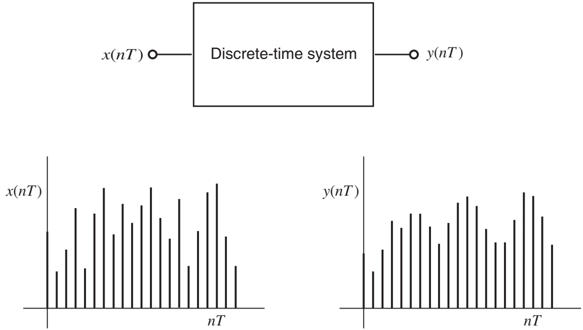
Chapter 2
DISCRETE-TIME SYSTEMS
2.5 Introduction to Time-Domain Analysis

Victoria, BC, Canada
Email: aantoniou@ieee.org

July 9, 2018

Introduction

Time-domain analysis is the process of finding the response of a system, $y(nT)$, to a given excitation, $x(nT)$.



Three different methods are available for the time-domain analysis of discrete-time systems:

Time-Domain Analysis

Three different methods are available for the time-domain analysis of discrete-time systems:

- Induction method
- State-space method
- z transform method

Induction Method

- The *induction method* involves solving the difference equation using induction.

Induction Method

- The *induction method* involves solving the difference equation using induction.
- The method is somewhat primitive and inefficient.

- The *induction method* involves solving the difference equation using induction.
- The method is somewhat primitive and inefficient.
- However, it is an intuitive method that demonstrates the mode of operation of a discrete-time system.

Induction Method

- The *induction method* involves solving the difference equation using induction.
- The method is somewhat primitive and inefficient.
- However, it is an intuitive method that demonstrates the mode of operation of a discrete-time system.
- It is useful as an introduction to time-domain analysis but it tends to become quite complicated in higher-order discrete-time systems.

State-Space Method

- The *state-space method* entails the manipulation of matrices.

State-Space Method

- The *state-space method* entails the manipulation of matrices.
- It is quite useful in applications where routines for the manipulation of matrices are available, e.g., in MATLAB.

State-Space Method

- The *state-space method* entails the manipulation of matrices.
- It is quite useful in applications where routines for the manipulation of matrices are available, e.g., in MATLAB.
- It is applicable to time-dependent systems.

Z Transform Method

- The *z transform method* is the most efficient and most frequently used method among the available methods.

Z Transform Method

- The *z transform method* is the most efficient and most frequently used method among the available methods.
- Its main disadvantage is that it cannot be applied to time-dependent or nonlinear systems.

Z Transform Method

- The *z transform method* is the most efficient and most frequently used method among the available methods.
- Its main disadvantage is that it cannot be applied to time-dependent or nonlinear systems.
- The details of the method can be found in Chap. 5.

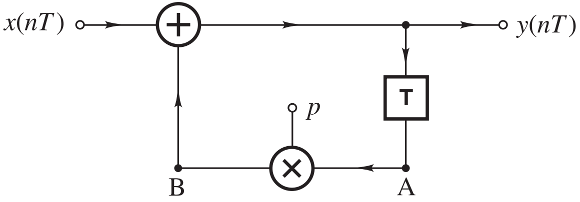
Induction Method *Cont'd*

- The induction method for time-domain analysis can be illustrated by finding the impulse, unit-step, and sinusoidal response of a simple recursive system.

  As will be shown, all that is necessary is simple algebra.

Example

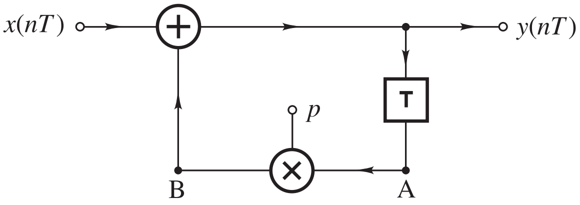
Find the impulse response of the recursive system



(a)

assuming an initially relaxed system.

. . .



(a)

Solution The difference equation is

$$y(nT) = x(nT) + py(nT - T)$$

Example *Cont'd*

...

$$y(nT) = x(nT) + py(nT - T)$$

If $x(nT) = \delta(nT)$, we have

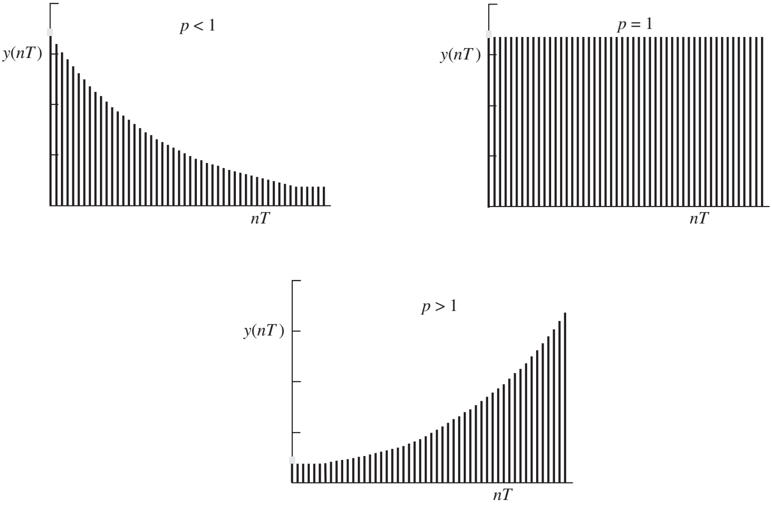
$$y(nT) = \delta(nT) + py(nT - T)$$

For an initially relaxed system, $y(nT) = 0$ for $n < 0$ and hence we have

$$\begin{aligned}
y(0) &= \delta(0) + py(-T) = 1 + 0 = 1 \\
y(T) &= \delta(T) + py(0) = 0 + p \times 1 = p \\
y(2T) &= \delta(2T) + py(T) = 0 + p \cdot p = p^2 \\
&\vdots \\
y(nT) &= u(nT)p^n \quad \blacksquare
\end{aligned}$$

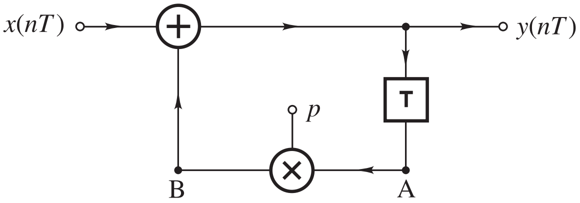
The unit-step $u(nT)$ is added to ensure that $y(nT) = 0$ for $n < 0$.

Example *Cont'd*



Example

Assuming that the system shown is initially relaxed, find the unit-step response:



(a)

Example *Cont'd*

Solution The difference equation is

$$y(nT) = x(nT) + py(nT - T)$$

If $x(nT) = u(nT)$, we have

$$y(nT) = u(nT) + py(nT - T)$$

For an initially relaxed system, $y(nT) = 0$ for $n < 0$ and hence we have

$$\begin{aligned}
y(0) &= u(0) + py(-T) = 1 + 0 = 1 \\
y(T) &= u(T) + py(0) = 1 + p \\
y(2T) &= u(2T) + py(T) = 1 + p + p^2 \\
&\vdots \\
y(nT) &= u(nT)\sum_{k=0}^{n} p^k
\end{aligned}$$

...

$$y(nT) = u(nT)\sum_{k=0}^{n} p^k$$

We can write

$$\begin{aligned} y(nT) &= u(nT)(1 + p + p^2 + \cdots + p^n) && \text{(A)} \\ py(nT) &= u(nT)(p + p^2 + \cdots + p^n + p^{(n+1)}) && \text{(B)} \end{aligned}$$

Subtracting Eq. (B) from Eq. (A), we get

$$y(nT) - py(nT) = u(nT)(1 - p^{(n+1)})$$

or

$$y(nT) = u(nT)\frac{1 - p^{(n+1)}}{1 - p}$$

...

$$y(nT) = u(nT)\frac{1 - p^{(n+1)}}{1 - p}$$

Therefore, there are three cases to consider:

(i) $p < 1$

(ii) $p = 1$

(iii) $p > 1$

. . .

$$y(nT) = u(nT)\frac{1 - p^{(n+1)}}{1 - p}$$

(i) For $p < 1$, the steady-state response is obtained by evaluating $y(nT)$ for $n \to \infty$, i.e.,

$$\lim_{n\to\infty} y(nT) = \frac{1}{1 - p} \quad \blacksquare$$

. . .

$$y(nT) = u(nT)\frac{1 - p^{(n+1)}}{1 - p}$$

(i) For $p < 1$, the steady-state response is obtained by evaluating $y(nT)$ for $n \to \infty$, i.e.,

$$\lim_{n\to\infty} y(nT) = \frac{1}{1-p} \quad \blacksquare$$

(ii) For $p = 1$, using l'Hôpital's rule we get

$$y(nT) = \lim_{p\to 1} \frac{d(1 - p^{(n+1)})/dp}{d(1-p)/dp} = n + 1$$

Hence

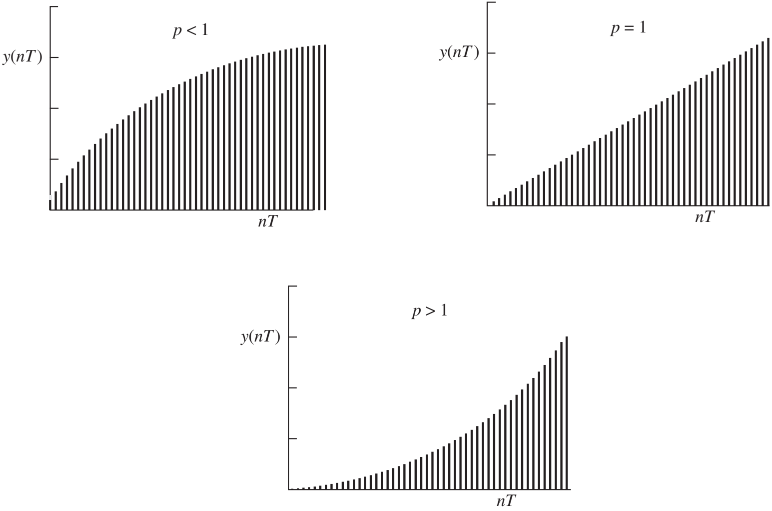
$$\lim_{n\to\infty} y(nT) \to \infty \quad \blacksquare$$

...

$$y(nT) = u(nT)\frac{1 - p^{(n+1)}}{1 - p}$$

(iii) For $p > 1$

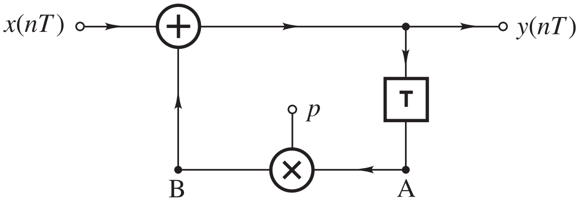
$$\lim_{n\to\infty} y(nT) \approx \frac{p^n}{p-1} \to \infty \quad \blacksquare$$



$p < 1$
$y(nT)$
nT
$p = 1$
$y(nT)$
nT
$p > 1$
$y(nT)$
nT

Example

Assuming zero initial conditions, find the response of the recursive system



(a)

to the sinusoidal excitation

$$x(nT) = u(nT)\sin\omega nT$$

Solution As before, the difference equation is

$$y(nT) = x(nT) + py(nT - T)$$

If $x(nT) = u(nT)\sin\omega nT$, we have

$$y(nT) = \mathcal{R}x(nT) = u(nT)\sin\omega nT + py(nT - T)$$

The system is linear and so

$$\begin{aligned} y(nT) &= \mathcal{R}[u(nT)\sin\omega nT] = \mathcal{R}\left[u(nT)\tfrac{1}{2j}(e^{j\omega nT} - e^{-j\omega nT})\right] \\ &= \tfrac{1}{2j}\left[\mathcal{R}u(nT)e^{j\omega nT} - \mathcal{R}u(nT)e^{-j\omega nT}\right] \\ &= \tfrac{1}{2j}[y_1(nT) - y_2(nT)] \end{aligned}$$

where

$$y_1(nT) = \mathcal{R}u(nT)e^{j\omega nT} \quad \text{and} \quad y_2(nT) = \mathcal{R}u(nT)e^{-j\omega nT}$$

...

$$y(nT) = x(nT) + py(nT - T)$$

$$y_1(nT) = \mathcal{R}u(nT)e^{j\omega nT} \quad \text{and} \quad y_2(nT) = \mathcal{R}u(nT)e^{-j\omega nT}$$

The partial response $y_1(nT)$ can be obtained as

$$y_1(nT) = \mathcal{R}\left[u(nT)e^{j\omega nT}\right] = u(nT)e^{j\omega nT} + py_1(nT - T)$$

Hence

$$\begin{aligned}
y_1(0) &= u(0)e^0 + py_1(-T) = 1 \\
y_1(T) &= e^{j\omega T} + py_1(0) = e^{j\omega T} + p \\
y_1(2T) &= e^{j2\omega T} + py_1(T) = e^{j2\omega T} + pe^{j\omega T} + p^2 \\
&\vdots \\
y_1(nT) &= u(nT)(e^{j\omega nT} + pe^{j\omega(n-1)T} + \cdots + p^{(n-1)}e^{j\omega T} + p^n) \\
&= u(nT)e^{j\omega nT}(1 + pe^{-j\omega T} + \cdots + p^n e^{(-jn\omega T)})
\end{aligned}$$

Example *Cont'd*

...

$$
\begin{aligned}
y_1(nT) &= u(nT)e^{j\omega nT}(1 + pe^{-j\omega T} + \cdots + p^n e^{(-jn\omega T)}) \\
&= u(nT)e^{j\omega nT}\sum_{k=0}^{n} p^k e^{(-jk\omega nT)}
\end{aligned}
$$

This is a geometric series in powers of $pe^{(-j\omega nT)}$ and its sum can be obtained as

$$
y_1(nT) = u(nT)\frac{e^{j\omega nT} - p^{(n+1)}e^{-j\omega T}}{1 - pe^{-j\omega T}}
$$

...

$$y_1(nT) = u(nT)\frac{e^{j\omega nT} - p^{(n+1)}e^{-j\omega T}}{1 - pe^{-j\omega T}} = \frac{e^{j\omega T}}{e^{j\omega T} - p} \times (e^{j\omega nT} - p^{(n+1)}e^{-j\omega T}$$

Now consider the function

$$H(e^{j\omega T}) = \frac{e^{j\omega T}}{e^{j\omega T} - p} = \frac{\cos\omega T + j\sin\omega T}{\cos\omega T + j\sin\omega T - p}$$

and let

$$H(e^{j\omega T}) = M(\omega)e^{j\theta(\omega)}$$

where $$M(\omega) = |H(e^{j\omega T})| = \frac{1}{\sqrt{1 + p^2 - 2p\cos\omega T}}$$

and $$\theta(\omega) = \arg H(e^{j\omega T}) = \omega T - \tan^{-1}\frac{\sin\omega T}{\cos\omega T - p}$$

...

$$y_1(nT) = \frac{e^{j\omega T}}{e^{j\omega T} - p} \times (e^{j\omega nT} - p^{(n+1)} e^{-j\omega T})$$

and

$$H(e^{j\omega T}) = M(\omega) e^{j\theta(\omega)}$$

where $\qquad M(\omega) = |H(e^{j\omega T})| = \dfrac{1}{\sqrt{1 + p^2 - 2p\cos\omega T}}$

and $\qquad \theta(\omega) = \arg H(e^{j\omega T}) = \omega T - \tan^{-1} \dfrac{\sin\omega T}{\cos\omega T - p}$

By using these relations, $y_1(\omega T)$ can be expressed as

$$y_1(nT) = u(nT) M(\omega) (e^{j[\theta(\omega) + \omega nT]} - p^{(n+1)} e^{j[\theta(\omega) - \omega T]})$$

Example *Cont'd*

...

$$y_1(nT) = u(nT)M(\omega)(e^{j[\theta(\omega)+\omega nT]} - p^{(n+1)}e^{j[\theta(\omega)-\omega T]})$$

By replacing ω by $-\omega$ in $y_1(nT)$, we get

$$y_2(nT) = u(nT)M(\omega)(e^{j[\theta(-\omega)-\omega nT]} - p^{(n+1)}e^{j[\theta(-\omega)+\omega T]})$$

By noting that $M(\omega)$ is an even function and $\theta(\omega)$ an odd function of ω, i.e.,

$$M(-\omega) = M(\omega) \quad \text{and} \quad \theta(-\omega) = -\theta(\omega)$$

we can readily show that

$$\begin{aligned} y(nT) &= \tfrac{1}{2j}[y_1(nT) - y_2(nT)] \\ &= u(nT)M(\omega)\sin[\omega nT + \theta(\omega)] \\ &\quad -u(nT)M(\omega)p^{(n+1)}\sin[\theta(\omega) - \omega T] \end{aligned}$$

. . .

$$\begin{aligned} y(nT) = {} & u(nT)M(\omega)\sin[\omega nT + \theta(\omega)] \\ & -u(nT)M(\omega)p^{(n+1)}\sin[\theta(\omega) - \omega T] \end{aligned}$$

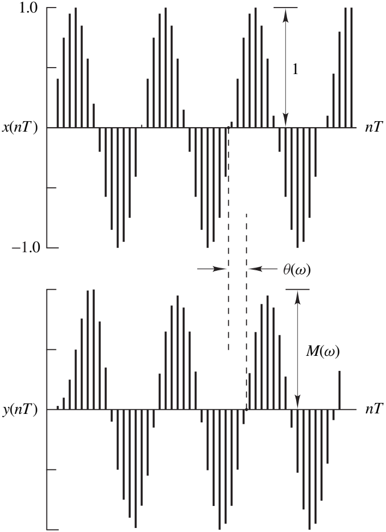
As can be seen, the sinusoidal response of the system consists of two components.

If $p < 1$, the second term represents a transient component that reduces to zero as $n \to \infty$. Therefore,

$$\tilde{y}(nT) = \lim_{n\to\infty} y(nT) = M(\omega)\sin[\omega nT + \theta(\omega)] \quad \blacksquare$$

If $p = 1$, the transient component is a constant. If $p > 1$ the transient component tends to infinity as $n \to \infty$ i.e.,

$$\tilde{y}(nT) = \lim_{n\to\infty} y(nT) \to \infty \quad \blacksquare$$



Sinusoidal response if $p < 1$

Sinusoidal Response

- The time-domain analysis has shown that the response of a first-order recursive system to a sinusoidal excitation of unity amplitude and zero phase angle, i.e.,

$$x(nT) = \sin(\omega nT)$$

is a sinusoid of amplitude $M(\omega)$ and angle $\theta(\omega)$, i.e.,

$$x(nT) = M(\omega)\sin[\omega nT + \theta(\omega)]$$

provided that the transient component decays to zero.

Sinusoidal Response

- The time-domain analysis has shown that the response of a first-order recursive system to a sinusoidal excitation of unity amplitude and zero phase angle, i.e.,

$$x(nT) = \sin(\omega nT)$$

is a sinusoid of amplitude $M(\omega)$ and angle $\theta(\omega)$, i.e.,

$$x(nT) = M(\omega)\sin[\omega nT + \theta(\omega)]$$

provided that the transient component decays to zero.

- It turns out that this is a general property of recursive as well as nonrecursive systems in general.

Sinusoidal Response

- The time-domain analysis has shown that the response of a first-order recursive system to a sinusoidal excitation of unity amplitude and zero phase angle, i.e.,

$$x(nT) = \sin(\omega nT)$$

is a sinusoid of amplitude $M(\omega)$ and angle $\theta(\omega)$, i.e.,

$$x(nT) = M(\omega)\sin[\omega nT + \theta(\omega)]$$

provided that the transient component decays to zero.

- It turns out that this is a general property of recursive as well as nonrecursive systems in general.

- The transient response of a discrete-time system will decay to zero only if the system is stable (see Sec. 2.7).

This slide concludes the presentation.
Thank you for your attention.