## The Theory of the Propagation of Plane Sound Waves in Tubes

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Abstract. The propagation of plane sound waves in gases in tubes can be divided into three main types, depending on the radius and frequency involved. These types are described as 'narrow' tube, 'wide' tube and 'very wide' tube propagation. The phase velocity, attenuation and cross section profile of particle relocity etc. are investigated theoretically, and their inter-relation pointed out. The factors affecting the validity of Kirchhoff's formulae are considered, and the theory is applied to some recent work.

## §1. INTRODUCTION

THIS paper is concerned with the theory for the propagation of sound waves of the principal mode in gases contained in cylindrical tubes of a large range of diameters. In particular the manner in which viscosity and thermal conductivity affect the motion is discussed for continuous waves.

Helmholtz (1863) was the first to investigate the problem, and quoted the formula for the velocity decrease due to viscosity. In his dust-tube experiment Kundt (1868) obtained fair agreement with the Helmholtz formula, but found the effect larger than predicted. He suggested heat conduction as a further cause, stimulating Kirchhoff (1868) to present the complete theory for the 'wide' tube, which is the most important case.

Kirchhoff's formulae are:

Phase velocity of sound:

$$c' = c[1 - \gamma'/r_w(2\omega)^{1/2}] = 3.434 \times 10^4 \{1 - 0.162/(r_w\sqrt{f})\} \text{ cm sec}^{-1}$$
. (1)

Amplitude attenuation constant:

$$m' = (\gamma'/cr_w)(\omega/2)^{1/2} = 2.964 \times 10^{-5} \sqrt{f/r_w} \text{ cm}^{-1}.$$
 (2)

The numerical values here and elsewhere are for air at 20°c and 76 cm Hg;  $\omega$  is the pulsatance, f the frequency,  $r_w$  the tube radius and r the general radius vector, and the following refer to the gaseous medium:

c = Laplacian adiabatic velocity of sound,	
$\gamma' = \sqrt{\nu} + (\gamma - 1)(\nu'/\gamma)^{1/2} = 0.574$ c.g.s. units = Kirchhoff's	
constant,	1
$\gamma = ratio$ of the principal specific heats,	ĺ
$\nu = \mu / \rho = \text{kinematic viscosity},$	
$\nu' = k/\rho c_{\rm v}$ = thermal diffusivity or thermometric	$\left.\right.$
conductivity,	
$\mu =$ shear viscosity coefficient,	
k = thermal conductivity,	
$\rho = \text{density},$	
$c_v =$ specific heat at constant volume.	J

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The author has made a critical examination of Kirchhoff's formulae in an effort to settle the controversy arising from the disagreement of many experimental results with theory. Using some results of Kirchhoff it may be shown that there are three main types of motion, the conditions for which are defined later (see table 1 and fig. 1):

(a) 'Narrow' tube. At very small radii and low frequencies the motion is isothermal and governed largely by viscous forces.

(b) 'Wide' tube. The sound energy is diffused evenly over the tube cross section, and the adiabatic motion approximates to that in an unbounded gas. There is, however, a narrow boundary layer at the wall, analogous to the Prandtl layer in continuous flow, where viscous and heat conduction processes lead to an energy loss.

(c) 'Very wide' tube. This differs from the 'wide' tube mainly in that the sound energy is concentrated near the walls.

Before giving the more formal analysis it is desirable to consider two alternative approaches to the problem—the resistance concept for 'narrow' tubes and the boundary layer idea for 'wide' tubes. Besides affording a clearer physical picture, these cover aspects additional to the main theory.

# § 2. The Resistance Concept for Narrow Tubes

where R is a coefficient of resistance and  $\partial p/\partial x$  the pressure gradient along the tube axis. This equation leads to the result

$$m^2 = i\omega R/P \qquad \dots \dots (5)$$

where m is the complex propagation constant and P the ambient pressure.

The value of R is usually calculated assuming that Poiseuille's law 1s applicable, viz.  $R = 8\mu/r_w^2$ . .....(6)

Thus for air, velocity  $c' = \frac{1}{2} c r_{w} (\omega/\gamma \nu)^{1/2} = 9.364 \times 10^{4} r_{w} \sqrt{f} \text{ cm sec}^{-1}$  .....(7) attenuation  $m' = 2(\gamma \nu \omega)^{1/2} / c r_{w} = 6.701 \times 10^{-5} \sqrt{f} / r_{w} \text{ cm}^{-1}$ . .....(8)

For the narrower tubes R may be modified for the effect of slip, and for very narrow tubes it may be found from Knudsen's formula for molecular streaming (see eqn. (72)).

## § 3. The Boundary Layer Concept for Wide Tubes

## (i) Viscosity and Temperature Waves

When a fluid is vibrating parallel to a plane solid surface the particle velocity u is given by (Stokes 1851)  $u = A e^{i\omega t} [1 - e^{-\alpha b(1+i)}]$  .....(9) where  $\alpha = (\omega/2\nu)^{1/2}$  is the viscous 'wave constant', and b is a coordinate measured normal to the solid surface. The first term represents the acoustic wave and the second term a 'viscous' wave.

It may easily be shown that (see e.g. Sexl 1930) the amplitude  $u_0$  of the particle velocity has a principal maximum equal to 1.067A at a distance from

the surface

$$\Delta_{\mu} = 2.2838 (2\nu/\omega)^{1/2} = 0.500 / \sqrt{f} \, \text{cm for air,} \qquad \dots \dots (10)$$

which is here defined as the boundary layer thickness.

Similarly it may be shown that the excess temperature (cf. Kirchhoff 1868, Rayleigh 1896, p. 322, Ballantine 1932)

$$\theta = B e^{i\omega t} [1 - e^{-\beta b(1+i)}] \qquad \dots \dots (11)$$

where  $\beta = (\omega \gamma / 2\nu')^{1/2}$  is the thermal 'wave constant'. The principal maximum of  $\theta_0$  occurs at

$$\Delta_{k} = 2.2838(2\nu'/\omega\gamma)^{1/2} = 0.596/\sqrt{f} \, \text{cm for air.} \qquad \dots \dots (12)$$

It may be noted that  $\Delta_{\mu} \sim \Delta_k \sim (\lambda \lambda')^{1/2}$ , the geometric mean of the wavelength and mean free path.

The shape of the wave front may also be calculated from eqn. (9). Thus

$$u = u_0 \cos\left(\omega t - m'' x + \phi\right) \qquad \dots \dots (13)$$

where

 $\phi$  has a maximum of  $\pi/2$  at the wall and a principal minimum of  $\phi = -0.00438\pi$ , corresponding to an appreciable forward displacement (about 0.2% wavelength) at  $b = 3.9408(2\nu/\omega)^{1/2}$ . .....(15)

 $\tan\phi = e^{-\alpha b} \sin \alpha b / (1 - e^{-\alpha b} \cos \alpha b)$ 

Figure 3 shows the velocity amplitude and wave front when the above theory 18 applied to a tube. The maxima due to viscous waves have been found experimentally by Richardson (1928), Carriere (1929), Richardson and Tyler (1929), and others.

### (ii) Velocity and Attenuation

Let the perimeter and cross-sectional area of the tube be E and S, and the excess pressure outside the layer

$$p = p_0 \exp(i\omega t - mx). \qquad \dots \dots (16)$$

The variable component of the particle velocity near the wall is

$$u = -(cp_0/\gamma P) \exp\{-\alpha b(1+i) + i\omega t - mx\}. \qquad (17)$$

From Newton's law the force on a cylindrical gas column of length  $\delta x$  is

$$E\delta x\mu\left(\frac{\partial u}{\partial b}\right)_{b=0} = \frac{Ep_0\delta x}{c}\left(\frac{\nu\omega}{2}\right)^{1/2}(1+i)\exp\left(i\omega t - mx\right). \quad \dots \dots (18)$$

The associated pressure is found to be

$$(E/S)(\nu/2\omega)^{1/2}(i-1)p.$$
 .....(19)

The variable component of the temperature excess near the wall is

$$\theta = -\{p_0(\gamma - 1)\Theta/\gamma P\}\exp\{-\beta b(1+i) + i\omega t - mx\} \qquad \dots (20)$$

where  $\Theta$  is the ambient temperature. The pressure arising is

$$-\{p_0(\gamma-1)/\gamma\}\exp\{-\beta b(1+i)+i\omega t-mx\}.$$
 (21)

This corresponds to a pressure, averaged over the cross section,

$$\frac{1}{S}\int_{0}^{\infty} -\frac{Ep_{0}(\gamma-1)}{\gamma}\exp\left\{-\beta b(1+i)+i\omega t-mx\right\}db = \frac{E}{S}\frac{\gamma-1}{\gamma^{3/2}}\left(\frac{\nu'}{2\omega}\right)^{1/2}(i-1)p.$$
.....(22)

If the total *fractional* change in p due to heat conduction is  $\epsilon$  there will be a corresponding fractional change  $\epsilon$  in the temperature excess in the bulk of the

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....(14)

tube. This produces part of the change  $\epsilon(\gamma - 1)/\gamma$  in p, from the constant volume relation. Thus  $\epsilon$  has two components:

$$\epsilon = \frac{E}{S} \frac{\gamma - 1}{\gamma^{3/2}} \left( \frac{\nu'}{2\omega} \right)^{1/2} (i - 1) + \frac{\gamma - 1}{\gamma} \epsilon. \qquad \dots \dots (23)$$

Solving eqn. (23)

$$\epsilon = \frac{E}{S} \frac{\gamma - 1}{\sqrt{\gamma}} \left( \frac{\nu'}{2\omega} \right)^{1/2} (i-1). \qquad \dots \dots (24)$$

The extra pressure due to both viscosity and heat conduction is

$$[E\gamma'/S(2\omega)^{1/2}](i-1)p$$
 .....(25)

and the wave equation is modified to

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2} \left[ 1 + \frac{E}{S} \frac{\gamma'}{(2\omega)^{1/2}} (i-1) \right]. \qquad \dots \dots (26)$$

The phase velocity and attenuation become

$$c' = c\{1 - E\gamma'/2S(2\omega)^{1/2}\}$$
 .....(27)

$$m' = (E\gamma'/2Sc)(\omega/2)^{1/2}.$$
 (28)

These reduce to (1) and (2) for the circular tube.

A version of the above theory for the effect of viscosity has been given by Rayleigh (1896) and others, but the heat conduction layer has not received such full attention (see however Ballantine 1932, Nielsen 1949a, b, Konstantinov 1939, Cremer 1948).

# §4. The More Complete Theory

## (i) General Equations

Kirchhoff (1868, see alternatively Rayleigh 1896), when analysing propagation in a tube, started from differential equations modified for both viscosity and thermal conduction. He derived expressions showing the variations across the tube of the particle velocity u parallel to the axis, the radial particle velocity q, and temperature excess  $\theta$ . These, however, involve constants A,  $A_1$ ,  $A_2$  of values to be determined by the boundary conditions.

$$u = AQ - A_{1}m(h/\lambda_{1} - \nu')Q_{1} - A_{2}m(h/\lambda_{2} - \nu')Q_{2} q = \frac{-Am}{h/\nu - m^{2}} \frac{dQ}{dr} - A_{1} \left(\frac{h}{\lambda_{1}} - \nu'\right) \frac{dQ_{1}}{dr} - A_{2} \left(\frac{h}{\lambda_{2}} - \nu'\right) \frac{dQ_{2}}{dr} \theta' = A_{1}Q_{1} + A_{2}Q_{2}$$
 (29)

where  $\theta' = \theta/(\gamma - 1)\Theta$ ,  $h = i\omega$  and  $Q = J_0\{r(m^2 - h/\nu)^{1/2}\}$   $Q_1 = J_0\{r(m^2 - \lambda_1)^{1/2}\}$   $Q_2 = J_0\{r(m^2 - \lambda_2)^{1/2}\}$  .....(30)  $\lambda_1$  and  $\lambda_2$  are the small and large roots of

$$h^{2} - \{c^{2} + h(\nu + \nu'' + \nu')\}\lambda + (\nu'/h)\{c^{2}/\gamma + h(\nu + \nu'')\}\lambda^{2} = 0 \quad \dots \dots (31)$$

 $\nu''$  is a constant which according to Stokes equals  $\nu/3$ .

At the walls of a rigid conducting tube, where  $r = r_w$ ,

$$u = q = \theta' = 0. \qquad \dots \dots (32)$$

The vanishing of the determinant (29) gives an equation for the propagation constant m used by Kirchhoff,

$$\frac{m^2h}{n/\nu - m^2} \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) \frac{d\ln Q}{dr_{\rm w}} + \left(\frac{h}{\lambda_1} - \nu'\right) \frac{d\ln Q_1}{dr_{\rm w}} - \left(\frac{h}{\lambda_2} - \nu'\right) \frac{d\ln Q_2}{dr_{\rm w}} = 0. \quad \dots (33)$$

The profiles, i.e. distributions of particle velocity etc. across the tube, may be expressed conveniently in terms of another constant B. The suffix w to a quantity denotes its value at the wall.

$$u = \left[ -h\left(\frac{1}{\lambda_{1}} - \frac{1}{\lambda_{2}}\right)Q_{1w}Q_{2w}Q + \left(\frac{h}{\lambda_{1}} - \nu'\right)Q_{w}Q_{2w}Q_{1} + \left(\frac{h}{\lambda_{2}} - \nu'\right)Q_{w}Q_{1w}Q_{2}\right]mB$$

$$q = \left[\frac{m^{2}h}{h/\nu - m^{2}}\left(\frac{1}{\lambda_{1}} - \frac{1}{\lambda_{2}}\right)Q_{1w}Q_{2w}\frac{dQ}{dr} + \left(\frac{h}{\lambda_{1}} - \nu'\right)Q_{w}Q_{2w}\frac{dQ_{1}}{dr} - \left(\frac{h}{\lambda_{2}} - \nu'\right)Q_{w}Q_{1w}\frac{dQ_{2}}{dr}\right]B$$

$$\theta' = \left[-Q_{2w}Q_{1} + Q_{1w}Q_{2}\right]Q_{w}B.$$
(34)

The form of solution depends on whether the moduli of  $Q_1, Q_2$  are small or large (see table 1 and fig. 1), and the various regions are now considered.

## Table 1. The Various Propagation Regions (at 20°c and 76 cm Hg)

'Very small' implies one term, and 'small' several terms in eqn. (38); 'large' implies several terms, and 'very large 'one term in eqn. (43).





Fig. 1. The various propagation regions in air at  $20^{\circ}$ c and 76 cm Hg showing where the correction terms in the formulae for the attenuation are less than 1%. Reported experimental work lies within the dotted line.

### (ii) The Narrow Tube

Cross section profiles of particle velocity etc.

The original treatment for waves in a narrow tube was given by Rayleigh (1896). First approximations from (31) are

$$\lambda_1 = h^2/c^2, \qquad \lambda_2 = h\gamma/\nu'. \qquad \dots \dots (35)$$

An approximation valid at all but the highest frequencies is  $h\nu/c^2 \ll 1$ , and for the narrow tube  $(\gamma'/r_w\sqrt{h} \gg 1)$ 

$$m^2 = 8\nu\gamma h/c^2 r_{\rm w}^2. \qquad \dots \dots (36)$$

Retaining only the leading terms the profiles may be calculated from (34)

$$u = (Bc/r_{\rm w})(\gamma h/2\nu)^{1/2}(r_{\rm w}^2 - r^2) q = -(Bhr/r_{\rm w}^2)(2\gamma - 1)(r_{\rm w}^2 - r^2) \theta' = -(Bh\gamma/4\nu')(r_{\rm w}^2 - r^2).$$
 (37)

The axial distribution is parabolic, as for continuous flow in tubes. Sexl (1928, 1930) also finds that his general expression, allowing for viscosity only, reduces to a Poiseuille type flow for narrow tubes. Note that  $q/u \sim (\omega \nu)^{1/2}/c$ , i.e. the radial is much less than the axial particle velocity. Compared with propagation in an unbounded gas  $\theta'/u$  is multiplied by the small factor  $r_{\rm w}(\omega/\nu)^{1/2}$ , so the flow may be called isothermal.

## Transition to the wide tube.

Starting from the two extremes it is possible to derive expressions applicable to the transition region between the narrow and wide tube. For the narrow tube approach, it may be shown, on developing the power expansion of a zero order Bessel function for small x, that

$$\frac{d\ln J_0(x)}{dx} = -\frac{x}{2} \left[ 1 + \frac{x^2}{8} + \frac{x^4}{48} + \frac{11x^6}{3072} \dots \right]. \qquad \dots \dots (38)$$

By making approximations in eqn. (33) as in table 1, but otherwise following Rayleigh, it is found that for air:

velocity  

$$\begin{aligned} c' &= (cr_{w}/2)(\omega/\nu\gamma)^{1/2}[1 - \gamma_{1}r_{w}^{2}\omega + (\gamma_{2} + \gamma_{1}^{2})r_{w}^{4}\omega^{2}] \\ &= 9\cdot364 \times 10^{4}r_{w}\sqrt{f}[1 - 2\cdot95r_{w}^{2}f + 1\cdot60r_{w}^{4}f^{2}] \text{ cm sec}^{-1} \qquad \dots (39) \\ \text{attenuation} \quad m' &= (2/cr_{w})(2\nu\gamma\omega)^{1/2}[1 - \gamma_{1}r_{w}^{2}\omega - \gamma_{2}r_{w}^{4}\omega^{2}] \\ &= 6\cdot701 \times 10^{-5}(\sqrt{f}/r_{w})[1 - 2\cdot95r_{w}^{2}f + 7\cdot11r_{w}^{4}f^{2}] \text{ cm}^{-1} \qquad \dots (40) \\ \text{where} \qquad \gamma_{1} &= \frac{1}{4}\left\{\frac{1}{3\nu} - \frac{\gamma - 1}{4\nu'}\right\} = 0\cdot470 \text{ c.g.s.} \\ &\qquad \gamma_{2} &= \frac{1}{32}\left\{-\frac{1}{8\nu^{2}} - \frac{\gamma - 1}{4\nu\nu'} + \frac{(\gamma - 1)(13\gamma + 3)}{48\nu'^{2}}\right\} = -0\cdot180 \text{ c.g.s.} \\ &\qquad \dots \dots (41) \end{aligned}$$

Kosten (1949) and Crandall (1927) have previously obtained a part of the correction term expressed in (40).

#### (iii) The Wide Tube

### Transition to the narrow tube.

When the boundary layer thickness  $\Delta$  becomes comparable with the tube radius Kirchhoff's assumptions begin to break down: in particular the moduli of Q and  $Q_2$  are no longer large. The discriminant between the two cases of adiabatic and isothermal flow is thus

$$r_{\mathrm{w}}(\omega/\nu)^{1/2} \propto r_{\mathrm{w}}/\Delta \propto \mathrm{moduli} \text{ of } Q, \ Q_2 \sim r_{\mathrm{w}}\sqrt{f}.$$
 (42)

For x large

$$\frac{d\ln J_0(x)}{dx} = -i\left(1 + \frac{1}{8x^2} - \frac{25}{128x^4}\dots\right) - \left(\frac{1}{2x} - \frac{1}{8x^3}\dots\right) \quad \dots \dots (43)$$

where for the conditions considered it has been possible to approximate  $\tan(x-\frac{1}{4}\pi)=i$ . By solving eqn. (33) with approximations as in table 1, but otherwise following Kirchhoff, it is found that for air:

velocity 
$$c' = c \left[ 1 - \frac{\gamma'}{r_{w}(2\omega)^{1/2}} + \frac{\gamma'^{2}}{2r_{w}^{2}\omega} - \frac{\gamma''' + \gamma'^{3}/2}{r_{w}^{3}(2\omega)^{3/2}} \right]$$
$$= 3 \cdot 434 \times 10^{4} \left[ 1 - \frac{0 \cdot 162}{r_{w}\sqrt{f}} + \frac{0 \cdot 0262}{r_{w}^{2}f} - \frac{0 \cdot 0124}{r_{w}^{3}f^{3/2}} \right] \text{ cm sec}^{-1} \dots (44)$$
$$\text{attenuation } m' = \frac{\gamma'}{r_{w}c} \left(\frac{\omega}{2}\right)^{1/2} + \frac{\gamma''}{r_{w}^{2}c} + \frac{\gamma'''}{r_{w}^{3}c(2\omega)^{1/2}}$$
$$= 10^{-5} \left[ 2 \cdot 964 \frac{\sqrt{f}}{r_{w}} + \frac{0 \cdot 473}{r_{w}^{2}} + \frac{0 \cdot 146}{r_{w}^{3}\sqrt{f}} \right] \text{ cm}^{-1} \dots (45)$$

where

$$\gamma'' = \nu + \frac{(\gamma - 1)}{\sqrt{\gamma}} (\nu \nu')^{1/2} - \frac{(\gamma - 1)}{2} \nu' = 0.162 \text{ c.g.s.}$$
  

$$\gamma''' = \frac{15}{8} \nu^{3/2} + \frac{4(\gamma - 1)}{\sqrt{\gamma}} \nu \nu'^{1/2} + \frac{3(\gamma - 1)(\gamma - 2)}{2\gamma} \nu^{1/2} \nu'$$
  

$$+ \frac{(\gamma - 1)(4\gamma^2 - 12\gamma + 7)}{8\gamma^{3/2}} \nu'^{3/2} = 0.182 \text{ c.g.s.}$$

The two sets of approximate expressions for velocity and attenuation, approaching from the wide and narrow tube limiting cases, have been plotted together in fig. 2. This transition region has also been investigated, using different methods, by Nielsen (1949a), Golay (1947), Daniels (1947, 1950), Zwikker and Kosten (1949), and Iberall (1950).

The relations between velocity and attenuation are of interest. For the narrow tube the amplitude falls to 1/e of its initial value after a phase advance of one radian. For the wide tube the fractional decrease in velocity is equal to the fractional decrease in amplitude after one radian. Bradfield (1951) has pointed out that a similar relation to the latter applies to other propagation processes.

The first order correction to Kirchhoff's expression for the attenuation <sup>18</sup>  $1+0\cdot160/r_w\sqrt{f}$ , and his expression for the velocity decrease involves  $1-0\cdot162/r_w\sqrt{f}$ . It is a coincidence that these two corrections are numerically close, so that they approximately cancel, and the attenuation per wavelength just inside the transition region is predicted correctly by the simple formula.

### Cross section profiles of particle velocity etc.

The boundary layer theory predicts certain profiles, which are confirmed on developing the exact equations (34). The modulus of  $Q_1$  is assumed small and those of Q and  $Q_2$  large, and the relation

$$m^2 = (h^2/c^2)(1 + 2\gamma'/r_w\sqrt{h})$$
 .....(47)

is used. Introducing a new constant B' the following expressions are obtained when near the wall:

$$u = B'[1 - (r/r_{w})^{1/2} \exp\{-\alpha b(1+i)\}]$$

$$q = \frac{B'\sqrt{\omega} e^{-i2\pi i/4}}{c} \left[ \frac{r}{r_{w}} \gamma' - \left(\frac{r}{r_{w}} \nu\right)^{1/2} \exp\{-\alpha b(1+i)\} - (\gamma - 1) \left(\frac{r\nu'}{r_{w}\gamma}\right)^{1/2} \exp\{-\beta b(1+i)\} \right]$$

$$\theta' = (B'/c)[1 - (r/r_{w})^{1/2} \exp\{-\beta b(1+i)\}]$$

$$(48)$$

The amplitudes of these quantities are plotted in fig. 3. It may be emphasized here that the enhanced particle velocity in the annulus shown in fig. 3 is a



Fig 2. Theoretical velocity c' and attenuation m' in a tube as a function of  $r_w \sqrt{f}$  for air at 20°c, 76 cm Hg.



Fig. 3. Cross section profiles in a wide tube. (a) amplitude of axial particle velocity or temperature fluctuation, (b) amplitude of radial particle velocity, (c) wave front or equiphase surface.

consequence of Kirchhoff's theory, and cannot be used to explain an experimental attenuation greater than the theoretical. The radial velocity has two components of the same order of magnitude travelling inwards from the wall, one with the velocity of viscosity waves and one with that of temperature waves. For continuous laminar flow it may be shown (Goldstein 1938, p. 51) that there is a velocity q in the layer normal to the surface such that  $q/u \sim \Delta/l$  where l is the distance traversed from the leading edge of a plate. It may be seen from (48) that this is also true for the acoustic layer, when l is the wavelength in the gas. There is not always a motion normal to the surface: for example q is zero in continuous or oscillatory rotational motion of a cylinder about its axis, since the phase does not vary over the surface. It may be inferred from an expression

due to Kirchhoff (see Rayleigh 1896, p. 322) that there is a particle velocity accompanying a temperature wave such that

$$q/\theta' = (\gamma - 1)(\omega \nu'/\gamma)^{1/2} \exp(-i3\pi/4)$$
 ..... (49)

agreeing with (48). Also  $u/\theta' = c$  outside the layer, which is the normal adiabatic relation.

Near the axis where the moduli of Q and  $Q_2$  are small the above analysis is not applicable. Modifying the analysis, only the first terms in (48) remain; this leads for example to the expression which satisfies the condition that the radial velocity falls to zero at the axis:

$$q = B' \sqrt{\omega} e^{-i 3\pi/4} \gamma' r/c r_{\rm w}. \qquad \dots \dots (50)$$

The significance of the radial particle velocity.

In the boundary layer theory for the velocity and attenuation the radial velocity was not considered, and this was permissible because the integral over the cross section of the pressure due to the velocity q must vanish.

Rewriting the temperature wave term in q from (48)

$$q = (\gamma - 1)(\omega \nu' / \gamma)^{1/2} e^{-i3\pi/4}\theta'$$
  
= - { $p_0(\gamma - 1)(\omega \nu')^{1/2}/P\gamma^{3/2}$ } exp { -  $i3\pi/4 - \beta b(1 + i) + i\omega t - mx$ }. .....(51)

The corresponding pressure excess

$$P\frac{d}{db}\int q\,dt = p_0\{(\gamma-1)/\gamma\}\exp\{-\beta b(1+i)+i\omega t - mx\}.$$
 (52)

This is equal in magnitude and opposite in sign to (21) so that altogether the temperature wave produces no pressure in the boundary layer. Similarly no pressure is developed in the layer by the viscosity wave. It should therefore be possible to calculate the extra pressure due to viscosity and heat conduction from the behaviour of q outside the layer, where from (50)

$$q = (p\gamma' r / P\gamma r_{\rm w}) \sqrt{\omega} e^{-i 3\pi/4}, \qquad \dots \dots (53)$$

and the corresponding displacement

$$\xi = (p\gamma' r/i\omega P\gamma r_{\rm w})\sqrt{\omega} e^{i\pi/4}. \qquad \dots \dots (54)$$

The corresponding pressure may be found using the adiabatic relation, since outside the layer q is associated with what is, essentially, an ordinary sound wave. The extra pressure is

$$-\gamma P[\partial \xi / \partial r + \xi / r] = \{2\gamma' / r_{\rm w}(2\omega)^{1/2}\}(i-1)p \qquad \dots \dots (55)$$

which agrees with (25). Thus the radial particle velocity has the effect of cancelling the extra forces at the boundary and distributing them evenly over the cross section.

### Transition to the very wide tube.

When the tube is so wide that the free gas attenuation begins to become comparable with the tube attenuation, Kirchhoff's assumption that the modulus of  $Q_1$  is very small breaks down. Substituting in (33) as did Kirchhoff, but without developing  $Q_1$ , we find

$$d \ln Q_1/dr_w = -h^{3/2}\gamma'/c^2.$$
 .....(56)

Using the first two terms of (38), (see table 1)

$$-(r_{\rm w}/2)(m^2-\lambda_1)[1+r_{\rm w}^2(m^2-\lambda_1)/8]=-h^{3/2}\gamma'/c^2.$$
 (57)

Hence

$$m^{2} - \lambda_{1} = (2h^{3/2}\gamma'/r_{w}c^{2}) - (h^{3}\gamma'^{2}/2c^{4}). \qquad (58)$$

A second approximation to  $\lambda_1$  is, from (31),

$$\lambda_1 = (h^2/c^2)[1 - (h/c^2)\{\nu + \nu'' + (\gamma - 1)\nu'/\gamma\}]. \quad (59)$$

The velocity and attenuation may be found from the complex propagation constant as before. The velocity is still predicted correctly by Kirchhoff's formula (1), but the attenuation is

$$m' = \frac{\gamma'}{r_{\rm w}c} \left(\frac{\omega}{2}\right)^{1/2} + \frac{\omega^2}{2c^3} \left\{ \nu + \nu'' + \frac{\gamma - 1}{\gamma} \nu' \right\} + \frac{\omega^2 \gamma'^2}{4c^3} = 2.964 \times 10^{-5} \sqrt{f/r_{\rm w}} + 1.403 \times 10^{-13} f^2 + 0.806 \times 10^{-13} f^2 \,\mathrm{cm}^{-1} \,\mathrm{for} \,\mathrm{air.} \dots (60)$$

The first term is the ordinary tube absorption, the second the ordinary free gas attenuation, and the third arises because the sound energy is not uniformly distributed over the tube but tends to concentrate near the walls. Bogert (1950) considering viscosity only and using an approximate method, obtained a similar result as far as the second term.

The profiles outside the boundary layer are

$$\begin{array}{l} u = B'[1 + \omega^{3/2} e^{-\imath \pi/4} r^2/2c^2 r_w] \\ q = (B'\gamma' \sqrt{\omega} e^{-\imath 3\pi/4} r/cr_w)[1 + \omega^{3/2} e^{-\imath \pi/4} r_w/4c^2] \\ \theta' = (B'/c)[1 + \omega^{3/2} e^{-\imath \pi/4} r^2/2c^2 r_w] \end{array} \right\} . \quad \dots \dots (61)$$

The axial velocity and temperature excess profiles have a parabolic curvature, being a minimum at the axis. The wave front is also parabolic, and convex towards +x.

(iv) The Very Wide Tube

Velocity and attenuation.

In the limit as the radius increases the modulus of  $Q_1$  becomes large so that formula (43) must be used, and (56) becomes

$$(\lambda_1 - m^2)^{1/2} - 1/2r_{\rm w} = -h^{3/2}\gamma'/c^2.$$
 (.....(62)

This leads to

$$c' = c \left\{ 1 - \frac{\gamma'}{2r_{\rm w}(2\omega)^{1/2}} \right\} = 3.434 \times 10^{4} \left\{ 1 - \frac{0.081}{r_{\rm w}\sqrt{f}} \right\} \, {\rm cm \, sec^{-1} \, for \, air} \qquad \dots \dots (63)$$
$$m' = \frac{\gamma'}{2cr_{\rm w}} \left(\frac{\omega}{2}\right)^{1/2} + \frac{\omega^{2}}{2c^{3}} \left\{ \nu + \nu'' + \frac{\gamma - 1}{\gamma} \nu' \right\} + \frac{\omega^{2}\gamma'^{2}}{2c^{3}}$$
$$= 1.482 \times 10^{-5} \sqrt{f/r_{\rm w}} + 1.403 \times 10^{-13} f^{2} + 1.612 \times 10^{-13} f^{2} \, {\rm cm^{-1} \, for \, air} \qquad \dots (64)$$

The velocity change and first absorption term have half their wide tube values. The boundary layer derivation given above is not valid because phase and intensity vary considerably over the cross section. The first attenuation term is now only a minor correction, the second is unchanged, and the last is doubled in comparison with (60).

## Cross section profiles of particle velocity etc.

The principal terms in the profiles (34) outside the boundary layer and away from the axis are (written in terms of a new constant B'')

$$u = (B''/\sqrt{r}) \exp \{r(1-i)\gamma'\omega^{3/2}/\sqrt{2c^2}\} q = (B''\gamma'\sqrt{\omega}/2c\sqrt{r}) \exp \{-i3\pi/4 + r(1-i)\gamma'\omega^{3/2}/\sqrt{2c^2}\} \theta' = (B''/c\sqrt{r}) \exp \{r(1-i)\gamma'\omega^{3/2}/\sqrt{2c^2}\}$$
.....(65)

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These are illustrated in fig. 4. The amplitude falls from a large value near the wall to a very small value near the axis. The thickness of the 'clinging' layer, measured to a point where the real part of the exponent is unity, is

$$\delta = \sqrt{2c^2/\gamma' \omega^{3/2}} = 1.84 \times 10^8 f^{-3/2} \text{ cm for air.}$$
 (66)

The discriminant between the wide and very wide tube types of propagation is thus

$$\frac{r_{\rm w}\omega^{3/2}\gamma'}{c^2} \propto \frac{r_{\rm w}}{\delta} \propto \text{ modulus of } Q_1 \propto \frac{\text{free gas absorption}}{\text{tube absorption}} \sim 10^{-8} r_{\rm w} f^{3/2}. \quad \dots \dots (67)$$

At a frequency of  $100 \text{ c/s} \delta$  is over a mile in thickness. Thus to make measurements in the wide to very wide tube transition region, tubes of the order of a centimetre radius and frequencies of the order of 10 kc/s are suitable. To keep below the cut-off frequency of the higher-order modes and also inside the transition region it would be useful to work at low pressures.



Fig. 4. Cross section profiles in a very wide tube, (a) amplitude of axial particle velocity or temperature fluctuation, (b) wave front or equiphase surface.

The wave front is inclined to the normal to the wall at an angle  $(\gamma'/c)(\omega/2)^{1/2}$ which is very small, and it is convex towards +x showing that the wave is divergent. The last term in m' is equal to the product of the 'attenuation' constant in the 'clinging' layer and the inclination of the wave front:

$$\gamma'^2 \omega^2 / 2c^3 = \gamma' \omega^{3/2} / \sqrt{2c^2} \times (\gamma'/c) (\omega/2)^{1/2}$$
 .....(68)

so that this extra attenuation corresponds to an enhanced energy transfer to the walls.

Rayleigh (1896, p. 327, 1901) has shown that there is a similar type of propagation between parallel rigid conducting plane walls, and obtains a term equivalent to the last in (64), but leaves out the second.

## (v) Other Solutions of the General Equation (33)

For very narrow tubes it is no longer permissible to neglect  $m^2$  in comparison with  $h/\nu$ . If we follow in other respects the narrow tube development we find

$$(r_{w}^{2}c^{2}/8\gamma h^{2})m^{4} - (r_{w}^{2}c^{2}/8\gamma \nu h)m^{2} + 1 = 0. \qquad \dots \dots (69)$$

There are three limiting cases, one expressed by (36), and the others

$$m^2 = 2\omega(2\gamma)^{1/2}/cr_w$$
 .....(70)  $m^2 = h/\nu$ . .....(71)

Equation (36) is Rayleigh's narrow tube solution, which passes over into (70) as the radius approaches the mean free path  $\lambda'$ . The classical continuum theory is

able to predict a correction at the radius (about  $10^{-5}$  cm) where its foundations begin to break down because the numerical value of  $\nu$  has been used, and this depends on  $\lambda'$ . There is a similar prediction when the wavelength approaches the mean free path. Using the resistance concept, and applying Knudsen's formula to (5), we find  $m^2 = 3h(2\pi\gamma)^{1/2}/8cr_w$  .....(72)

which is very similar to (70). The solution expressed by (71) suggests that viscosity waves propagating along the tube axis can co-exist with acoustic waves. There are also other solutions corresponding to temperature waves and to higher order modes in the tube.

# § 5. The Validity of Kirchhoff's Formulae

The assumptions which Kirchhoff made in deriving the wide tube formulae (1) and (2), in addition to those noted in table 1, are briefly stated here.

(a) Kirchhoff assumes a homogeneous medium, and only considers the extra effects due to viscosity and heat conduction.

(b) The sound amplitude is assumed small, so that there is no circulation or turbulence etc. (see Binder 1943).

(c) All the parameters are assumed to vary as  $e^{i\omega t - mx}$ , so that the formulae do not apply to pulses. Henry (1931) has pointed out that for a wave train the tube correction for the group velocity is only half that for the phase velocity. Boussinesq (1891) has applied boundary layer theory to give a comprehensive account of the propagation of pulses in tubes.

(d) The theory is for an indefinitely long tube, and it may take some distance for the velocity profile etc. to become stable. Thissen (1907) has investigated the extra heat conduction effects at the end of a tube. Schweikert (1915, 1917) has considered the positions of the nodes in a Kundt's tube, and predicts in some circumstances an apparent velocity reduction due to absorption and proportional to  $f^{-3/2}$ . This agrees with the experiments of Schneebeli (1869) and Seebeck (1870), which have long been difficult to interpret.

(e) Circular symmetry is assumed, so that the wall surface is given by  $r = r_w$ , the surface must also be perfectly smooth, impervious and rigid. Fissures parallel to the axis will increase the tube effect by increasing the ratio E/S (see eqn. (28)), and corrugations may cause an irregular gas motion at the wall. An imperfect wall surface is a frequent cause of experimental disagreement with Kirchhoff's formulae.

(f) The motion is assumed to vanish at the walls. The effect of non-rigid walls has been investigated by many workers (e.g. Korteweg 1878), and may be large at resonance or for a rubber-like tube. Henry (1931) considers the effect of a relative normal motion of tube and gas at the wall, such as might occur for a porous material etc. He concludes that this will lead to only a small velocity change, but may cause appreciable extra absorption. Henry also allows for molecular slip by modifying the last part of Kirchhoff's analysis, and his result together with that for the temperature discontinuity effect (paragraph (g)) may be written

$$c' = c \left\{ 1 - \frac{\gamma'}{r_{w}(2\omega)^{1/2}} + \left(\frac{\pi}{2P\rho}\right)^{1/2} \frac{\mu}{r_{w}} \frac{2 - f'}{f'} + \frac{(\gamma - 1)\sqrt{\gamma k}}{r_{w}\zeta} \right\}$$
  
=  $3 \cdot 434 \times 10^{4} \left\{ 1 - \frac{0 \cdot 162}{r_{w}\sqrt{f}} + \frac{6 \cdot 5 \times 10^{-6}}{r_{w}} \frac{2 - f'}{f'} + \frac{4 \cdot 4 \times 10^{-6}}{r_{w}} \frac{2 - g'}{g'} \right\} \text{ cm sec}^{-1} \text{ for air}$   
.....(73)

where according to Knudsen the thermal permeability

$$\zeta \sim (k/1 \cdot 9\lambda') \{ 2g'/(2-g') \} \qquad \dots \dots (74)$$

and  $\lambda'$  is the gaseous mean free path, f' and g' the coefficients of slip and accommodation. The correction terms in the attenuation formula due to slip, temperature discontinuity, and a cross-product term are all found to contain  $\omega\nu/c^2$ , and are therefore negligible at normal frequencies.

The corresponding expressions for a narrow tube are given by eqn. (5), using a value of R modified for slip:

$$c' = cr_{w} \left(\frac{\pi f}{2\gamma\nu}\right)^{1/2} \left\{ 1 + \left(\frac{2\pi}{P\rho}\right)^{1/2} \frac{\mu}{r_{w}} \frac{2-f'}{f'} \right\}$$
  
= 9.346 × 10<sup>4</sup>  $r_{w}\sqrt{f} \{1 + (13 \times 10^{-6}/r_{w})(2-f')/f'\} \operatorname{cm} \operatorname{sec}^{-1} \operatorname{for} \operatorname{air} \dots \dots (75)$   
 $m' = \frac{2(2\pi\gamma\nu f)^{1/2}}{cr_{w}} \left\{ 1 - \left(\frac{2\pi}{P\rho}\right)^{1/2} \frac{\mu}{r_{w}} \frac{2-f'}{f'} \right\}$   
= 6.701 × 10<sup>-5</sup>  $(\sqrt{f}/r_{w}) \{1 - (13 \times 10^{-6}/r_{w})(2-f')/f'\} \operatorname{cm}^{-1} \operatorname{for} \operatorname{air} \dots \dots (76)$ 

The correction term due to slip is twice as great in (75) as in (73).

(g) The gas temperature is assumed constant at the walls. Henry takes into account the finite temperature discontinuity, but makes some errors in an algebraic transformation. Thus the predicted effect on the attenuation is too large by several orders of magnitude, and the corrected velocity may be written as in (73). Zwikker (1941) also performs part of this calculation but it is not clear where he obtains the low value for  $\zeta$ , which invalidates his conclusions.

There is also a small temperature variation in the wall at its surface. Since it has a finite thermal conductivity  $k_w$  and finite volume specific heat  $\rho_w c_w$ , rapidly damped temperature waves are propagated into the wall thickness. Henry (1931) and Nielsen (1949*a*) have shown independently that for both attenuation and velocity  $\gamma'$  is changed very slightly

$$\gamma' = \sqrt{\nu + (\gamma - 1)(\nu'/\gamma)^{1/2}} \{ 1 - (k\rho c_{\rm p}/k_{\rm w}\rho_{\rm w}c_{\rm w})^{1/2} \}.$$
(77)

(*k*) Kirchhoff assumes that the velocity of propagation of viscous effects  $(2\nu\omega)^{1/2}$  or of temperature waves  $(2\nu'\omega/\gamma)^{1/2}$  is much less than the velocity of sound *c*; i.e.  $\omega\nu/c^2 \ll 1$  and  $\omega\nu'/\gamma c^2 \ll 1$ . This is again equivalent to the sound wavelength being much greater than the mean free path, which approximation breaks down at about 10° c/s for normal pressures.

(i) Kirchhoff's formulae only apply to waves of the principal mode. Rayleigh (1896, p. 161) has shown that a wave must eventually become plane if the frequency be lower than the lowest natural transverse frequency, so for a circular tube it is desirable that  $\lambda > 3.413r_{w}$ . .....(78)

### § 6. Application of the Theory to Lawley's Results

The application of the theory to the recent work of Lawley (1952) is of interest, especially to the part with varying pressure. He measures the attenuation in oxygen at 120 kc/s, in a tube of diameter 1.5 mm, and at pressures from 5 to 130 cm Hg. That part of the absorption proportional to  $P^{-1/2}$  (tube absorption,  $K_{\rm T}$ ) can be separated from that proportional to  $P^{-1}$  (free gas absorption,  $K_{\rm G}$ ) by plotting  $m'P^{1/2}$  against  $P^{-1/2}$ .

Examination of eqns. (45) and (60) shows that although the tube is 'wide' all the time, the corrections arising because of the transition to both the narrow and very wide tube types are of a magnitude comparable with the free gas absorption.

The complete expression is

$$m' = \frac{\gamma'(\pi f)^{1/2}}{cr_{w}} + \frac{\gamma''}{cr_{w}^{2}} + \frac{2\pi}{c^{3}} \left\{ \nu + \nu'' + \frac{\gamma - 1}{\gamma} \nu' \right\} f^{2} + \frac{\pi^{2} \gamma'^{2} f^{2}}{c^{3}} \qquad \dots \dots (79)$$
  
$$n' P^{1/2} = K_{T} + K_{G} P^{-1/2}$$
  
$$= 1.25 + (0.068 + 0.175 + 0.101) P^{-1/2} \qquad \dots \dots (80)$$

or n

$$1 \cdot 25 + (0 \cdot 068 + 0 \cdot 175 + 0 \cdot 101)P^{-1/2} \qquad \dots \dots (80)$$

for oxygen at  $27^{\circ}$ c, where *P* is in cm Hg.

It is now found that the theoretical value for  $K_{G}$  is greater than the experimental figure (table 2). Probably the main effect, as yet unconsidered, is the molecular

#### Table 2. Values of Constants in equation (80)

	$K_{ m T}$	$K_{ m G}$
Experimental (Lawley)	1.38	0.246
Theoretical	1.25	0.344

absorption, in particular the manner in which it varies with pressure. The latter depends on the relaxation frequency at atmospheric pressure.

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