial t^2). \tag{3}$$

The amplitude factors A_m are determined by the requirement that the total wave satisfy the boundary conditions at the surface of the cylinder, $r=a$. This boundary condition will be taken as a normal impedance. Specifically we postulate an inward-looking, modal, specific acoustic impedance, Z_m, which relates the radial particle velocity and pressure at $r=a$ by the equation

$$p_m(a)=-\dot{w}_m(a)Z_m, \tag{4}$$

where $\dot{w}_m$ and p_m are the mth terms in the series expansions for radial velocity and pressure.

[2] E. H. Kennard, J. Appl. Mech. 20, 33–40 (1953).

[3] The assumption will be discussed in a moment.

Such a modal impedance cannot be defined in all cases. Its existence requires that the motion of the cylinder in response to a modal force p_m, with angular distribution $\cos m\phi$, shall have the same angular distribution. Obviously the presence of angular periodicities in the cylinder will negate this possibility; e.g., with a rigid diagonal strut in the shell, pressures in the $m=0$ mode will cause motions in the $m=2$ and other modes. However, the initial assumption that the cylinder is uniform justifies the use of these modal impedances. In just the same manner, the assumed uniformity in the axial direction justifies the assumption, made in connection with Eq. (2), that all responses to the incoming plane wave will have an axial dependence of the same form as that wave.

The previous equations can now be combined and solutions obtained for the scattered wave amplitudes A_m. Straightforward algebraic manipulation yields the expressions

$$A_m=-P_0\epsilon_m(-j)^m\frac{J_m'(x_r)}{H_m^{(2)\prime}(x_r)}\frac{z_m+z_m{}^i(x_r)}{z_m+z_m{}^s(x_r)}, \tag{5}$$

where $x_r=k_ra=\omega a/c_r$; $c_r=c/\cos\phi$; and z_m, $z_m{}^i$, $z_m{}^s$ are modal acoustic impedance ratios for the cylinder, incident wave, and scattered wave respectively. These impedance ratios are defined

$$z_m=Z_m/\rho c_r;\quad z_m{}^i=-jJ_m/J_m';\quad z_m{}^s=-jH_m^{(2)}/H_m^{(2)\prime}. \tag{6}$$

The primes on the Bessel and Hankel functions indicate derivatives with respect to the argument.

This expression for the amplitudes A_m differs from expressions in previous analyses for a normally-incident wave primarily in that it employs impedance ratios in place of admittance ratios; however Eq. (5) can readily be converted into admittances to show the equivalence with standard expressions. The preference for impedances is based on the fact that the impedance of the combination of thin shell and its contents is the sum of the impedances of the parts. The other difference from analyses for normally-incident waves is the appearance of the velocity c_r in place of c; the velocity c_r is the phase velocity of the incoming plane wave in a direction normal to the axis of the shell and in the plane of that axis and the direction of propagation of the wave. These similarities to standard analyses mean that standard tables[4] are applicable and, indeed, that the results of previous analyses for normally-incident waves are directly transferable, if c_r is used in place of c, in so far as the impedance z_m is independent of angle of incidence. This latter condition is not satisfied by elastic shells and cylinders.

In the present analysis we are specifically interested

[4] Among others, see reference 1, and Lowan, Morse, Feshbach, and Lax, "Scattering and radiation from circular cylinders and spheres," U. S. Navy Department, July, 1946, reprinted for the Office of Naval Research, March, 1954.

in the power absorbed by the shell and its contents. Therefore we shall develop the expression for the absorption cross section of the cylinder, or, more simply, for the cross-sectional absorption coefficient which equals the absorption cross section divided by the diameter, $2a$. The cross-sectional absorption coefficient Q^a is defined as the ratio of the power absorbed (per unit length) to the power in the incoming plane wave which is incident upon a unit length of the longitudinal section (the plane $\phi=\pm\pi/2$). The approach will be very similar to that presented by Lax and Feshbach,[1] except for the use here of impedances.

The power absorbed, per unit length, is

$$\Pi^a=-(\tfrac{1}{2})\int_{\phi=0}^{2\pi}\mathrm{Re}(p\dot{w}^*)ad\phi, \tag{7}$$

where the minus sign is necessary to obtain flux *into* the cylinder. The superscript star indicates the complex conjugate. The power incident on the longitudinal section, per unit length is simply

$$\Pi^i=P_0^2a\cos\theta/\rho c. \tag{8}$$

Equation (7) must be evaluated from the series expansions for pressure and radial velocity; it is convenient to introduce the modal impedances in the series for pressure. Cross terms, involving modes with different m, vanish in the integration. Straightforward manipulation leads eventually to an expression for the cross-sectional absorption coefficient in the form

$$\begin{aligned}Q^a&=\Sigma Q_m{}^a\\ Q_m{}^a&=(2\epsilon_m/x_r)r_mr_m{}^s/[(r_m+r_m{}^s)^2+(\chi_m+\chi_m{}^s)^2],\end{aligned} \tag{9}$$

where

$$\begin{aligned}r_m&=\mathrm{Re}(z_m),\quad \chi_m=Im(z_m),\\ r_m{}^s&=\mathrm{Re}(z_m{}^s)=(2/\pi x_r)/(J'^2+N'^2)\\ \chi_m{}^s&=\mathrm{Im}(z_m{}^s)=-(JJ'+NN')/(J'^2+N'^2).\end{aligned} \tag{10}$$

The radiation resistance ratios $r_m{}^s$ and reactance ratios $\chi_m{}^s$ are perhaps most readily computed from functions defined and tabulated by Lowan, Morse *et al.*[4] through the equations

$$\begin{aligned}r_m{}^s&=(C_m/C_m')\sin(\delta_m-\delta_m')\\ \chi_m{}^s&=(C_m/C_m')\cos(\delta_m-\delta_m').\end{aligned} \tag{11}$$

They have been computed and presented in graphical form by Junger.[5]

The denominator of the expression for $Q_m{}^a$ in Eq. (9) will be recognized as the square of the modal impedance presented to an ideal membrane vibrating on the surface of the cylinder, including both the impedance of the cylinder and the radiation impedance.

[5] Miguel C. Junger, J. Acoust. Soc. Am. 24, 288–289 (1952).

FORCED WAVES IN A CYLINDRICAL SHELL

It is desired to determine the impedance of the shell to a radial driving force, as a function of the axial wavelength and the angular dependence of the force. We assume that only normal, radial stresses are applied to the surfaces of the shell, conditions appropriate to a shell immersed in a fluid medium.

In order to meet the boundary conditions, the radial component of displacement of the shell must have a functional dependence upon the z and ϕ coordinates identical with that for the particle velocity in the surrounding fluid medium. In general that dependence can be written as an infinite series, but the analysis can be carried out for each mode (single term of the series) separately. Let us now consider the forced motion of a single mode in a thin shell, under the action of normal radial stresses, with the purpose of determining the modal impedance Z_m.

The displacements from equilibrium of the median surface of the shell wall in the axial, tangential, and radially outward directions can be written for the mth mode in the form

$$\begin{aligned}u&=jU\cos m\phi e^{-jk_zz}e^{j\omega t}\\ v&=V\sin m\phi e^{-jk_zz}e^{j\omega t}\\ w&=W\cos m\phi e^{-jk_zz}e^{j\omega t}.\end{aligned} \tag{12}$$

The expression for w is chosen to match terms in the series used in the previous section; the expressions for u and v have been chosen, after a preliminary look at the differential equation of shell motion, to yield a complete solution in which the ratios of amplitudes U/W and V/W would be real. The pressure acting on the outside and inside surfaces of the shell will be written

$$\begin{aligned}p^+&=P^+\cos n\phi e^{-jk_zz}e^{j\omega t}\\ p^-&=P^-\cos n\phi e^{-jk_zz}e^{j\omega t}.\end{aligned} \tag{13}$$

When Eqs. (12) and (13) are substituted in the dynamical equations for motion of a thin shell, three simultaneous, linear algebraic equations are obtained relating the amplitudes of motion U, V, W and the amplitudes of the external pressures P^+ and P^-. These equations may be solved by the method of determinants for the amplitude of radial motion W as a function of the pressures.

The solution obtained is quite complicated since the dynamical equations are quite precise in their description of motions of thin shells. Indeed, the equations show that the simple concept of a radial modal impedance for the shell is not *precisely* correct. This simple concept is one in which the impedance at the outer surface of the shell is the sum of one contribution uniquely determined by the shell and a second term determined by the fluid medium inside the shell. Fundamentally, this concept is based on the failure to recognize differences in the radial motions of the inner and outer surfaces of the shell or to discriminate between the effects on shell motion of pressures inside and outside. These errors in the simple concept appear in the more precise solution described above by the presence of factors of the form $[1+\mathcal{O}(h/a)]$. The correction terms, of the order (h/a), are not always negligible compared with unity even in a thin shell where (h/a) is small. However, a careful study of the equations reveals that the correction terms become important only when the impedance predicted by the simple concept becomes very large. Fortunately, these are the regions in which we have least interest; in an analysis of the transmission of sound, interest is naturally concentrated in the regions of small impedance where the sound transmission is large.

Therefore in this analysis we shall neglect the small correction terms in order to retain the simple concept of impedance for the shell, with its appurtenant advantages that a single solution for the shell, by itself, can be used in all situations. Although this approach discards some terms in (h/a), it is not to be considered as based on the "membrane theory" of shells; the effects of bending, for example, are correctly included, except in those regions where the impedance is very large. The modal, inward-looking radial impedance of the shell which results from this analysis can be written

$$Z^{sh}{}_m=\frac{P^+-P^-}{-j\omega W}=j\rho^{sh}c_L(2h/a)(D/x_LD_{33}), \tag{14}$$

where D is the whole determinant and D_{33} is the (3–3) cofactor of the array:

$$\begin{vmatrix} \left\{-x_L^2+x_z^2+\dfrac{1-\nu}{2}m^2\right\} & \dfrac{1+\nu}{2}mx_z & \nu x_z \\ \dfrac{1+\nu}{2}mx_z & \left\{-x_L^2+\dfrac{1-\nu}{2}x_z^2+m^2\right\} & \left\{m-\dfrac{h^2}{2a^2}\dfrac{\nu}{1-\nu}m(m^2-1)\right\} \\ \nu x_z & m & \left\{1-x_L^2+\dfrac{h^2}{3a^2}\left[(x_z^2+m^2)^2-\dfrac{4-\nu}{2(1-\nu)}m^2+\dfrac{2+\nu}{2(1-\nu)}\right]\right\} \end{vmatrix}.$$

In this equation, c_L is a velocity characteristic of the shell material so that $x_L=\omega a/c_L$ is a frequency parameter. The quantity $x_z=\omega a/c_z=2\pi a/\lambda_z$, where λ_z is the axial wavelength, is fundamentally a measure of the *axial trace wavelength* of the incoming plane wave. It was shown in the previous sections that $c_z=c/\sin\theta$; therefore x_z varies with frequency and angle of incidence.

The impedance $Z_m{}^{sh}$ vanishes when the determinant D vanishes. The conditions for this occurrence have been considered in another paper[6]; they are identical with the conditions for the existence of *free* waves in the shell. The coincidence is fundamental; the free waves in the shell are motions in which there is radial motion (except in the case of the axisymmetrical shear, or torsion, wave) in the absence of external forces. It follows directly that an external force with the same frequency and wavelengths as the free wave would "see" no impedance.

The impedance $Z_m{}^{sh}$ becomes infinite when the cofactor D_{33} vanishes and in the limit as frequency (i.e., x_L) approaches zero. The conditions for the vanishing of D_{33} can be written analytically in the forms

$$x_L^2=x_z^2+m^2, \quad \text{or} \quad (1-\nu)x_L^2/2=x_z^2+m^2. \tag{15}$$

There is a very simple interpretation of these expressions. The wave for which the displacements assume the form given in Eq. (12) can be interpreted as the sum of two waves traveling around the shell in two helical paths with opposite twists.[6] The sum of the two helical waves forms the vibration considered here, which is a wave standing in the ϕ direction and traveling in the z direction. The helical waves are traveling in a direction making an angle $\theta=\pm\tan^{-1}(m/x_z)$ with the generators of the cylinder. The phase constant $k_h=(\omega/c_h)$ of each helical wave in its direction of propagation is such that $k_h^2a^2=(x_z^2+m^2)$. Equations (15) are therefore equivalent to statements that the shell is being forced to move in helical waves traveling with a velocity equal either to the low-frequency velocity of a longitudinal wave in a flat plate (corresponding to x_L^2) or to the velocity of a shear wave in a flat plate (corresponding to $(1-\nu)x_L^2/2$). Consider the second case. The shear wave in a flat plate is a vibration in which there is no motion normal to the surface. This is a type of motion which is not natural to the curved plate or shell (except in the axisymmetrical case). To force the shell to move in this way requires external constraining forces. But the lack of a corresponding normal motion signifies that the normal impedance must be infinite. A similar argument can be made in the first case, except that there, although the *median* surface of the plate executes no normal motion, the inner and outer surfaces move very slightly in a normal direction due to Poisson contraction. Because of the slight motion, the normal impedance will properly be large but not infinite; this error is due to the approxi-

[6] P. W. Smith, Jr., J. Acoust. Soc. Am. 27, 1065–1075 (1955).

mation discussed earlier. These considerations also demonstrate that an impedance, in the form shown in Eq. (14), cannot be specified in this case since the normal motion of the outer surfaces will depend on the *sum* of the external pressures; it is the sum of pressures which tends to contract the thickness of the shell.

The impedance function defined in Eq. (14) can be written, after much rearranging of terms, in the form[7]

$$Z_m{}^{sh}=j\rho^{sh}c_L(2h/a)(1/x_L)\left\{1-x_L{}^2+\frac{[x_L{}^2-(1-\nu)x_h{}^2/2][m^2+\nu^2x_z{}^2]-(1-\nu^2)(1-\nu)m^2x_z{}^2/2}{D_{33}}\right.$$
$$\left.+\frac{h^2}{3a^2}\left[x_h{}^4-\frac{4-\nu}{2(1-\nu)}m^2-\frac{3\nu}{2(1-\nu)}m^2(m^2-1)\left(\frac{x_L{}^2-(1-\nu)x_h{}^2/2-(1-\nu^2)x_z{}^2/2}{D_{33}}\right)\right]\right\} \quad (16)$$

where $x_h{}^2=(m^2+x_z{}^2)$, and D_{33} is the cofactor of the determinant having a value

$$D_{33}=[x_L{}^2-x_h{}^2][x_L{}^2-(1-\nu)x_h{}^2/2].$$

This rather formidable expression can be considered as made up of two parts; within the braces, the first three terms are due to the membrane action of the shell while the term multiplied by $(h^2/3a^2)$ is a bending term due to the finite thickness of the shell wall. The bending term is generally unimportant except when x_h is very large or when the membrane terms are very small. Numerous approximations for these terms can be made under various conditions; however, extreme care must be exercised in order that the process of approximation be consistent.[8]

RADIAL IMPEDANCE OF A THIN SHELL

In this section we shall present some results of computation from Eq. (16) of the radial modal impedance of the shell. The profusion of variables and parameters in the equation makes it necessary to be somewhat arbitrary in the choice of variables for plotting results. In the graphs shown here, we have used the frequency parameter x_L as abscissa and the velocity ratio $(c_z/c_L)=(x_L/x_z)$ as ordinate. Each graph pertains to a single value of mode number m; all graphs have been computed for the same values of Poisson's ratio (*viz.*, $\nu=\frac{1}{3}$) and of the ratio of wall thickness to diameter (*viz.*, $h/a=1/100$). The curves plotted are contours in the $(c_z/c_L)-x_L$ plane along which the shell impedance is constant. Each curve is labeled with the value of a normalized shell *reactance* $\bar{X}_m{}^{sh}$ defined as

$$\bar{X}_m{}^{sh}=Z_m{}^{sh}/[j\rho^{sh}c_L(2h/a)]. \quad (17)$$

Positive values of this reactance indicate that the inward-looking impedance of the shell is massive in character, while negative values correspond to impedances which are spring-like. The graphs for $m=0,1,2$ are shown in Figs. 2, 3, and 4. If the results had been plotted in three dimensions, with shell reactance as the third coordinate plotted vertically, one would have obtained a surface for each m which, at any point, slopes upward in a more or less "northeasterly" direction (i.e., for increasing values of (c_z/c_L) and x_L). The most striking features of the surfaces are the infinities of reactance which were discussed earlier. The reactance has a negative infinite value at $x_L=0$, $m\neq 1$, or at $(c_z/c_L)=0$. The other infinities occur along the lines defined in Eq. (15); there the reactance surfaces have the form of a vertical escarpment, plunging from a positive infinite value to a negative infinite value.

A study of the reactance charts reveals that there are no *broad* regions of low impedance except at very low frequencies when $m>0$. In reaching this conclusion, it must be remembered that the contours are labeled with the value of the *normalized* shell reactance (see Eq. (17)); for the specific case of an aluminum shell immersed in air $(h/a=1/100)$, the normalization factor is such that a unit value of normalized reactance corresponds to 2.9×10^4 rayls (cgs), or some 700 times the value of ρc of air. Except at low frequencies, the impedance will be small only at points very close to the lines of zero impedance; that is, only for frequencies and angles of incidence (value of c_z) very close to certain critical values.

However, at low frequencies there are broad regions in which the impedance is low. This characteristic becomes much clearer when the data in the figures are replotted with a logarithmic frequency scale, so that the low-frequency region is not unduly compressed. At the same time, we shall change the ordinate scale from normalized axial velocity (c_z/c_L) to angle of incidence θ since, by definition in the present problem of sound transmission, the axial velocity is related to angle of incidence by the equation $c_z=c/\sin\theta$, where c is the

[8] Indeed, it may be said that a significant proportion of the literature on shells is based on the possibility of making similar approximations in different fashions.

[7] This impedance is very similar to the expression derived in a recent, excellent paper by L. Cremer, Acustica 5, 245–256 (1955). Professor Cremer's Eqs. (41) and (42) are identical with Eq. (16) except in the following particulars: (1) his impedance, being "outward-looking", is the negative of Eq. (16); (2) the terms in square brackets multiplied by the factor $(h^2/3a^2)$ include, in Professor Cremer's results, only the first term shown here, *viz.* $x_h{}^4$. This second difference becomes significant particularly at low frequencies (x_L small) when x_h is not too large. For a specific example, consider the frequencies for inextensional (bending-mode) resonances of a cylindrical shell, with no motion in the axial direction. Equation (16) yields the classical formula for the resonant frequency given by Rayleigh (*Theory of Sound*, second edition, 1894, par. 235 g), which can be written $x_L{}^2=(h^2/3a^2)(m^3-m)^2/(m^2+1)$. (See reference 7.) The corresponding result from Cremer's work would be $x_L{}^2=(h^2/3a^2)m^6/(m^2+1)$. The difference is seen to be large only for the smaller values of m.

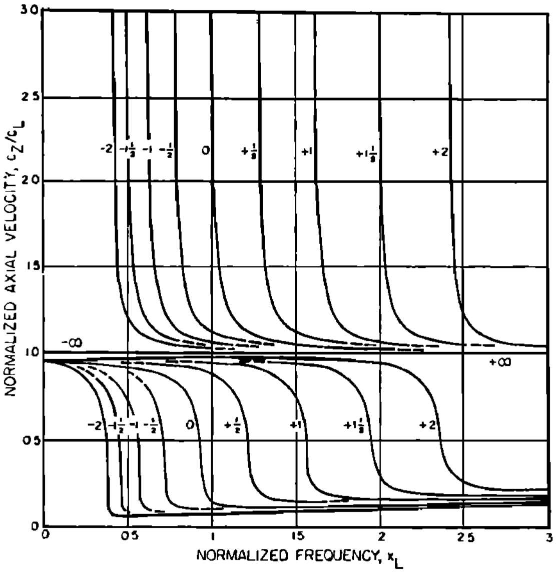

FIG. 2. Contours of normalized shell reactance for $m=0$, $\nu=\frac{1}{3}$, $h/a=10^{-2}$.

velocity of sound in the surrounding fluid. This transformation has been made by assuming a ratio $(c/c_L)=6.36\times10^{-2}$, which is correct for an aluminum or steel shell immersed in air. The reactance contours are labeled, as before, with the value of the normalized shell reactance defined in Eq. (17). The resulting plots for $m=0, 1, 2$ are shown in Figs. 5, 6, and 7.

The fairly broad regions (in angle and frequency) of low impedance for the cases $m=1$ and 2 are apparent from a study of these figures. The graph for $m=3$ would be quite similar to that for $m=2$ except that the frequency at which the zero reactance contour has a vertical asymptote would be higher. The minimum in the zero reactance curve for $m=2$, which is also evident

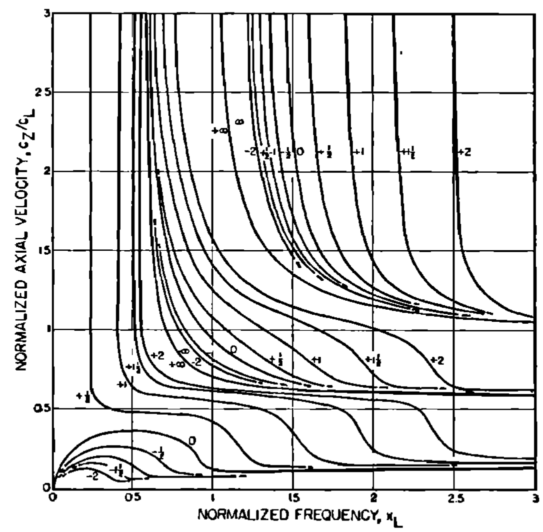

FIG. 3. Countours of normalized shell reactance for $m=1$, $\nu=\frac{1}{3}$, $h/a=10^{-2}$.

in Fig. 4, is of some interest. It corresponds to a minimum in the axial phase velocity of a free wave; the minimum value and the frequency of occurrence are dependent on the shell thickness through the formulas

$$\left(\frac{c_z}{c_L}\right)^2_{\min}\doteq\frac{2h}{3^{\frac{1}{2}}a}(1-\nu^2)^{\frac{1}{2}}\left(\frac{m^2-1}{m^2+1}\right)$$
$$(x_L)_{\min}\doteq\frac{2^{\frac{1}{2}}h}{3^{\frac{1}{2}}a}\frac{m(m^2-1)}{(m^2+1)^{\frac{1}{2}}}. \tag{18}$$

A second minimum in the zero reactance contours is apparent at higher frequencies. This minimum, which is approximately the same for all values of m, is defined

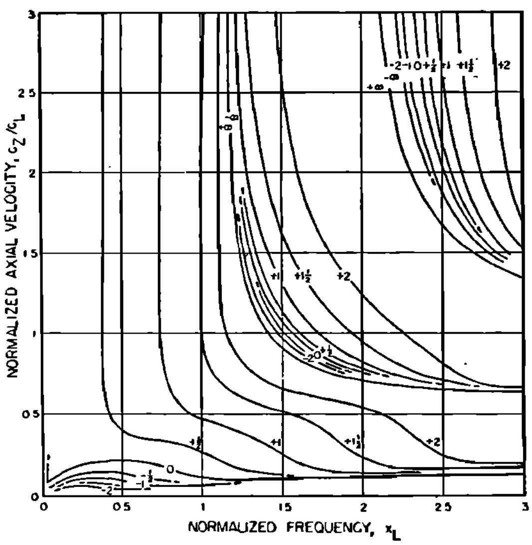

FIG. 4. Contours of normalized shell reactance for $m=2$, $\nu=\frac{1}{3}$, $h/a=10^{-2}$.

by the formulas[9]

$$\left(\frac{c_z}{c_L}\right)^2_{\min}\doteq\frac{2h}{3^{\frac{1}{2}}a}(1-\nu^2)^{\frac{1}{2}},$$
$$(x_L)_{\min}\doteq 2^{\frac{1}{2}}(1-\nu^2)^{\frac{1}{2}}. \tag{19}$$

CROSS-SECTIONAL ABSORPTION COEFFICIENT

The analysis of the transmission of sound through a cylindrical shell, from outside to inside, faces peculiar difficulties in the attempt to define a useful measure of the insulating properties of the shell, i.e. a "transmission loss." In the case of a plane partition, these problems do not arise; the transmission loss is defined as the ratio of energy transmitted by the wall to the energy incident upon it (the ratio being averaged over

[9] A semiquantitative discussion of the noise transmission problem with comments on these minima is to be found in a letter by Junger and Smith, Acustica 5, 47–48 (1955). The formulas presented here are more accurate approximations which differ slightly from the earlier expressions.

angle of incidence). This definition implies the absence of reflections on the "receiving" side of the wall. In practical measurements or applications the spaces on each side are bounded by partly reflective walls, but the spaces are sufficiently large and irregular that corrections can be made in terms of reverberation times or room constants.[10] In the case of a cylindrical shell the inside "receiving" space is finite, regular, and—in many cases of interest—relatively small. Therefore the acoustical reaction, considered either as reflections or as an acoustical impedance, is quite resonant and markedly dependent on frequency. There is a temptation to postulate some sort of "diffusing structure" inside the cylindrical shell so that a "room constant" or "reverberation time" might be used. However such an approach would face many difficulties, due, first, to the possibly small size of the receiving space and, secondly, to the coupling between modes that would thereby be introduced in the responsive motion of the shell. Such mode coupling was expressly forbidden in the previous analysis (see section on

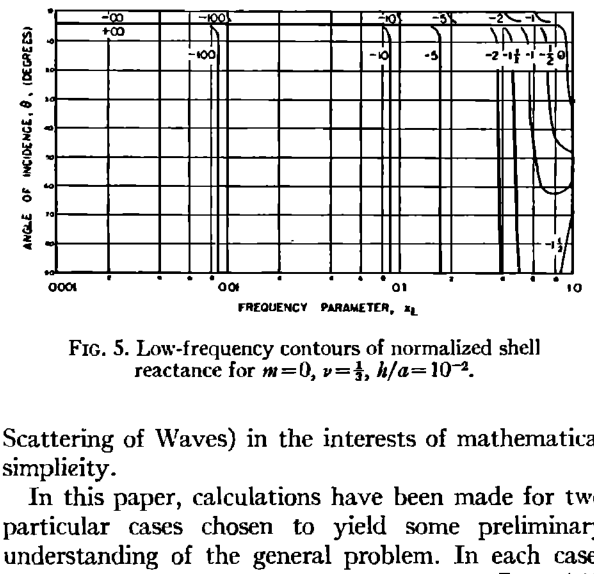

FIG. 5. Low-frequency contours of normalized shell reactance for $m=0$, $\nu=\frac{1}{3}$, $h/a=10^{-2}$.

Scattering of Waves) in the interests of mathematical simplicity.

In this paper, calculations have been made for two particular cases chosen to yield some preliminary understanding of the general problem. In each case, the cross-sectional absorption coefficient Q^a [Eq. (9)] has been used as the measure of the sound insulating properties of the shell. It will be remembered that this coefficient was defined as the ratio of power absorbed to the power incident in the plane wave. Since no losses in the shell have been postulated (they will be discussed later), all the power absorbed must be transmitted to the interior. In the first case, it was assumed that the normal specific acoustic impedance of the inside space is numerically equal to the value of (ρc) for air, independent of mode and angle of incidence. Analytically, this results in the expression for the impedance at the outer surface of the shell [see Eqs.

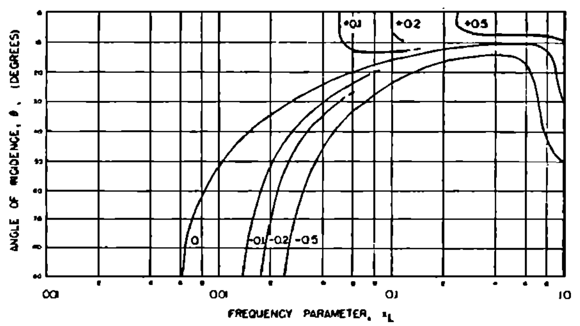

FIG. 6. Low-frequency contours of normalized shell reactance for $m=1$, $\nu=\frac{1}{3}$, $h/a=10^{-2}$.

(4) and (14)]

$$Z_m = Z_m^{sh} + \rho c. \tag{20}$$

In the second case, it was assumed that there is some undefined mechanism inside the shell such that only the inward-traveling wave exists in that space. This assumption is made in an obvious analogy to the situation for which the transmission loss of a flat plate is defined. The inward-traveling wave solution of the wave equation in cylindrical coordinates is well known to be a Hankel function of the first kind. It is quite easy to show that the analytical expression for the impedance at the outer surface of the shell, in this second case, is

$$Z_m = Z_m^{sh} + j\rho c_r [H_m^{(1)}(x_r)/H_m^{(1)\prime}(x_r)]. \tag{21}$$

The second term—the impedance of the inward-traveling wave—has both resistive and reactive parts. [See Eq. (6).]

The results of computation in these two cases are presented in Figs. 8 and 9. The graphs show the *modal* absorption coefficients Q_m^a plotted in decibels re unity (i.e., $10 \log_{10} Q_m^a$) as a function of the angle of incidence of the plane wave. Each graph represents the results at a single low frequency, specifically $x_L = 0.02$. The total absorption coefficient Q^a is the algebraic sum of the modal coefficients [see Eq. (9)] and therefore a curve of Q^a as a function of angle lies close to the modal curve which is highest at any given angle.

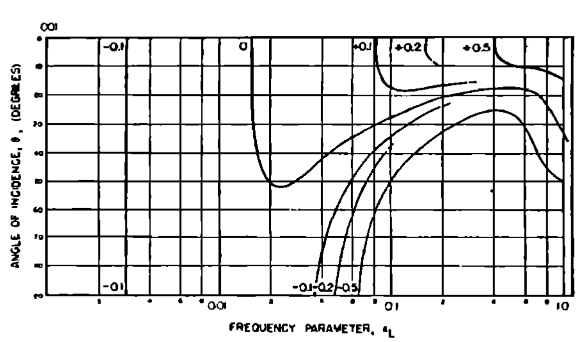

FIG. 7. Low-frequency contours of normalized shell reactance for $m=2$, $\nu=\frac{1}{3}$, $h/a=10^{-2}$.

[10] See, e.g., Leo L. Beranek, *Acoustics* (McGraw-Hill Book Company, Inc., New York, 1954), part XXV.

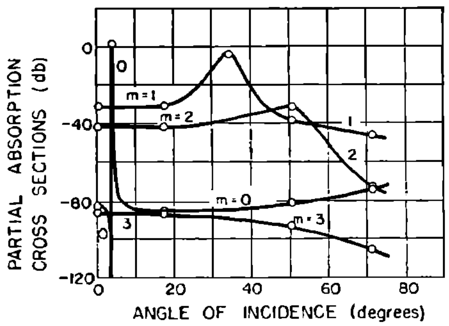

FIG. 8. Partial absorption coefficients of shell as function of angle of incidence θ assuming a ρc normal impedance inside. Results computed for a low frequency ($x_L=0.02$) for a shell with $\nu=\frac{1}{3}$, $h/a=10^{-2}$.

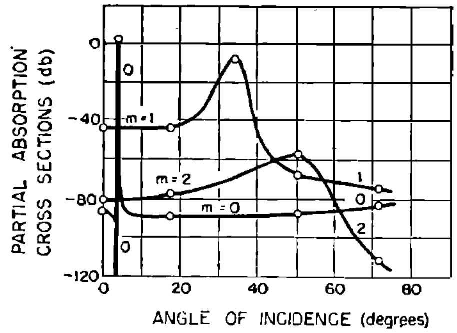

FIG. 9. Partial absorption coefficients of shell as function of angle of incidence θ assuming as inside termination an ingoing wave. Results computed for a low frequency ($x_L=0.02$) for a shell with $\nu=\frac{1}{3}$, $h/a=10^{-2}$.

DISCUSSION

The results presented in the two Figs. 8 and 9 are remarkably similar in nature in spite of the very different assumptions concerning the acoustic impedance inside the shell. They both exhibit a very sharp peak in the curve for $m=0$ which would be of little importance in the over-all picture, i.e., if the sound energy were distributed smoothly over all angles of incidence. On the other hand, the curves for $m=1$ and 2 exhibit broad peaks which are of major importance; the angle at which the peaks appear is approximately the angle for which the shell reactance vanishes. [See Figs. 6 and 7.] The relative values of the modal absorption coefficients for different values of m decrease as m increases because the amplitudes of the different modes in the series expansion for the incoming plane wave [Eq. (1)] decrease with increasing m at this low frequency. Furthermore the absorption coefficients for corresponding values of m in the two figures are lower in Fig. 9 principally because of the very small values of the resistance in that case.

The results were computed for the single frequency corresponding to $x_L=0.02$; this frequency was chosen because, first, it lies in the region of low shell reactance and, secondly, because it is a round number. However, it is quite easy to deduce from the reactance contours Figs. 5 to 7 the nature of the variation of the absorption coefficients with frequency. If the frequency is increased from $x_L=0.02$, the values for $m=0$ would become generally larger due to a decrease in the values of reactance. On the other hand, the values for $m=1$ and 2 would become generally smaller, particularly for values of θ above the value for peak absorption; however the peak value would not be strongly affected, so that the peaks would tend to become sharper as functions of θ. As the frequency is decreased from $x_L=0.02$, the opposite tendencies would be observed. Near $x_L=0.16$ the reactance for $m=2$ very nearly vanishes for a large band of angles θ and the absorption would exhibit a correspondingly wide and high plateau. Below that frequency, the absorption curve for $m=2$ would not have a peak but would resemble the $m=3$ curve of Fig. 8 in shape; as frequency is further decreased this $m=2$ absorption curve would become gradually less important.

It was assumed in the analysis of impedance of the shell that there were no losses in the shell vibration. However the possibility of shell losses is implicit in the derivation of the expression for the absorption coefficients [Eq. (9)], and their effect, whatever their origin, can be deduced from that expression. The presence of shell losses requires that the resistance ratio of the cylinder r_m be taken as the sum of two parts,

$$r_m = r_m^{sh} + r_m^c, \tag{22}$$

where the first part is the contribution of the shell and the second part is the contribution of the contents of the shell; previously r_m^c had been the only resistive term. Now, the total power absorbed by the cylinder is absorbed individually by the shell and the contents in proportion to their resistances. Since our interest is confined to the sound power that reaches the interior, it is necessary to modify the expression for the absorption coefficients [Eq. (9)]; the modified modal absorption coefficient is therefore

$$\begin{aligned} Q_m^{a\prime} &= r_m^c Q_m^a / r_m \\ &= (2\epsilon_m/x_r) r_m^c r_m^s / [(r_m + r_m^s)^2 + (\chi_m + \chi_m^s)^2]. \end{aligned} \tag{23}$$

This coefficient expresses the ratio of the power (in mode m) transmitted to and absorbed in the contents of the shell to the power in the incident plane wave which is directed toward the longitudinal section of the cylinder. It is seen that the introduction of losses in the shell affects the modified coefficient only in the value of the denominator, through the term r_m. As previously pointed out, the denominator is the square of the total impedance to vibration at the surface of the cylinder. Unless the quality factor "Q" of the cylinder is quite small, the introduction of shell losses will be apparent only near resonance, i.e. when $(\chi_m+\chi_m^s)=0$. However these resonances are the regions of highest values of the coefficients Q_m^a so that the result is very useful. If the shell losses are made to predominate over other (radiation, etc.) losses, then the peaks in the modified coefficients $Q_m^{a\prime}$ would decrease 6 db with each doubling of the shell resistance.

In conclusion, comments can be made concerning the possibility of extention of the present method of analysis to more complicated shells, made up of structural members and an attached skin. If the frequency is low enough that the skin would partake of the motion of the structure, it should be possible to replace the real shell by an equivalent uniform shell. Modification of the shell analysis would be necessary in cases when the effective elastic constants are different in the axial and circumferential directions. However, it would be expected that many of the conclusions of this analysis would be qualitatively applicable to the more complicated case.

APPENDIX

Certain asymptotic and approximate forms for the modal absorption coefficients can be derived for use in special cases. At low frequencies, when the parameter x_r is small, it can be shown[4] that the modal radiation resistance ratio $r_m^s(x_r)$ approaches

$$r_m^s\rightarrow\frac{4\pi}{(m!)^2}\left(\frac{x_r}{2}\right)^{2m+1},\quad m\neq 0,\ x_r^2\ll 2m+1;$$

$$r_0^s\rightarrow\frac{\pi x_r}{2}\left(1+\frac{x_r^2\ln x_r}{2}\right),\quad x_r^2\ll 1.$$

It will be noted that $x_r=k_ra=\omega a\cos\theta/c$.

At the resonant frequency or angle, defined by $(\chi_m+\chi_m^s)=0$, the modified absorption coefficient [Eq. (23)] assumes a quite simple form. Subject to the approximation of a low frequency and the further approximation that the resistance r_m, of shell and contents, predominates over the radiation resistance r_m^s, then

$$\text{at resonance:}\quad Q^a{}_m{}'\doteq\frac{8\pi}{(m!)^2}\left(\frac{x_r}{2}\right)^{2m}\frac{r_m^c}{r_m^2}.$$

Off the resonant frequency, the reactance of the shell will usually predominate over the resistance terms and the other reactances; subject to this approximation, at low frequencies $Q_m^{a\prime}$ is given by the expression

$$\text{off resonance:}\quad Q^a{}_m{}'\doteq\frac{8\pi}{(m!)^2}\left(\frac{x_r}{2}\right)^{2m}\frac{r_m^c}{(\chi_m^{sh})^2}.$$

THE JOURNAL OF THE ACOUSTICAL SOCIETY OF AMERICA VOLUME 29, NUMBER 6 JUNE, 1957

Vibrations of Ferroelectric Cylindrical Shells with Transverse Isotropy. I. Radially Polarized Case

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(Received December 14, 1956)

Expressions for the coupled mechanical vibrations and electrical admittance of ferroelectric tubes having transverse isotropy are derived and the results supported with experimental data. Coupled modes in radially polarized barium titanate tubes are examined as a function of length to radius ratio. A frequency-dependent expression is given for the admittance in terms of electromechanical coupling, capacitance, and tube length. Results permit close prediction of resonant frequencies. An optimum length to radius ratio of 2.9 is indicated for maximum electromechanical coupling with the gravest radial mode in a radially polarized tube.

1. INTRODUCTION

THIS paper is concerned with the vibrations and electrical characteristics of an electrically driven cylindrical tube of a ferroelectric ceramic. The mechanical resonance of thin-walled cylindrical tubes for the isotropic case is given by Love.[1] Later a similar result was obtained by Giebe and Blechschmidt[2] using an entirely different approach, but arriving at the same formula for mechanical resonance. Recently Stephenson[3] has worked out formulas for both the mechanical and electrical properties of short hollow cylinders of barium titanate.

We will demonstrate how the formula for mechanical resonance is valid for a radially polarized ferroelectric

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[1] A. E. H. Love, *Mathematical Theory of Elasticity* (Dover Publications, New York, 1944), fourth edition, p. 546.

[2] E. Giebe and E. Blechschmidt, Ann. Physik, Ser. 5, No. 5, 18, 417 (1933).

[3] C. V. Stephenson, J. Acoust. Soc. Am. 28, 51-56 (1956).