Automatica 127 (2021) 109496

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Convergence time of average consensus with heterogeneous random link failures[☆]

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ARTICLE INFO

Article history:
Received 25 June 2019
Received in revised form 27 October 2020
Accepted 8 January 2021
Available online 12 February 2021

Keywords:
Average consensus
Random link failures
Convergence time
Event-triggered communication

ABSTRACT

Average consensus algorithms have attracted considerable attention in recent years. Communication impairments make the network suffer from random link failures. In this paper, for networks with heterogeneous random link failures, we investigate the convergence time of the average consensus problem and consider the impacts of event-triggered communication on the consensus performances. We introduce (α, γ)-convergence time to evaluate how fast the algorithm converges in probability γ to the value at most α away from the average. And we derive the dynamical range of the consensus parameter that guarantees the convergence, together with the upper and lower bounds of the closed-form expression for (α, γ)-convergence time. Then, the impact of the event-trigger communication on the convergence accuracy is analyzed, and an upper bound of finite (α, γ)-convergence time is further derived, which can be employed to calculate the triggering parameters with guaranteed convergence time. Finally, extensive simulations are conducted to verify the results.

1. Introduction

Distributed algorithms to achieve average consensus in networks have been widely investigated (Carli & Zampieri, 2014; Franceschelli & Gasparri, 2019; Olfati-Saber & Murray, 2004; Ren, Beard, & Atkins, 2007; Schenato & Fiorentin, 2011; Xiao & Boyd, 2004; Xie & Lin, 2017; Xie & Wang, 2007). For average consensus on static (i.e., time-invariant, fixed) digraphs, Olfati-Saber and Murray (2004) and Xiao and Boyd (2004) justified that a balanced and strongly connected topology was necessary and sufficient to guarantee convergence. Weight-balanced digraphs are essential in distributed averaging. In Rikos, Charalambous, and Hadjicostis (2014), Rikos et al. proposed distributed algorithms over static topologies to solve the weight balancing problem when the weights were arbitrary non-negative real numbers. Given a general strongly connected digraph, Gharesifard and Cortés (2012) considered how to find corresponding weight-balanced digraphs, and proposed two algorithms to achieve this goal by selecting the out-edge weights to balance the in-degrees and out-degrees.

Randomized designed network communication protocol, interferences in wireless sensor networks and ad hoc networks, imperfect fading channels, and etc., make the topology (connectivity) vary with time. In these scenarios, it is desirable to quantify the effects of randomness on the performance of average consensus algorithms. To the best of our knowledge, the first work on the average consensus over random networks is Hatano and Mesbahi (2005), with several works followed in different settings (Aysal, Yildiz, Sarwate, & Scaglione, 2009; Boyd, Ghosh, Prabhakar, & Shah, 2006; Fagnani & Zampieri, 2007; Patterson, Bamieh, & El Abbadi, 2010; Pereira & Pagès-Zamora, 2010; Tahbaz-Salehi & Jadbabaie, 2008, 2009). In Boyd et al. (2006), Boyd et al. analyzed the averaging problem for an arbitrary network graph with pairwise gossip constraints, and derived the upper and lower bounds of the ϵ-averaging time, which was depended on the second largest eigenvalue of a doubly stochastic matrix. Pereira and Pagès-Zamora (2010) investigated the almost sure convergence in the mean square sense for wireless sensor networks with random asymmetric topologies, and analyzed the mean square error (MSE) of the state in probabilistic term from two different cases: links with equal and different probabilities of connection. Aysal et al. (2009) studied an asynchronous broadcasted gossiping algorithm to calculate the (possibly weighted) average of the initial measurements of the nodes in the network, and showed that the broadcast gossip algorithm converges almost

☆ This work is supported by National Natural Science Foundation of China (NSFC) under Grant Nos. 51874205, 61903328, 61803279, 61876121 and 61672371; Jiangsu Qinglan Project. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Julien M. Hendrickx under the direction of Editor Christos G. Cassandras.

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https://doi.org/10.1016/j.automatica.2021.109496
0005-1098/

surely to a consensus value, which was in expectation equal to the desired average of initial measurements. Patterson et al. (2010) defined the convergence in terms of the variance of deviation from the average, and derived the expressions for the mean square convergence rate in the asymptotic limits of a small failure probability for large networks. In Fagnani and Zampieri (2007), Fagnani and Zanpieri relaxed the requirement on symmetric communication graph and focused on the probability consensus analysis of random protocols with homogeneous link failures. They derived conditions to achieve probability consensus, and analyzed the mean square performance of the normalized distance from the consensus. Tahbaz-Salehi and Jadbabaie (2009) considered the convergence of consensus algorithms when the underlying graphs of the network were generated by an ergodic and stationary random process. They proved that the consensus algorithms converge almost surely, if and only if, the expected graph of the network contains a directed spanning tree.

Event-triggered communication mechanism offers an opportunity for improving efficiency while maintaining system performance (see, e.g., Dimarogonas, Frazzoli, & Johansson, 2012; Ding, Wang, Shen, & Wei, 2015; Seyboth, Dimarogonas, & Johansson, 2013; Yi, Yang, Wu, & Johansson, 2019 and the references therein), and a recent overview on event-triggered consensus and control of multi-agent systems can be referred to in Nowzari, Garcia, and Cortés (2019). For networks of a single integrator agent with or without communication delays, and networks of double-integrator agents, Seyboth et al. (2013) presented a control strategy for multi-agent coordination with event-based broadcasting. Ding et al. (2015) focused on the event-triggered consensus control problem for a class of discrete-time stochastic multi-agent systems with state-dependent noises. In Yi et al. (2019), an event-triggered consensus protocol for first-order continuous-time multi-agent systems with input saturation was considered, and there was no requirement of any prior knowledge on the global network parameters. It should be pointed out that most of the existing works focus on the event-triggered consensus mechanism design for networks with deterministic links, where the randomness of links is neglected. We fill this gap by studying the convergence time problem of event-triggered average consensus over random networks, which is more practical given the random behavior of communication links, particularly in the wireless domain. When considering the event-triggered average consensus for a network with random link failures, the following open issues should be addressed carefully. Topologies of networks with random link failures are relatively sparser, and the introduction of the event-triggered communication will make the topology even sparser, is it feasible to introduce the event-triggered communication to those kinds of networks? If so, how to quantify the impact of the event-triggered communication on the consensus performances, and how to design the triggering function such that both the convergence and convergence accuracy can be guaranteed?

In this paper, inspired by Fagnani and Zampieri (2007) and Tahbaz-Salehi and Jadbabaie (2009), we investigate the average consensus over random networks with heterogeneous random link failures, and focus on the analysis of convergence efficiency. We define a new notion of (α, γ)-convergence time, rather than the ϵ-averaging time in Aysal et al. (2009) and Boyd et al. (2006), to evaluate how fast the algorithm converges in probability γ to the value at most α away from the average. The random network we considered is a special case of those proposed in Fagnani and Zampieri (2007) and Tahbaz-Salehi and Jadbabaie (2009), and we try to derive an explicit result on the convergence time. Moreover, Fagnani and Zampieri (2007) assumed the link probabilities in the network to be homogeneous. Also, the prearranged network we considered is different from Pereira and Pagès-Zamora (2010). We assume it to be connected undirected or strongly connected directed, while the one in Pereira and Pagès-Zamora (2010) was assumed to be fully connected, which may not always be satisfied in practice, especially for large-scale networks. Furthermore, inspired by the existing event-triggered mechanism for continuous-time cases in Seyboth et al. (2013), we further study the event-triggered scenario to qualify the performance and reveal the impact of the event-triggered mechanism.

The main contributions are listed as follows: (1) We provide the average consensus model in networks with heterogeneous random link failures, analyze the condition on consensus parameter for convergence, and derive the analytical expression of the lower and upper bounds of (α, γ)-convergence time. (2) We consider the impacts of event-triggered mechanism to the average consensus with heterogeneous random link failures, derive the upper bound of the steady mean square convergence error, and the expression of the finite upper bound (α, γ)-convergence time.

The remainder of this paper is organized as follows. Section 2 introduces the basic notations and concepts. The problem formulation is introduced in Section 3. Section 4 presents the analysis of the convergence and the performance-based design of the triggering parameters. Simulation results and concluding remarks are given in Section 5 and Section 6, respectively.

2. Preliminaries

2.1. Notation

Let 1_n and 0_n be the vector consisting of n ones and zeros, respectively. For a matrix $A \in R^{n\times n}$, its transpose and spectral radius are denoted by A^{T} and $\rho(A)$, respectively, while its Euclidean-norm is denoted by $\|A\| = \sqrt{\lambda_{max}(A^{\mathrm{T}}A)}$, where $\lambda_{max}(A)$ is the maximum eigenvalue of A. We use notation $\mathrm{diag}(a_i)$ to denote a diagonal matrix with a_i being its (i, i)th entry. Matrix A is row stochastic if it is non-negative and satisfies $A1_n = 1_n$. Moreover, A is doubly stochastic if both A and A^{T} are row stochastic. For a given vector $x \in R^n$, its cardinality and Euclidean norm are denoted by $|x|$ and $\|x\|$, respectively. Symbols $\Pr\{\cdot\}$ and $E[\cdot]$ denote the probability and expectation, respectively. The conditional expectation of X given Y is denoted by $E\big[X|Y\big]$. I_n is the identity matrix with dimension $n \times n$. For any real number r, its ceiling and floor functions are denoted as $\lceil r\rceil$ and $\lfloor r\rfloor$, respectively. Matrix Π_n is denoted by $1_n 1_n^{\mathrm{T}}/n$.

2.2. Graph theory

The information flow communicated among nodes of a fixed network can be modeled by a digraph $G = (V, \mathcal{E})$, where $V = \{1, 2, \ldots, N\}$ is the nonempty set of N nodes, and $\mathcal{E} = \big\{(j, i) : i, j \in V\big\}$ denotes the directed edge set. A directed edge $(j, i) \in \mathcal{E}$ means that node i can obtain information from node j. Let $A = [A_{ij}]$ be the adjacency matrix of network G. If $(j, i) \in \mathcal{E}$, one has $A_{ij} = 1$; otherwise $A_{ij} = 0$. For $i \neq j$, if $(j, i) \in \mathcal{E}$ implies $(i, j) \in \mathcal{E}$, the graph is said to be undirected, and it is directed, otherwise. A directed path in a digraph is an ordered sequence of nodes so that any two consecutive vertices in the sequence are an edge of the digraph. A digraph is strongly connected if, for any distinct nodes i and j, there exists a directed path that connects them. The in-degree and out-degree of node i are defined as $d_i = \sum_{j=1}^{N} A_{ij}$ and $d_i^o = \sum_{j=1}^{N} A_{ji}$, respectively. A digraph is balanced if the in-degree and out-degree are equal for all nodes. The in-neighbor set of node i is denoted as $N_i = \big\{j \in V, j \neq i, (j, i) \in \mathcal{E}\big\}$. The Laplacian matrix of a graph is defined as $L = D - A$, where $D = \mathrm{diag}(d_i)$ is the in-degree matrix. For a connected graph, 0 is the simple eigenvalue of L with the associated eigenvector 1_N, and the eigenvalues of matrix L can be ordered in the ascending form as $0 = \big|\lambda_1(L)\big| < \big|\lambda_2(L)\big| \leq \cdots \leq \big|\lambda_N(L)\big|$.

3. Problem formulation

The average consensus in networks with heterogeneous random link failures and the event-triggered communication are formulated in this section.

3.1. Average consensus over networks with heterogeneous link failures

We consider the average consensus in a prearranged network $G = (V, \mathcal{E})$ with heterogeneous random link failures, where the adjacency and Laplacian matrices are denoted as A and L, respectively. The random network caused by link failures at time k $(k = 0, 1, 2, \ldots)$ is denoted as $G(k) = \big(V, \mathcal{E}(k)\big)$, and the (i, j)-entry of its adjacency matrix $A(k)$ is written as

$$A_{ij}(k) = A_{ij}\delta_{ij}(k), \tag{1}$$

where $\delta_{ij}(k) = 0$ denotes the link failure from node j to i at k. In random network $G(k)$, let $0 < p_{ij} \leq 1$ be the link probability of $(j, i) \in \mathcal{E}$, and one has

$$\delta_{ij}(k) = \begin{cases} 1, & with\ p_{ij}, \\ 0, & with\ 1 - p_{ij}. \end{cases} \tag{2}$$

Here, we assume that for all $k, l = 0, 1, 2, \ldots$ and $k \neq l$, $\delta_{ij}(k)$, $\delta_{ij}(l)$ are independent and identically distributed (i.i.d.). The in-degree and Laplacian matrices of $G(k)$ are denoted by $D(k) = \text{diag}\big(d_i(k)\big)$ and $L(k) = D(k) - A(k)$, respectively, where $d_i(k) = \sum_{j=1}^{N} A_{ij}(k)$. Moreover, the expected network $\bar{G}$ is defined as the one associated with the expected adjacency matrix $\bar{A} = E\big[A(k)\big]$ and Laplacian matrix $\bar{L} = E\big[L(k)\big]$.

The update of state $x_i(k)$ for node i is designed as

$$x_i(k+1) = x_i(k) - \mu \sum_{j\in V} A_{ij}(k)\big(x_i(k) - x_j(k)\big), \tag{3}$$

where $\mu > 0$ is the parameter to weight the disagreement among neighboring nodes and its design will be presented in Section 4.1. Let $x(k) = \big[x_1(k), \ldots, x_N(k)\big]^{\mathrm{T}}$, one can obtain the matrix form of (3) as

$$x(k+1) = \big(I_N - \mu L(k)\big)x(k) \triangleq W(k)x(k), \tag{4}$$

where $W(k) = I_N - \mu L(k)$.

3.2. Event-triggered communication over random networks

For node i $(i \in V)$, we use $\hat{x}_i(k)$ to denote the latest broadcasted state, and k_l^i to denote its lth triggering instant. Assume that for $k \in [k_l^i, k_{l+1}^i)$, $\hat{x}_i(k) = \hat{x}_i(k_l^i)$ and node i keeps its in-neighbor's state $x_j(k_l^j)$ as $\hat{x}_j(k)$ before a new information reception. Then, (3) is changed into

$$x_i(k+1) = x_i(k) - \mu \sum_{j\in V} A_{ij}(k)\big(\hat{x}_i(k) - \hat{x}_j(k)\big). \tag{5}$$

Let $e_i(k) = \hat{x}_i(k) - x_i(k)$ be the difference between the latest broadcasted and current state of node i. Motivated by Seybold et al. (2013), the distributed triggering function $f_i\big(e_i(k)\big)$ is designed with one exponentially decreasing threshold β^k and one constant offset c, shown as

$$f_i\big(e_i(k)\big) = |e_i(k)| - (c + \beta^k), \tag{6}$$

where $c > 0$ and $\beta > 0$ are triggering parameters to be designed in Section 4.2. Specifically, the event times for node i are defined by

$$k_{l+1}^i = \min\Big\{k \geq k_l^i \Big| f_i\big(e_i(k)\big) \geq 0\Big\}.$$

3.3. Some definitions

Let $x_{ave} = \Pi_N x(0)$ be the initial average vector of N nodes. The definition of convergence to the average is given as follows.

Definition 1 (*Convergence* Boyd et al., 2006)**.** A sequence of random vectors $\big\{x(k)\,(k = 0, 1, 2, \ldots)\big\}$ converges to the average x_{ave}, (i) in expectation, if $\lim_{k\to\infty} E\big[x(k)\big] = x_{ave}$; (ii) in mean square, if $E\big[\|x(k)\|^2\big] < \infty$, $E\big[\|x_{ave}\|^2\big] < \infty$, and $\lim_{k\to\infty} E\big[\|x(k) - x_{ave}\|^2\big] = 0$.

It should be mentioned that (3) and (5) are randomized because of the link failures and/or event-triggered communications. Then $x(k)$ might not converge exactly to the initial average x_{ave}, but to a neighborhood of it (Fagnani & Zampieri, 2007). This is captured by the notion of convergence in (α, γ)-probability, and the earliest time for achieving α accuracy with γ probability is defined as (α, γ)-convergence time.

Definition 2 (*Convergence in (α, γ)-probability* Ding et al., 2015)**.** One consensus protocol achieves convergence in (α, γ)-probability, if for any $\gamma \in [0, 1]$ and $\alpha \in R \geq 0$, the sequence of random vectors $\big\{x(k)\,(k = 0, 1, 2, \ldots)\big\}$ converges to a ball centered at x_{ave} with radius α and probability at least γ, i.e., $\lim_{k\to\infty} \Pr\big\{\|x(k) - x_{ave}\| \leq \alpha\big\} \geq \gamma$ holds.

Definition 3 (*(α, γ)-convergence Time*)**.** For a sequence of random vectors $\{x(k)\,(k = 0, 1, 2, \ldots)\}$, given any $\gamma \in [0, 1]$ and $\alpha \in R \geq 0$, suppose that $\|x(0) - x_{ave}\| \neq 0$, its (α, γ)-convergence time T_k is defined as

$$T_k = \inf\left\{k : \Pr\left\{\frac{\|x(k) - x_{ave}\|}{\|x(0) - x_{ave}\|} \leq \alpha\right\} \geq \gamma\right\},$$

indicating the least iterations needed for $x(k)$ to be α close to x_{ave} with probability at least γ.

4. Main results

In this section, we will illustrate the results of convergence analysis and the triggering parameter design for average consensus over networks with heterogeneous random link failures.

4.1. Convergence analysis

We first make some assumptions and provide a lemma.

Assumption 1. The prearranged network G is undirected connected. For a pair of communicating nodes i and j, they suffer the same link failures, i.e., $\delta_{ij}(k) = \delta_{ji}(k)$ and $p_{ij} = p_{ji}$. Therefore, $W(k)$ and $\bar{W} = E\big[W(k)\big]$ are symmetric.

Assumption 2. All $W(k)$ $(k = 0, 1, 2, \ldots)$ are i.i.d. Therefore, $E\big[W(k)W(l)\big] = \bar{W}^2$ $(k \neq l)$ holds.

Lemma 1 (Boyd et al., 2006)**.** *Let X be a random variable such that $0 \leq X \leq B$, then for any $0 < \alpha < B$, one has*

$$\Pr\big\{X \geq \alpha\big\} \geq \frac{E[X] - \alpha}{B - \alpha}.$$

Let $\tilde{x}(k) = x(k) - x_{ave}$ be the difference between $x(k)$ and the initial average x_{ave}. Under Assumption 1, $W(k)$ is symmetric, then $W(k)1_N = 1_N$ and $1_N^{\mathrm{T}}W(k) = 1_N^{\mathrm{T}}$ hold. Hence, one has $\Pi_N x(k) = x_{ave}$. Then the evolution of $\tilde{x}(k)$ can be written as

$$\begin{aligned} \tilde{x}(k+1) &= \big(W(k) - \Pi_N\big)\big(\tilde{x}(k) + \Pi_N x(k)\big) \\ &= M(k)\tilde{x}(k) = \prod_{l=0}^{k} M(k-l)\tilde{x}(0), \end{aligned} \tag{7}$$

where $M(k) = W(k) - \Pi_N$. Let $\bar{M}$ be $E[M(k)]$. Under Assumption 2, one has $E\left[\tilde{x}(k)\right] = \bar{M}^k \tilde{x}(0)$.

The results on convergence and convergence time are given in Lemma 2 and Theorem 1, respectively.

Lemma 2 (*Fagnani & Zampieri, 2007; Tahbaz-Salehi & Jadbabaie, 2009*). *Under Assumptions 1–2, if $\rho(\bar{M}) < 1$, then average consensus protocol (4) over prearranged network G achieves its convergence in expectation.*

Theorem 1. *For average consensus protocol (4) over prearranged network G under Assumptions 1–2, then, (i) $\rho(C_W) < 1$ guarantees its convergence in mean square; (ii) if $\rho(C_W) < 1$ and $\rho(\bar{M}) < 1$, the finite (α, γ)-convergence time T_k is upper bounded by*

$$T_{ku} = \left\lceil \frac{\log\left(\alpha^2(1-\gamma)\right)}{\log \rho(C_W)} \right\rceil, \tag{8}$$

and there exists an $x(0)$ such that T_k is lower bounded by

$$T_{kl} = \left\lfloor \frac{\log(1-\gamma+\gamma\alpha^2)}{2\log \rho(\bar{M})} \right\rfloor, \tag{9}$$

where $C_W = E[M^{\mathrm{T}}(k)M(k)]$.

Proof. We first give the convergence analysis of (4) in mean square. From (7), one has

$$\begin{aligned} &E\left[\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)\middle|x(k-1)\right] \\ &= E\left[\tilde{x}^{\mathrm{T}}(k-1)M^{\mathrm{T}}(k-1)M(k-1)\tilde{x}(k-1)\right] \\ &= \tilde{x}^{\mathrm{T}}(k-1)E\left[M^{\mathrm{T}}(k-1)M(k-1)\right]\tilde{x}(k-1) \\ &\le \rho(C_W)\tilde{x}^{\mathrm{T}}(k-1)\tilde{x}(k-1), \end{aligned} \tag{10}$$

where $C_W = E\left[M^{\mathrm{T}}(k)M(k)\right]$. With repeatedly conditioning and by using (10), one finally obtains

$$E\left[\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)\right] \le \rho^k(C_W)\tilde{x}^{\mathrm{T}}(0)\tilde{x}(0). \tag{11}$$

Thus if $\rho(C_W) < 1$, one has $\lim_{k\to\infty} E\left[\|\tilde{x}(k)\|^2\right] = 0$. Hence, the convergence of (4) in mean square is achieved.

For the upper bound of (α, γ)-convergence time, according to Markov's inequality, for any non-zero $\|\tilde{x}(0)\|$, from (11), one has

$$\Pr\left\{\frac{\|\tilde{x}(k)\|}{\|\tilde{x}(0)\|} \ge \alpha\right\} \le \frac{E\left[\|\tilde{x}(k)\|^2\right]}{\alpha^2\|\tilde{x}(0)\|^2} \le \frac{\rho^k(C_W)}{\alpha^2}.$$

Then the upper bounded (α, γ)-convergence time T_{ku} is derived by finding the ceil of the solution to $\rho^k(C_W) = (1-\gamma)\alpha^2$, shown as (8). Then, for $k \ge T_{ku}$, we have $\Pr\left\{\frac{\|x(k)-x_{ave}\|}{\|\tilde{x}(0)\|} \le \alpha\right\} \ge \gamma$.

Note that $\bar{W}$ is symmetric under Assumption 1, then $\bar{M} = \bar{W} - \Pi_N$ has N real eigenvalues. If $\rho(\bar{M}) < 1$, the N real eigenvalues of $\bar{M}$ can be written in the ascending order, i.e., $0 = \lambda_1(\bar{M}) < \lambda_2(\bar{M}) \le \cdots \le \lambda_N(\bar{M}) = \rho(\bar{M}) < 1$, with the corresponding orthonormal eigenvectors denoted as $1_N, \nu_2, \ldots, \nu_N$. Thus, if the initial state is chosen as (Boyd et al., 2006)

$$x(0) = \frac{1_N}{\sqrt{N}} + \nu_N, \tag{12}$$

we have $\tilde{x}(0) = \nu_N$ since $\Pi_N \nu_N = 0_N$. Then $E\left[\tilde{x}(k)\right] = \lambda_N^k(\bar{M})\nu_N$ holds. By using Jensen's inequality, one has

$$\begin{aligned} E\left[\|\tilde{x}(k)\|_2^2\right] &= E\left[\sum_{i=1}^N \tilde{x}_i(k)^2\right] \ge \sum_{i=1}^N E\left[\tilde{x}_i(k)\right]^2 \\ &= E\left[\tilde{x}(k)\right]^{\mathrm{T}} E\left[\tilde{x}(k)\right] = \rho^{2k}(\bar{M}). \end{aligned} \tag{13}$$

Note that if $W(k)$ and $x(0)$ are independent, and $E\left[\tilde{x}(k)\right] = \bar{M}^k\tilde{x}(0)$ holds under Assumption 2, then one has

$$E\left[\|\tilde{x}(k)\|^2\right] < E\left[\|\tilde{x}(0)\|^2\right] = 1$$

due to $\rho(\bar{M}) < 1$. According to Lemma 1, for any $\alpha < 1$, one can calculate that

$$\Pr\left\{\frac{\|\tilde{x}(k)\|}{\|\tilde{x}(0)\|} \ge \alpha\right\} \ge \frac{\rho^{2k}(\bar{M}) - \alpha^2}{1-\alpha^2}.$$

Let $\frac{\rho^{2k}(\bar{M})-\alpha^2}{1-\alpha^2} = 1-\gamma$, the lower bound of (α, γ)-convergence time is shown as (9), and it follows that for $k \le T_{kl}$, $\Pr\left\{\frac{\|\tilde{x}(k)\|}{\|\tilde{x}(0)\|} \le \alpha\right\} \le \gamma$ holds. This concludes the proof. □

Inspired by Fagnani and Zampieri (2007), Pereira and Pagès-Zamora (2010), and Xiao and Boyd (2004), for (4) over random networks with heterogeneous link failures, we will illustrate the design for consensus parameter μ, together with the relationship between $\rho(C_W)$ and $\rho(\bar{M})$.

Lemma 3. *Under Assumptions 1–2, if μ is designed as*

$$0 < \mu \le \frac{2}{\lambda_N(L) + \lambda_2(L)}, \tag{14}$$

then, (i) $\rho(\bar{M}) = 1 - \mu\lambda_2(\bar{L}) < 1$, the convergence of (4) in expectation is achieved; (ii) $\rho(C_W) \le \rho(\bar{M}) < 1$, indicating that convergence in expectation guarantees convergence in mean square.

Proof. The proof to Lemma 3 can be found in Appendix A.

Remark 1. In (14), since for a connected network G, one has $2d_{max} \ge \lambda_N(L) + \lambda_2(L)$. Then if μ is selected as $0 < \mu < \frac{1}{d_{max}}$(Bullo, 2019; Garin & Schenato, 2010), our results in Lemma 3 hold. Also, under Assumption 1, $W(k)$ $(k = 0, 1, 2, \ldots)$ and $\bar{W}$ are doubly stochastic.

Next, we extend our results to balanced directed networks by relaxing Assumption 1.

Assumption 3. The prearranged network G is a strongly connected digraph. For a random network $G(k)$, $\forall i \in V$ and $\forall k = 0, 1, 2, \ldots$, $\sum_{j=1}^N A_{ij}p_{ij} = \sum_{j=1}^N A_{ji}p_{ji}$ and $\sum_{j=1}^N A_{ij}\delta_{ij}(k) = \sum_{j=1}^N A_{ji}\delta_{ji}(k)$ hold.

Lemma 4. *Under Assumptions 2–3, for average consensus protocol (4), we have, (i) $\rho(\bar{M}) < 1$ and $\rho(C_W) < 1$ guarantee the average consensus in expectation and mean square, respectively; (ii) its (α, γ)-convergence time T_k is upper bounded by (8); (iii) if $\bar{W}$ is further symmetric, there exists an initial vector $x(0)$ such that (α, γ)-convergence time T_k is lower bounded by (9).*

Proof. The proof to Lemma 4 can be found in Appendix B.

4.2. Performance-based triggering parameters design

In this section, we will extend the analysis on convergence time of (4) over random networks to the event-triggered case, and the main results on the design of the triggering parameters are given in Theorem 2.

Theorem 2. *Consider random network $G(k)$ with random links (1) over any connected prearranged network G, for event-triggered average consensus protocol (5), with triggering function designed as (6), if $\rho(C_W) < 1$ and $0 < \beta \le \rho^{\frac{1}{2}}(C_W)$ hold, then, (i) the steady-state convergence error is upper-bounded by $R(c, \bar{L}) = \frac{\mu c\sqrt{N}\rho^{\frac{1}{2}}(C_L)}{1-\rho^{\frac{1}{2}}(C_W)}$; (ii) its convergence in $\left(\alpha, 1 - \frac{R(c,\bar{L})}{\alpha}\right)$ probability is achieved if $R(c, \bar{L}) < \alpha$; (iii) if triggering parameter c is designed as*

$$0 < c < \frac{\alpha(1-\gamma)\left(1-\rho^{\frac{1}{2}}(C_W)\right)\|\tilde{x}(0)\|}{\mu\sqrt{N}\rho^{\frac{1}{2}}(C_L)}, \tag{15}$$

its finite (α, γ)-convergence time is upper bound by

$$T_{ku}^{e} = \left\lceil \frac{\boldsymbol{W}(y)}{\ln \rho^{\frac{1}{2}}(C_W)} - \frac{1}{\zeta} \right\rceil, \tag{16}$$

where $C_L = E[L^{\mathrm{T}}(k)L(k)]$, $C_W = E[M^{\mathrm{T}}(k)M(k)]$, $\zeta = \frac{\mu\sqrt{N}\rho^{\frac{1}{2}}(C_L)}{\|\tilde{x}(0)\|\rho^{\frac{1}{2}}(C_W)}$, $y = \left(1-\gamma-\frac{R(c,\bar{L})}{\alpha\|\tilde{x}(0)\|}\right)\frac{\alpha\rho^{\frac{1}{2\zeta}}(C_W)}{\zeta}\ln\rho^{\frac{1}{2}}(C_W)$, *and* $\boldsymbol{W}(y)$ *is the solution to* $ke^k = y$.

Proof. Since $e_i(k) = x_i(k) - \hat{x}_i(k)$, (5) can be rewritten as

$$x(k+1) = W(k)x(k) + \mu L(k)e(k), \tag{17}$$

where $e(k) = \left[e_1(k), \ldots, e_N(k)\right]^{\mathrm{T}}$. Under Assumptions 1 and 3, one has

$$\tilde{x}(k) = \prod_{l=0}^{k-1} M(k-1-l)\tilde{x}(0) + \mu \sum_{l=0}^{k-1}\prod_{i=0}^{l-1} M(k-1-i)L(k-1-l)e(k-1-l).$$

Let C_L and M_k^l be $E\left[L^{\mathrm{T}}(k)L(k)\right]$ and $\prod_{i=0}^{l-1}E\left[\|M(k-1-i)\|\right]E\left[\|L(k-1-l)\|\right]E\left[\|e(k-1-l)\|\right]$, respectively. Noting that triggering condition $f_i(k) = |e_i(k)| - (c+\beta^k) \geq 0$ enforces $E\left[\|e(k)\|\right] \leq \sqrt{N}(c+\beta^k)$, one has

$$\begin{aligned}
E\left[\|\tilde{x}(k)\|\right] &\leq \prod_{l=0}^{k-1} E\left[\|M(k-1-l)\|\right]\|\tilde{x}(0)\| + \mu\sum_{l=0}^{k-1} M_k^l \\
&\leq \rho^{\frac{k}{2}}(C_W)\|\tilde{x}(0)\| + \mu\sqrt{N}\rho^{\frac{1}{2}}(C_L)\sum_{l=0}^{k-1}\rho^{\frac{l}{2}}(C_W)(c+\beta^{k-1-l}) \\
&= \rho^{\frac{k}{2}}(C_W)\|\tilde{x}(0)\| + \mu c\sqrt{N}\rho^{\frac{1}{2}}(C_L)\frac{1-\rho^{\frac{k}{2}}(C_W)}{1-\rho^{\frac{1}{2}}(C_W)} \\
&\quad + \mu\sqrt{N}\rho^{\frac{1}{2}}(C_L)\sum_{l=0}^{k-1}\rho^{\frac{l}{2}}(C_W)\beta^{k-1-l}.
\end{aligned} \tag{18}$$

If the threshold β in (6) is chosen as $0 < \beta \leq \rho^{\frac{1}{2}}(C_W)$, (18) can be further rewritten as

$$\begin{aligned}
E\left[\|\tilde{x}(k)\|\right] \leq& \rho^{\frac{k}{2}}(C_W)\|\tilde{x}(0)\| + \mu c\sqrt{N}\rho^{\frac{1}{2}}(C_L)\frac{1-\rho^{\frac{k}{2}}(C_W)}{1-\rho^{\frac{1}{2}}(C_W)} \\
&+ \mu\sqrt{N}\rho^{\frac{1}{2}}(C_L)k\rho^{\frac{k-1}{2}}(C_W).
\end{aligned} \tag{19}$$

By taking limits to $E[\|\tilde{x}(k)\|]$ as $k \to \infty$, if $\rho(C_W) < 1$ holds, one can obtain

$$\begin{aligned}
\lim_{k\to\infty} E\left[\|\tilde{x}(k)\|\right] &\leq \frac{\mu c\sqrt{N}\rho^{\frac{1}{2}}(C_L)}{1-\rho^{\frac{1}{2}}(C_W)} + \lim_{k\to\infty}\frac{\mu\sqrt{N}\rho^{\frac{1}{2}}(C_L)k}{\left(\rho^{-\frac{1}{2}}(C_W)\right)^{k+1}} \\
&= \frac{\mu c\sqrt{N}\rho^{\frac{1}{2}}(C_L)}{1-\rho^{\frac{1}{2}}(C_W)} = R(c,\bar{L}).
\end{aligned}$$

Using Markov's inequality to (19) yields

$$\begin{aligned}
&\lim_{k\to\infty}\Pr\left\{\|\tilde{x}(k)\| \geq \alpha\right\} \\
&\leq \lim_{k\to\infty}\frac{\rho^{\frac{k}{2}}(C_W)}{\alpha}\left\{\|\tilde{x}(0)\| - \frac{\mu c\sqrt{N}\rho^{\frac{1}{2}}(C_L)}{1-\rho^{\frac{1}{2}}(C_W)} + \frac{\mu\sqrt{N}\rho^{\frac{1}{2}}(C_L)k}{\rho^{\frac{1}{2}}(C_W)}\right\} \\
&\quad + \frac{\mu c\sqrt{N}\rho^{\frac{1}{2}}(C_L)}{\alpha\left(1-\rho^{\frac{1}{2}}(C_W)\right)} = \frac{R(c,\bar{L})}{\alpha},
\end{aligned}$$

where the last equality holds since $\rho(C_W) < 1$. Therefore, if $R(c,\bar{L}) < \alpha$, the convergence of (5) in $(\alpha, 1-\frac{R(c,\bar{L})}{\alpha})$-probability is achieved.

For the finite (α,γ)-convergence time analysis, one has

$$\begin{aligned}
&\Pr\left\{\frac{\|\tilde{x}(k)\|}{\|\tilde{x}(0)\|} \geq \alpha\right\} \leq \frac{E\left[\|\tilde{x}(k)\|\right]}{\alpha\|\tilde{x}(0)\|} \\
&\leq \frac{\rho^{\frac{k}{2}}(C_W)}{\alpha}\left(1 - \frac{\mu c\sqrt{N}\rho^{\frac{1}{2}}(C_L)}{\left(1-\rho^{\frac{1}{2}}(C_W)\right)\|\tilde{x}(0)\|} + \frac{\mu\sqrt{N}\rho^{\frac{1}{2}}(C_L)k}{\|\tilde{x}(0)\|\rho^{\frac{1}{2}}(C_W)}\right) \\
&\quad + \frac{R(c,\bar{L})}{\alpha\|\tilde{x}(0)\|} \\
&\leq \frac{\rho^{\frac{k}{2}}(C_W)}{\alpha}\left(1 + \frac{\mu\sqrt{N}\rho^{\frac{1}{2}}(C_L)k}{\|\tilde{x}(0)\|\rho^{\frac{1}{2}}(C_W)}\right) + \frac{R(c,\bar{L})}{\alpha\|\tilde{x}(0)\|} \\
&= \frac{\zeta}{\alpha\rho^{\frac{1}{2\zeta}}(C_W)}\rho^{\frac{1}{2}(k+\frac{1}{\zeta})}(C_W)(k+\frac{1}{\zeta}) + \frac{R(c,\bar{L})}{\alpha\|\tilde{x}(0)\|},
\end{aligned} \tag{20}$$

where $\zeta = \frac{\mu\sqrt{N}\rho^{\frac{1}{2}}(C_L)}{\|\tilde{x}(0)\|\rho^{\frac{1}{2}}(C_W)}$. In (20), if $R(c,\bar{L}) < \alpha(1-\gamma)\|\tilde{x}(0)\|$ further holds, indicating that c satisfies (15), then the upper bound of the finite (α,γ)-convergence time of (5) can be derived by setting the right-side term in the last equality as $1-\gamma$. Denoting $\left(1-\gamma-\frac{R(c,\bar{L})}{\alpha\|\tilde{x}(0)\|}\right)\frac{\alpha\rho^{\frac{1}{2\zeta}}(C_W)}{\zeta}$ as z, one then has

$$\left(k+\frac{1}{\zeta}\right)\left(\rho^{\frac{1}{2}}(C_W)\right)^{k+\frac{1}{\zeta}} = z. \tag{21}$$

As a result, one can calculate

$$k = \frac{\boldsymbol{W}(y)}{\ln\rho^{\frac{1}{2}}(C_W)} - \frac{1}{\zeta},$$

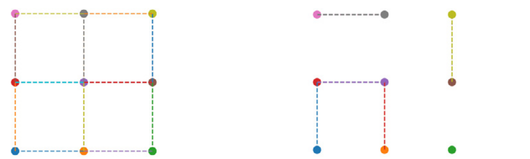
where $y = z\ln\rho^{\frac{1}{2}}(C_W)$, and $\boldsymbol{W}(y)$ is the Lambert function that is the solution to $ke^k = y$. Hence, the finite (α,γ)-convergence time of (5) is upper bounded by (16). This concludes the proof.

Remark 2. The triggering parameter c in (6) determines how frequently the triggering events happen during the final iterations. A larger c results in a lower communication rate, indicating a higher communication efficiency, but the convergence error $R(c,\bar{L})$ will get increased. Therefore, in (16), to guarantee a finite (α,γ)-convergence time, the triggering parameter c should be chosen carefully such that (15) holds.

5. Simulations

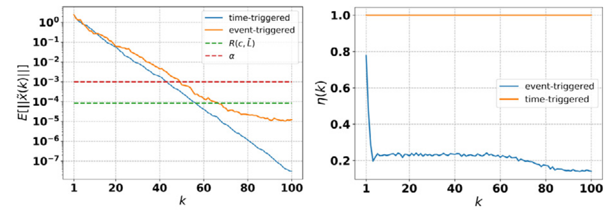
The correctness of our analysis is illustrated by extensive simulations. All figures are drawn from the results of the average over $r = 10^3$ runs. Simulation parameters are set as: $N = 9$, $\gamma = 0.9$, and $\alpha = 10^{-3}$. During each simulation, we randomly choose $p_{ij} \in [0.5, 0.8]$. The topologies of the prearranged network G and a sample case of random network $G(k)$ with heterogeneous link failures are shown in Fig. 1.

The initial state vector $x(0)$ and parameter μ are chosen according to (12) and (14), respectively. During the simulation, Assumption 1 is satisfied. The convergence of $E[x_i(k)]$ and $E\left[\|\tilde{x}(k)\|\right]$ are shown in Fig. 2(a) and (b), respectively, where $\tilde{x}_{\alpha,\gamma}(k)$ denotes the iteration of $\tilde{x}(k)$ with α accuracy and γ probability, while $\tilde{x}(k)$, $\tilde{x}_u(k)$ and $\tilde{x}_l(k)$ denote the iteration of the simulated, theoretical upper and lower bounds of $E\left[\|\tilde{x}(k)\|\right]$, respectively. It can be seen from Fig. 2 that, (i) convergence of average consensus protocol (4) in expectation and mean square are achieved; (ii) theoretical results for the upper and lower bounds in (8), (9) can be used to estimate tightly the convergence time of protocol (4).



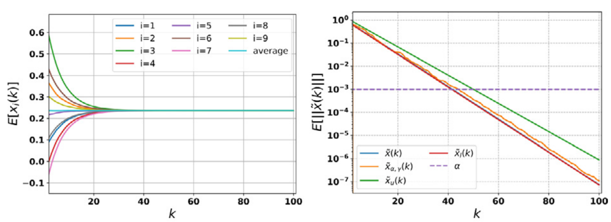
(a) Prearranged network G (b) A sample case of $G(k)$

Fig. 1. Topology of an undirected network with $N = 9$, where solid circles and dashed lines indicate nodes and the undirected communication edges, respectively.



(a) $E[\|\tilde{x}(k)\|]$ (b) $\eta(k)$.

Fig. 4. Effects of the event-triggered communication with $c = 10^{-5}$ and $\beta = 0.95\rho^{0.5}(C_W)$.



(a) $E[x_i(k)]$ (b) $E\left[\|\tilde{x}(k)\|\right]$

Fig. 2. Iterations of $E\left[x_i(k)\right]$ and $E\left[\|\tilde{x}(k)\|\right]$.

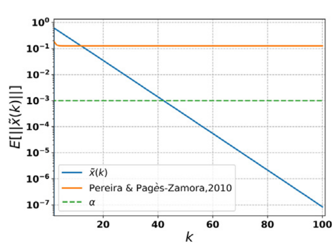


Fig. 3. Comparison on convergence accuracy with Pereira and Pagès-Zamora (2010).

For average consensus with random link failures, Pereira and Pagès-Zamora (2010) have derived the upper bound of the asymptotic MSE shown as its (27) with the adjacency matrix constructed as its (1). Also, the prearranged network G is fully connected. Fig. 3 shows the comparison on convergence accuracy between (Pereira & Pagès-Zamora, 2010) and the one proposed in this paper. It can be seen from Fig. 3 that, our analysis can achieve a high convergence accuracy while the one in Pereira and Pagès-Zamora (2010) can only converge to 0.034. It means that the upper bound of $\mathrm{MSE}(x(k))$ in Pereira and Pagès-Zamora (2010) is less tight than our results since Pereira and Pagès-Zamora (2010) consider network with asymmetric link probabilities.

We define the average network communication rate at iteration k as $\eta(k) = \sum_{i=1}^{N} \eta_i(k)/N$ (Wu, Jia, Johansson, & Shi, 2013), where $\eta_i(k)$ is 1 or 0, indicating whether or not there is a communication event for node i at iteration time k. Fig. 4 shows the impact of event-triggered communication to convergence accuracy and communication rate $\eta(k)$. From Fig. 4, by properly choosing triggering parameters c and β, we have, i) the event-triggered average consensus protocol has non-zero steady-state convergence error; ii) convergence in (α, γ)-probability is achieved, with a larger finite (α, γ)-convergence time than that of the time-triggered one; iii) the average broadcasted events for each node of the event-triggered mechanism is far less than the one of the time-triggered one. Therefore, with triggering parameters being properly designed, the event-triggered communication mechanism is effective in the energy efficiency of a network with heterogeneous random link failures.

6. Conclusions

In this paper, we have investigated the convergence time of the average consensus with heterogeneous random link failures, together with the analysis of the impact of event-triggered communication on convergence performance. We have derived the conditions on convergence in expectation and mean square, and the closed-form expression of the lower- and upper-bounds of the (α, γ)-convergence time. We also have analyzed the impact of the event-trigger communication on the convergence error and (α, γ)-convergence time. Finally, some numerical simulations have verified the effectiveness of the results.

Acknowledgments

This work is supported by National Natural Science Foundation of China (NSFC) under Grant nos. 51874205, 61903328, 61803279, 61876121 and 61672371; Jiangsu Qinglan Project, China.

Appendix A. Proof to Lemma 3

Proof. We first show that (14) guarantees $\rho(\bar{M}) < 1$. When the prearranged network G is undirected connected, under Assumption 1, the expected network $\bar{G}$ is connected and $\bar{W}$ is symmetric. Therefore, 1_N is the eigenvector of $\bar{L}$ and $\bar{M}$ with eigenvalues 0 and 1, respectively. The rest of the $N-1$ nonzero real eigenvalues of matrix $\bar{M}$ can be written as $1-\mu\lambda_N(\bar{L}), \ldots, 1-\mu\lambda_2(\bar{L})$. Since $0 < p_{ij} \le 1$, one has $\lambda_i(\bar{L}) \le \lambda_i(L)$ $(i = 2, \ldots, N)$. Hence if μ is selected as (14), $\mu < \frac{2}{\lambda_N(L)} \le \frac{2}{\lambda_N(\bar{L})}$ holds, and then $\rho(\bar{M}) < 1$ is guaranteed.

In (14), if $0 < \mu \le \frac{1}{\lambda_N(L)}$, i.e., $\mu\lambda_N(L) \le 1$, then one has $\rho(\bar{M}) = 1 - \mu\lambda_2(\bar{L})$. Otherwise when μ is selected such that $\frac{1}{\lambda_N(L)} < \mu \le \frac{2}{\lambda_N(L)+\lambda_2(L)}$, then one has $\mu\lambda_N(L) > 1$. Since $\lambda_N(\bar{L}) \le \lambda_N(L)$, then if $\mu\lambda_N(\bar{L}) \le 1$ holds, one also has $\rho(\bar{M}) = 1 - \mu\lambda_2(\bar{L})$. For the case when $\mu\lambda_N(\bar{L}) \ge 1$, one has

$$1 - \mu\lambda_2(\bar{L}) \ge 1 - \mu\lambda_2(L),$$
$$1 - \mu\lambda_2(L) \ge \mu\lambda_N(L) - 1,$$
$$\mu\lambda_N(L) - 1 \ge \mu\lambda_N(\bar{L}) - 1 > 0,$$

and then we can yield that $\rho(\bar{M}) = 1 - \mu\lambda_2(\bar{L})$.

We continue to show $\rho(C_W) \leq \rho(\bar{M}) < 1$ when μ is designed as (14). Since $C_W = E\left[M^{\mathrm{T}}(k)M(k)\right]$, under Assumptions 1–2, one has

$$C_W = I_N - \Pi_N - 2\mu\bar{L} + \mu^2 C_L. \tag{A.1}$$

where $C_L = E\left[L^{\mathrm{T}}(k)L(k)\right]$. Let $\bar{L}_{ij}$ and e_{ij} be the (i,j)th entry of matrices $\bar{L}$ and C_L, respectively. Then $\bar{L}_{ij}$ and e_{ij} can be written as

$$\bar{L}_{ij} = \begin{cases} \sum_{l=1}^{N} A_{il}p_{il}, & j = i, \\ -A_{ij}p_{ij}, & j \neq i, \end{cases}$$

and

$$e_{ij} = \begin{cases} E\left[\left(\sum_{l=1}^{N} A_{il}\delta_{il}(k)\right)^2 + \sum_{l=1, l\neq i}^{N} \left(A_{li}\delta_{li}(k)\right)^2\right], & j = i, \\ -E\left[A_{ij}\delta_{ij}(k)\sum_{l=1}^{N}\left(A_{il}\delta_{il}(k)\right)\right] \\ -E\left[A_{ji}\delta_{ji}(k)\sum_{l=1}^{N}\left(A_{jl}\delta_{jl}(k)\right)\right] \\ +E\left[\sum_{l=1, l\neq i,j}^{N} \left(A_{li}\delta_{li}(k)A_{lj}\delta_{lj}(k)\right)\right], & j \neq i. \end{cases} \tag{A.2}$$

In (A.2), according to (2), $\delta_{il}(k)$ is equal to 1 with probability p_{il} and 0 with probability $1-p_{il}$. Then $E[\delta_{il}^2(k)] = p_{il}$ holds. Note that, under Assumption 2, for $j \neq l$, $\delta_{il}(k)$ and $\delta_{ij}(k)$ are independent, then one has $E[\delta_{il}(k)\delta_{ij}(k)] = p_{il}p_{ij}$. Also, $A_{ij}^2 = A_{ij}$ since A_{ij} is 1 or 0. As there is no self-loop in the prearranged network G, under Assumption 1, for all $i, l \in V$ and $i \neq l$, $A_{il}p_{il} = A_{li}p_{li}$ holds. One has

$$e_{ij} = \begin{cases} \sum_{l=1}^{N}(\bar{L}_{il}\bar{L}_{li}) + 2\sum_{l=1}^{N}(A_{il}p_{il}) - 2\sum_{l=1}^{N}(A_{il}p_{il}^2), & j = i, \\ \sum_{l=1}^{N}(\bar{L}_{il}\bar{L}_{lj}) - 2A_{ij}p_{ij} + 2A_{ij}p_{ij}^2, & j \neq i. \end{cases} \tag{A.3}$$

Therefore, C_L in (A.1) can be summarized into $\bar{L}^2 + 2\bar{L} - 2B$, where $B \in R^{N\times N}$ has its (i,j)th entry shown as

$$B_{ij} = \begin{cases} \sum_{l=1}^{N}(A_{il}p_{il}^2), & j = i, \\ -A_{ij}p_{ij}^2, & j \neq i. \end{cases}$$

Here B can be viewed as one special Laplacian matrix of expected network $\bar{G}$ with link probabilities $p_{ij}^2 \in (0,1]$. Therefore, matrix B is positive semi-definite with $\lambda_1(B) = 0$. Substituting $C_L = \bar{L}^2 + 2\bar{L} - 2B$ into (A.1), using $\Pi_N\bar{M} = \bar{M}\Pi_N = \Pi_N$ under Assumption 1, one obtains

$$C_W = \bar{M}^2 + 2\mu^2\bar{L} - 2\mu^2 B. \tag{A.4}$$

In (A.4), we write the eigenvalues of $C_W, \bar{M}, \bar{L}, \bar{M}^2 + 2\mu^2\bar{L}$, and B in ascending order. Note that (14) guarantees $\lambda_N(\bar{M}) = 1 - \mu\lambda_2(\bar{L})$, and $\lambda_1(B) = \lambda_1(\bar{L}) = 0$ for a connected G. According to Weyl's inequality, one has

$$\lambda_N(C_W) \leq \lambda_N(\bar{M}^2 + 2\mu^2\bar{L}) = \lambda_N^2(\bar{M}) + 2\mu^2\lambda_2(\bar{L}),$$

where the last equality holds since $\lambda_N(\bar{M}^2) = \lambda_N^2(\bar{M})$. Hence, one has

$$\lambda_N(C_W) - \lambda_N(\bar{M}) \leq -\mu\lambda_2(\bar{L})\left[1 - \mu\left(\lambda_2(\bar{L}) + 2\right)\right].$$

Note that, for a connected prearranged undirected network G with size $N > 2$, $\lambda_N(L) \geq 2$ holds (Bullo, 2019). One has

$$\lambda_2(L) + \lambda_N(L) \geq \lambda_2(L) + 2 \geq \lambda_2(\bar{L}) + 2.$$

Therefore, if μ is chosen as (14), one has $\mu(\lambda_2(\bar{L}) + 2) \leq 1$, then $\lambda_N(C_W) - \lambda_N(\bar{M}) \leq 0$ holds. When all the link probabilities p_{ij} satisfy $0 < p_{ij} \leq 1$, the matrix $\bar{L} - B$ is positive semi-definite, then C_W has non-negative eigenvalues, and thus one has $\rho(C_W) = \lambda_N(C_W)$. Therefore, $\rho(C_W) \leq \rho(\bar{M})$ holds with μ satisfying (14). This completes the proof. □

Appendix B. Proof to Lemma 4

Proof. Under Assumptions 2–3, one has $\Pi_N\bar{A} = \bar{A}\Pi_N$ and $\Pi_N A(k) = A(k)\Pi_N$, indicating that the expected network $\bar{G}$ and random networks $G(k)$ for all k are weight balanced directed graphs. Hence, $\bar{W}$ and all $W(k)$s are symmetric. Combining with the proof of Theorem 1, we complete the proof for (i) and (ii).

For the proof for (iii), under Assumption 3, if $\bar{W}$ is further symmetric, $\bar{M}$ has N real eigenvalues. Thus, when $\rho(\bar{M}) < 1$, there exists a $x(0)$ satisfying (12) such that $E\left[\|\tilde{x}(k)\|_2^2\right] \geq \nu_N^{\mathrm{T}}\rho^{2k}(\bar{M})\nu_N$, where ν_N is the eigenvector corresponding to the largest eigenvalue of $\bar{M}$. Therefore, the lower bound of (α,γ)-convergence time is shown as (9). Thus, we have completed the proof. □

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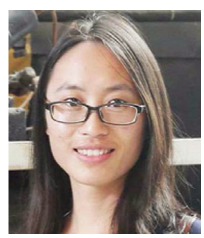

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