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Multiconductor Transmission-line Theory in the TEM Approximation

Abstract: Starting with Maxwell's equations, the transmission line equations are derived for a system consisting of an arbitrary number of conductors. The derivation is rigorous for long lossless conductors embedded in a uniform perfect dielectric. The presentation is essentially tutorial, most of the results being well known, at least for two- and three-conductor systems. The novelty lies in the point of view adopted in obtaining a systematic generalization to the case of an arbitrary number of conductors. Explicit expressions are obtained for the electric and magnetic fields in the dielectric surrounding the conductors, and a rigorous formulation is given for the problem of calculating the coefficients of capacitance and inductance.

Introduction

Interest in the theory of mutually coupled, multiple conductor, parallel transmission lines extends over the past forty years. With the exception of early works by Levin [1] and Pipes [2,3], and more recent investigations by Kuznetsov and Stratonovich [4], by Amemiya [5], and by Matsumoto [10], most of the published work has focused on the theory of two parallel, mutually coupled transmission lines with considerable attention given to the subject of directional coupling [6,9]. Papers treating arbitrary numbers of conductors fall into two groups. Papers of the first group [2,5,10] take the generalized transmission line equations as a starting point and assume the inductance and capacitance matrices given. Papers of the second group [1,3,4] take Maxwell's equations for the electromagnetic field as a starting point. This paper belongs to the second group.

The purpose of the present paper is essentially tutorial. Most of the results arrived at are well-known, at least for two- and three-conductor systems, and can be found in the papers cited above as well as in the widely known texts of King [11] and Collin [12]. It is hoped, however, that some novelty will be found in the point of view adopted here to obtain a systematic generalization of familiar results to systems of arbitrary numbers of conductors.

There are two motivations for such an investigation. The first of these has to do with the generalization of the familiar two-conductor result, $LC = 1/v^2$, to multiconductor systems. In the multiconductor case L and C are matrices of inductances and capacitances per unit length, respectively; v is the velocity of propagation, and the unity element is a unit matrix of the same dimensions as L and C. This generalization has been applied explicitly to multiconductor TEM propagations by several authors [5,10,13] and is implicit in the work of several others [6–9]. In the usual justification, stated quite nicely by Amemiya [5], it is shown that the solution of the generalized transmission line equations for an $N+1$ conductor system consists of the superposition of N independent modes whose propagation velocities are the square roots of the reciprocals of the eigenvalues of the product of the L and C matrices. Then an ad hoc assumption is made that for TEM systems, all these propagation velocities must be equal. It follows that the product of the L and C matrices must be $1/v^2$ times a unit matrix. It will be shown that to assume constant propagation velocities is unnecessary; it can instead be derived rigorously from Maxwell's equations. Similarly, it will be shown to be a strict consequence of Maxwell's equations that the product of the L and C matrices is $1/v^2$ times a unit matrix for TEM systems. The main results are contained in Eqs. (8) and (69)–(71).

The second motivation for the investigation is to find an equation for the electric and magnetic fields produced by a given distribution of currents and voltages on a multiconductor TEM transmission line system. The result is embodied in Eqs. (5) and (6). A typical application of this equation would be to estimate the stray magnetic field at an unselected bit position in a thin magnetic film memory when several adjacent lines are pulsed. Granted that a memory plane is not a TEM system, the TEM solution nevertheless serves as a useful starting point for the computation.

The subsequent discussion is limited to propagation on lossless systems in a uniform dielectric; in short, to TEM systems. By making such a restriction it is possible to derive results which apply to a wide variety of conductor geometries. To include losses or multilayer dielectrics would soon force the analysis to a discussion of a single system with fixed geometry. By way of justification it might be added that practical analysis of real systems is based on TEM assumptions. Also a thorough understanding of the properties of lossless systems certainly is a prerequisite for the understanding of the more complicated systems encountered in practice.

TEM solutions of Maxwell's equations

The specific configuration to be considered in this paper consists of $N+1$ parallel, infinitely long, lossless conductors. The cross sections of these conductors may vary from conductor to conductor, but the cross section of any given conductor must be uniform over its entire length. It is supposed that the space surrounding the conductors is filled with a uniform perfect insulator of constant permittivity ϵ and permeability μ. The system is to be operated so that the sum of the currents in the $N+1$ conductors is zero at all times. In this case, any one of the conductors can be selected as the common reference conductor for the system.

In particular, let the conductor chosen as the reference conductor be called the $N+1$st conductor and the remaining conductors be numbered from 1 to N, in any order. Throughout this paper, the common reference conductor will be referred to as the $N+1$st conductor. All voltages will be measured with respect to this conductor.

Since the space outside the conductors is assumed to be a perfect insulator, there can be no flow of charge into or out of this region. If this space is uncharged initially, it remains uncharged. Hence, it can be assumed that both the conduction current density and charge density vanish outside the conductors. Thus, in the space outside the conductors, the electric and magnetic fields satisfy Maxwell's equations:

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}; \tag{1}$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}; \tag{2}$$

$$\nabla \cdot \mathbf{E} = 0; \tag{3}$$

$$\nabla \cdot \mathbf{H} = 0. \tag{4}$$

The boundary conditions at the surfaces of the conductors are that the tangential component of **E** and the normal component of $\mathbf{B} = \mu\mathbf{H}$ must be continuous across the conductor surfaces. Since the conductors are assumed lossless, both **E** and **H** vanish in their interiors. Thus, the boundary conditions reduce to the requirement that the tangential component of **E** and the normal component of **H** vanish at all conductor surfaces.

The $N+1$ conductor configuration just described is capable of sustaining a TEM field, that is, a field in which the **E** and **H** vectors lie in planes perpendicular to the direction of propagation. It is the TEM solutions of Maxwell's equations that are of particular interest in transmission line theory. If one chooses a rectangular coordinate system, with the z axis parallel to the conductors and the x and y axes perpendicular to the conductors, and looks for solutions of Maxwell's equations representing wave propagation in a direction parallel to the conductors, then the requirement for a TEM solution is that the z component of **E** and **H** vanish everywhere in the space outside the conductors. The primary objective of this paper is to demonstrate that, for the $N+1$ conductor transmission line system described, such a TEM solution of Maxwell's equations exists in the form

$$\mathbf{E} = -\sum_{i=1}^{N} V_i(z,t)\nabla\phi_i(x,y); \tag{5}$$

$$\mathbf{H} = -\sum_{i=1}^{N} \left(\frac{1}{\mu}\sum_{j=1}^{N} L_{ij} I_j(z,t)\right)\mathbf{k} \times \nabla\phi_i(x,y); \tag{6}$$

where **k** is a unit vector parallel to and directed along the positive z axis, the z axis being parallel to the conductors (Fig. 1).

The N functions $V_i(z,t)$ appearing in Eq. (5) are assumed to be linearly independent, at least once differentiable, functions of z and t as are the N functions $I_i(z,t)$ appearing in Eq. (6). Both the $V_i(z,t)$ and $I_i(z,t)$ are assumed to be independent of the variables x and y. The N functions $\phi_i(x,y)$ appearing in Eqs. (5) and (6) are assumed to be at least twice differentiable functions of x and y but independent of the variables z and t. The N^2 quantities L_{ij} in Eq. (6) are constants. The following five *requirements* are to be placed on the fields given by Eqs. (5) and (6).

1. The tangential component of **E** and normal component of **H** must vanish at all conductor surfaces.
2. The function $V_i(z,t)$ of Eq. (5) is the voltage of the ith conductor, with respect to the $N+1$st (reference) conductor.
3. The function $I_i(z,t)$ of Eq. (6) is the current carried by the ith conductor.
4. The sum of the currents carried by all $N+1$ conductors must vanish for all times t and positions z, and

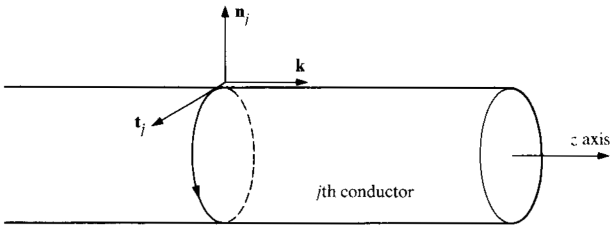

Figure 1 Orientation of unit vectors.

E and H must vanish at an infinite distance from the conductors.

5. The electric field **E** given by Eq. (5) and the magnetic field **H** given by Eq. (6) must satisfy Maxwell's equations (1) through (4).

Necessary and sufficient conditions will be obtained for the fields given by Eqs. (5) and (6) to satisfy these five requirements. For each of the five requirements there are five such necessary and sufficient *conditions*:

1. The functions $\phi_i(x,y)$ assume constant values on all conductor surfaces.
2. If the constant value of $\phi_i(x,y)$ on the surface of the jth conductor is denoted by $\phi_i(j)$ for $j = 1, 2, \cdots, N+1$, then

$$\phi_i(j) = \delta_{ij} + \phi_i(N+1) \tag{7}$$

for $i = 1, 2, \cdots, N$. Here, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

3. The constants L_{ij} of Eq. (6) must be chosen so that

$$\sum_{j=1}^{N} C_{ij} L_{jk} = \epsilon\mu\delta_{ik}\,, \tag{8}$$

where the constants C_{ij} are defined by

$$C_{ij} = -\epsilon \oint_i \mathbf{n}_i \cdot \nabla\phi_j \, dl_i\,, \tag{9}$$

where $\mathbf{n}_i$ is a unit vector normal to the surface and directed out of the ith conductor. The path of integration lies in a plane of constant z; encircles the ith conductor once, in a counterclockwise sense with respect to the positive z axis; and lies arbitrarily close to the conductor surface. The quantity dl_i is an element of arc length along this path.

4. For $i = 1, 2, \cdots, N$

$$\mathbf{r} \cdot \nabla\phi_i \to 0 \text{ as } |\mathbf{r}| \to \infty\,, \tag{10}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, where $\mathbf{i}$ is a unit vector parallel to and directed along the positive x axis, and $\mathbf{j}$ is a unit vector parallel and directed along the positive y axis.

5. For $i = 1, 2, \cdots, N$, the functions $\phi_i(x,y)$ satisfy the two-dimensional Laplace equation

$$\nabla^2\phi_i = \frac{\partial^2\phi_i}{\partial x^2} + \frac{\partial^2\phi_i}{\partial y^2} = 0\,, \tag{11}$$

and the functions $V_i(z,t)$ and $I_i(z,t)$ satisfy the transmission line equations

$$-\frac{\partial V_i}{\partial z} = \sum_{j=1}^{N} L_{ij}\frac{\partial I_j}{\partial t}\,; \tag{12}$$

$$-\frac{\partial I_i}{\partial z} = \sum_{j=1}^{N} C_{ij}\frac{\partial V_j}{\partial t}\,. \tag{13}$$

It will now be shown that the electric field given by Eq. (5) and the magnetic field given by Eq. (6) satisfy the five requirements if, and only if, the five conditions are satisfied. Consider the first requirement. Let $\mathbf{n}_j$ be a unit vector normal to the surface of the jth conductor and directed outward from the conductor. Let $\mathbf{t}_j$ be a unit vector lying in a plane of constant z and tangential to the surface of the jth conductor. Suppose $\mathbf{t}_j$ is oriented so that

$$\mathbf{t}_j = \mathbf{k} \times \mathbf{n}_j; \qquad \mathbf{n}_j = \mathbf{t}_j \times \mathbf{k}\,, \tag{14}$$

as shown in Fig. 1. Here, $\mathbf{k}$ is a unit vector parallel to, and directed along, the positive z axis. The first requirement says that $\mathbf{E} \cdot \mathbf{t}_j$ and $\mathbf{H} \cdot \mathbf{n}_j$ must vanish everywhere on the surface of the jth conductor for $j = 1, 2, \cdots, N+1$.

From Eq. (5) one obtains

$$\mathbf{E} \cdot \mathbf{t}_j = -\sum_{i=1}^{N} V_i \mathbf{t}_j \cdot \nabla\phi_i\,, \tag{15}$$

while from Eq. (6) one finds, with the aid of Eq. (14),

$$\mathbf{H} \cdot \mathbf{n}_j = \sum_{i=1}^{N} \left(\frac{1}{\mu}\sum_{j=1}^{N} L_{ij} I_j\right) \mathbf{t}_j \cdot \nabla\phi_i\,. \tag{16}$$

The right sides of Eqs. (15) and (16) must vanish everywhere on the surface of the jth conductor. Clearly, a sufficient condition is that

$$\mathbf{t}_j \cdot \nabla\phi_i = 0 \tag{17}$$

on the surface of the jth conductor for $j = 1, 2, \cdots, N+1$ and $i = 1, 2, \cdots, N$. From the linear independence of the functions V_i and the linear independence of the functions I_i, it follows that this is also a necessary condition. But Eq. (17) holds at the conductor surfaces for all i and j if, and only if, all the functions ϕ_i assume constant values on the conductor surfaces, which is precisely the first condition.

Consider the second requirement. If V_j is the voltage of the jth conductor with respect to the $N+1$st (reference) conductor, then V_j is related to the electric field by the relation

$$V_j(z,t) = -\int_{N+1}^{j} \mathbf{E} \cdot d\mathbf{r}\,, \tag{18}$$

where the path of integration is any path lying in a plane of constant z which connects a point on the surface of the jth conductor to a point on the surface of the $N+1$st conductor. Putting (5) into (18), one obtains

$$V_j(z,t) = \sum_{i=1}^{N} V_i(z,t) \int_{N+1}^{j} \nabla\phi_i \cdot d\mathbf{r}$$
$$= \sum_{i=1}^{N} V_i(z,t)[\phi_i(j) - \phi_i(N+1)], \qquad (19)$$

where $\phi_i(j)$ is the constant value of ϕ_i on the surface of the jth conductor, and $\phi_i(N+1)$ is the constant value of ϕ_i on the reference conductor. But Eq. (19) will be satisfied if, and only if,

$$\phi_i(j) = \delta_{ij} + \phi_i(N+1), \qquad (20)$$

which is precisely Eq. (7) of the second condition.

Now consider the third requirement. If I_j is the current carried by the jth conductor, then I_j is related to the magnetic field by means of the relation

$$I_j(z,t) = \oint_j \mathbf{H} \cdot \mathbf{t}_j dl_j, \qquad (21)$$

where $\mathbf{t}_j$ is the unit target vector of Eq. (14). The path of integration, shown in Fig. 1, lies in a plane of constant z; encircles the jth conductor once, in a counterclockwise sense with respect to the positive z axis; and lies arbitrarily close to the conductor surface. Note that $\mathbf{t}_j$ is tangent to the path of integration. The quantity dl_j is an element of arc length along the path of integration.

Writing m instead of j for the second summation index in Eq. (6), one finds, upon substituting (6) into (21),

$$I_j(z,t) = -\sum_{i=1}^{N} \left(\frac{1}{\mu} \sum_{m=1}^{N} L_{im} I_m(z,t)\right) \oint_j \mathbf{n}_j \cdot \nabla\phi_i dl_j, \qquad (22)$$

where use has been made of Eq. (14). Defining a set of constants C_{ji} by

$$C_{ji} = -\epsilon \oint_j \mathbf{n}_j \cdot \nabla\phi_i dl_j, \qquad (23)$$

one finds that Eq. (22) can be rewritten as

$$I_j(z,t) = \sum_{m=1}^{N} \left(\frac{1}{\epsilon\mu} \sum_{i=1}^{N} C_{ji} L_{im}\right) I_m(z,t). \qquad (24)$$

But equation (24) will be satisfied if, and only if,

$$\sum_{i=1}^{N} C_{ji} L_{im} = \epsilon\mu\delta_{jm}. \qquad (25)$$

Apart from a renaming of indices, Eqs. (25) and (23) are, respectively, the same as Eqs. (8) and (9) of the third condition.

Next, consider the fourth requirement. One begins by observing that the sum of the currents carried by the $N+1$ conductors can be expressed in terms of the magnetic field by means of the relation

$$\sum_{j=1}^{N+1} I_j(z,t) = \oint \mathbf{H} \cdot \mathbf{t} dl, \qquad (26)$$

where the path of integration lies in a plane of constant z and is a circle of radius $r = |\mathbf{r}|$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, and where r is sufficiently large for the circle to include all $N+1$ conductors. The vector $\mathbf{t}$ is a unit vector tangent to the circle of radius r, and $dl = rd\theta$ is an element of arc length along this circle. The direction of $\mathbf{t}$ is in the counterclockwise sense, with respect to the positive z axis, and the integration is carried out in this direction. It should be noted that the integral on the right side of Eq. (26) can yield only currents carried by the conductors because any displacement current density that might arise from a time varying electric field cannot have a z component since the electric field itself has no z component.

Putting Eq. (6) into Eq. (26), observing that

$$(\mathbf{k} \times \nabla\phi_i) \cdot \mathbf{t} = (\mathbf{t} \times \mathbf{k}) \cdot \nabla\phi_i = \frac{1}{r} \mathbf{r} \cdot \nabla\phi_i \qquad (27)$$

and noting that $dl = rd\theta$, one obtains

$$\sum_{j=1}^{N+1} I_j(z,t) = -\sum_{i=1}^{N} \left(\frac{1}{\mu} \sum_{j=1}^{N} L_{ij} I_j(z,t)\right) \int_{\theta=0}^{2\pi} (\mathbf{r} \cdot \nabla\phi_i) d\theta. \qquad (28)$$

Clearly, Eq. (28) must hold for all sufficiently large $r = |\mathbf{r}|$. Particularly, it must hold in the limit as $|\mathbf{r}| \to \infty$. But, according to the fourth requirement, the sum of the currents on the $N+1$ conductors must vanish. Clearly, a sufficient condition for this to happen is to have

$$\mathbf{r} \cdot \nabla\phi_i \to 0 \text{ as } |\mathbf{r}| \to \infty, \qquad (29)$$

for $i = 1, 2, \cdot\cdot\cdot, N$. Furthermore, if Eq. (29) is satisfied $\nabla\phi_i$ vanishes as $|\mathbf{r}| \to \infty$ and, from Eqs. (5) and (6), **E** and **H** also vanish as $|\mathbf{r}| \to \infty$. To prove the necessity of Eq. (29), one needs the unproved result that all the ϕ_i satisfy the two-dimensional Laplace equation, (11). If one assumes this result, it follows that if ϕ_i is expressed in polar coordinates, then, for sufficiently large values of r, the function ϕ_i must be of the form

$$\phi_i = b_0 + a_0 \ln(r) + \sum_{n=1}^{\infty} (a_n r^n + b_n r^{-n})$$
$$\times (c_n \cos n\theta + d_n \sin n\theta), \qquad (30)$$

where a_n, b_n, c_n and d_n are constants. It follows from Eqs. (5) and (6) that if **E** and **H** are to vanish as $r \to \infty$, then $\partial\phi_i/\partial r$ and $(1/r)\partial\phi_i/\partial\theta$ must vanish as $r \to \infty$ which, in turn, implies that $a_n = 0$ for $n \geq 1$. Furthermore, if the sum of the currents on the $N+1$ conductors is to vanish, it follows from Eq. (28) and the linear independence of I_j that for $i = 1, 2, \cdot\cdot\cdot, N$,

$$\int_0^{2\pi} (\mathbf{r} \cdot \nabla\phi_i) d\theta = 0. \qquad (31)$$

Putting Eq. (30) into Eq. (31) and evaluating the integrals, one finds that (31) is satisfied if, and only if, $a_0 = 0$. Hence, if Eq. (11) is true, then the fourth requirement implies that for sufficiently large r, the function ϕ_i has the form

$$\phi_i(r,\theta) = b_0 + \sum_{n=1}^{\infty} b_n r^{-n} (c_n \cos n\theta + d_n \sin n\theta) . \quad (32)$$

But Eq. (32) satisfies the condition of Eq. (29). It will soon be proved that all the ϕ_i must satisfy Eq. (11), the two-dimensional Laplace equation, and the proof will be completely independent of the condition expressed by Eq. (29). At the time the proof of Eq. (11) is completed, the necessity of Eq. (29) will have been established, thus establishing both the necessity and sufficiency of the fourth condition.

Now consider the last requirement. One begins by writing out the components of Eqs. (5) and (6) and (1) through (4). Upon substituting Eqs. (5) and (6) into (1) through (4), one finds that (1) through (4) are satisfied if, and only if,

$$\sum_{i=1}^{N} \frac{\partial\phi_i}{\partial y} \left(\sum_{j=1}^{N} L_{ij} \frac{\partial I_j}{\partial z} + \epsilon\mu \frac{\partial V_i}{\partial t}\right) = 0 ; \quad (33)$$

$$\sum_{i=1}^{N} \frac{\partial\phi_i}{\partial x} \left(\sum_{j=1}^{N} L_{ij} \frac{\partial I_j}{\partial z} + \epsilon\mu \frac{\partial V_i}{\partial t}\right) = 0 ; \quad (34)$$

$$\sum_{i=1}^{N} \left(\sum_{j=1}^{N} L_{ij} I_j\right) \left(\frac{\partial^2\phi_i}{\partial x^2} + \frac{\partial^2\phi_i}{\partial y^2}\right) = 0 ; \quad (35)$$

$$\sum_{i=1}^{N} \frac{\partial\phi_i}{\partial y} \left(\frac{\partial V_i}{\partial z} + \sum_{j=1}^{N} L_{ij} \frac{\partial I_j}{\partial t}\right) = 0 ; \quad (36)$$

$$\sum_{i=1}^{N} \frac{\partial\phi_i}{\partial x} \left(\frac{\partial V_i}{\partial z} + \sum_{j=1}^{N} L_{ij} \frac{\partial I_j}{\partial t}\right) = 0 ; \quad (37)$$

$$\sum_{i=1} V_i \left(\frac{\partial^2\phi_i}{\partial x^2} + \frac{\partial^2\phi_i}{\partial y^2}\right) = 0 . \quad (38)$$

Clearly sufficient conditions for the satisfaction of Eqs. (33) through (38) are obtained by requiring

$$\frac{\partial^2\phi_i}{\partial x^2} + \frac{\partial^2\phi_i}{\partial y^2} = 0 ; \quad (39)$$

$$-\frac{\partial V_i}{\partial z} = \sum_{j=1}^{N} L_{ij} \frac{\partial I_j}{\partial t} ; \quad (40)$$

$$-\sum_{j=1}^{N} L_{ij} \frac{\partial I_j}{\partial t} = \epsilon\mu \frac{\partial V_i}{\partial t} . \quad (41)$$

The necessity of Eq. (39) follows from (35) and the linear independence of the functions I_j, and from (38) and the linear independence of the functions V_i. To establish the necessity of Eq. (40), consider the function ψ defined by

$$\psi = \sum_{i=1}^{N} (\phi_i(x,y) - \phi_i(N+1)) \left(\frac{\partial V_i}{\partial z} + \sum_{j=1}^{N} L_{ij} \frac{\partial I_j}{\partial t}\right), \quad (42)$$

where $\phi_i(N+1)$ is the constant value of ϕ_i on the $N+1$st (reference) conductor. Equation (37) implies that $\partial\psi/\partial x = 0$, and Eq. (36) implies that $\partial\psi/\partial y = 0$. Hence, ψ is independent of x and y. Thus, the functions $\phi_i(x,y)$ on the right side of Eq. (42) can be evaluated at any point (x,y) without affecting the value of ψ. In particular, evaluating $\phi_i(x,y)$ for a point on the surface of the $N+1$st conductor, one finds that $\psi = 0$. Thus, since ψ is independent of x and y, it follows from Eq. (42) that

$$\sum_{j=1}^{N} [\phi_i(x,y) - \phi_i(N+1)] \left(\frac{\partial V_i}{\partial z} + \sum_{j=1}^{N} L_{ij} \frac{\partial I_j}{\partial t}\right) = 0 \quad (43)$$

for all x and y. Evaluating Eq. (43) for a point on the surface of the mth conductor and using Eq. (7), one finds that Eq. (43) reduces to

$$\frac{\partial V_m}{\partial z} + \sum_{j=1}^{N} L_{mj} \frac{\partial I_j}{\partial t} = 0 \quad (44)$$

and that this must be true for all values of $m = 1, 2, \cdots, N$. Hence, Eq. (40) is necessary, as well as sufficient, for the satisfaction of Eqs. (36) and (37). The necessity of Eq. (41) is established in a similar manner.

Equations (39) and (40) are identical to Eqs. (11) and (12) of the fifth condition. To convert (41) into (13) of the fourth condition, it is necessary only to multiply (41) by C_{mi} and sum over i to obtain, with the aid of Eq. (8),

$$-\frac{\partial I_m}{\partial z} = \sum_{i=1}^{N} C_{mi} \frac{\partial V_i}{\partial t} . \quad (45)$$

Apart from a renaming of indices, Eq. (45) is the same as Eq. (13).

It has been established that the fields given by Eqs. (5) and (6) satisfy the five requirements if, and only if, the five conditions are satisfied. At this point, an observation is in order. It should be noted that there is no loss of generality if the constants $\phi_i(N+1)$ in Eq. (7) are chosen so that

$$\phi_i(N+1) = 0 \quad (46)$$

for $i = 1, 2, \cdots, N$. This is readily seen by considering the functions $\phi_i'(x,y)$ defined by

$$\phi_i'(x,y) = \phi_i(x,y) - \phi_i(N+1) . \quad (47)$$

First, observe that $\phi_i'(x,y)$ satisfies Eqs. (10) and (11). Instead of Eq. (7), $\phi_i'(x,y)$ satisfies the boundary conditions

$$\phi_i'(j) = \phi_{ij} , \quad (48)$$

where $\phi_i'(j)$ is the value of $\phi'(x,y)$ on the surface of the jth conductor. Furthermore

$$\nabla\phi_i' = \nabla\phi_i , \quad (49)$$

so that ϕ_i' gives rise to the same electric and magnetic fields as ϕ_i [see Eqs. (5) and (6)]. Furthermore, ϕ_i' gives rise to the same set of constants C_{ij} and L_{ij}, as ϕ_i [see Eqs. (8) and (9)]. In all subsequent calculations, it can be assumed without loss of generality that $\phi_i(N+1)=0$ for $i=1,\cdot\cdot\cdot, N+1$.

Symmetry of the capacitance and inductance matrices

Proofs of the symmetry of the capacitance and inductance matrices are given in this section. The starting point is the identity

$$\nabla\cdot(\phi_i'\nabla\phi_j'-\phi_j'\nabla\phi_i')=\phi_i'\nabla^2\phi_j'-\phi_j'\nabla^2\phi_i', \tag{50}$$

where

$$\phi_i'(x,y)=\phi_i(x,y)-\phi_i(N+1) \tag{51}$$

for $i=1,2,\cdot\cdot\cdot,N$. Here, $\phi_i(N+1)$ is the constant value of ϕ_i on the surface of the $N+1$st (reference) conductor. It follows from Eqs. (11) and (51) that the right side of Eq. (50) vanishes everywhere in the space outside the conductors for all values of i and j. Now, suppose one forms a volume integral using the left side of Eq. (50) as the integrand. The volume of integration consists of that part of space lying between two planes of constant z, separated by a distance Δz, and inside a cylinder of radius $r=|\mathbf{r}|$, where r is sufficiently large so that the cylinder encloses all $N+1$ conductors, but excludes the volume occupied by the $N+1$ conductors themselves. Thus, since the right side of Eq. (50) vanishes, one obtains

$$\iiint\nabla\cdot(\phi_i'\nabla\phi_j'-\phi_j'\nabla\phi_i')\,dx\,dy\,dz=0, \tag{52}$$

where the integration is over the volume just described. The volume integral in Eq. (52) can be converted to a surface integral over the surface enclosing the volume by means of Gauss' theorem. Thus, noting from (51) that $\nabla\phi_i'=\nabla\phi_i$ for $i=1,2,\cdot\cdot\cdot,N$, one obtains

$$\iint(\phi_i'\nabla\phi_j-\phi_j'\nabla\phi_i)\cdot d\mathbf{A}=0, \tag{53}$$

where the integration is over the entire surface that bounds the volume. The vector element of area $d\mathbf{A}$ is normal to the surface of integration and is directed out of the volume enclosed by the surface. Since $\nabla\phi_i$ and $\nabla\phi_j$, for all i and j are perpendicular to $d\mathbf{A}$ on the two planes of constant z, these two planes make no contribution to the integral. Thus, it is necessary to evaluate Eq. (53) only over the curved portions of the surface. The curved part of the surface has $N+2$ parts as shown in Fig. 2. First, there is the surface of the cylinder of radius $r=|\mathbf{r}|$, and then there are $N+1$ surfaces consisting of the surfaces of the $N+1$ conductors. On the surface of the kth conductor,

$$d\mathbf{A}=-\mathbf{n}_k\,dl_k\,dz, \tag{54}$$

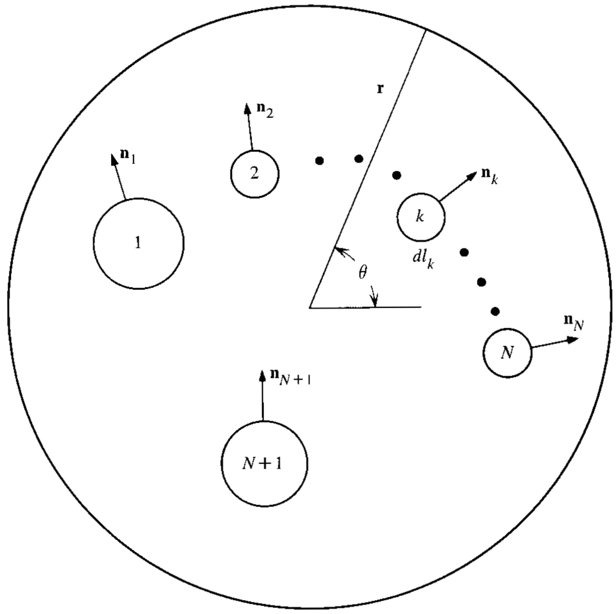

Figure 2 Surface integration for energy calculation.

where $\mathbf{n}_k$ is a unit vector normal to the surface of the kth conductor and directed away from the conductor into the surrounding space. (Since $d\mathbf{A}$ must be directed out of the volume enclosed by the surface of integration, its direction must be into the conductors, and hence in the direction $-\mathbf{n}_k$.) The quantity dl_k is an element of arc length along the perimeter of the kth conductor, the perimeter lying in a plane of constant z. On the surface of the cylinder of radius r, which encloses the $N+1$ conductors,

$$d\mathbf{A}=\mathbf{r}\,d\theta\,dz, \tag{55}$$

where θ is the polar angle of the radius vector $\mathbf{r}$.

The z integration is trivial since none of the integrands depends on z. Thus, performing the z integration and dividing by Δz one obtains, from Eq. (53),

$$\int_0^{2\pi}(\phi_i'\nabla\phi_j-\phi_j'\nabla\phi_i)\cdot\mathbf{r}\,d\theta)$$
$$=\sum_{k=1}^{N+1}\oint_k(\phi_i'\nabla\phi_j-\phi_j'\nabla\phi_i)\cdot\mathbf{n}_k\,dl_j\,. \tag{56}$$

The contour integrals on the right side go all the way around the kth conductor for each value of k, and the integral on the left side goes all the way around the circle of radius r. The particular value of this radius is immaterial, provided that it is large enough to enclose all $N+1$ conductors. In particular, one can let $|\mathbf{r}|\to\infty$. But, from Eq. (32) ϕ_i and ϕ_j are bounded as $|\mathbf{r}|\to\infty$, and from Eq. (10) $\mathbf{r}\cdot\nabla\phi_i\to 0$ and $\mathbf{r}\cdot\nabla\phi_j\to 0$ as $|\mathbf{r}|\to\infty$. It follows that the integrand, and the integral on the left side of (56), vanishes in the limit as $|\mathbf{r}|\to\infty$.

Combining Eq. (7) with Eq. (51) one obtains

$$\phi_i'(k) = \delta_{ik} \tag{57}$$

for $i = 1, 2, \cdots, N$ and $k = 1, 2, \cdots, N + 1$. Putting Eq. (57) into Eq. (56) and evaluating the sum over k, one obtains

$$\oint_i \mathbf{n}_i \cdot \nabla\phi_j dl_i - \oint_j \mathbf{n}_j \cdot \nabla\phi_i dl_j = 0 . \tag{58}$$

Upon comparing Eq. (58) with Eq. (9), which defines the constants C_{ij}, it is seen that (58) is equivalent to the statement

$$C_{ji} = C_{ij} . \tag{59}$$

That is, C_{ij} forms a symmetric matrix. Furthermore, according to Eq. (8), the inductance matrix is $\epsilon\mu$ times the inverse of the capacitance matrix. Since the inverse of a symmetric matrix is also symmetric, it follows that the inductance matrix is symmetric. That is,

$$L_{ji} = L_{ij} . \tag{60}$$

With the assumption that $\phi_i\ (N + 1) = 0$ for all i, Eqs. (7) through (11), for a fixed value of i, determine the unique solution to the electrostatic potential problem wherein the ith conductor is raised to a static potential of one volt while all other conductors are maintained at zero potential. The electrostatic field determined by the potential ϕ_i is given by $-\nabla\phi_i$ with all field lines originating on the ith condcutor and terminating on the remaining conductors. Thus, $-\nabla\phi_i$ is in the same direction as the unit vector $\mathbf{n}_i$ at the surface of the ith conductor and is in the opposite direction to the unit vectors $\mathbf{n}_j$ on the surface of the remaining conductors. Thus,

$$-\mathbf{n}_i \cdot \nabla\phi_i > 0 \tag{61}$$

on the surface of the ith conductor and

$$-\mathbf{n}_j \cdot \nabla\phi_i < 0 \tag{62}$$

on the surface of the jth conductor for $j \neq i$. From the definition of the C_{ij} given by (9), it follows that

$$C_{ii} > 0 ;$$

$$C_{ij} < 0 (i \neq j) . \tag{63}$$

These relations hold for all values of i and j.

Further properties

Further insight into the meaning of the capacitance and inductance matrices can be gained from the following considerations. Let $q_i(z,t)$ be the charge per unit length at position z and time t on the ith conductor of the transmission line system. The charge per unit length is given in the limit as Δz becomes arbitrarily small, by the equation

$$q_i(z,t)\Delta z = \epsilon \int_z^{z+\Delta z} \oint_i \mathbf{E} \cdot \mathbf{n}_i dl_i dz , \tag{64}$$

where the unit vector $\mathbf{n}_i$ is shown in Fig. 1 and the inner integration is taken around the perimeter of the ith conductor, again as shown in Fig. 1. By substituting (5) into (64) and using (9) it is found, on dividing both sides by Δz and letting $\Delta z \to 0$ that

$$q_i(z,t) = \sum_{j=1}^{N} C_{ij} V_j(z,t) . \tag{65}$$

Equation (65) thus confirms the fact that the matrix C_{ij} has been interpreted properly as the matrix of capacitive coefficients.

To get analogous results for the inductance matrix let $\Phi_i(t,t)$ be the magnetic flux per unit length linking the circuit formed by the ith conductor and conductor $N + 1$, the common reference conductor. The flux per unit length is given in terms of the magnetic field by the equation

$$\Phi_i(z,t)\Delta z = \mu \int_z^{z+\Delta z} \int_i^{N+1} \mathbf{H} \cdot \mathbf{k} \times d\mathbf{r} dz , \tag{66}$$

where the path of integration is any path starting at the surface of conductor i, ending at the surface of conductor $N + 1$, and lying in a plane of constant z.

By substituting (6) into (66) and noting that

$$(\mathbf{k} \times \nabla\phi_j) \cdot (\mathbf{k} \times d\mathbf{r}) = \nabla\Phi_j \cdot d\mathbf{r} ,$$

it is found with the aid of the boundary condition (7) that

$$\Phi_i(z,t) = \sum_{j=1}^{N} L_{ij} I_j(z,t) \tag{67}$$

where, again, the limit as $\Delta z \to 0$ has been taken.

Similarly, starting from the expression

$$\tfrac{1}{2}(\epsilon\mathbf{E} \cdot \mathbf{E} + \mu\mathbf{H} \cdot \mathbf{H})$$

for the energy per unit volume stored in an electromagnetic field, it can be shown that the energy per unit length at position z and time t for the transmission line system under consideration is given by

$$W(z,t) = \tfrac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (L_{ij} I_i I_j + C_{ij} V_i V_j) . \tag{68}$$

The somewhat cumbersome proof will be omitted.

Computational aspects

Computation of the fields set up by signals propagating on a multiconductor transmission line system requires two separate calculations. First, it is necessary to solve the N two-dimensional electrostatic potential problems defined by (7), (10), and (11) and then calculate the capacitance and inductance matrix elements from (9) and (8), respectively. Except for a relatively trivial class of problems, it is not possible to solve the required potential problems by analytical means and one must resort to numerical methods. An extensive bibliography on this subject is given in Ref. [14].

The second problem is, given the capacitance and inductance matrices, to solve (12) and (13) for the currents and voltages of the N lines. If only the currents and voltages are of interest, the potentials $\phi_i(x,y)$ are not needed. Only the capacitance and inductance matrices are required. The elements of the capacitance matrix can be determined approximately by numerical computation [14] or by direct measurements [13] on the transmission line system in question. The elements of the inductance matrix can be calculated from (8) once the capacitance matrix is known. Equation (8) states that the capacitance matrix is $\epsilon\mu$ times the unit matrix. Thus, the inductance matrix is $\epsilon\mu$ times the inverse of the capacitance matrix. This is a direct generalization of the familiar results for two-wire lines.

The solution of the transmission line equations (12) and (13) presents few problems. If $f_i(t)$ and $g_i(t)$ with $i = 1, 2, \cdots, N$, are any $2N$ at least once differentiable functions, then (12) and (13) are satisfied by

$$I_i = f_i(t - x/v) - g_i(t - x/v)\,; \tag{69}$$

$$V_i = \sum_{j=1}^{N} Z_{ij}[f_j(t - x/v) + g_i(t + x/v)]\,, \tag{70}$$

where

$$v = 1/\sqrt{\epsilon\mu} \tag{71}$$

and

$$Z_{ij} = vL_{ij} = v^{-1}(C^{-1})ij\,. \tag{72}$$

The functions f_i represent forward waves and the functions g_i backward waves. The quantity v is the propagation velocity. Note that all $2N$ waves propagate with the same velocity, v; it is unnecessary to assume a common propagation velocity. The matrix of coefficients Z_{ij} plays the same role as the characteristic impedance in the theory of the two-wire lines. This matrix properly can be called the characteristic impedance matrix. The functions f_i and g_i are determined by boundary conditions applied at the ends of each conductor. The procedure is similar to that used in the solution of two-wire transmission line problems and will not be discussed here.

Conclusions

The relationship between Maxwell's equations and the transmission line equations for a lossless, multiconductor system has been explored in detail. In particular, explicit expressions were derived for the electric and magnetic fields in the space surrounding the conductors. These expressions are given by Eqs. (5) and (6), respectively. A rigorous treatment [(7) through (11)] of the calculation of capacitance and inductance coefficients was given. Equation (8) is of particular interest. It is the generalization to the multiconductor case from a well-known result for two-wire lines: the product of capacitance per unit length and inductance per unit length is equal to the reciprocal of the square of the velocity of propagation.

Since reliable methods are available for calculating the coefficients of capacitance, (8) provides a simple way to determine the coefficients of inductance for systems in which the assumption of TEM propagation is valid. Equation (72) then provides an explicit representation of the characteristic impedance matrix.

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