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ON THE PROPAGATION OF SOUND WAVES IN CYLINDRICAL TUBES

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It is shown that the two main parameters governing the propagation of sound waves in gases contained in rigid cylindrical tubes, are the shear wave number, $s = R\sqrt{\rho_s \omega/\mu}$, and the reduced frequency, $k = \omega R/a_0$. It appears possible to rewrite the most significant analytical solutions for the propagation constant, Γ, as given in the literature, as simple expressions in terms of these two parameters. With the aid of these expressions the various solutions are put in perspective and their ranges of applicability are indicated.

It is demonstrated that most of the analytical solutions are dependent only on the shear wave number, s, and that they are covered completely by the solution obtained for the first time by Zwikker and Kosten (1949).

The full solution of the problem has been obtained by Kirchhoff (1868) in the form of a complicated, complex transcendental equation. In the present paper this equation is rewritten in terms of the mentioned basic parameters and brought in the attractive form $F\langle\Gamma, s, k\rangle = 0$, which is solved numerically by using the Newton–Raphson procedure. As first estimate in this procedure the value of Γ according to the solution of Zwikker and Kosten is taken. Results are presented for a wide range of s and k values.

1. INTRODUCTION

The problem of the propagation of sound waves in gases contained in cylindrical tubes is a classical one, to which famous names are connected like Helmholtz [1], Kirchhoff [2] and Rayleigh [3]. Since then many papers have been written on the subject, often in relation to studies dealing with the dynamic response of pressure transmission lines [4–34].

The *analytical* solutions given in the literature can be divided roughly into two groups. The first group comprises solutions obtained by analytical approximations of the full Kirchhoff solution, which is given in the form of a very complicated transcendental equation [2, 3]. The solutions of the second group have been derived directly from the basic equations governing the problem, by the introduction of one or more simplifying assumptions.

The first approximations of the full Kirchhoff solution were produced by Kirchhoff [2] himself for "wide" tubes and by Rayleigh [3] for "narrow" tubes. Later on, higher order approximations have been given by Weston [4], who derived formulae for the transitions "narrow-wide", "wide-narrow", "wide-very-wide" and for "very wide" tubes.

Analytical solutions of the second group obtained directly from more or less simplified basic equations have been presented by, among others, Kerris [5], Zwikker and Kosten [6], Iberall [7], Rohmann and Grogan [8], and by Karam and Franke [9].

Numerical solutions of the problem were published by Tsao [10], Gerlach and Parker [11], Scarton and Rouleau [12], and by Shields, Lee and Wiley [13]. Tsao [10] based his finite difference solution on the same simplified basic equations as used by Zwikker and Kosten [6], and Iberall [7]. In the papers of Gerlach and Parker [11], and Scarton and Rouleau [12] the basic equations are simplified considerably by the assumption that thermal effects are negligible. This simplification made it possible to derive a set of two decoupled equations, which have been solved by means of the method of eigenvalues. In references [11] and [12] the existence of higher order modes is shown and in reference [12] the first thirty-two modes are presented.

The numerical solution for the fundamental mode of Shields *et al.* [13] has been obtained by an iterative solution of the full Kirchhoff equation and thus can be considered as the most complete solution so far available.

In the subsequent sections of this paper it first is shown, by a consideration of the basic equations, that the problem of small amplitude, sinusoidal motions of a fluid column in a rigid cylinder is completely determined by the following four parameters (a list of symbols is given in Appendix D):

$s = R\sqrt{\rho_s \omega/\mu}$, the shear wave number, sometimes also referred to as the Stokes number,

$k = \omega R/a_0$, the reduced frequency,

$\sigma = \sqrt{\mu C_p/\lambda}$, the square root of the Prandtl number,

$\gamma = C_p/C_v$, the ratio of specific heats.

As for a given gas σ and γ often can be considered as constants, the two main parameters are the *shear wave number* and the *reduced frequency*. Subsequently, the most significant analytical solutions of the propagation constant Γ† given in the literature are rewritten as simple expressions in s and k and collected in Table 1, which for the sake of completeness also contains the solution of Helmholtz [1], obtained from a Poiseuille-like consideration of the motion in axial direction.

It is demonstrated that nearly all the approximate solutions are covered completely by the solution obtained for the first time by Zwikker and Kosten [6], and also to be found in the later references [7] and [14]–[17]. This solution may be designated as the "low reduced frequency solution", because the solution depends only on the shear wave number and is valid only for $k \ll 1$ and $k/s \ll 1$.

To the author's knowledge analytical solutions without the restriction $k \ll 1$ have been published only by Weston in reference [4], which solutions are valid only for very large values of the shear wave number. It further appears that the exact results of Shields *et al.* [13]—once they are expressed in the main parameters s and k—cover only a rather limited range of these parameters. To remove these limitations a new numerical solution of the full Kirchhoff equation is presented. This "exact" solution can be obtained relatively easily by rewriting the full Kirchhoff equation in terms of the four non-dimensional parameters mentioned before. This procedure leads to the attractive equation

$$F\langle \Gamma, s, k, \sigma, \gamma \rangle = 0,$$

which can be solved numerically by making use of the Newton–Raphson procedure. As a first estimate the value of Γ resulting from the "low reduced frequency solution" has been taken.

† Defined in section 3.

Numerical values of the propagation constant are given for a wide range of the main parameters s and k, which may be helpful in further studies and calculations.

The results reveal that the propagation constant strongly depends on the reduced frequency in the range of relatively low shear wave numbers ($s \lesssim 4$) and that in this area a strong dispersion occurs.

2. FORMULATION OF THE PROBLEM

The equations describing the motion of a fluid column in a circular cylinder are the Navier–Stokes equations (one in the axial and one in the radial direction), the equation of continuity, the equation of state and the energy equation, giving the balance between thermal and kinetic energy.

These five equations suffice to obtain the solution for the same number of unknown quantities: namely, the velocities in the axial and radial directions, the density, the temperature and the pressure.

The aforementioned equations can be simplified to (see Appendix A)

$$\mathrm{i}u = -\frac{1}{\gamma}\frac{\partial p}{\partial \xi} + \frac{1}{s^2}\left[\left(k^2\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial u}{\partial \eta}\right) + \frac{1}{3}k\frac{\partial}{\partial \xi}\left(k\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} + \frac{v}{\eta}\right)\right], \tag{2.1}$$

$$\mathrm{i}kv = -\frac{1}{\gamma}\frac{\partial p}{\partial \eta} + \frac{k}{s^2}\left[\left(\frac{\partial^2 v}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial v}{\partial \eta} - \frac{v}{\eta^2} + k^2\frac{\partial^2 v}{\partial \xi^2}\right) + \frac{1}{3}\frac{\partial}{\partial \eta}\left(k\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} + \frac{v}{\eta}\right)\right], \tag{2.2}$$

$$\mathrm{i}k\rho = -\left(k\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} + \frac{v}{\eta}\right), \tag{2.3}$$

$$p = \rho + T, \tag{2.4}$$

$$\mathrm{i}T = \mathrm{i}\frac{\gamma - 1}{\gamma}p + \frac{1}{\sigma^2 s^2}\left(\frac{\partial^2 T}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial T}{\partial \eta} + k^2\frac{\partial^2 T}{\partial \xi^2}\right), \tag{2.5}$$

if the following assumptions are introduced:

(a) homogeneous medium, which means that the wave length and the tube radius must be large in comparison with the mean free path; for air of normal atmospheric temperature and pressure, this condition breaks down for $f > 10^8$ Hz and $R < 10^{-5}$ cm;
(b) no steady flow;
(c) small amplitude, sinusoidal perturbations (no circulation and no turbulence);
(d) tube long enough, so that end effects are negligible.

To obtain the solution for a rigid tube of constant circular cross-section, equations (2.1)–(2.5) have to satisfy the following boundary conditions and assumptions:

(a) at the rigid tube wall the axial and radial velocity must be zero: i.e.,

$$\text{at } \eta = 1, \qquad u = 0 \quad \text{and} \quad v = 0; \tag{2.6}$$

(b) the radial velocity must be zero at the tube axis due to the axisymmetry of the problem: i.e.,

$$\text{at } \eta = 0, \qquad v = 0 \tag{2.7}$$

and u, p, ρ and T have to remain finite;

(c) the heat conductivity of the tube wall is large in comparison with the heat conductivity of the fluid: i.e.,

$$\text{at } \eta = 1, \qquad T = 0 \quad \text{(isothermal walls)}. \tag{2.8}$$

3. REVIEW OF ANALYTICAL SOLUTIONS GIVEN IN THE LITERATURE

The solution, for the pressure perturbation, p, of equations (2.1)–(2.5) in general can be put in the form

$$p = \{A\langle\eta\rangle \, e^{\Gamma\xi} + B\langle\eta\rangle \, e^{-\Gamma\xi}\} \, e^{i\omega t},$$

with η and ξ being the dimensionless co-ordinates (see Figure 1).

The propagation constant occurring in this solution consists of a real part, Γ', representing the attenuation over a unit distance in the ξ direction, and an imaginary part, Γ'', being the phase shift over the same distance.

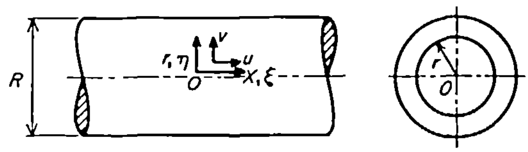

Figure 1. Co-ordinate system. Dimensionless co-ordinates: $\xi = \omega x / a_0$; $\eta = r/R$.

As stated already in the introduction, the most significant analytical solutions for Γ, known in the literature, can be expressed in terms of the four basic parameters (see Table 1).

A closer examination of the formulae in Table 1 reveals that the expressions do not contain the reduced frequency, k, except for the "wide-very-wide" and the "very wide" approximations of Weston [4]. In these cases the expressions for the attenuation, among other things, contain the term

$$\Gamma'_{\text{class}} = \frac{1}{2} \frac{k^2}{s^2} \left(\frac{4}{3} + \frac{\gamma - 1}{s^2} \right),$$

which equals the attenuation of plane waves in free air [3] and which is independent of the tube radius.

In the "wide" tube solution of Kirchhoff the parameters s and σ are present, which indicates that both viscosity effects and heat conductivity have been accounted for. Putting $\gamma = 1$ (isothermal conditions) reduces the Kirchhoff solution to that of Helmholtz. For both solutions the result $\lim_{s \to \infty} \Gamma = i$ holds, this being the solution for plane waves in free air without absorption.

The solution which Rayleigh obtained for the "narrow" tube, by assuming the diameter so small that heat is transferred freely from the center to the wall, does not contain the parameter σ, which means that only viscosity effects are involved.

As mentioned in the introduction, Weston's formulae are higher order approximations of the full Kirchhoff solution and therefore it is not surprising that the first terms of the "narrow-wide" approximation equal the Rayleigh solution, whereas the first terms of the "wide-narrow" and "wide-very-wide" transitions show the characteristics of the "wide" tube of Kirchhoff.

The analytical solutions of the second group, obtained directly from simplified basic equations, are of a different type, with the exception of the "high frequency solution" of Karam and Franke [9], which looks very similar to that of Kirchhoff.

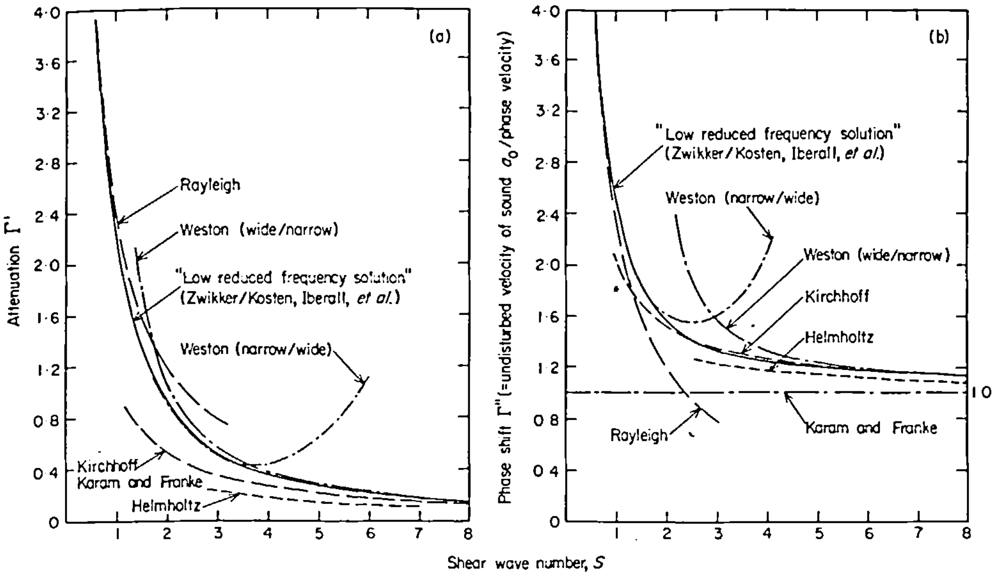

Figure 2. (a) Attenuation Γ' and (b) phase shift Γ'' as functions of shear wave number.

The mutual relationship of all the analytical solutions to one another, except Weston's "wide-very-wide" and "very wide" approximations, is shown in Figures 2(a) and (b), where the attenuation, Γ', and the phase shift, Γ'' are plotted as functions of the shear wave number, s.

A point of interest revealed by Figures 2(a) and (b) is that the solution obtained for the first time by Zwikker and Kosten [6], and designated as the "low reduced frequency solution", passes continuously from Rayleigh's solution into the solution of Kirchhoff.

The "low reduced frequency solution", to be discussed in more detail in the next section, can be shown to be valid over the complete range of shear wave numbers in the case $k \ll 1$ and $k/s \ll 1$. The solution therefore covers all the solutions in Table 1, except the "wide-very-wide" and "very wide" approximations of Weston.

Another point of interest revealed by Figures 2(a) and (b) is that "narrow" tube solutions are valid for low values of s and "wide" tube solutions for high values of this parameter.

As a large value of $s(=R\sqrt{\rho_s\omega/\mu})$ can be obtained not only for large tube radii but also for high frequencies, large mean densities or pressure and small viscosity, it is clear that names like "narrow", "wide" or "high frequency solution", as used in the literature, are somewhat

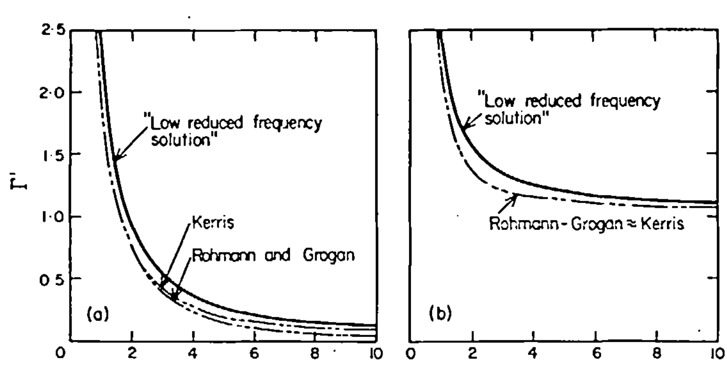

Figure 3. Comparison of the solutions of Kerris and of Rohmann and Grogan with the "low reduced frequency solution".

TABLE 1

Review of significant analytical solutions, expressed in s, σ, γ and k

Author	Year	Formula for the constant of propagation $\Gamma = \Gamma' + \mathrm{i}\Gamma''$
Helmholtz	1863[1]	$\Gamma = 1 + \frac{(1+\mathrm{i})}{\sqrt{2}} \frac{1}{s}$
Kirchhoff ("wide tube")	1868 [2]	$\Gamma = \mathrm{i} + \frac{(1+\mathrm{i})}{\sqrt{2}} \left[\frac{\gamma - 1 + \sigma}{\sigma s}\right]$
Rayleigh ("narrow tube")	1896 [3]	$\Gamma = 2(1+\mathrm{i}) \frac{\sqrt{\gamma}}{s}$
Kerris	1939 [5]	$\Gamma = \sqrt{\frac{J_0\langle \mathrm{i}^{3/2} s\rangle}{J_2\langle \mathrm{i}^{3/2} s\rangle}}$
Zwikker and Kosten (independently also Iberall, 1950)	1949 [6] [7]	$\Gamma = \sqrt{\frac{J_0\langle \mathrm{i}^{3/2} s\rangle}{J_2\langle \mathrm{i}^{3/2} s\rangle}} \sqrt{\frac{\gamma}{n}}$ with $n = \left[1 + \frac{\gamma - 1}{\gamma} \frac{J_2\langle \mathrm{i}^{3/2} \sigma s\rangle}{J_0\langle \mathrm{i}^{3/2} \sigma s\rangle}\right]^{-1}$
Weston Transition "narrow-wide"†	1953 [4]	$\begin{cases} \Gamma' \approx 2\frac{\sqrt{\gamma}}{s}\left\{1 - \frac{1}{48}s^2\left(4 - 3\frac{\gamma-1}{\gamma}\sigma^2\right) - \frac{1}{1536}s^4\left(-6 - 12\frac{\gamma-1}{\gamma}\sigma^2 + 13\frac{\gamma^2-1}{\gamma^2}\sigma^4\right)\right\} \\ \Gamma'' \approx 2\frac{\sqrt{\gamma}}{s}\left[1 - \frac{1}{48}s^2\left(4 - 3\frac{\gamma-1}{\gamma}\sigma^2\right) + \frac{1}{1536}s^4\left(-6 - 12\frac{\gamma-1}{\gamma}\sigma^2 + 13\frac{\gamma^2-1}{\gamma^2}\sigma^4\right) + \frac{1}{2304}s^4\left(4 - 3\frac{\gamma-1}{\gamma}\sigma^2\right)^2\right]^{-1} \end{cases}$

Transition "wide-narrow"		$\begin{cases} \Gamma' = \frac{1}{\sqrt{2}}\left(\frac{\gamma-1+\sigma}{\sigma s}\right) + \frac{1}{s^2}\left\{\frac{\gamma-1+\sigma}{\sigma} - \frac{\gamma(\gamma-1)}{2\sigma^2}\right\} + \frac{1}{8\sqrt{2}}\frac{1}{s^3}\left\{15 + 32\frac{\gamma-1}{\sigma} + \frac{12(\gamma-1)(\gamma-2)}{\sigma^2} + \right. \\ \left. \quad + (\gamma-1)(4\gamma^2-12\gamma+7)\frac{1}{\sigma^3}\right\} \\ \Gamma'' = \left[1 - \frac{1}{\sqrt{2}}\left(\frac{\gamma-1+\sigma}{\sigma s}\right) + \frac{1}{(\sqrt{2})^2}\left(\frac{\gamma-1+\sigma}{\sigma s}\right)^2 - \frac{1}{(\sqrt{2})^3}\left(\frac{\gamma-1+\sigma}{\sigma s}\right)^3 - \frac{1}{8\sqrt{2}}\frac{1}{s^3}\left\{15 + 32\frac{\gamma-1}{\sigma} + \right.\right. \\ \left.\left. \quad + \frac{12(\gamma-1)(\gamma-2)}{\sigma^2} + (\gamma-1)(4\gamma^2-12\gamma+7)\frac{1}{\sigma^3}\right\}\right]^{-1} \end{cases}$
Transition "wide-very wide"		$\begin{cases} \Gamma' = \frac{1}{\sqrt{2}}\left(\frac{\gamma-1+\sigma}{\sigma s}\right) + \frac{1}{2}\frac{k^2}{s^2}\left(\frac{4}{3} + \frac{\gamma-1}{\sigma^2}\right) + \frac{1}{4}k^2\left(\frac{\gamma-1+\sigma}{\sigma s}\right)^2 \\ \Gamma'' = \left[1 - \frac{1}{\sqrt{2}}\frac{\gamma-1+\sigma}{\sigma s}\right]^{-1} \end{cases}$
"Very wide"		$\begin{cases} \Gamma' = \frac{1}{2\sqrt{2}}\left(\frac{\gamma-1+\sigma}{\sigma s}\right) + \frac{1}{2}\frac{k^2}{s^2}\left(\frac{4}{3} + \frac{\gamma-1}{\sigma^2}\right) + \frac{1}{4}k^2\left(\frac{\gamma-1+\sigma}{\sigma s}\right)^2 \\ \Gamma'' = \left[1 - \frac{1}{2\sqrt{2}}\left(\frac{\gamma-1+\sigma}{\sigma s}\right)\right]^{-1} \end{cases}$
Rohmann and Grogan	1957 [8]	$\Gamma = \sqrt{\mathrm{i}\frac{8}{s^2} - (1 + \frac{1}{3}\mathrm{e}^{-s^2/32\pi})}$
Karam and Franke	1969 [9]	$\Gamma = \mathrm{i} + \frac{1}{\sqrt{2}}\left[\frac{\gamma-1+\sigma}{\sigma s}\right]$

† In reference [4] the solution for Γ' is a factor $\sqrt{2}$ in error.

misleading, the more so as most of the formulae of Table 1 in their original versions are written explicitly in terms of radius and frequency.

The importance of the shear wave number as the governing parameter, already recognized by Zwikker and Kosten [6], has been stressed by Iberall in his discussion appended to reference [8] and later also by the author in reference [18] and by Goldschmied in reference [19].

To make the picture of the analytical solutions for $k \ll 1$ complete, the results of Kerris [5], who assumed isentropic conditions, and of Rohmann and Grogan [8], who produced an approximate solution of already simplified basic equations, are shown in Figure 3. A comparison of both solutions with the "low reduced frequency solution" reveals that the formulae of references [5] and [8] are not too accurate.

4. MAIN RESULTS OF THE "LOW REDUCED FREQUENCY SOLUTION"

4.1. GENERAL

In this section the properties of the "low reduced frequency solution" will be discussed in somewhat more detail, because it is the solution which has proven to be suitable for most of the practical applications [9, 17, 19–23] and which, moreover, serves as the first approximation in the procedure to determine the "exact" solution, to be discussed in section 5.

The derivation of the "low reduced frequency solution", including the complete set of acoustic variables, is given in Appendix B. It can be verified *a posteriori* that this solution is valid if the following conditions are fulfilled:

$$k \ll 1 \quad \text{and} \quad k/s \ll 1.\dagger$$

4.2. PROPAGATION CONSTANT

The solution for the propagation constant, Γ, yields

$$\Gamma = \sqrt{\frac{J_0\langle i^{3/2} s\rangle}{J_2\langle i^{3/2} s\rangle}}\ \sqrt{\frac{\gamma}{n}}, \tag{4.1a}$$

with

$$n = \left[1 + \frac{\gamma - 1}{\gamma}\,\frac{J_2\langle i^{3/2}\sigma s\rangle}{J_0\langle i^{3/2}\sigma s\rangle}\right]^{-1}. \tag{4.1b}$$

As a solution identical to equation (4.1a) can be obtained if the equation of state (A4) and the energy equation (A5) together are replaced by the polytropic relation $\bar{p}/\bar{\rho}^n = \text{constant}$, the thermodynamic process within the tube evidently can be considered to occur polytropically, with the (complex) polytropic constant, n, given by equation (4.1b). This polytropic constant is a function of the product σs, which means that the constant does not depend on the viscosity and only accounts for the effect of heat conduction. The development of the polytropic constant with σs is shown in Figure 4. Asymptotic values are $\lim_{s\to 0} n = 1$ and $\lim_{s\to\infty} n = \gamma$, corresponding to isothermal and isentropic conditions, respectively.

If one puts $n = \gamma$, independently of σs, the solution of Kerris [5] is obtained. This implies that the difference between the "low reduced frequency solution" and the solution of Kerris given in Figure 3, is caused by the losses due to heat conduction.

† These conditions differ from the condition given by Rott [24], who states that it is sufficient to assume that the radius of the tube is much smaller than its length. He therefore proposed the name "long tube" approximation.

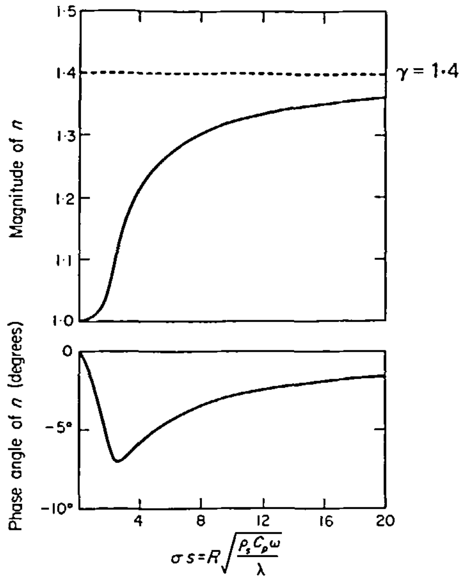

Figure 4. "Polytropic constant", n, as a function of σs.

When the isothermal conditions are assumed, i.e., $n = 1$, the formula of D'Souza and Oldenburger [25] for hydraulic lines is obtained.

In Appendix C it is shown that solution (4.1) converges to the "wide" tube solution of Kirchhoff for very large values of the shear wave number. At very small values of this parameter solution (4.1) passes over into Rayleigh's "narrow" tube solution. Moreover, it is shown that at vanishingly small values of the shear wave number the law of Poiseuille is obtained.

4.3. VELOCITY DISTRIBUTIONS IN AN INFINITELY LONG TUBE

Next consider the velocity distributions in axial and radial direction for an outgoing wave. in an infinitely long tube. The velocity profiles (magnitudes) are shown in Figures 5(a) and (b).

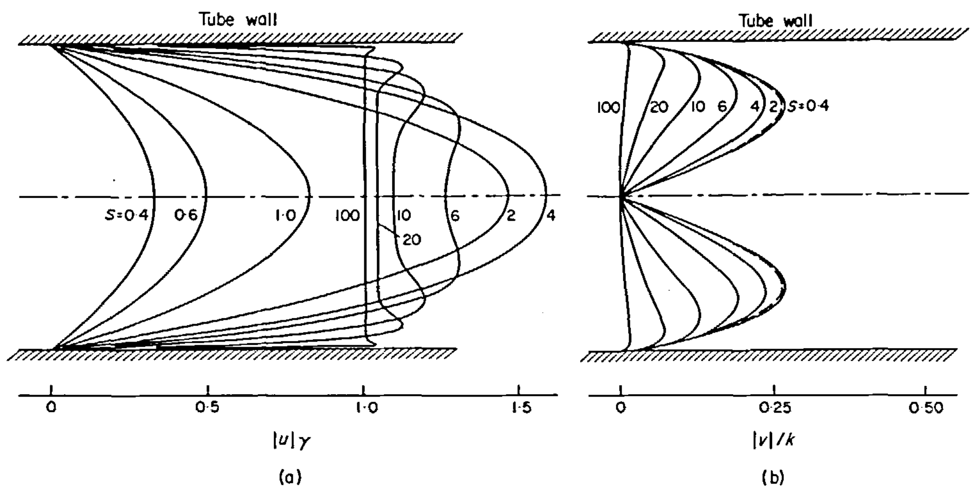

Figure 5. Velocity distribution in (a) axial direction and (b) radial direction according to the "low reduced frequency solution".

At low values of the shear wave number the axial velocity shows a parabolic profile. At higher values the amplitudes of the velocity in the central part of the tube become smaller and the profile becomes more and more uniform. At very high values of the shear wave number the velocity profile is almost completely flat, with small peaks close to the tube wall.

The velocity profiles, as described in the early work of Crandall [26] and of Sexl [27], have been verified experimentally by Harris, Peev and Wilkinson [28] for a liquid-filled tube.

The distribution of the radial velocity (Figure 5(b)), which in the "low reduced frequency solution" appears to be directly proportional to the reduced frequency, k (see Appendix B), reveals that maximum radial velocities occur at relatively small shear wave numbers.

4.4. PHASE VELOCITY AND GROUP VELOCITY

Another point of interest concerns the phase velocity, W, which in terms of the present notation can be expressed as

$$W = a_0/\Gamma''. \tag{4.2}$$

The phase velocity can be considered as the speed of propagation of an infinitely long sinusoidal wave. The present case deals with a so-called dispersive medium, because the phase velocity depends on the frequency, which means that sinusoidal waves with different frequencies propagate with different velocities. This implies that the propagation of a signal consisting of two or more sinusoidal waves, with different frequencies, is accompanied by a change in the signal shape.

For instance, if one considers the propagation of a carrier wave with a superimposed modulation, one can distinguish the phase velocity, W, being the motion of the elementary wavelets, and the group velocity, U, being the velocity with which the modulation is propagated (see Figure 6).

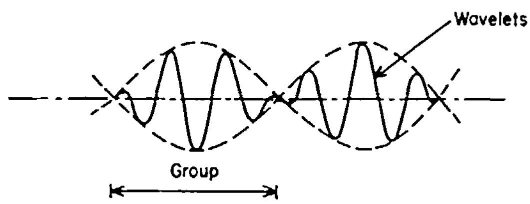

Figure 6. Sketch of a modulated wave showing group and wavelet components.

Between the group velocity, U, and the phase velocity, W, the following relation exists [35, 36]:

$$U = W + \kappa \frac{\partial W}{\partial \kappa}, \tag{4.3}$$

with $\kappa = \omega/W$ being the wave number. The velocities U and W are identical for non-absorbing media, but they may differ considerably in dispersive media.

For the present case of a gas-filled tube it can be shown that

$$\frac{U}{a_0} = \frac{2}{\Gamma''\langle s\rangle + \dfrac{\partial}{\partial s}\{s\,.\,\Gamma''\langle s\rangle\}}. \tag{4.4}$$

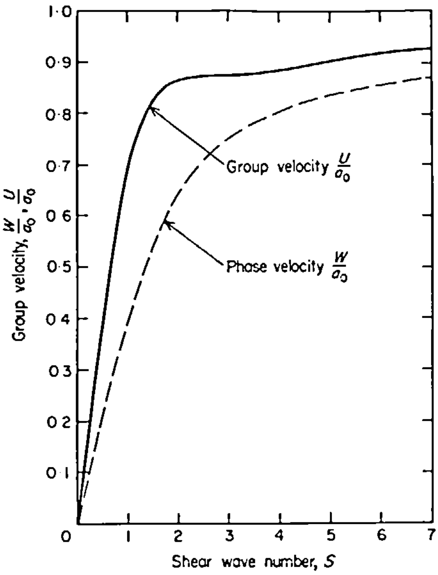

Figure 7. Phase velocity and group velocity according to the "low reduced frequency solution".

This result, as obtained by numerical differentiation of the imaginary part of the "low reduced frequency solution", is plotted in Figure 7. It can be seen that high frequency waves propagate faster than low frequency waves. In Figure 7 also the phase velocity of the "low reduced frequency solution" is given. There exists a considerable difference between phase velocity and group velocity. The fact that $U > W$ implies, for instance, that the elementary wavelets in the example of a modulated carrier wave are building up in front of the modulation and disappear at the rear end.

In Figure 8 a comparison is presented of the group velocity following from the "low reduced frequency solution" and the group velocities according to the approximate solutions

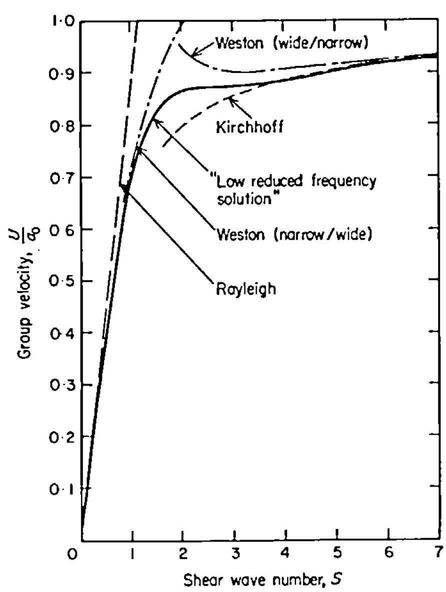

Figure 8. Comparison of various solutions for the group velocity.

of Rayleigh, Kirchhoff and Weston. It can be observed that the approximate solutions are not able to close the gap between $\sim 0{\cdot}5 < s < \sim 4$.

Now consider the propagation of a short isolated succession of wavelets, with the system at rest before the signal arrives and also after it has passed. Then, after a certain length of propagation through the dispersive medium, first very weak signals, the so-called forerunners, will appear. Thereafter, the main signal, with intensities of the order of magnitude of the input signal, will arrive. The velocity at which the forerunners propagate is called the front velocity and the main signal arrives with the signal velocity. According to Brillouin [35] the signal velocity is practically equal to the group velocity, except in regions with very large absorption. In that case the group velocity loses its meaning as a signal velocity. The maximum speed with which the forerunners can propagate equals the undisturbed velocity of sound, a_0.

In reference [29] Holmboe and Rouleau, who experimentally investigated the propagation of a short pulse in a liquid-filled tube, mention the unexpected appearance of a high frequency disturbance that preceded the pulse at the receiving transducer. The above consideration justifies the thought that this phenomenon most likely has to do with the dispersive behaviour in a tube filled with gas or liquid and that the observed high frequency signals are the forerunners.

5. NUMERICAL SOLUTION OF THE FULL KIRCHHOFF EQUATION

The full solution of the basic equations (2.1)–(2.5), without assumptions other than those discussed in section 2, is given in the original paper of Kirchhoff [2]. The solution is obtained in the form of a transcendental complex frequency equation, which does not lend itself to further analytical treatment. The derivation of the transcendental equation is not repeated here, because a very detailed account of the Kirchhoff paper is given in reference [3].

By using the knowledge that the problem is governed by the four basic parameters s, k, σ and γ, the original Kirchhoff solution can be brought into a much more attractive form, if the quantities used are rewritten in terms of these basic parameters.

After some manipulation the solution of Kirchhoff (see reference [3], p. 324, equation (11)) can be rewritten as

$$\mathrm{i}Z\left(Z-\mathrm{i}\frac{s^2}{k^2}\right)^{-1}\left(\frac{1}{x_1}-\frac{1}{x_2}\right)\left\{\frac{\mathrm{d}\ln Q}{\mathrm{d}\eta}\right\}_w+\left(\frac{\gamma k^2}{\sigma^2 s^2}-\mathrm{i}\frac{1}{x_1}\right)\left\{\frac{\mathrm{d}\ln Q_1}{\mathrm{d}\eta}\right\}_w$$

$$-\left(\frac{\gamma k^2}{\sigma^2 s^2}-\mathrm{i}\frac{1}{x_2}\right)\left\{\frac{\mathrm{d}\ln Q_2}{\mathrm{d}\eta}\right\}_w=0 \tag{5.1a}$$

with

$$Q=\mathrm{J}_0\langle \eta k(Z-\mathrm{i}s^2/k^2)^{1/2}\rangle,$$

$$Q_1=\mathrm{J}_0\langle \eta k(Z-x_1)^{1/2}\rangle,$$

$$Q_2=\mathrm{J}_0\langle \eta k(Z-x_2)^{1/2}\rangle.$$

x_1 and x_2 are the small and large roots of

$$1+x\left\{1+\mathrm{i}\frac{k^2}{s^2}\left(\frac{4}{3}+\frac{\gamma}{\sigma^2}\right)\right\}+\mathrm{i}\frac{\gamma k^2}{\sigma^2 s^2}\left(\frac{1}{\gamma}+\mathrm{i}\frac{4}{3}\frac{k^2}{s^2}\right)x^2=0 \tag{5.1b}$$

and $Z = \Gamma^2$. The suffix w denotes the value of a quantity at the wall ($\eta = 1$). A more convenient form of equation (5.1a) is:

$$F\langle Z\rangle = \mathrm{i}Z\left(Z - \mathrm{i}\frac{s^2}{k^2}\right)^{-1/2}\left(\frac{1}{x_1} - \frac{1}{x_2}\right)\frac{\mathrm{J}_1\langle\alpha_1\rangle}{\mathrm{J}_0\langle\alpha_1\rangle}$$

$$+\left(\frac{\gamma k^2}{\sigma^2 s^2} - \mathrm{i}\frac{1}{x_1}\right)(Z - x_1)^{1/2}\frac{\mathrm{J}_1\langle\alpha_2\rangle}{\mathrm{J}_0\langle\alpha_2\rangle}$$

$$-\left(\frac{\gamma k^2}{\sigma^2 s^2} - \mathrm{i}\frac{1}{x_2}\right)(Z - x_2)^{1/2}\frac{\mathrm{J}_1\langle\alpha_3\rangle}{\mathrm{J}_0\langle\alpha_3\rangle} = 0, \tag{5.2}$$

with

$$\alpha_1 = k(Z - \mathrm{i}s^2/k^2)^{1/2}, \qquad \alpha_2 = k(Z - x_1)^{1/2}, \qquad \alpha_3 = k(Z - x_2)^{1/2}.$$

From equation (5.2) one can easily derive

$$\frac{\partial F}{\partial Z} = F'\langle Z\rangle = \mathrm{i}\left(Z - \mathrm{i}\frac{s^2}{k^2}\right)^{-1}\left(\frac{1}{x_1} - \frac{1}{x_2}\right)\left[\frac{\mathrm{J}_1\langle\alpha_1\rangle}{\mathrm{J}_0\langle\alpha_1\rangle}\left(Z - \mathrm{i}\frac{s^2}{k^2}\right)^{1/2}\right.$$

$$\left. - \frac{1}{2}Z\left(Z - \mathrm{i}\frac{s^2}{k^2}\right)^{-1/2}\frac{\mathrm{J}_1\langle\alpha_1\rangle}{\mathrm{J}_0\langle\alpha_1\rangle} + \tfrac{1}{2}kZ\left(1 - \frac{\mathrm{J}_1\langle\alpha_1\rangle}{\alpha_1\mathrm{J}_0\langle\alpha_1\rangle} + \frac{\mathrm{J}_1^2\langle\alpha_1\rangle}{\mathrm{J}_0^2\langle\alpha_1\rangle}\right)\right]$$

$$+\frac{1}{2}\left(\frac{\gamma k^2}{\sigma^2 s^2} - \mathrm{i}\frac{1}{x_1}\right)\left[(Z - x_1)^{-1/2}\frac{\mathrm{J}_1\langle\alpha_2\rangle}{\mathrm{J}_0\langle\alpha_2\rangle} + k\left(1 - \frac{\mathrm{J}_1\langle\alpha_2\rangle}{\alpha_2\mathrm{J}_0\langle\alpha_2\rangle} + \frac{\mathrm{J}_1^2\langle\alpha_2\rangle}{\mathrm{J}_0^2\langle\alpha_2\rangle}\right)\right]$$

$$-\frac{1}{2}\left(\frac{\gamma k^2}{\sigma^2 s^2} - \mathrm{i}\frac{1}{x_2}\right)\left[(Z - x_2)^{-1/2}\frac{\mathrm{J}_1\langle\alpha_3\rangle}{\mathrm{J}_0\langle\alpha_3\rangle} + k\left(1 - \frac{\mathrm{J}_1\langle\alpha_3\rangle}{\alpha_3\mathrm{J}_0\langle\alpha_3\rangle} + \frac{\mathrm{J}_1^2\langle\alpha_3\rangle}{\mathrm{J}_0^2\langle\alpha_3\rangle}\right)\right]. \tag{5.3}$$

Equation (5.1) can be solved numerically with respect to Z, if s and k are given. An effective procedure to solve the equation is the Newton–Raphson procedure:

$$Z_{n+1} = Z_n - \frac{F\langle Z_n\rangle}{F'\langle Z_n\rangle}. \tag{5.4}$$

As start value of the procedure that value of $Z = \Gamma^2$ is taken which can be obtained from the "low reduced frequency solution" for the desired value of the shear wave number, s. Convergence is obtained in a few steps.

For the Bessel functions involved (first kind, integer order and arbitrary complex arguments), an efficient algorithm was developed by Simons [37]. His algorithm is based on an integral representation for the Bessel functions and evaluation of the integral with the trapezoidal rule, which appears to have a remarkably small error.

Once the solution of Z has been obtained, the axial and radial velocity profiles can be calculated from

$$\frac{u}{B} = -\sqrt{Z}\left[\mathrm{i}\left(\frac{1}{x_1}-\frac{1}{x_2}\right)\mathrm{J}_0\langle\eta\alpha_1\rangle\,\mathrm{J}_0\langle\alpha_2\rangle\,\mathrm{J}_0\langle\alpha_3\rangle\right.$$

$$+\left(\frac{\gamma k^2}{\sigma^2 s^2}-\mathrm{i}\frac{1}{x_1}\right)\mathrm{J}_0\langle\alpha_1\rangle\,\mathrm{J}_0\langle\eta\alpha_2\rangle\,\mathrm{J}_0\langle\alpha_3\rangle$$

$$\left.-\left(\frac{\gamma k^2}{\sigma^2 s^2}-\mathrm{i}\frac{1}{x_2}\right)\mathrm{J}_0\langle\alpha_1\rangle\,\mathrm{J}_0\langle\alpha_2\rangle\,\mathrm{J}_0\langle\eta\alpha_3\rangle\right] \tag{5.5}$$

and

$$\frac{v}{B} = \mathrm{i}Z\left(Z-\mathrm{i}\frac{s^2}{k^2}\right)^{-1/2}\left(\frac{1}{x_1}-\frac{1}{x_2}\right)\mathrm{J}_1\langle\eta\alpha_1\rangle\,\mathrm{J}_0\langle\alpha_2\rangle\,\mathrm{J}_0\langle\alpha_3\rangle$$

$$+\left(\frac{\gamma k^2}{\sigma^2 s^2}-\mathrm{i}\frac{1}{x_1}\right)(Z-x_1)^{1/2}\,\mathrm{J}_0\langle\alpha_1\rangle\,\mathrm{J}_1\langle\eta\alpha_2\rangle\,\mathrm{J}_0\langle\alpha_3\rangle$$

$$-\left(\frac{\gamma k^2}{\sigma^2 s^2}-\mathrm{i}\frac{1}{x_2}\right)(Z-x_2)^{1/2}\,\mathrm{J}_0\langle\alpha_1\rangle\,\mathrm{J}_0\langle\alpha_2\rangle\,\mathrm{J}_1\langle\eta\alpha_3\rangle. \tag{5.6}$$

Expressions (5.5) and (5.6) were obtained from the corresponding expressions of reference [3], rewritten in terms of the basic parameters used here. B is a constant, which depends on the boundary conditions to be applied.

6. RESULTS OF THE NUMERICAL SOLUTION

6.1. DISCUSSION OF RESULTS

The results of the numerical solution of the full Kirchhoff solution for the propagation constant, Γ, are given in Tables 2 and 3 for a wide range of s and k values and are also presented in Figures 9(a) and (b). These figures show a large effect of the reduced frequency on both attenuation and phase shift, especially in the range of relatively low values of s. For small values of k the exact solution approaches the "low reduced frequency solution", discussed in the preceding sections. From Figures 10(a) and (b), presenting the attenuation and phase shift as functions of k for constant shear wave numbers, it becomes clear that for shear wave numbers below about 4 a strong dispersion occurs. Among other things, this is reflected in the phase velocity, W, which in certain regions becomes larger than the undisturbed velocity of sound, a_0.

The large influence of the reduced frequency for $s \lesssim 4$ also is reflected in the exact solutions for the velocity profiles, calculated from equations (5.5) and (5.6) and presented in Figure 11. For small values of k the shapes of the velocity profiles in Figure 11 are exactly the same as those obtained in the "low reduced frequency solution", shown in Figures 5(a) and (b). For higher values of k the influence is largest on the velocity profiles for $s \lesssim 4$.

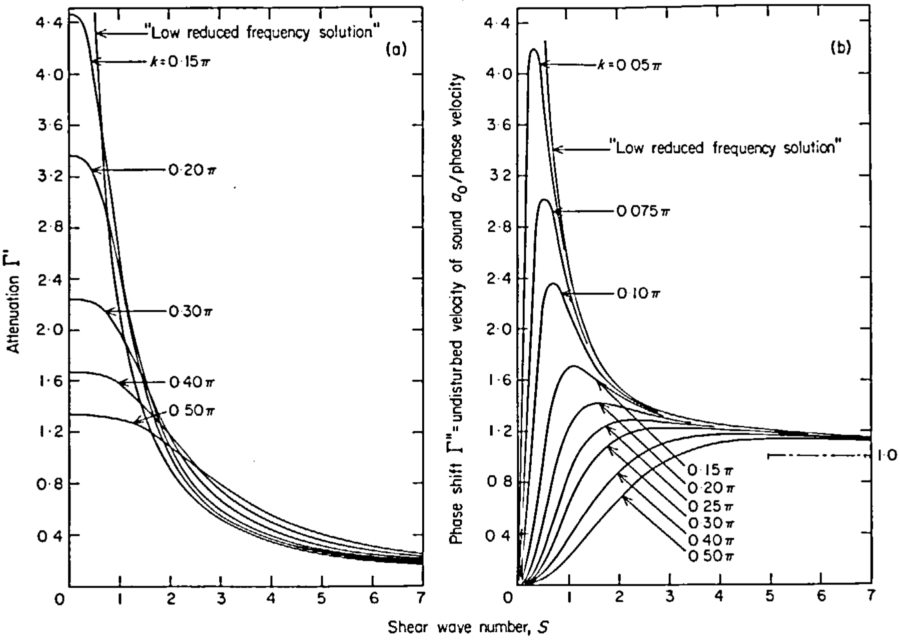

Figure 9. Exact solution of (a) Γ' and (b) Γ'' as functions of shear wave number and reduced frequency.

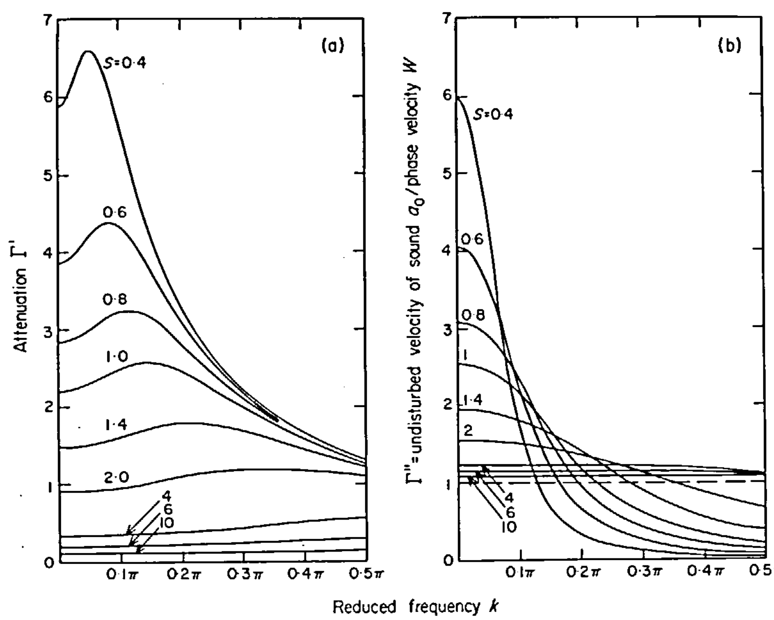

Figure 10. Exact solution of (a) Γ' and (b) Γ'' as functions of reduced frequency.

TABLE 2

Numerical solution of Γ' as a function of s and k.

s	Long wave solution, $k \ll 1$	$k = 0{\cdot}025\pi$	$0{\cdot}05\pi$	$0{\cdot}10\pi$	$0{\cdot}15\pi$	$0{\cdot}20\pi$	$0{\cdot}25\pi$	$0{\cdot}30\pi$	$0{\cdot}40\pi$	$0{\cdot}50\pi$
0·2	11·7988	13·2834	11·4410	6·6030	4·4585	3·3518	2·6833	2·2367	1·6778	1·3423
0·4	5·8497	6·1420	6.6042	5.7033	4·2663	3.2960	2·6623	2·2271	1·6750	1·3412
0·6	3·8456	3·9387	4·1666	4·3454	3·7840	3·1188	2·5885	2·1920	1·6640	1·3367
0·8	2·8288	2·8696	2·9807	3·2306	3·1602	2·8203	2·4442	2·1179	1·6395	1·3264
1·0	2·2083	2·2299	2·2911	2·4743	2·5671	2·4599	2·2401	2·0021	1·5981	1·3086
1·2	1·7875	1·8004	1·8378	1.9641	2·0830	2·0985	2·0034	1·8530	1·5384	1·2823
1·4	1·4824	1·4908	1·5156	1·6044	1·7106	1·7746	1·7613	1·6850	1·4620	1·2470
1·6	1·2512	1·2572	1·2745	1·3392	1·4264	1·5023	1·5336	1·5121	1·3731	1·2031
1·8	1·0709	1·0752	1·0880	1·1367	1·2072	1·2801	1·3314	1·3455	1·2770	1·1519
2·0	0·9272	0·9305	0·9404	0·9782	1·0354	1·1008	1·1581	1·1925	1·1785	1·0955
2·2	0·8112	0·8138	0·8215	0.8518	0·8988	0.9558	1·0124	1·0566	1·0816	1·0359
2·4	0·7163	0·7184	0·7247	0·7495	0·7886	0·8381	0·8910	0·9385	0·9893	0·9751
2·6	0·6380	0·6397	0·6470	0·6656	0·6987	0·7416	0·7899	0·8371	0·9033	0·9145
2·8	0·5728	0·5743	0·5787	0·5962	0·6245	0·6620	0·7055	0·7506	0·8248	0·8553

3·0	0·5181	0·5194	0·5232	0·5382	0·5627	0·5956	0·6347	0·6768	0·7539	0·7985
3·2	0·4719	0·4730	0·4763	0·4894	0·5108	0·5398	0·5749	0·6138	0·6905	0·7445
3·4	0·4326	0·4336	0·4365	0·4480	0·4668	0·4926	0·5241	0·5598	0·6342	0·6939
3·6	0·3990	0·3999	0·4024	0·4126	0·4293	0·4523	0·4807	0·5134	0·5843	0·6468
3·8	0·3701	0·3708	0·3731	0·3822	0·3971	0·4177	0·4434	0·4733	0·5401	0·6035
4·0	0·3450	0·3457	0·3477	0·3558	0·3693	0·3878	0·4111	0·4385	0·5011	0·5638
5	0·2583	0·2587	0·2600	0·2651	0·2735	0·2852	0·3001	0·3181	0·3619	0·4128
6	0·2073	0·2076	0·2084	0·2119	0·2176	0·2256	0·2359	0·2484	0·2795	0·3177
7	0·1732	0·1734	0·1741	0·1766	0·1807	0·1865	0·1940	0·2031	0·2261	0·2549
8	0·1487	0·1489	0·1493	0·1512	0·1544	0·1588	0·1644	0·1714	0·1889	0·2112
9	0·1302	0·1304	0·1307	0·1322	0·1347	0·1381	0·1426	0·1480	0·1618	0·1795
10	0·1158	0·1159	0·1162	0·1174	0·1194	0·1222	0·1257	0·1301	0·1412	0·1555
12	0·0948	0·0949	0·0951	0·0959	0·0973	0·0992	0·1016	0·1046	0·1123	0·1222
14	0·0803	0·0803	0·0805	0·0811	0·0821	0·0834	0·0852	0·0874	0·0930	0·1002
16	0·0696	0·0696	0·0697	0·0702	0·0709	0·0720	0·0734	0·0750	0·0793	0·0847
18	0·0614	0·0614	0·0615	0·0619	0·0625	0·0633	0·0644	0·0657	0·0690	0·0733
20	0·0549	0·0550	0·0550	0·0553	0·0558	0·0565	0·0573	0·0584	0·0611	0·0645
40	0·0268	0·0268	0·0268	0·0269	0·0270	0·0271	0·0273	0·0276	0·0283	0·0291
60	0·0177	0·0177	0·0177	0·0177	0·0178	0·0179	0·0179	0·0181	0·0184	0·0187
80	0·0132	0·0132	0·0132	0·0132	0·0133	0·0133	0·0134	0·0134	0·0136	0·0138
100	0·0105	0·0105	0·0105	0·0106	0·0106	0·0106	0·0106	0·0107	0·0108	0·0109

TABLE 3

Numerical solution of Γ'' as a function of s and k.

s	Long wave solution, $k \ll 1$	$k = 0{\cdot}025\pi$	$0{\cdot}05\pi$	$0{\cdot}10\pi$	$0{\cdot}15\pi$	$0{\cdot}20\pi$	$0{\cdot}25\pi$	$0{\cdot}30\pi$	$0{\cdot}40\pi$	$0{\cdot}50\pi$
0·2	11·8657	8·8573	3·4827	0·6245	0·1998	0·0899	0·0494	0·0310	0·0156	0·0097
0·4	5·9835	5·6422	4·4849	1·7901	0·7158	0·3434	0·1932	0·1223	0·0621	0·0385
0·6	4·0461	3·9501	3·6279	2·3516	1·2479	0·6814	0·4057	0·2636	0·1369	0·0857
0·8	3·0958	3·0567	2·9296	2·3570	1·5791	0·9938	0·6410	0·4349	0·2342	0·1488
1·0	2·5411	2·5217	2·4600	2·1779	1·7034	1·2185	0·8551	0·6119	0·3470	0·2249
1·2	2·1849	2·1740	2·1399	1·9857	1·7027	1·3483	1·0238	0·7742	0·4672	0·3107
1·4	1·9423	1·9357	1·9152	1·8236	1·6501	1·4045	1·1421	0·9100	0·5863	0·4026
1·6	1·7708	1·7666	1·7534	1·6955	1·5847	1·4179	1.2164	1·0158	0·6979	0·4964
1·8	1·6463	1·6434	1·6346	1·5962	1·5226	1·4078	1·2580	1·0933	0·7975	0·5881
2·0	1·5540	1·5520	1·5459	1·5194	1·4688	1·3884	1.2779	1·1470	0·8832	0·6746
2·2	1·4843	1·4829	1·4785	1·4596	1.4238	1.3662	1·2846	1·1822	0·9544	0·7536
2·4	1·4308	1·4298	1·4265	1·4128	1·3867	1·3446	1·2836	1·2041	1·0117	0·8239
2·6	1·3889	1·3882	1·3857	1·3754	1·3560	1·3246	1·2785	1·2166	1·0565	0·8849
2·8	1·3556	1·3550	1·3532	1·3452	1·3306	1·3067	1·2713	1·2229	1·0907	0·9368

3·0	1·3287	1·3282	1·3267	1·3206	1·3092	1·2907	1·2632	1·2249	1·1162	0·9802
3·2	1·3064	1·3061	1·3049	1·3001	1·2911	1·2765	1·2548	1·2243	1·1347	1·0159
3·4	1·2878	1·2875	1·2866	1·2827	1·2755	1·2639	1·2464	1·2219	1·1479	1·0448
3·6	1·2720	1·2718	1·2710	1·2679	1·2620	1·2525	1·2384	1·2184	1·1570	1·0679
3·8	1·2583	1·2581	1.2575	1.2549	1.2500	1·2423	1·2307	1·2142	1·1630	1·0862
4·0	1·2462	1·2461	1·2456	1·2434	1·2394	1·2329	1·2233	1·2095	1·1666	1·1004
5	1·2012	1·2011	1·2009	1·1999	1·1980	1·1951	1·1907	1·1846	1·1649	1·1325
6	1·1699	1·1699	1·1698	1·1692	1·1682	1·1666	1·1643	1·1611	1·1509	1·1337
7	1·1466	1·1466	1·1465	1·1462	1·1456	1·1447	1·1433	1·1415	1·1356	1·1257
8	1·1288	1·1288	1·1287	1·1285	1·1281	1·1275	1·1267	1·1255	1·1219	1·1157
9	1·1148	1·1148	1·1147	1·1146	1·1143	1·1139	1·1134	1·1126	1·1102	1·1061
10	1·1035	1·1035	1·1035	1·1034	1·1032	1·1029	1·1025	1·1019	1·1003	1·0975
12	1·0865	1·0865	1·0864	1·0864	1·0863	1·0861	1·0859	1·0856	1·0847	1·0833
14	1·0742	1·0742	1·0742	1·0742	1·0741	1·0740	1·0739	1·0737	1·0732	1·0723
16	1·0650	1·0650	1·0650	1·0650	1·0649	1·0649	1·0648	1·0647	1·0643	1·0638
18	1·0578	1·0578	1·0578	1·0578	1·0578	1·0577	1·0577	1·0576	1·0574	1·0570
20	1·0520	1·0520	1·0520	1·0520	1·0520	1·0520	1·0519	1·0519	1·0517	1·0515
40	1·0261	1·0261	1·0261	1·0261	1·0261	1·0261	1·0261	1.0260	1·0260	1·0260
60	1·0174	1·0174	1·0174	1·0174	1·0174	1·0174	1·0174	1·0174	1·0174	1·0174
80	1·0130	1·0130	1·0130	1·0130	1·0130	1·0130	1·0130	1·0130	1·0130	1·0130
100	1·0104	1·0104	1·0104	1·0104	1·0104	1·0104	1·0104	1·0104	1·0104	1·0104

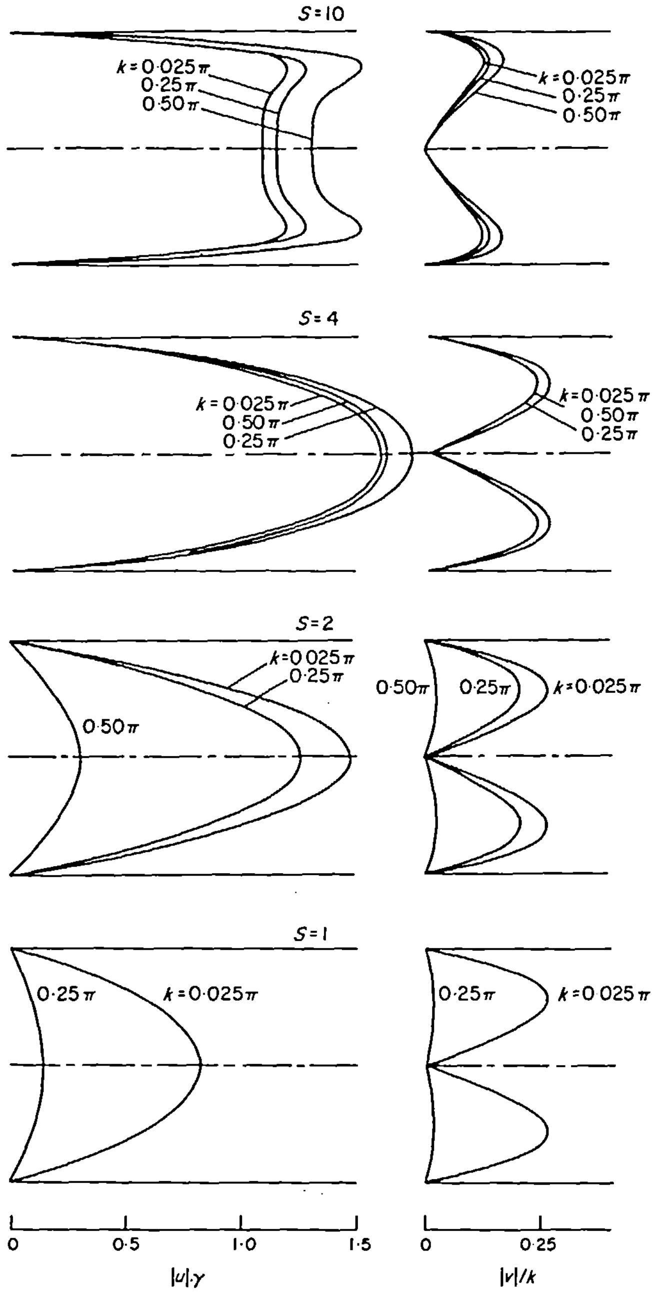

Figure 11. Velocity distributions in radial and axial directions.

6.2. COMPARISON WITH OTHER NUMERICAL SOLUTIONS

As already mentioned numerical solutions for the propagation of sound waves in cylindrical rigid tubes have been obtained by Tsao [10], Gerlach and Parker [11], Scarton and Rouleau [12], and by Shields, Lee and Wiley [13].

The solution of Tsao [10] is based on the same simplified basic equations as used for the "low reduced frequency solution" and therefore this solution, which makes use of a finite

TABLE 4

Comparison of numerical results derived from reference [13] *and results of present method*

s	$k \ll 1$ Low reduced frequency solution	$k = 0{\cdot}05\pi$ [13]	$k = 0{\cdot}05\pi$ Present	$k = 0{\cdot}15\pi$ [13]	$k = 0{\cdot}15\pi$ Present	$k = 0{\cdot}25\pi$ [13]	$k = 0{\cdot}25\pi$ Present	$k = 0{\cdot}35\pi$ [13]	$k = 0{\cdot}35\pi$ Present
(a) *Attenuation* (Γ')									
2·6069	0·6355	0·6478	0·6425	0·7017	0·6959	0·7930	0·7867	0·8792	0·8735
2·9794	0·5233	0·5323	0·5285	0·5728	0·5686	0·6461	0·6414	0·7289	0·7251
3·4759	0·4193	0·4253	0·4229	0·4545	0·4519	0·5098	0·5068	0·5813	0·5782
4·1711	0·3261	0·3300	0·3286	0·3499	0·3483	0·3886	0·3868	0·4426	0·4407
5·2139	0·2453	0·2476	0·2469	0·2600	0·2592	0·2846	0·2837	0·3205	0·3194
6·9519	0·1746	0·1759	0·1755	0·1827	0·1822	0·1962	0·1957	0·2164	0·2158
10·4278	0·1106	0·1111	0·1110	0·1141	0·1139	0·1199	0·1197	0·1287	0·1284
20·8556	0·0526	0·0527	0·0527	0·0534	0·0534	0·0548	0·0548	0·0569	0·0569
(b) *Phase shift* (Γ'')									
2·6069	1·3877	1·3842	1·3845	1·3540	1·3551	1·2752	1·2782	1·1365	1·1419
2·9794	1·3312	1·3285	1·3292	1·3100	1·3112	1·2615	1·2640	1·1701	1·1743
3·4759	1·2816	1.2796	1·2804	1·2691	1·2701	1·2415	1.2433	1·1870	1.1902
4·1711	1·2370	1·2359	1·2364	1·2303	1·2310	1·2160	1·2172	1·1875	1·1893
5·2139	1.1937	1·1932	1·1933	1·1906	1·1909	1·1841	1·1846	1·1712	1·1720
6·9519	1·1476	1·1475	1·1475	1·1465	1·1466	1·1441	1·1442	1·1394	1·1397
10·4278	1·0993	1·0993	1·0993	1·0990	1·0990	1·0983	1·0984	1·0972	1·0973
20·8556	1·0499	1·0499	1·0499	1·0499	1·0499	1·0498	1·0498	1·0497	1·0497

difference method, does not contribute new points of view for the case of sinusoidal perturbations.

In the solutions of references [11] and [12] the additional assumption is made that thermal effects are negligible, which means that the temperature perturbation no longer appears as an unknown. The way of solution followed in references [11] and [12] with the help of the method of eigenvalues will become much more complicated, if not impossible, in case thermal effects also have to be included.

To the author's knowledge, a solution for the fundamental mode of the complete set of equations has been published previously only by Shields *et al.* [13], who also started from the transcendental equation derived by Kirchhoff (equation (5.2) of the present paper, but not expressed in terms of the basic parameters). They solved the Kirchhoff equation iteratively, using as a first approximation of the propagation constant the value

$$\Gamma = \left[\alpha_{t_{\text{class}}} + \mathrm{i}\frac{\omega}{v_t} \right] \frac{a_0}{\omega}, \tag{6.1}$$

with

$$\alpha_{t_{\text{class}}} = \alpha_t + \alpha_{\text{class}}. \tag{6.2}$$

α_t and v_t are the absorption and phase velocity according to the "wide tube" solution of Kirchhoff, which in the present notation reads

$$\alpha_t = \frac{1}{\sqrt{2}} \left(\frac{\gamma - 1 + \sigma}{\sigma s} \right), \tag{6.3}$$

$$v_t = \left[1 + \frac{1}{\sqrt{2}} \left(\frac{\gamma - 1 + \sigma}{\sigma s} \right) \right]^{-1} a_0. \tag{6.4}$$

In reference [13], instead of equation (6.4), the formula

$$v_t = \left[1 - \frac{1}{\sqrt{2}} \left(\frac{\gamma - 1 + \sigma}{\sigma s} \right) \right] a_0, \tag{6.5}$$

has been used, which, however, is valid only for high values of s. Rayleigh [3] has given equation (6.4) as well as equation (6.5) without any comment. Clearly equation (6.4) is the proper one. The quantity α_{class} is the free air absorption, which in the notation here can be written as

$$\alpha_{\text{class}} = \frac{1}{2} \frac{k^2}{s^2} \left(\frac{4}{3} + \frac{\gamma - 1}{s^2} \right) \frac{\omega}{a_0}. \tag{6.6}$$

To solve equation (5.2) Shields *et al.* substitute the first value of $Z = \Gamma^2$, obtained from equation (6.1), into the arguments of all Bessel functions and furthermore into the first two terms of equation (5.2). From the third term of the latter equation a new value of Z is calculated and so on.

Shields *et al.* present tables of the fractional error in the Kirchhoff phase velocity

$$\Delta v' = \frac{v - v_t}{a_0 - v_t} \tag{6.7}$$

and the fractional error in the absorption

$$\Delta\alpha' = \frac{\alpha - \alpha_{t_{\text{class}}}}{\alpha_{t_{\text{class}}}}. \tag{6.8}$$

These quantities are given as functions of

$$\Delta v_t = 1 - \frac{v_t}{a_0} \tag{6.9}$$

and a reduced frequency parameter, which is equivalent to k/π.

With the help of expressions (6.1)–(6.9) the propagation constant can be converted into a function of the basic parameters k and s.†

Table 4 shows that after this conversion, a good agreement is obtained between the results of reference [13] and the present results. Moreover, it becomes clear that Shields *et al.* have covered the shear wave number range between 2·6069 and 20·8556. In this range of shear wave numbers and for already small reduced frequencies, the main conclusion reached in reference [13] is that "the exact value of the phase velocity differs about the same amount from the phase velocity in Kirchhoff's "wide tube" approximation as the latter velocity differs from the undisturbed velocity of sound".

The present results—compare Figures 2(b) and 9(b)—show that this conclusion is incorrect and that within the mentioned range of shear wave numbers the "wide tube" approximation is rather accurate. The wrong conclusion was drawn in reference [13] because equation (6.5) was used instead of equation (6.4), which accounts for the largest part of the observed differences.

7. CONCLUSIONS

The main conclusions to be drawn from the present investigation are as follows.

(1) The problem of sound propagation through cylindrical tubes is governed by the following parameters: the shear wave number, $s = R\sqrt{\rho_s \omega/\mu}$, the reduced frequency, $k = \omega R/a_0$, the square root of the Prandtl number, σ, and the ratio of specific heats, γ. For a given gas the shear wave number, s, and the reduced frequency, k, are the two main parameters.

(2) Rewriting the existing analytical solutions for the propagation constant, Γ, in terms of s and k offers a good means to compare the various solutions and to determine their range of validity.

(3) Most of the analytical solutions depend on the shear wave number, s, only and are covered completely by the "low reduced frequency solution", obtained for the first time by Zwikker and Kosten.

(4) An exact solution for the propagation constant, Γ, is presented, obtained by a numerical solution of the full Kirchhoff equation with the help of the Newton–Raphson procedure.

(5) The exact solution shows a larger influence of the reduced frequency, k, for relatively small values of the shear wave number ($s \lesssim 4$).

† For conversion of the results of reference [13], of course, the approximate expression (6.5) is used instead of the correct expression (6.4).

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APPENDIX A: BASIC EQUATIONS

The basic equations for the motion of a fluid in a circular tube (Figure 1) are
(a) the Navier–Stokes equations, for a constant value of the viscosity, μ,

$$\bar{\rho}\left[\frac{\partial\bar{u}}{\partial t}+\bar{v}\frac{\partial\bar{u}}{\partial r}+\bar{u}\frac{\partial\bar{u}}{\partial x}\right]=-\frac{\partial\bar{p}}{\partial x}+\mu\left\{\left[\frac{\partial^2\bar{u}}{\partial x^2}+\frac{\partial^2\bar{u}}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{u}}{\partial r}\right]+\frac{1}{3}\frac{\partial}{\partial x}\left[\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial r}+\frac{\bar{v}}{r}\right]\right\},\tag{A1}$$

$$\bar{\rho}\left[\frac{\partial\bar{v}}{\partial t}+\bar{v}\frac{\partial\bar{v}}{\partial r}+\bar{u}\frac{\partial\bar{v}}{\partial x}\right]=-\frac{\partial\bar{p}}{\partial r}+\mu\left\{\left[\frac{\partial^2\bar{v}}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{v}}{\partial r}-\frac{\bar{v}}{r^2}+\frac{\partial^2\bar{v}}{\partial x^2}\right]+\frac{1}{3}\frac{\partial}{\partial r}\left[\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial r}+\frac{\bar{v}}{r}\right]\right\},\tag{A2}$$

(b) the equation of continuity,

$$\frac{\partial\bar{\rho}}{\partial t}+\bar{u}\frac{\partial\bar{\rho}}{\partial x}+\bar{v}\frac{\partial\bar{\rho}}{\partial r}+\bar{\rho}\left[\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial r}+\frac{\bar{v}}{r}\right]=0,\tag{A3}$$

(c) the equation of state for an ideal gas,

$$\bar{p} = \bar{\rho} R_0 \bar{T}, \tag{A4}$$

(d) the energy equation,

$$\bar{\rho} C_p \left[\frac{\partial \bar{T}}{\partial t} + \bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial r}\right] = \lambda\left[\frac{\partial^2 \bar{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{T}}{\partial r} + \frac{\partial^2 \bar{T}}{\partial x^2}\right] + \frac{\partial \bar{p}}{\partial t} + \bar{u}\frac{\partial \bar{p}}{\partial x} + \bar{v}\frac{\partial \bar{p}}{\partial r} + \mu\phi, \tag{A5}$$

where ϕ is the dissipation function, representing the heat transfer due to internal friction,

$$\phi = 2\left[\left(\frac{\partial \bar{u}}{\partial x}\right)^2 + \left(\frac{\partial \bar{v}}{\partial r}\right)^2 + \left(\frac{\bar{v}}{r}\right)^2\right] + \left[\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial r}\right]^2 - \frac{2}{3}\left[\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial r} + \frac{\bar{v}}{r}\right]^2.$$

Upon assuming

$$\begin{aligned}
\bar{u} &= a_0 u\langle x,r\rangle e^{i\omega t},\\
\bar{v} &= a_0 v\langle x,r\rangle e^{i\omega t},\\
\bar{p} &= p_s(1 + p\langle x,r\rangle e^{i\omega t}) = \frac{\rho_s a_0^2}{\gamma}(1 + p\langle x,r\rangle e^{i\omega t}),\\
\bar{\rho} &= \rho_s(1 + \rho\langle x,r\rangle e^{i\omega t}),\\
\bar{T} &= T_s(1 + T\langle x,r\rangle e^{i\omega t}),
\end{aligned} \tag{A6}$$

with u, v, p, ρ and T being small sinusoidal perturbations, and by introduction of the dimensionless co-ordinates

$$\xi = \omega x/a_0, \qquad \eta = r/R, \tag{A7}$$

the equations (A1)-(A5) can be rewritten as

$$\begin{aligned}
iu = -\frac{1}{\gamma}\frac{\partial p}{\partial \xi} + \frac{\mu}{\rho_s\omega}\frac{1}{R^2}\Bigg\{&\left[\left(\frac{\omega R}{a_0}\right)^2\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial u}{\partial \eta}\right] +\\
&+ \frac{1}{3}\left(\frac{\omega R}{a_0}\right)\frac{\partial}{\partial \xi}\left[\left(\frac{\omega R}{a_0}\right)\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} + \frac{v}{\eta}\right]\Bigg\},
\end{aligned} \tag{A8}$$

$$\begin{aligned}
iv = -\frac{1}{\gamma}\left(\frac{a_0}{\omega R}\right)\frac{\partial p}{\partial \eta} + \frac{\mu}{\rho_s\omega}\frac{1}{R^2}\Bigg\{&\left[\frac{\partial^2 v}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial v}{\partial \eta} - \frac{v}{\eta^2} + \left(\frac{\omega R}{a_0}\right)^2\frac{\partial^2 v}{\partial \xi^2}\right] +\\
&+ \frac{1}{3}\frac{\partial}{\partial \eta}\left[\left(\frac{\omega R}{a_0}\right)\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} + \frac{v}{\eta}\right]\Bigg\}
\end{aligned} \tag{A9}$$

$$i\rho = -\left(\frac{a_0}{\omega R}\right)\left[\left(\frac{\omega R}{a_0}\right)\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} + \frac{v}{\eta}\right], \tag{A10}$$

$$p = \rho + T, \tag{A11}$$

$$\mathrm{i}T = \left(\frac{\lambda}{\mu C_p}\right)\left(\frac{\mu}{\rho_s \omega}\,\frac{1}{R^2}\right)\left[\frac{\partial^2 T}{\partial \eta^2} + \frac{1}{\eta}\,\frac{\partial T}{\partial \eta} + \left(\frac{\omega R}{a_0}\right)^2 \frac{\partial^2 T}{\partial \xi^2}\right] + \mathrm{i}\,\frac{\gamma - 1}{\gamma}\,p. \tag{A12}$$

In these equations the four parameters can be distinguished: namely,

$s = R\sqrt{\rho_s \omega / \mu}$, the shear wave number,

$\sigma = \sqrt{\mu C_p / \lambda}$, the square root of the Prandtl number,

$k = \omega R / a_0$, the reduced frequency, being proportional to the ratio of tube radius to wave length,

$\gamma = C_p / C_v$, the ratio of specific heats.

In terms of these parameters the basic equations become equations (2.1)–(2.5) of section 2.

APPENDIX B: DERIVATION OF THE "LOW REDUCED FREQUENCY SOLUTION"

When the internal tube radius is small in comparison with the wave length and the radial velocity component, v, is small with respect to the axial velocity, u (i.e., $\omega R/a_0 \ll 1$ and $v/u \ll 1$), the basic equations (2.1)–(2.5) can be reduced to

$$\mathrm{i}u = -\frac{1}{\gamma}\,\frac{\partial p}{\partial \xi} + \frac{1}{s^2}\left[\frac{\partial^2 u}{\partial \eta^2} + \frac{1}{\eta}\,\frac{\partial u}{\partial \eta}\right], \tag{B1}$$

$$0 = -\frac{1}{\gamma}\,\frac{\partial p}{\partial \eta}, \tag{B2}$$

$$\mathrm{i}k\rho = -\left[k\,\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} + \frac{v}{\eta}\right], \tag{B3}$$

$$p = \rho + T, \tag{B4}$$

$$\mathrm{i}T = \mathrm{i}\,\frac{\gamma - 1}{\gamma}\,p + \frac{1}{\sigma^2 s^2}\left[\frac{\partial^2 T}{\partial \eta^2} + \frac{1}{\eta}\,\frac{\partial T}{\partial \eta}\right], \tag{B5}$$

with the boundary conditions

$$\text{at } \eta = 1, \quad u = 0, \quad v = 0, \quad T = 0,$$

$$\text{at } \eta = 0, \quad v = 0. \tag{B6}$$

From equation (B2) it follows immediately that the amplitude of the pressure perturbation, p, depends only on the axial co-ordinate, ξ.

Upon putting $u=f\langle\xi\rangle h\langle z\rangle$, with $z=\mathrm{i}^{3/2}\eta s$, equation (B1) can be rewritten as

$$\frac{\partial^2 h}{\partial z^2}+\frac{1}{z}\frac{\partial h}{\partial z}+h=\frac{\mathrm{i}}{\gamma f\langle\xi\rangle}\frac{\mathrm{d}p}{\mathrm{d}\xi}, \tag{B7}$$

with the solution

$$h\langle z\rangle=C_1\,\mathrm{J}_0\langle z\rangle+C_2\,\mathrm{Y}_0\langle z\rangle+\frac{\mathrm{i}}{\gamma f\langle\xi\rangle}\frac{\mathrm{d}p}{\mathrm{d}\xi}. \tag{B8}$$

To maintain a finite value of u for $\eta=0$, the constant C_2 has to be zero.

From the condition $u=0$ for $\eta=1$ it follows that

$$f\langle\xi\rangle=-\frac{\mathrm{i}}{\gamma}\frac{1}{C_1\,\mathrm{J}_0\langle\mathrm{i}^{3/2}s\rangle}\frac{\mathrm{d}p}{\mathrm{d}\xi}, \tag{B9}$$

wherefrom

$$u=f\langle\xi\rangle h\langle z\rangle=\frac{\mathrm{i}}{\gamma}\frac{\mathrm{d}p}{\mathrm{d}\xi}\left[1-\frac{\mathrm{J}_0\langle\mathrm{i}^{3/2}\eta s\rangle}{\mathrm{J}_0\langle\mathrm{i}^{3/2}s\rangle}\right]. \tag{B10}$$

Equation (B5) for the temperature perturbation, T, can be solved in a very similar way as the equation for the axial velocity. The solution that fulfils the requirement that T remains finite for $\eta=0$ and vanishes for $\eta=1$ yields is

$$T=\frac{\gamma-1}{\gamma}\,p\left[1-\frac{\mathrm{J}_0\langle\mathrm{i}^{3/2}\sigma\eta s\rangle}{\mathrm{J}_0\langle\mathrm{i}^{3/2}\sigma s\rangle}\right]. \tag{B11}$$

Substitution of equations (B10) and (B11) into equation (B4) gives

$$\rho=p-T=p\left[1-\frac{\gamma-1}{\gamma}\left\{1-\frac{\mathrm{J}_0\langle\mathrm{i}^{3/2}\sigma\eta s\rangle}{\mathrm{J}_0\langle\mathrm{i}^{3/2}\sigma s\rangle}\right\}\right]. \tag{B12}$$

Finally the equation of continuity (B3) has to be satisfied; thus

$$\frac{1}{\eta}\frac{\partial(v.\eta)}{\partial\eta}=k\left[\mathrm{i}\rho+\frac{\partial u}{\partial x}\right]. \tag{B13}$$

By using equations (B10) and (B12) this can be expressed as

$$\frac{1}{\eta}\frac{\partial(v.\eta)}{\partial\eta}=\mathrm{i}k\left[p\left\{1-\frac{\gamma-1}{\gamma}\left(1-\frac{\mathrm{J}_0\langle\mathrm{i}^{3/2}\sigma\eta s\rangle}{\mathrm{J}_0\langle\mathrm{i}^{3/2}\sigma s\rangle}\right)\right\}+\frac{1}{\gamma}\frac{\mathrm{d}^2p}{\mathrm{d}\xi^2}\left(1-\frac{\mathrm{J}_0\langle\mathrm{i}^{3/2}\eta s\rangle}{\mathrm{J}_0\langle\mathrm{i}^{3/2}s\rangle}\right)\right]. \tag{B14}$$

After integration with respect to η one has

$$v\eta = \mathrm{i}k\left[p\left\{\tfrac{1}{2}\eta^2 - \frac{\gamma-1}{\gamma}\left(\tfrac{1}{2}\eta^2 - \frac{\eta}{\mathrm{i}^{3/2}\sigma s}\,\frac{\mathrm{J}_1\langle \mathrm{i}^{3/2}\sigma\eta s\rangle}{\mathrm{J}_0\langle \mathrm{i}^{3/2}\sigma s\rangle}\right)\right\} + \right.$$

$$\left. + \frac{1}{\gamma}\,\frac{\mathrm{d}^2 p}{\mathrm{d}\xi^2}\left(\tfrac{1}{2}\eta^2 - \frac{\eta}{\mathrm{i}^{3/2}s}\,\frac{\mathrm{J}_1\langle \mathrm{i}^{3/2}\eta s\rangle}{\mathrm{J}_0\langle \mathrm{i}^{3/2}s\rangle}\right) + F\langle\xi\rangle\right]. \tag{B15}$$

From the boundary condition $v = 0$ at $\eta = 1$ it follows that

$$-F\langle\xi\rangle = \tfrac{1}{2}p\left\{1 + \frac{\gamma-1}{\gamma}\,\frac{\mathrm{J}_2\langle \mathrm{i}^{3/2}\sigma s\rangle}{\mathrm{J}_0\langle \mathrm{i}^{3/2}\sigma s\rangle}\right\} - \frac{1}{2\gamma}\,\frac{\mathrm{d}^2 p}{\mathrm{d}\xi^2}\,\frac{\mathrm{J}_2\langle \mathrm{i}^{3/2}s\rangle}{\mathrm{J}_0\langle \mathrm{i}^{3/2}s\rangle}. \tag{B16}$$

Due to the axial symmetry, $\lim_{\eta\to 0} v = 0$. This requirement is fulfilled if $F\langle\xi\rangle = 0$, or

$$p\left\{1 + \frac{\gamma-1}{\gamma}\,\frac{\mathrm{J}_2\langle \mathrm{i}^{3/2}\sigma s\rangle}{\mathrm{J}_0\langle \mathrm{i}^{3/2}\sigma s\rangle}\right\} - \frac{1}{\gamma}\,\frac{\mathrm{d}^2 p}{\mathrm{d}\xi^2}\,\frac{\mathrm{J}_2\langle \mathrm{i}^{3/2}s\rangle}{\mathrm{J}_0\langle \mathrm{i}^{3/2}s\rangle} = 0. \tag{B17}$$

From the equation (B17) one can solve for p:

$$p = A\,\mathrm{e}^{\Gamma\xi} + B\,\mathrm{e}^{-\Gamma\xi}, \tag{B18a}$$

with

$$\Gamma = \sqrt{\frac{\mathrm{J}_0\langle \mathrm{i}^{3/2}s\rangle}{\mathrm{J}_2\langle \mathrm{i}^{3/2}s\rangle}}\;\sqrt{\frac{\gamma}{n}} \tag{B18b}$$

and

$$n = \left[1 + \frac{\gamma-1}{\gamma}\,\frac{\mathrm{J}_2\langle \mathrm{i}^{3/2}\sigma s\rangle}{\mathrm{J}_0\langle \mathrm{i}^{3/2}\sigma s\rangle}\right]^{-1}. \tag{B18c}$$

The solution for the other acoustic variables becomes

$$u = \frac{\mathrm{i}\Gamma}{\gamma}\left[1 - \frac{\mathrm{J}_0\langle \mathrm{i}^{3/2}\eta s\rangle}{\mathrm{J}_0\langle \mathrm{i}^{3/2}s\rangle}\right][A\,\mathrm{e}^{\Gamma\xi} - B\,\mathrm{e}^{-\Gamma\xi}], \tag{B19}$$

$$v = \mathrm{i}k\left[\tfrac{1}{2}\eta\left\{1 + \frac{\mathrm{J}_0\langle \mathrm{i}^{3/2}s\rangle}{\mathrm{J}_2\langle \mathrm{i}^{3/2}s\rangle}\,\frac{\gamma}{n}\right\} + \frac{\gamma-1}{\mathrm{i}^{3/2}\sigma s}\,\frac{\mathrm{J}_1\langle \mathrm{i}^{3/2}\sigma\eta s\rangle}{\mathrm{J}_0\langle \mathrm{i}^{3/2}\sigma s\rangle}\right.$$

$$\left. - \frac{\gamma}{\mathrm{i}^{3/2}\eta s}\,\frac{\mathrm{J}_1\langle \mathrm{i}^{3/2}\eta s\rangle}{\mathrm{J}_2\langle \mathrm{i}^{3/2}s\rangle}\right][A\,\mathrm{e}^{\Gamma\xi} + B\,\mathrm{e}^{-\Gamma\xi}], \tag{B20}$$

$$\rho = \left[1 - \frac{\gamma-1}{\gamma}\left\{1 - \frac{\mathrm{J}_0\langle \mathrm{i}^{3/2}\sigma\eta s\rangle}{\mathrm{J}_0\langle \mathrm{i}^{3/2}\sigma s\rangle}\right\}\right][A\,\mathrm{e}^{\Gamma\xi} + B\,\mathrm{e}^{-\Gamma\xi}], \tag{B21}$$

$$T = \frac{\gamma-1}{\gamma}\left[1 - \frac{\mathrm{J}_0\langle \mathrm{i}^{3/2}\sigma\eta s\rangle}{\mathrm{J}_0\langle \mathrm{i}^{3/2}\sigma s\rangle}\right][A\,\mathrm{e}^{\Gamma\xi} + B\,\mathrm{e}^{-\Gamma\xi}]. \tag{B22}$$

The constants A and B can be determined by specifying additional boundary conditions at both ends of the tube.

From the solution for the radial velocity, v, it can be verified that the condition $v/u \ll 1$ is fulfilled if $k \ll 1$ and $k/s \ll 1$.

APPENDIX C: BEHAVIOUR OF THE "LOW REDUCED FREQUENCY SOLUTION" AT VERY LARGE AND AT VERY LOW VALUES OF THE SHEAR WAVE NUMBER

The solution for the pressure perturbation, p, according to the "low reduced frequency solution", is given in Appendix B, equations (B18a)–(B18c). For large values of the shear wave number, s, the solution can be approximated by the use of the following expansions of the Bessel functions involved [38]:

$$J_0\langle i^{3/2} x\rangle = \mathrm{ber}\langle x\rangle + i\,\mathrm{bei}\langle x\rangle,$$

with

$$\mathrm{ber}\langle x\rangle = \frac{e^{x/\sqrt{2}}}{x\sqrt{2\pi}}\,[L_0\cos\theta - M_0\sin\theta], \tag{C1a}$$

$$\mathrm{bei}\langle x\rangle = \frac{e^{x/\sqrt{2}}}{x\sqrt{2\pi}}\,[M_0\cos\theta + L_0\sin\theta] \tag{C1b}$$

and

$$\theta = \frac{x}{\sqrt{2}} - \frac{\pi}{8},$$

$$L_0 = 1 + \frac{1}{1!8x}\cos\frac{\pi}{4} + \cdots,$$

$$M_0 = -\frac{1^2}{1!8x}\sin\frac{\pi}{4} + \cdots; \tag{C1c}$$

$$J_2\langle i^{3/2} x\rangle = \mathrm{ber}_2\langle x\rangle + i\,\mathrm{bei}_2\langle x\rangle,$$

with

$$\mathrm{ber}_2\langle x\rangle = \frac{e^{x/\sqrt{2}}}{x\sqrt{2\pi}}\,[-L_2\cos\theta + M_2\sin\theta], \tag{C2a}$$

$$\mathrm{bei}_2\langle x\rangle = \frac{e^{x/\sqrt{2}}}{x\sqrt{2\pi}}\,[-M_2\cos\theta - L_2\sin\theta] \tag{C2b}$$

and

$$L_2 = 1 - \frac{4\cdot 2^2 - 1}{1!8x}\cos\frac{\pi}{4} + \cdots,$$

$$M_2 = \frac{4\cdot 2^2 - 1}{1!8x}\sin\frac{\pi}{4} + \cdots. \tag{C2c}$$

With these expansions it can be shown that

$$\frac{J_0\langle i^{3/2}x\rangle}{J_2\langle i^{3/2}x\rangle} \approx -1 - \frac{\sqrt{2}}{x} + i\frac{\sqrt{2}}{x}. \tag{C3}$$

By using expression (C3), equation (B18c) for n can be written as

$$n = \gamma - \frac{\gamma-1}{\gamma}\frac{\sqrt{2}}{\sigma s} + i\frac{\gamma-1}{\gamma}\frac{\sqrt{2}}{\sigma s} + \cdots. \tag{C4}$$

From expression (C4) it follows directly that $\lim_{s\to\infty} n = \gamma$. (C5)

The expression for Γ can be approximated by

$$\Gamma \approx \left[i + \frac{(1+i)}{\sqrt{2}}\left(\frac{\gamma - 1 + \sigma}{\sigma s}\right)\right], \tag{C6}$$

which equals the "wide" tube solution of Kirchhoff.

For very large values of s,

$$\lim_{s\to\infty}\Gamma = i, \tag{C7}$$

being the solution for a plane wave, without viscosity and heat effects. For small values of the shear wave number the following approximations of the Bessel functions are valid:

$$J_0\langle i^{3/2}x\rangle \approx 1 - \frac{x^4}{2^6} + \cdots + i\left(\frac{x^2}{x^4} - \frac{x^6}{2^6.(3!)^2} + \cdots\right), \tag{C8}$$

$$J_2\langle i^{3/2}x\rangle \approx \frac{x^4}{2^4.3!} + \cdots - i\left(\frac{x^2}{2^3} + \cdots\right). \tag{C9}$$

The ratio then can be approximated by

$$\frac{J_2\langle i^{3/2}x\rangle}{J_0\langle i^{3/2}x\rangle} \approx -i\frac{x^2}{8}. \tag{C10}$$

Using expression (C10) in equation (B18c) gives

$$n \approx 1 + i\frac{\gamma-1}{\gamma}\frac{(\sigma s)^2}{8} + \cdots, \tag{C11}$$

wherefrom

$$\lim_{s\to 0} n = 1. \tag{C12}$$

The approximated expression for the propagation constant becomes

$$\Gamma \approx \left(-\frac{\gamma}{i s^2/8}\right)^{1/2} = 2(1+i)\frac{\sqrt{\gamma}}{s}. \tag{C13}$$

This result is identical to the solution of Rayleigh.

The velocity in the axial direction has been derived as (Appendix B, equation (B10))

$$\bar{u} = a_0 u = \frac{ia_0}{\gamma}\frac{dp}{d\xi}\left[1 - \frac{J_0\langle i^{3/2}\eta s\rangle}{J_0\langle i^{3/2}s\rangle}\right] = \frac{1}{i\omega\rho_s}\frac{d\bar{p}}{dx}\left[1 - \frac{J_0\langle i^{3/2}\eta s\rangle}{J_0\langle i^{3/2}s\rangle}\right]. \tag{C14}$$

The mean velocity is equal to

$$\bar{u}_m = \frac{1}{\pi R^2}\int_0^1 u 2\pi\eta\, d\eta. \tag{C15}$$

Substitution of equation (C14) into equation (C15) leads to

$$\bar{u}_m = \frac{1}{i\omega\rho_s}\frac{J_2\langle i^{3/2}s\rangle}{J_0\langle i^{3/2}s\rangle}\frac{d\bar{p}}{dx}. \tag{C16}$$

Introduction of the approximation (C10) gives

$$\bar{u}_m = -\frac{1}{i\omega\rho s}\frac{is^2}{8}\frac{d\bar{p}}{dx}$$

or

$$\bar{u}_m = \frac{D^2}{32}\frac{1}{\mu}\frac{d\bar{p}}{dx}. \tag{C17}$$

This is the well known expression for the Poiseuille flow.

APPENDIX D: LIST OF SYMBOLS

$a_0 = \sqrt{\gamma p_s/\rho_s}$	undisturbed velocity of sound
C_p	specific heat at constant pressure
C_v	specific heat at constant volume
f	frequency (Hz)
$i = \sqrt{-1}$	imaginary unit
J_n	Bessel function of first kind of order n
$k = \omega R/a_0$	reduced frequency
n	kind of polytropic constant, given in equation (4.1b)
$\bar{p} = p_s(1 + pe^{i\omega t})$	pressure
p_s	mean pressure
p	amplitude of pressure perturbation
r	co-ordinate in radial direction
R	internal tube radius
R_0	gas constant
$s = R\sqrt{\rho_s\omega/\mu}$	shear wave number
t	time
$\bar{T} = T_s(1 + Te^{i\omega t})$	temperature
T_s	mean temperature
T	amplitude of temperature perturbation
U	group velocity
$\bar{u} = ua_0e^{i\omega t}$	velocity component in axial direction
u	amplitude of velocity perturbation in axial direction
$\bar{v} = va_0e^{i\omega t}$	velocity component in radial direction

v	amplitude of velocity perturbation in radial direction
$W = a_0/\Gamma''$	phase velocity
x	co-ordinate in axial direction
Y_n	Neumann function of first kind of order n
$\Gamma = \Gamma' + i\Gamma''$	propagation constant
Γ'	attenuation per unit distance in ξ direction
Γ''	phase shift per unit distance in ξ direction
$\gamma = C_p/C_v$	ratio of specific heats
$\eta = r/R$	dimensionless co-ordinate in radial direction
$\kappa = \omega/W$	wave number
λ	thermal conductivity
μ	absolute fluid viscosity
$\xi = \omega x/a_0$	dimensionless co-ordinate in axial direction
$\bar{\rho} = \rho_s(1 + \rho e^{i\omega t})$	density
ρ_s	mean density
ρ	amplitude of density perturbation
$\sigma^2 = \mu C_p/\lambda$	Prandtl number
$\omega = 2\pi f$	frequency (rad/s)

Note: in the numerical examples the value of $\gamma = 1{\cdot}4$ and $\sigma^2 = 0{\cdot}71$.