# A Concise Formulation of Huygens' Principle for the Electromagnetic Field*

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THIS communication is to direct attention to the fact that Huygens' principle for the electromagnetic field can be stated in a concise mathematical form by using the unified electromagnetic field vector, together with the dyadic Green's function.

We consider the Maxwellian equations for a harmonically oscillating field defined in a homogeneous and isotropic medium, *i.e.*,

$$\nabla \times \vec{E} = i\omega\mu\vec{H} \tag{1}$$

$$\nabla \times \vec{H} = \vec{J} - i\omega\epsilon\vec{E}. \tag{2}$$

Following Bateman[1] and Itoh,[2] we introduce the unified electromagnetic field vector defined by

$$\vec{F} = \vec{E} + \Gamma\vec{H} \tag{3}$$

where

$$\Gamma = \pm i(\mu/\epsilon)^{1/2} = ih\eta. \tag{4}$$

The ambiguity sign $\pm 1$, denoted by $h$, will be used as a separation operator similar to $\sqrt{-1}$ or $i$ in complex number theory. We adopt the rule that $h^2 = 1$. By multiplying (2) by $\Gamma$ and adding it to (1) we obtain

$$\nabla \times \vec{F} = \kappa\vec{F} + \Gamma\vec{J} \tag{5}$$

where

$$\kappa = -i\omega\epsilon\Gamma = hk \tag{6}$$

with $k = \omega\epsilon\eta = 2\pi/\lambda$. Eqs. (1) and (2) can be recovered from (5) by separating the parts with and without $h$. A function, such as

$$f = A + hB, \tag{7}$$

therefore plays a similar role as the function $u + iv$ in complex variable theory. For convenience, we shall call $A$ of (7) the transversal part, and $B$ the longitudinal part.

To integrate (5), we first take the curl of that equation that yields

$$\nabla \times \nabla \times \vec{F} - k^2\vec{F} = \vec{f} \tag{8}$$

where

$$\vec{f} = \Gamma(\kappa\vec{J} + \nabla \times \vec{J}). \tag{9}$$

We introduce now the free-space dyadic Green's function[3] defined by

$$\overleftrightarrow{G}_0 = \frac{1}{4\pi}\left(\overleftrightarrow{I} + \frac{1}{k^2}\nabla\nabla\right)\frac{\exp ik|\vec{r} - \vec{r}'|}{|\vec{r} - \vec{r}'|}, \tag{10}$$

which satisfies the equation

$$\nabla \times \nabla \times \overleftrightarrow{G}_0 - k^2\overleftrightarrow{G}_0 = \overleftrightarrow{I}\delta(\vec{r}|\vec{r}'), \tag{11}$$

where $\overleftrightarrow{I}$ denotes the unit dyadic, and $\delta(\vec{r}/\vec{r}')$ the three dimensional delta function. Applying now the vector Green's identity[4] to (8) and (11), we obtain

$$\vec{F}(\vec{r}') = \iiint_V \vec{f}\cdot\overleftrightarrow{G}_0 dV + \oint\!\!\oint_S (\vec{F} \times \hat{n}) \cdot(\nabla \times \overleftrightarrow{G}_0 + \kappa\overleftrightarrow{G}_0)dS. \tag{12}$$

For the case that $\vec{f}$ does not exist inside $V$, (12) reduces to

$$\vec{F}(\vec{r}') = \oint\!\!\oint_S (\vec{F} \times \hat{n})\cdot(\nabla \times \overleftrightarrow{G}_0 + \kappa\overleftrightarrow{G}_0)dS. \tag{13}$$

Eq. (13) represents perhaps the most compact expression in describing Huygens' principle as applied to a harmonically oscillating electromagnetic field. By taking the transversal part and the longitudinal part of (13), one obtains, respectively, the expressions of $\boldsymbol{E}$ $(r)$ and $\boldsymbol{H}$ $(r)$ derivable by the conventional method. The formulation described here, of course, does not yield any new information about the Huygens' principle. It merely assembles our known knowledge in a more concise form. It is obvious that the technique outlined here can readily be extended to problems involving dyadic Green's functions which satisfy certain specific boundary conditions, such as the vector Dirichelet and the vector Neumann conditions.[5]

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