The propagation of plane sound waves in narrow and wide circular tubes, and generalization to uniform tubes of arbitrary cross-sectional shape

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The general Kirchhoff theory of sound propagation in a circular tube is shown to take a considerably simpler form in a regime that includes both narrow and wide tubes. For tube radii greater than $r_w = 10^{-3}$ cm and sound frequencies f such that $r_w f^{3/2} < 10^6$ cm s$^{-3/2}$, the Kirchhoff solution reduces to the approximate solution suggested by Zwikker and Kosten. In this regime, viscosity and thermal conductivity effects are treated separately, within complex density and complex compressibility functions. The sound pressure is essentially constant through each cross section, and the excess density and sound pressure (when scaled by the equilibrium density and pressure of air, respectively) are comparable in magnitude. These last two observations are assumed to apply to uniform tubes having *arbitrary* cross-sectional shape, and a generalized theory of sound propagation in narrow and wide tubes is derived. The two-dimensional wave equation that results can be used to describe the variation of either particle velocity or excess temperature over a cross section. Complex density and compressibility functions, propagation constants, and characteristic impedances may then be calculated. As an example, this procedure has been used to determine the propagation characteristics for a tube of rectangular cross section.

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INTRODUCTION

The propagation of sound in a uniform, circular tube is a fundamental problem that arises in many areas of acoustics. The exact solution, given many years ago by Kirchhoff[1] (also, see Rayleigh[2]), accounts for the effects of both air viscosity and thermal conductivity in tubes of arbitrary diameter. While generally true, the equations obtained from this theory are unnecessarily complicated for many applications. More recently, Zwikker and Kosten[3] introduced a simpler, approximate treatment. The effects of viscosity and thermal conductivity are treated separately and summarized in terms of complex density and compressibility functions. Many researchers have since adopted this approximate theory.[4–7]

The validity of the Zwikker and Kosten approach was originally justified only for the extremes of "low" and "high" frequencies;[3] an intermediate band of frequencies could not be treated. Moreover, another regime of behavior (at very high frequency or very large radius) was not considered. Tijdeman[8] has examined the applicability of the Zwikker and Kosten approach more carefully. Propagation constants for a cylindrical tube were obtained through a numerical implementation of the exact Kirchhoff theory. These calculated results were compared to the Zwikker and Kosten solution and found to be in agreement in the limit of small "reduced frequency."

An alternate approach will be considered in this paper. The equations that make up the exact Kirchhoff solution will be used as a starting point. These equations will be shown to reduce analytically to the Zwikker and Kosten solution when simplifying approximations, appropriate for certain choices of tube radius and sound frequency, are applied. The range of values of radius and frequency that permit this simplification delineates the regime for which the Zwikker and Kosten approach is valid.

The complex density and compressibility functions given by Zwikker and Kosten have been selected as our target approximations because of the explicit comparisons, already made by these authors, to the Kirchhoff theory. However, other approximate treatments, presented in different formats, have been discussed by various authors. Crandall[9] has considered the velocity distribution through a cross section of a cylinder when thermal conductivity is assumed negligible; with this assumption, laminar flow is obtained and propagation characteristics can be calculated. Daniels[10] has considered the thermal conduction inside cylindrical tubes and derived a relationship between temperature change and sound pressure; this analysis assumes, though, that thermal gradients along the tube axis may be ignored. These approximate velocity and thermal conduction aspects may be brought together, with the propagation of sound in cylinders being described in terms of equivalent series impedances and shunt admittances.[11–13]

The circular tube is a prototypal geometry for many endeavors, and extension of the theory to tubes having noncircular cross sections is desirable. One example is in the study of sound propagation in porous materials.[6,14] Pores are rarely circular and, in modeling real materials, microstructural factors representing the departure from a circular cross section must be introduced.

Certain features of the solution for the circular tube will be identified as being applicable to tubes of arbitrary cross-sectional shape, and a general procedure will be derived for the calculation of their propagation characteristics. To demonstrate the utility of the procedure, the propagation characteristics of a tube of rectangular cross section will be derived.

I. KIRCHHOFF THEORY FOR THE CIRCULAR TUBE

The theory presented by Kirchhoff is outlined here. This provides a basis for the discussion and approximations in the later sections.

We consider a tube of radius r_w containing an ideal gas of viscosity μ and thermal conductivity κ. The axial direction $\hat{z}$ and radial direction $\hat{r}$ are as indicated in Fig. 1. Several quantities are involved in the description of the state of the gas: These are the pressure P, the temperature T, the density ρ, and the particle velocity $\mathbf{V}$. The first-order relations between these variables are given by the linearized Navier–Stokes force equation (e.g., Refs. 15 and 16), the mass continuity equation, and an equation describing thermal conduction within the gas:[1]

$$\rho_0 \frac{\partial \mathbf{V}}{\partial t} = -\nabla P + \frac{4}{3}\mu \nabla(\nabla\cdot\mathbf{V}) - \mu \nabla\times\nabla\times\mathbf{V}, \quad (1)$$

$$\frac{\partial \rho}{\partial t} = -\rho_0 \nabla\cdot\mathbf{V}, \quad (2)$$

$$\kappa \nabla^2 T = \frac{T_0}{P_0}\left(\rho_0 C_v \frac{\partial P}{\partial t} - P_0 C_p \frac{\partial \rho}{\partial t}\right), \quad (3)$$

where ρ_0, T_0, and P_0 are the equilibrium density, temperature, and pressure of air, C_v is the specific heat (per unit mass) at constant volume, and C_p is the specific heat at constant pressure. As well, for an ideal gas, the equation of state may be written as

$$\frac{\partial P}{\partial t} = \frac{P_0}{\rho_0 T_0}\left(\rho_0 \frac{\partial T}{\partial t} + T_0 \frac{\partial \rho}{\partial t}\right). \quad (4)$$

An $\exp(i\omega t)$ time dependence will be assumed for all variables, where ω is the angular frequency and i is $(-1)^{1/2}$. Complex quantities p, δ, τ, and $\mathbf{v}$, representing the sound pressure, the excess density, the excess temperature, and the particle velocity, respectively, are then introduced through

$$\begin{aligned} P(t) &= P_0 + \mathrm{Re}\{pe^{i\omega t}\}, \\ \rho(t) &= \rho_0 + \mathrm{Re}\{\delta e^{i\omega t}\}, \\ T(t) &= T_0 + \mathrm{Re}\{\tau e^{i\omega t}\}, \\ V(t) &= \mathrm{Re}\{\mathbf{v}e^{i\omega t}\}. \end{aligned} \quad (5)$$

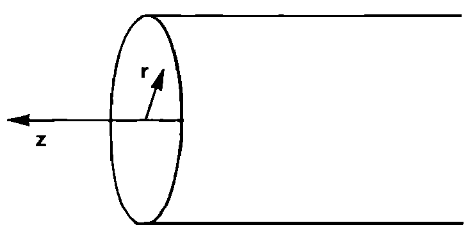

FIG. 1. The cylindrical tube for which the exact Kirchhoff solution is applicable. The axial direction $\hat{z}$, along which sound is assumed to propagate, and the radial direction $\hat{r}$ are indicated.

With these forms, Eqs. (1)–(4) become

$$i\omega\rho_0 \mathbf{v} = -\nabla p + \tfrac{4}{3}\mu \nabla(\nabla\cdot\mathbf{v}) - \mu \nabla\times\nabla\times\mathbf{v}, \quad (6)$$

$$i\omega\delta = -\rho_0 \nabla\cdot\mathbf{v}, \quad (7)$$

$$\kappa \nabla^2\tau = (i\omega T_0/P_0)(\rho_0 C_v p - P_0 C_p \delta), \quad (8)$$

$$p = (P_0/\rho_0 T_0)(\rho_0\tau + T_0\delta). \quad (9)$$

The boundary conditions require that the velocity and excess temperature be zero on the tube wall so equations will be written in terms of $\mathbf{v}$ and τ. Eliminating p and δ from these equations, we obtain

$$i\omega\rho_0 \mathbf{v} = -\frac{P_0}{T_0}\nabla\tau + \left(\frac{P_0}{i\omega} + \frac{4}{3}\mu\right)\nabla(\nabla\cdot\mathbf{v}) - \mu \nabla\times\nabla\times\mathbf{v}, \quad (10)$$

$$\kappa \nabla^2\tau = i\omega\rho_0 C_v \tau + \rho_0 T_0 (C_p - C_v)\nabla\cdot\mathbf{v}. \quad (11)$$

Introducing the constants ν and ν' through

$$\nu = \mu/\rho_0, \quad \nu' = \kappa/(\rho_0 C_v), \quad (12)$$

and making use of the relation, valid for ideal gases,

$$C_p - C_v = P_0/(\rho_0 T_0), \quad (13)$$

then Eqs. (10) and (11) can be written as

$$i\omega\mathbf{v} = -\frac{c^2}{\gamma T_0}\nabla\tau + \left(\frac{4}{3}\nu + \frac{c^2}{i\omega\gamma}\right)\nabla(\nabla\cdot\mathbf{v}) - \nu \nabla\times\nabla\times\mathbf{v}, \quad (14)$$

$$\nu' \nabla^2\tau = i\omega\tau + (\gamma - 1)T_0 \nabla\cdot\mathbf{v}. \quad (15)$$

Here, γ is the ratio C_p/C_v of specific heats, and the adiabatic sound speed c has been introduced through

$$\rho_0 c^2 = \gamma P_0. \quad (16)$$

Once the velocity and excess temperature have been computed, the sound pressure and excess density may be calculated using Eqs. (7) and (9).

Equations (14) and (15) are general, for ideal gases, and could be examined for a variety of possible tube cross-sectional shapes. In this section, though, only tubes of circular cross section will be considered. The solution, in this case, has been given by Kirchhoff.[1] Let $\mathbf{v}$ be composed of radial and axial components, i.e.,

$$\mathbf{v} = q\hat{r} + u\hat{z}. \quad (17)$$

Then, traveling wave solutions are given by

$$\begin{aligned} u = [&AQ - A_1 m(i\omega/\lambda_1 - \nu')Q_1 \\ &- A_2 m(i\omega/\lambda_2 - \nu')Q_2]e^{mz}, \end{aligned} \quad (18)$$

$$\begin{aligned} q = \Bigg[&-\frac{Am}{(i\omega/\nu) - m^2}\frac{dQ}{dr} - A_1\left(\frac{i\omega}{\lambda_1} - \nu'\right)\frac{dQ_1}{dr} \\ &- A_2\left(\frac{i\omega}{\lambda_2} - \nu'\right)\frac{dQ_2}{dr}\Bigg]e^{mz}, \end{aligned} \quad (19)$$

$$\tau = (\gamma - 1)T_0(A_1 Q_1 + A_2 Q_2)e^{mz}, \quad (20)$$

where the functions Q, Q_1, and Q_2 are given by

$$\begin{aligned} Q &= J_0[r(m^2 - i\omega/\nu)^{1/2}], \\ Q_1 &= J_0[r(m^2 - \lambda_1)^{1/2}], \\ Q_2 &= J_0[r(m^2 - \lambda_2)^{1/2}], \end{aligned} \quad (21)$$

and where λ_1 and λ_2 are the small and large roots of

$$\lambda^2\left(\frac{c^2\nu'}{i\omega\gamma}+\frac{4}{3}\nu\nu'\right)-\lambda\left[c^2+i\omega\left(\frac{4}{3}\nu+\nu'\right)\right]-\omega^2=0. \tag{22}$$

The factor $\exp(mz)$ is common to both components of velocity and the excess temperature. The parameter m is the propagation constant. It is evaluated by setting u, q, and τ equal to zero at the tube wall. This leads to an expression that can be solved (numerically) to give m:

$$\frac{i\omega m^2}{(i\omega/\nu)-m^2}\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)\frac{d\ln Q}{dr_w}+\left(\frac{i\omega}{\lambda_1}-\nu'\right)\frac{d\ln Q_1}{dr_w}-\left(\frac{i\omega}{\lambda_2}-\nu'\right)\frac{d\ln Q_2}{dr_w}=0. \tag{23}$$

In this expression, the derivatives are taken with respect to r and evaluated at the wall, i.e., at $r=r_w$.

Following Weston[17] (with the correction of a sign), by setting $u=\tau=0$ in Eqs. (18) and (20), the constants A, A_1, and A_2 are determined, giving

$$u=mB\left[-i\omega\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)Q_{1w}Q_{2w}Q+\left(\frac{i\omega}{\lambda_1}-\nu'\right)Q_wQ_{2w}Q_1-\left(\frac{i\omega}{\lambda_2}-\nu'\right)Q_wQ_{1w}Q_2\right]e^{mz}, \tag{24}$$

$$q=B\left[\frac{i\omega m^2}{(i\omega/\nu)-m^2}\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)Q_{1w}Q_{2w}\frac{dQ}{dr}+\left(\frac{i\omega}{\lambda_1}-\nu'\right)Q_wQ_{2w}\frac{dQ_1}{dr}-\left(\frac{i\omega}{\lambda_2}-\nu'\right)Q_wQ_{1w}\frac{dQ_2}{dr}\right]e^{mz}, \tag{25}$$

$$\tau=B(\gamma-1)T_0Q_w(-Q_{2w}Q_1+Q_{1w}Q_2)e^{mz}, \tag{26}$$

where the subscript w indicates an evaluation of the term Q, Q_1, or Q_2 at the tube wall, and where the constant B is

$$B=-A_1/(Q_wQ_{2w}). \tag{27}$$

Using Eqs. (7) and (9), the solution given above may be extended to describe explicitly the excess density,

$$\delta=\rho_0BQ_w[(\nu'\lambda_1/i\omega-1)Q_{2w}Q_1-(\nu'\lambda_2/i\omega-1)Q_{1w}Q_2]e^{mz}, \tag{28}$$

and the sound pressure,

$$p=P_0BQ_w[(\nu'\lambda_1/i\omega-\gamma)Q_{2w}Q_1-(\nu'\lambda_2/i\omega-\gamma)Q_{1w}Q_2]e^{mz}, \tag{29}$$

Equations (24)–(29) have been used to calculate the position dependence of the state quantities p, τ, and δ, and of the velocity components q and u. The key step of the numerical calculation is the determination of the propagation constant m through Eq. (23). A Newton's method was found to work well for all choices of frequency and radius; this iterative calculation was relatively insensitive to initial choice of m, and an arbitrary starting value of $(0.02+i0.06)$ cm^{-1} was used in all cases. In Fig. 2, results are shown for a tube having a diameter of 0.01 cm, at a sound frequency of 100 Hz. Shown in panels (a)–(e) are the magnitudes of u, q, τ/T_0, p/P_0, and δ/ρ_0; all five quantities are displayed as a function of the radial position r. The viscous boundary layer thickness, calculated using $d_v=(2\nu/\omega)^{1/2}$, is 0.022 cm, so that this tube may be considered narrow. In Fig. 3, a similar set of results is shown for a tube of 0.1-cm diam, for a frequency of 10 kHz. The boundary layer thickness, $d_v=0.0022$ cm, is much less than the tube radius so the tube may be considered wide. The variations in axial velocity u,

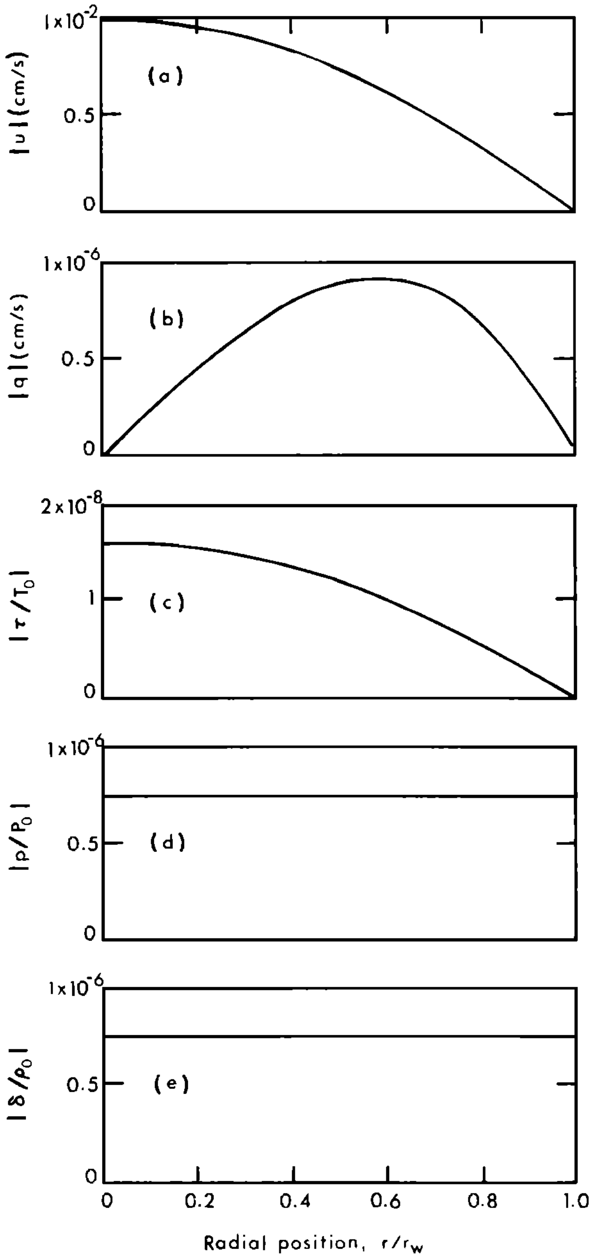

FIG. 2. Variation of several propagation parameters along a radius of a tube, from the center $\hat{z}$ axis to the tube wall at $r=r_w$. The magnitudes of (a) axial velocity u; (b) radial velocity q; (c) excess temperature τ, relative to the ambient T_0; (d) sound pressure p, relative to the ambient P_0; and (e) excess density, relative to the ambient ρ_0 are shown. The results are for a tube of radius 0.01 cm, at a sound frequency of 100 Hz. With a viscous boundary layer thickness of 0.022 cm, this is a "narrow" tube.

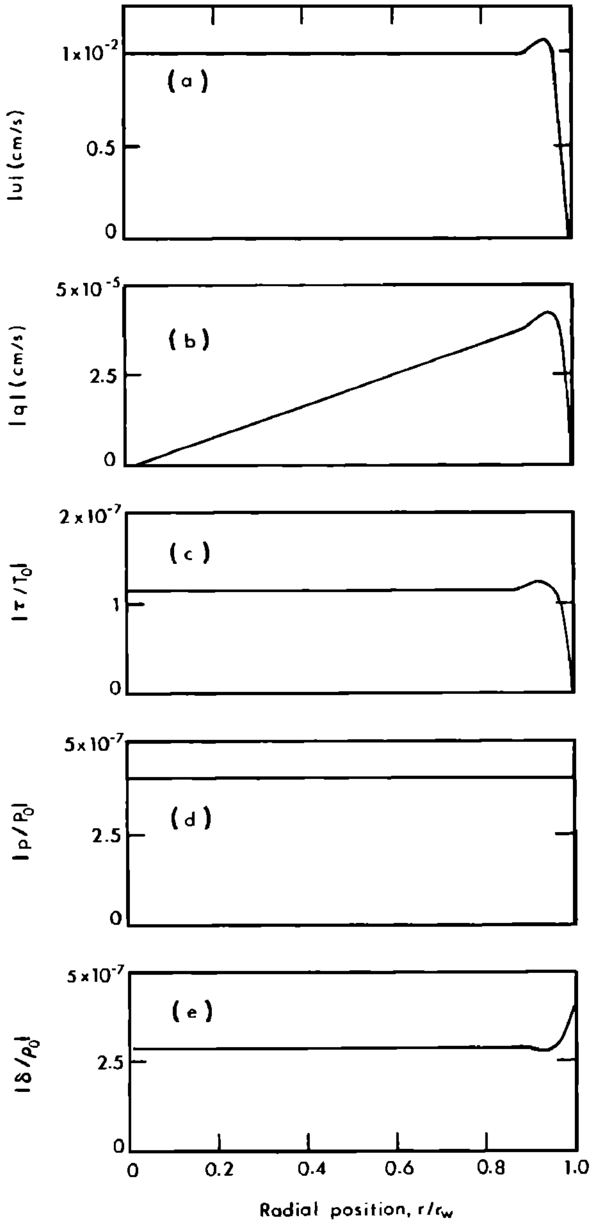

FIG. 3. As in Fig. 2, but for a tube of radius 0.1 cm, at a sound frequency of 10 kHz. The viscous boundary layer thickness is 0.0022 cm, so the tube is considered "wide."

excess temperature τ, and excess density δ appear primarily within a few d_v of the tube wall.

Several features in Figs. 2 and 3 are of particular interest. The sound pressure is essentially constant through the cross section of the tube, for both wide and narrow cases. The only significant variation of pressure is the $\exp(mz)$ axial dependence, along the length of the tube. The excess density and the sound pressure are comparable in magnitude, when scaled by ρ_0 and P_0, respectively. In Fig. 2, $|\delta/\rho_0|$ is very nearly equal to $|p/P_0|$; from Eq. (9), this suggests that the sound propagation is isothermal, and indeed, $|\tau/T_0|$ is considerably smaller. In Fig. 3, $|\delta/\rho_0|$ is smaller than $|p/P_0|$ by a factor $\lambda = 1.4$ over most of the cross section, but the two terms become equal at the tube wall; from Eq. (8), the propagation is essentially adiabatic in the central core of this "wide" tube. For both tubes, the radial velocity is considerably smaller than the axial velocity.

In many applications, including the study of porous materials, the detailed variation of velocity, density, and temperature, through a cross section, is not required. The final desired result is the *average* of the quantity over the cross section of a single pore, or tube. Hence, for a quantity $\xi(\mathbf{s})$, we calculate the average $\langle \xi \rangle$ using

$$\langle \xi \rangle = \frac{1}{A_T} \oint \xi(\mathbf{s}) dA, \tag{30}$$

where $\mathbf{s}$ is a general position vector within the cross-sectional plane and A_T is the total cross-sectional area. For a circular tube of radius r_w, this average is simply

$$\langle \xi \rangle = \frac{1}{\pi r_w^2} \int_0^{r_w} 2\pi r \xi(r) dr. \tag{31}$$

For the discussion in Sec. II, the velocity and excess density, averaged through the cross section, will be required. Integrating according to Eq. (31), Eqs. (24) and (28) give

$$\begin{aligned} \langle u \rangle = -\frac{2mB}{r_w} \Bigg[& i\omega \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right) \frac{Q_{1w} Q_{2w} R_w}{(m^2 - i\omega/\nu)^{1/2}} \\ & - \left(\frac{i\omega}{\lambda_1} - \nu' \right) \frac{Q_w Q_{2w} R_{1w}}{(m^2 - \lambda_1)^{1/2}} \\ & + \left(\frac{i\omega}{\lambda_2} - \nu' \right) \frac{Q_w Q_{1w} R_{2w}}{(m^2 - \lambda_2)^{1/2}} \Bigg] e^{mz} \end{aligned} \tag{32}$$

and

$$\begin{aligned} \langle \delta \rangle = \rho_0 \frac{2BQ_w}{i\omega r_w} \Bigg(& \frac{\lambda_1 \nu' - i\omega}{(m^2 - \lambda_1)^{1/2}} Q_{2w} R_{1w} \\ & - \frac{\lambda_2 \nu' - i\omega}{(m^2 - \lambda_2)^{1/2}} Q_{1w} R_{2w} \Bigg) e^{mz}, \end{aligned} \tag{33}$$

where the functions R, R_1, and R_2 are Bessel functions of order 1:

$$\begin{aligned} R &= J_1 [r(m^2 - i\omega/\nu)^{1/2}], \\ R_1 &= J_1 [r(m^2 - \lambda_1)^{1/2}], \\ R_2 &= J_1 [r(m^2 - \lambda_2)^{1/2}], \end{aligned} \tag{34}$$

with the subscript w indicating evaluation of these functions at $r = r_w$.

II. APPROXIMATE SOLUTION FOR NARROW AND WIDE CIRCULAR TUBES

The Kirchhoff solution for the propagation of sound in circular tubes is generally true. However, the equations are complicated and difficult to apply. Simpler, approximate expressions were proposed by Zwikker and Kosten,[3] but justification of these expressions was given only for the extremes of "low" and "high" frequency. Numerical comparison of the two approaches, though, reveals very good agreement over a wide range of frequency and tube radius.[8] In this section, the exact Kirchhoff theory will be examined critically, and the applicability of approximate solutions determined.

Weston[17] has studied the solution presented by Kirchhoff and identified three main types of propagation, distinguished by the following two discriminants

$$d_1 = 2r_w f^{1/2}, \quad d_2 = 10^{-8} r_w f^{3/2}. \tag{35}$$

The *narrow* tube has $d_1 \ll 1$ cm s$^{-1/2}$; the *wide* tube has $d_1 \gg 1$ cm s$^{-1/2}$, and $d_2 \ll 1$ cm s$^{-3/2}$; and the *very wide* tube has $d_2 \gg 1$ cm s$^{-3/2}$. Weston's classification excludes very narrow tubes with radii less than 10^{-3} cm, for which the radius approaches the mean-free path, and frequencies greater than 10^8 Hz, for which the wavelength approaches the mean-free path. In this and the following sections, we will restrict our attention to the regime composed of *both* narrow and wide tubes, i.e., the broad range of frequencies and radii encompassed by

$$r_w f^{3/2} < 10^6 \text{ cm s}^{-3/2} \text{ and } r_w > 10^{-3} \text{ cm}. \tag{36}$$

The solution given by Kirchhoff will be examined and shown to reduce to a much simpler form in this regime.

Some approximations can be made at the outset. With $\nu = 0.151$ cm^2/s and $c = 34300$ cm/s, it is found that

$$\omega\nu/c^2 \ll 1 \tag{37}$$

for frequencies below 10^8 Hz, and so the roots of Eq. (23) are

$$\lambda_1 \approx -\omega^2/c^2, \quad \lambda_2 \approx i\omega\gamma/\nu'. \tag{38}$$

For air, $\gamma = 1.4$ and $\nu' = 0.30$ cm^2/s, giving $\lambda_1 \ll \lambda_2$.

Weston states that the argument of the function Q_1, i.e., $r(m^2 - \lambda_1)^{1/2}$, is small for both narrow tubes and wide tubes. This is confirmed in Fig. 4, in which the argument, evaluated at $r = r_w$, is shown as a function of frequency for several tube radii. The dashed curve indicates the maximum extent of the regime being considered, i.e., $r_w f^{3/2} = 10^6$ cm s$^{-3/2}$. The argument of Q_1 is less than 0.1, within the regime that includes both narrow and wide tubes. Thus terms that involve this argument may be approximated. We have

$$Q_1 \approx 1, \tag{39}$$

to better than 0.25%, and

$$\frac{dQ_1}{dr} = -(m^2 - \lambda_1)^{1/2} R_1 \approx -\tfrac{1}{2}(m^2 - \lambda_1) r, \tag{40}$$

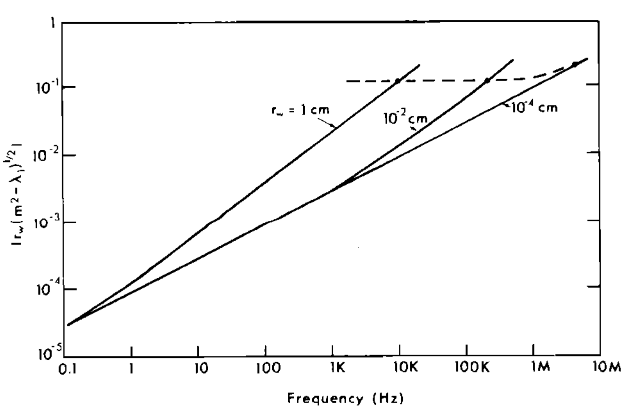

FIG. 4. The magnitude of the term $r_w(m^2 - \lambda_1)^{1/2}$, which appears in Kirchhoff's theory, for several tube radii. The dashed curve indicates the limit $r_w f^{3/2} = 10^6$ cm s$^{-3/2}$, above which the tube approaches the "very wide" regime. Below this limit curve, for both narrow and wide tubes, the term is small, leading to significant simplifications in the theory.

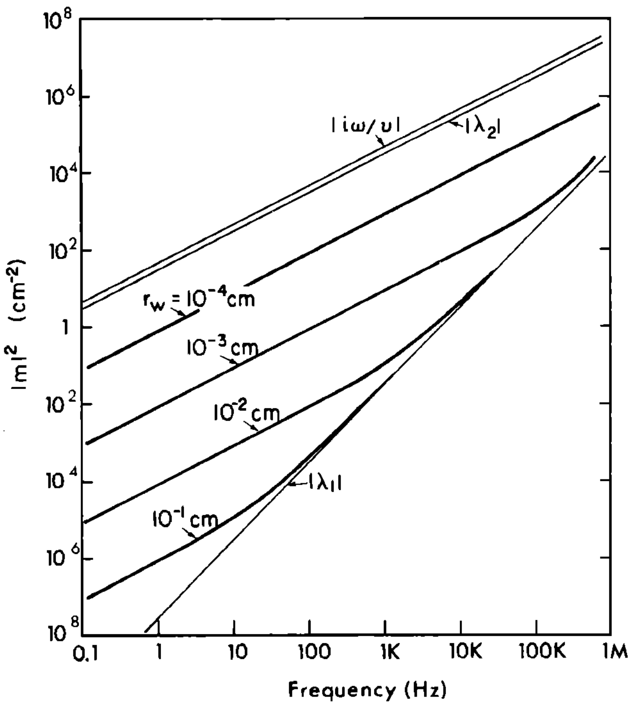

FIG. 5. Comparison of the propagation constant m to three terms involved in Kirchhoff's theory. Here, $|m|^2$ has been calculated for several values of tube radius, as a function of frequency, and plotted as the thick curves. It may be compared to the terms $|i\omega/\nu|$, $|\lambda_1|$, and $|\lambda_2|$, which are shown as the thinner lines. For tube radius greater than 10^{-3} cm, m^2 is much smaller in magnitude than $i\omega/\nu$ and λ_2.

to better than 0.125%. With these approximations, the sound pressure, in Eq. (29), takes a simple form

$$p \approx -P_0 B\gamma Q_w Q_{2w} e^{mz}. \tag{41}$$

None of the terms in this expression depend on the radial variable r. The sound pressure is essentially constant through the cross section of the tube, as was observed in Figs. 2 and 3.

Simplifications of other expressions are possible on considering the magnitude of the propagation constant m. In Fig. 5, the magnitude of m^2 has been plotted as a function of frequency for various tube radius. These may be compared with the terms $|\lambda_1|$, $|\lambda_2|$, and $|i\omega/\nu|$, which have been plotted on the same graph as the thinner lines. We note that

$$|m^2| \ll |\lambda_2| \text{ and } |m^2| \ll |i\omega/\nu|, \tag{42}$$

provided that the radius is greater than 10^{-3} cm. This is the regime that has been assumed in this section. Thus we may ignore m^2 relative to both λ_2 and $i\omega/\nu$ in the expressions of Sec. I.

Applying these approximations to Eq. (23), a simpler expression is obtained for the propagation constant

$$m^2 = -(\omega^2/c^2)\{1 + 2(\gamma - 1)(-i\omega\gamma/\nu')^{-1/2} \times G[r_w(-i\omega\gamma/\nu')^{1/2}]/r_w\} \times \{1 - 2(-i\omega/\nu)^{-1/2} G[r_w(-i\omega/\nu)^{1/2}]/r_w\}^{-1}, \tag{43}$$

where the function G is defined according to

$$G[\zeta] = J_1(\zeta)/J_0(\zeta). \tag{44}$$

It is important to note in Eq. (43) that the effects of viscosity and the effects due to thermal conduction are now separate. The first term enclosed by braces { } depends only on the thermal conduction properties, through the term ν', and is independent of viscosity. Conversely, the term enclosed by the second set of braces depends only on the air viscosity, through the term ν.

The same approximations can be applied to Eqs. (32) and (33), giving, for the average velocity and the excess density,

$$\langle u \rangle = (mBc^2/i\omega)Q_{2w} \times [Q_w - 2(-i\omega/\nu)^{1/2}R_w/r_w]e^{mz}, \quad (45)$$

$$\langle \delta \rangle = -\rho_0 BQ_w[Q_{2w} + 2(\gamma - 1) \times (-i\omega/\nu')^{-1/2}R_{2w}/r_w]e^{mz}. \quad (46)$$

If a *complex density* function $\rho(\omega)$ is defined through

$$i\omega\rho(\omega)\langle u \rangle = -\frac{dp}{dz}, \quad (47)$$

then Eqs. (42) and (45) may be brought together to give

$$\rho(\omega) = \rho_0\{1 - 2(-i\omega/\nu)^{-1/2} \times G[r_w(-i\omega/\nu)^{1/2}]/r_w\}^{-1}. \quad (48)$$

This complex density includes both inertial and viscous contributions. It is noted, though, that thermal conduction effects are not present. In a similar fashion, we define a *complex compressibility* function $C(\omega)$ using

$$C(\omega) = \langle \delta \rangle / \rho_0 p. \quad (49)$$

Then, combining Eqs. (42) and (46),

$$C(\omega) = (1/\gamma P_0)\{1 + 2(\gamma - 1)(-i\omega\gamma/\nu')^{-1/2} \times G[r_w(-i\omega\gamma/\nu')^{1/2}]/r_w\}. \quad (50)$$

The complex compressibility is a function of thermal conductivity, but not of viscosity.

The propagation constant may be written in a simple form. Using Eqs. (43), (48), and (50), we obtain

$$m^2 = -\omega^2\rho(\omega)C(\omega). \quad (51)$$

The main results of this analysis are the approximate expressions obtained for the propagation constant, the complex density, and the complex compressibility, as given by Eqs. (43), (48), and (50), respectively. They represent considerable simplification over the exact results obtained by Kirchhoff, but they have application over a broad range of sound frequencies and tube radii. Numerical calculations confirm that these approximate expressions and the complete Kirchhoff theory give nearly identical results for frequencies and tube radii in the stated range.

These approximate expressions were presented by Zwikker and Kosten[3] previously, but the verification of the results was given only for the extremes of low and high frequency. In their work, the velocity problem and the thermal conduction problem were treated separately; the expression for complex density was obtained with thermal conductivity assumed zero and the compressibility calculated with viscosity assumed zero. Approximations to these functions, in the low- and high-frequency cases, were compared with corresponding approximate terms from the exact Kirchhoff theory and found to be the same. However, Zwikker and Kosten were not able to demonstrate that their results were consistent with the exact theory at intermediate frequencies. The work in this section (and the numerical comparisons performed by Tijdeman[8]) provides the verification for the intermediate frequency range and explicitly defines the range of sound frequency and tube radius over which the Zwikker and Kosten equations are applicable, i.e., through Eq. (36).

III. GENERALIZATION TO TUBES OF ARBITRARY CROSS-SECTIONAL SHAPE

A general procedure is developed here for the determination of the propagation characteristics of tubes that are uniform along their length but have arbitrary cross-sectional shape, as indicated in Fig. 6. The procedure is appropriate for tubes in the narrow and wide tube regimes.

Equations (6)–(9) are general and not restricted to circular tubes. However, determining a solution to these general equations, for a tube of arbitrary cross-sectional shape, would be a formidable challenge. In the case of the circular tube, the solution obtained by Kirchhoff was exact and thus valid for narrow, wide, and very wide tubes. We have seen, though, that considerable simplification of the Kirchhoff solution is obtained if we restrict our attention to only narrow and wide tubes. It is reasonable to suppose that considerable simplification of the *general equations* might also be obtained in the same regime of narrow and wide tubes. These simplified equations would then be applicable to tubes of arbitrary cross-sectional shape.

Three features of the solution for the circular tube, within the narrow and wide regimes, are of particular interest:

(1) As seen in Figs. 2 and 3, the sound pressure p (and hence, dp/dz as well) does not vary significantly through a cross-sectional slice. Calculations confirm that variations are very small, being proportional to $r_w f^{3/2}$ and reaching 0.1% only when $r_w f^{3/2}$ is at its maximum of 10^6 cm s$^{-3/2}$. We will assume that p contains only the axial $\exp(mz)$ dependence, and is constant through each cross section of noncircular tubes.

(2) The excess density and the sound pressure are of comparable magnitude when scaled by ρ_0 and P_0, respectively. As suggested by the examples of Figs. 2 and 3, the two terms will be equal in magnitude for isothermal conditions and will differ by a factor of $\gamma = 1.4$ for adiabatic conditions. We will assume, then, that

$$\delta/\rho_0 \approx p/P_0. \quad (52)$$

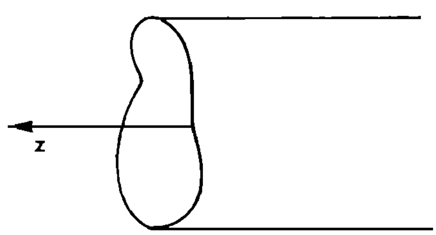

FIG. 6. Sketch of a possible tube geometry for which the generalized theory would be appropriate. The tube cross section is arbitrary in shape, but constant along the propagation axis.

(3) For the regime that includes both narrow and wide circular tubes, the propagation constant m is much smaller than certain other terms. In accordance with Eq. (42), it will be assumed that $|m^2|$ is negligible relative to $|i\omega/\nu|$ and to $|i\omega\gamma/\nu'|$ for tubes with arbitrary cross-sectional shape.

These three assumptions will be adopted for tubes having arbitrary cross-sectional shape. It will be noted that we are *not* assuming laminar flow. This assumption is not necessary for the development of the theory. The radial component of velocity, though, is considerably smaller than the axial component, reaching a maximum of 2% only when $r_w f^{3/2}$ is at its upper limit of 10^6 cm s$^{-3/2}$.

Consider, first, the general force expression, Eq. (6). The $\hat{z}$ component of this equation may be rewritten as

$$\nabla^2 u - \frac{i\omega}{\nu} u = \frac{1}{\mu}\frac{\partial}{\partial z}\left(p - \frac{1}{3}\mu\,\nabla\cdot\mathbf{v}\right). \tag{53}$$

The term involving $\nabla\cdot\mathbf{v}$ is negligible. To see this, we use Eqs. (7), (12), and (16), and the general assumption of Eq. (52) to obtain

$$-\tfrac{1}{3}\mu\,\nabla\cdot\mathbf{v} = \tfrac{1}{3}(i\omega\mu/\rho_0)\delta$$
$$\approx\tfrac{1}{3}(i\omega\nu\gamma/c^2)p. \tag{54}$$

By the general approximation expressed by Eq. (37), then, this term is much less than p and may be ignored in Eq. (53). Noting that both p and u vary with z only as $\exp(mz)$ for the assumed traveling wave solution, and with $|m^2| \ll |i\omega/\nu|$, Eq. (53) becomes

$$\nabla_c^2 u - (i\omega/\nu)u = (m/\mu)p, \tag{55}$$

where the ∇_c^2 term is the part of the Laplacian operator ∇^2 representing differentiation within the cross section, i.e.,

$$\nabla^2 \equiv \nabla_c^2 + \frac{\partial^2}{\partial z^2}. \tag{56}$$

Thus the force equation has been cast in the form of a two-dimensional boundary value problem, with the sound pressure acting as a driving term.

A corresponding equation can be obtained for the excess temperature τ. Eliminating δ between Eqs. (8) and (9), making use of the definitions in Eqs. (12) and (13), and with $|m^2| \ll |i\omega\gamma/\nu'|$, we find

$$\nabla_c^2\tau - (i\omega\gamma/\nu')\tau = -(i\omega/\kappa)p. \tag{57}$$

It is evident that the equations for the velocity and for the excess temperature have the *same form*. Moreover, the boundary conditions are the same, i.e., both u and τ must be identically zero on the perimeter of the two-dimensional cross section. Thus we are able to consolidate the two boundary value problems into a single form. Considering the velocity equation first, a generalized variable ψ is introduced through

$$u = -(mp/i\omega\rho_0)\psi, \tag{58}$$

with which Eq. (55) becomes

$$\nabla_c^2\psi - (i\omega/\eta)\psi = -i\omega/\eta, \tag{59}$$

where the constant η is just ν. The temperature equation can be treated in a similar fashion. If we set

$$\tau = (\nu' p/\gamma\kappa)\psi, \tag{60}$$

substitute this into Eq. (57), and take $\eta = \nu'/\gamma$, then we obtain the same equation, Eq. (59), that was obtained for the velocity.

Equation (59) provides the basis for calculating the propagation characteristics of tubes of arbitrary cross-sectional shape. Suppose that a solution ψ has been found, satisfying the boundary condition that $\psi = 0$ on the perimeter of the cross section. We are interested in the average of this solution over the cross section. Hence, using Eq. (30), an average $\langle\psi\rangle$ may be calculated. This average quantity retains a dependence on η which we indicate symbolically by introducing a function $F(\eta)$ through

$$F(\eta) = \langle\psi\rangle. \tag{61}$$

We are then able to obtain several necessary quantities in terms of this function:

$$\langle u\rangle = -(mp/i\omega\rho_0)F(\nu), \tag{62}$$

$$\langle\tau\rangle = (\nu' p/\gamma\kappa)F(\nu'/\gamma), \tag{63}$$

and, using Eq. (9),

$$\langle\delta\rangle = \rho_0 p/P_0 - (\rho_0\nu' p/T_0\gamma\kappa)F(\nu'/\gamma). \tag{64}$$

The complex density and complex compressibility functions have been defined in Eqs. (47) and (49), respectively. With Eqs. (62)–(64) above, and with Eq. (13), they may be expressed in terms of the function F as

$$\rho(\omega) = \rho_0/F(\nu) \tag{65}$$

and

$$C(\omega) = (1/\gamma P_0)[\gamma - (\gamma - 1)F(\nu'/\gamma)]. \tag{66}$$

The propagation constant may be evaluated by considering the equation for mass continuity, Eq. (7). Averaging over the cross section and assuming only an axial component of velocity,

$$i\omega\langle\delta\rangle = -\rho_0 m\langle u\rangle. \tag{67}$$

Thus we have

$$m^2 = -\omega^2\rho(\omega)C(\omega)$$
$$= -(\omega^2/c^2)\{[\gamma - (\gamma - 1)F(\nu'/\gamma)]/F(\nu)\} \tag{68}$$

in accordance with Eq. (51), which was obtained for the circular tube.

The characteristic impedance of the tube may also be expressed in terms of the functions $F(\eta)$. Keeping in mind that the assumed $\exp(mz)$ dependence of all quantities corresponds to a wave traveling in the $-\hat{z}$ direction, we define the characteristic specific impedance as $Z = -p/\langle u\rangle$. Then, using Eqs. (47) and (68), we get

$$Z = [\rho(\omega)/C(\omega)]^{1/2}$$
$$= \rho_0 c F(\nu)^{-1/2}[\gamma - (\gamma - 1)F(\nu'/\gamma)]^{-1/2}. \tag{69}$$

IV. APPLICATION OF THE GENERAL PROCEDURE

Section III provides a procedure for the determination of the propagation characteristics of sound in uniform tubes of arbitrary cross-sectional shape. Here, we consider two examples of the application of the method.

A. The circular tube

The simplest example is a tube of circular cross section. The solution is already known for the narrow and wide tube regimes, having been presented in Sec. II. The general procedure of Sec. III should give the same result.

For a circular geometry, with no angular dependence of velocity or excess temperature, Eq. (59) is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right)-\frac{i\omega}{\eta}\psi=-\frac{i\omega}{\eta}. \quad (70)$$

This has a solution

$$\psi(r)=1-J_0[r(-i\omega/\eta)^{1/2}]/J_0[r_w(-i\omega/\eta)^{1/2}], \quad (71)$$

which, when averaged over the cross section, gives

$$F(\eta)=1-2(-i\omega/\eta)^{-1/2}G[r_w(-i\omega/\eta)^{1/2}]/r_w, \quad (72)$$

where G is as defined in Eq. (44).

It is easily verified that using this $F(\eta)$ with Eqs. (65), (66), and (68) gives the same complex density, complex compressibility, and propagation constant, respectively, that were obtained earlier as Eqs. (48), (50), and (43).

B. Rectangular tube

A tube of rectangular cross section is considered next. Coordinates x and y, in the cross-sectional plane, are introduced, as indicated in Fig. 7. The width of the tube (in the x direction) is $2a$ and the height (in the y direction) is $2b$.

Equation (59) takes the form

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}-\frac{i\omega}{\eta}\psi=-\frac{i\omega}{\eta}. \quad (73)$$

We assume for the solution ψ an expansion of the form

$$\psi(x,y)=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}A_{kn}\cos\alpha_k x\cos\beta_n y, \quad (74)$$

where the constants α_k and β_n are given by

$$\alpha_k=(k+\tfrac{1}{2})\pi/a,$$
$$\beta_n=(n+\tfrac{1}{2})\pi/b. \quad (75)$$

Each term of the expansion satisfies the boundary condition $\psi=0$ on $x=\pm a$ and on $y=\pm b$, and together they form a

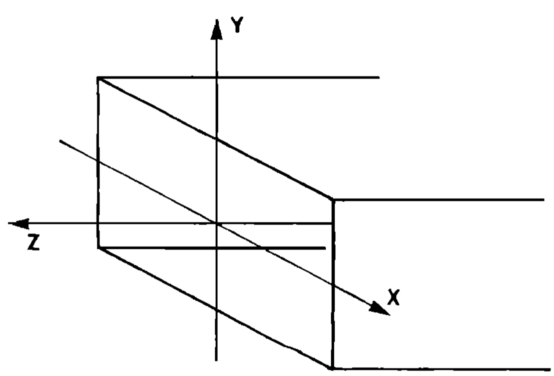

FIG. 7. Rectangular tube, with the coordinates used in the analysis. The tube has a width $2a$, along the x direction, and a height $2b$, along the y direction.

complete set of basis functions. The expansion of Eq. (74) is substituted into Eq. (73) to give

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}A_{kn}\left(\alpha_k^2+\beta_n^2+\frac{i\omega}{\eta}\right)\cos\alpha_k x\cos\beta_n y=\frac{i\omega}{\eta}. \quad (76)$$

The coefficients A_{kn} are evaluated by making use of the orthogonality of the cosine functions. Both sides of Eq. (76) are multiplied by $(\cos\alpha_{k'}x\cos\beta_{n'}y)$, then integrated over x from $-a$ to $+a$ and over y from $-b$ to $+b$, leading to

$$A_{kn}=\frac{i\omega}{\eta ab}\frac{4(-1)^k(-1)^n}{\alpha_k\beta_n(\alpha_k^2+\beta_n^2+i\omega/\eta)}. \quad (77)$$

The solution of Eq. (73) is then

$$\psi=\frac{4i\omega}{\eta ab}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^k(-1)^n\cos\alpha_k x\cos\beta_n y}{\alpha_k\beta_n(\alpha_k^2+\beta_n^2+i\omega/\eta)}. \quad (78)$$

The function $F(\eta)$, appropriate to a rectangular cross section, is obtained by averaging Eq. (78) over the cross section of the tube. With Eqs. (30) and (61), then,

$$F(\eta)=\frac{4i\omega}{\eta a^2b^2}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left(\alpha_k^2\beta_n^2(\alpha_k^2+\beta_n^2+\frac{i\omega}{\eta})\right)^{-1}. \quad (79)$$

The complex density and complex compressibility functions follow, using Eqs. (65) and (66):

$$\rho(\omega)=\rho_0\frac{\nu a^2b^2}{4i\omega}\left\{\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left[\alpha_k^2\beta_n^2\times\left(\alpha_k^2+\beta_n^2+\frac{i\omega}{\nu}\right)\right]^{-1}\right\}^{-1}, \quad (80)$$

$$C(\omega)=\frac{1}{P_0}\left\{1-\frac{4i\omega(\gamma-1)}{\nu' a^2b^2}\times\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left[\alpha_k^2\beta_n^2\left(\alpha_k^2+\beta_n^2+\frac{i\omega\gamma}{\nu'}\right)\right]^{-1}\right\}. \quad (81)$$

The propagation constant and characteristic impedance may be calculated using Eqs. (68) and (69). Results equivalent to these, i.e., Eqs. (73)–(81), have been obtained recently by Roh *et al.*[18]

The implications of these results for the modeling of porous, rigid-framed materials will be considered in a future paper. We will consider here only one limiting case of the theory, a limit for which results exist in the literature.

Consider the behavior when $a \gg b$, i.e., the narrow slit extreme. In this limit, the α_k^2 term within the parentheses in Eq. (79) may be dropped. With the definitions in Eq. (75), this simplification gives

$$F(\eta)\approx\frac{4i\omega b^2}{\eta\pi^6}\sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)^{-2}\times\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{-2}\left[\left(n+\frac{1}{2}\right)^2+\frac{i\omega b^2}{\eta\pi^2}\right]^{-1}. \quad (82)$$

The two sums can be evaluated, giving

$$F(\eta)=1-(i\omega b^2/\eta)^{-1/2}\tanh(i\omega b^2/\eta)^{1/2} \quad (83)$$

for the narrow slit. Introducing a dimensionless parameter λ as

$$\lambda = b(\omega/\nu)^{1/2}, \tag{84}$$

and the Prandtl number N by

$$N = \gamma\nu/\nu', \tag{85}$$

we obtain

$$\rho(\omega) = \rho_0 [1 - \tanh(i^{1/2}\lambda)/(i^{1/2}\lambda)]^{-1}, \tag{86}$$

$$C(\omega) = (1/\gamma P_0)[1 + (\gamma - 1) \times \tanh(i^{1/2}N^{1/2}\lambda)/(i^{1/2}N^{1/2}\lambda)]. \tag{87}$$

These equations are the same as the expressions given by Attenborough[6] for the narrow slit (after making allowance for a difference in assumed time convention).

V. CONCLUDING REMARKS

The Zwikker and Kosten[3] approximation for sound propagation in a circular tube has been validated for a broad range of tube radius and sound frequency, given by $r_w > 10^{-3}$ cm and $r_w f^{3/2} < 10^6$ cm s$^{-3/2}$. This regime includes both narrow and wide tubes, according to Weston's classification.[17] This validation was based on a critical application of the exact theory given by Kirchhoff.[1] The effects of viscosity are contained solely within a complex density term, and the effects of thermal conductivity solely within a complex compressibility term.

The theory has been generalized to describe sound propagation in uniform tubes having cross sections of *arbitrary* shape. Examination of the solution for the circular tube reveals two key features that we assume are generally true, in the narrow and wide tube regime. First, the sound pressure is constant through each cross section. Second, the sound pressure and the excess density, when scaled by the values of ambient pressure and temperature, respectively, are of comparable magnitude. Using these two assumptions, a general approach for tubes of arbitrary cross-sectional shape has been derived. Separate equations are obtained describing the variation of velocity and excess temperature over a cross section. Both equations have the same form, though, and the solution of a single two-dimensional boundary value problem [Eq. (59)] allows both complex density and complex compressibility functions to be determined; the propagation constant and characteristic impedance for the tube may be calculated from these functions. As an example, the general procedure has been used to determine the sound propagation characteristics in tubes of rectangular cross section.

The work presented here is applicable to the study of the acoustical properties of porous materials. Various theoretical models[6,14] introduce "shape factors" to accommodate the departure of pore cross sections from a circular shape. Using the formalism discussed in Sec. III, we are able to calculate directly the acoustical properties of samples containing pores with specific cross-sectional shapes, and appropriate shape factors may be inferred.

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