# Chapter 2
# DISCRETE-TIME SYSTEMS
# 2.6 Convolution Summation
# 2.7 Stability


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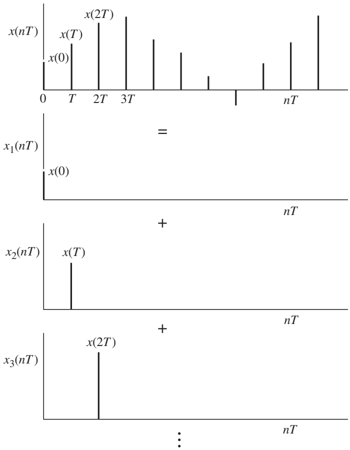
# Introduction

- The *convolution summation* is of considerable importance for the characterization, representation, analysis, and design of discrete-time systems.
- This presentation will deal with the derivation, properties, and applications of the convolution summation.

# Derivation

- An arbitrary excitation $x(nT)$ can be considered to be made up of a series of impulses as shown:



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$$x(nT) = \sum_{k=-\infty}^{\infty} x_k(nT)$$

where

$$x_k(nT) = \begin{cases} x(kT) & \text{for } n = k \\ 0 & \text{otherwise} \end{cases}$$
$$= x(kT)\delta(nT - kT)$$

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- Hence

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- From Eq. (A), we have

$$y(nT) = \mathcal{R}x(nT) = \mathcal{R} \sum_{k=-\infty}^{\infty} x(kT)\delta(nT - kT)$$

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- Since the system is *linear*,

$$y(nT) = \sum_{k=-\infty}^{\infty} x(kT)\mathcal{R}\delta(nT - kT)$$

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- The system is also *time-invariant* and hence we get

$$y(nT) = \sum_{k=-\infty}^{\infty} x(kT)h(nT - kT)$$

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- The system is also *time-invariant* and hence we get

$$y(nT) = \sum_{k=-\infty}^{\infty} x(kT)h(nT - kT)$$

- This relation is known as the *convolution summation*.

# Alternative Form

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$$k \to \infty \quad \text{then} \quad k' \to -\infty$$

and if

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- Hence the convolution summation can also be expressed as

$$y(nT) = \sum_{k'=\infty}^{-\infty} x(nT - k'T)h(k'T)$$

...

$$y(nT) = \sum_{k'=\infty}^{-\infty} x(nT - k'T)h(k'T)$$

- Dropping the primes and reversing the order of summation, we obtain the identity

$$y(nT) = \sum_{k=-\infty}^{\infty} x(kT)h(nT - kT) = \sum_{k=-\infty}^{\infty} h(kT)x(nT - kT)$$

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$$y(nT) = \sum_{k=-\infty}^{n} x(kT)h(nT - kT) = \sum_{k=0}^{\infty} h(kT)x(nT - kT)$$

- If, in addition, $x(nT) = 0$ for $n < 0$, we have

$$y(nT) = \sum_{k=0}^{n} x(kT)h(nT - kT) = \sum_{k=0}^{n} h(kT)x(nT - kT)$$

...

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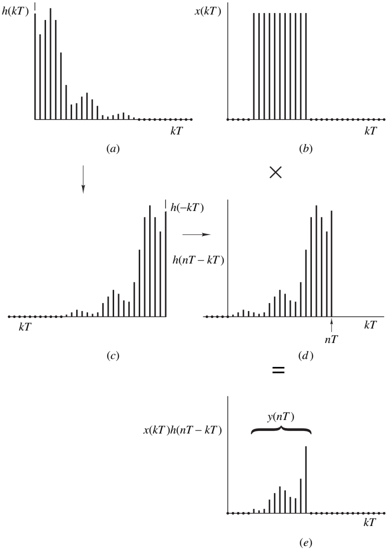
Evidently, *if the impulse response $h(nT)$ of a discrete-time system is known, its response to an arbitrary excitation can be readily determined* by using the convolution summation.

# Graphical Representation



# Example

Using the convolution summation, find the unit-step response of a discrete-time system characterized by the equation

$$y(nT) = x(nT) + py(nT - T)$$

The system has an impulse response

$$h(nT) = u(nT)p^n$$

and is initially relaxed (i.e., $y(nT) = 0$ for $n < 0$).

**Solution** The convolution summation gives

$$y(nT) = \mathcal{R}u(nT) = \sum_{k=-\infty}^{\infty} u(kT)p^k u(nT - kT)$$

$$= \cdots + \overbrace{u(-T)p^{-1}u(nT + T)}^{k=-1} + \overbrace{u(0)p^0 u(nT)}^{k=0} + \overbrace{u(T)p^1 u(nT - T)}^{k=1}$$

$$+ \cdots + \overbrace{u(nT)p^n u(0)}^{k=n} + \overbrace{u(nT + T)p^{n+1} u(-T)}^{k=n+1} + \cdots$$

For $n < 0$, the unit step assumes a value of zero and hence we get $y(nT) = 0$ since all the terms are zero.

# Example *Cont'd*

...

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$$+ \cdots + \overbrace{u(nT)p^n u(0)}^{k=n} + \overbrace{u(nT+T)p^{n+1}u(-T)}^{k=n+1} + \cdots$$

For $n \geq 0$, we obtain

$$y(nT) = 1 + p^1 + p^2 + \cdots + p^n = 1 + \sum_{n=1}^{n} p^n = \frac{1 - p^{(n+1)}}{1 - p}$$

since this is a geometric series with a common ratio $p$.

# Example *Cont'd*

...

For $n < 0$, $y(nT) = 0$.

For $n \geq 0$,

$$y(nT) = \frac{1 - p^{(n+1)}}{1 - p}$$

Therefore, the response can be expressed in closed form as

$$y(nT) = u(nT)\frac{1 - p^{(n+1)}}{1 - p} \quad \blacksquare$$

# Example

$$y(nT) = \mathcal{R}u(nT) = u(nT)\frac{1-p^{(n+1)}}{1-p}$$

and is initially relaxed (i.e., $y(nT) = 0$ for $n < 0$).

Find the response produced by the excitation

$$x(nT) = \begin{cases} 1 & \text{for } 0 \leq n \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

# Example *Cont'd*

**Solution** We observe that

$$x(nT) = \begin{cases} 1 & \text{for } 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases} = u(nT) - u(nT - 5T)$$

and so

$$y(nT) = \mathcal{R}x(nT) = \mathcal{R}u(nT) - \mathcal{R}u(nT - 5T)$$

Since

$$y(nT) = \mathcal{R}u(nT) = u(nT)\frac{1 - p^{(n+1)}}{1 - p}$$

we get

$$y(nT) = u(nT)\frac{1 - p^{(n+1)}}{1 - p} - u(nT - 5T)\frac{1 - p^{(n-4)}}{1 - p} \qquad \blacksquare$$

# Example

An initially relaxed causal nonrecursive system was tested with an input

$$x(nT) = \begin{cases} 0 & \text{for } n < 0 \\ n & \text{for } n \geq 0 \end{cases}$$

and found to have the response given by the following table:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $y(nT)$ | 0 | 1 | 4 | 10 | 20 | 30 | 40 | 50 |

$(a)$ Find the impulse response of the system for values of $n$ over the range $0 \leq n \leq 5$.

$(b)$ Using the result in part $(a)$, find the unit-step response for $0 \leq n \leq 5$.

# Example *Cont'd*

**Solution** ($a$) Since the system is causal and $x(nT) = 0$ for $n < 0$, the convolution summation assumes the form

$$\begin{aligned} y(nT) = \mathcal{R}x(nT) &= \sum_{k=0}^{n} x(kT)h(nT - kT) \\ &= x(0)h(nT) + x(T)h(nT - T) + \cdots + h(0)x(nT) \end{aligned}$$

# Example *Cont'd*

**Solution** ($a$) Since the system is causal and $x(nT) = 0$ for $n < 0$, the convolution summation assumes the form

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Evaluating $y(nT)$ for $n = 1,\ 2, \ldots,$ we get

$$\begin{aligned} y(T) &= x(0)h(T) + x(T)h(0) = 0 \cdot h(T) + 1 \cdot h(0) = 1 \quad \text{or} \quad h(0) = 1 \\ y(2T) &= x(0)h(2T) + x(T)h(T) + x(2T)h(0) \\ &= 0 \cdot h(2T) + 1 \cdot h(T) + 2 \cdot h(0) \\ &= 0 + h(T) + 2 = 4 \quad \text{or} \quad h(T) = 2 \\ y(3T) &= x(0)h(3T) + x(T)h(2T) + x(2T)h(T) + x(3T)h(0) \\ &= 0 \cdot h(3T) + 1 \cdot h(2T) + 2 \cdot h(T) + 3 \cdot h(0) \\ &= h(2T) + 2 \cdot 2 + 3 \cdot 1 = 10 \quad \text{or} \quad h(2T) = 3 \end{aligned}$$

# Example *Cont'd*

$$
\begin{aligned}
y(4T) &= x(0)h(4T) + x(T)h(3T) + x(2T)h(2T) + x(3T)h(T) \\
&\quad +x(4T)h(0) \\
&= 0 \cdot h(4T) + 1 \cdot h(3T) + 2 \cdot h(2T) + 3 \cdot h(T) + 4 \cdot h(0) \\
&= h(3T) + 2 \cdot 3 + 3 \cdot 2 + 4 \cdot 1 = 20 \quad \text{or} \quad h(3T) = 4 \\
y(5T) &= x(0)h(5T) + x(T)h(4T) + x(2T)h(3T) + x(3T)h(2T) \\
&\quad + x(4T)h(T) + x(5T)h(0) \\
&= 0 \cdot h(5T) + 1 \cdot h(4T) + 2 \cdot h(3T) + 3 \cdot h(2T) + 4 \cdot h(T) + 5 \cdot h(0) \\
&= 0 + h(4T) + 2 \cdot 4 + 3 \cdot 3 + 4 \cdot 2 + 5 \cdot 1 = 30 \quad \text{or} \quad h(4T) = 0 \\
y(6T) &= x(0)h(6T) + x(T)h(5T) + x(2T)h(4T) + x(3T)h(3T) \\
&\quad + x(4T)h(2T) + x(5T)h(T) + x(6T)h(0) \\
&= 0 \cdot h(6T) + 1 \cdot h(5T) + 2 \cdot h(4T) + 3 \cdot h(3T) + 4 \cdot h(2T) \\
&\quad +5 \cdot h(T) + 6 \cdot h(0) \\
&= h(5T) + 2 \cdot 0 + 3 \cdot 4 + 4 \cdot 3 + 5 \cdot 2 + 6 \cdot 1 \\
&= 40 \quad \text{or} \quad h(5T) = 0
\end{aligned}
$$

# Example *Cont'd*

Summarizing the results so far, we have

$$\begin{aligned} h(0) &= 1 \quad h(T) = 2 \quad h(2T) = 3 \\ h(3T) &= 4 \quad h(4T) = 0 \quad h(5T) = 0 \quad \blacksquare \end{aligned}$$

($b$) Using the convolution summation again, we obtain the unit-step response as follows:

$$y(nT) = \mathcal{R}x(nT) = \sum_{k=0}^{n} u(kT)h(nT - kT) = \sum_{k=0}^{n} h(nT - kT)$$

Hence

$$\begin{aligned}
y(0) &= h(0) = 1 \\
y(T) &= h(T) + h(0) = 2 + 1 = 3 \\
y(2T) &= h(2T) + h(T) + h(0) = 3 + 2 + 1 = 6 \\
y(3T) &= h(3T) + h(2T) + h(T) + h(0) = 10 \\
y(4T) &= h(4T) + h(3T) + h(2T) + h(T) + h(0) = 15 \\
y(5T) &= h(5T) + h(4T) + h(3T) + h(2T) + h(T) + h(0) = 21 \quad \blacksquare
\end{aligned}$$

# Alternative Classification of Discrete-Time Systems

Discrete-time systems can also be classified on the basis of the duration of the impulse response as

- finite-duration impulse response (FIR) systems
- infinite-duration impulse response (IIR) systems

# Alternative Classification *Cont'd*

- If the impulse response of a discrete-time system is of finite duration such that $h(nT) = 0$ for $n > N$, then the convolution summation gives

$$y(nT) = \sum_{k=0}^{N} h(kT)x(nT - kT)$$

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$$y(nT) = \sum_{k=0}^{N} h(kT)x(nT - kT)$$

- This equation is of the same form as the difference equation of a nonrecursive system, i.e.,

$$y(nT) = \sum_{i=0}^{N} a_i x(nT - iT)$$

with

$$h(0) = a_0,\ h(T) = a_1,\ \ldots,\ h(NT) = a_N$$

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- the impulse response of a nonrecursive system is always of finite duration, and
- given an impulse response of finite duration, a nonrecursive system can be obtained.

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- An impulse response of infinite duration could be achieved with a nonrecursive system of infinite order or with a recursive system.
- Since infinite-order systems are not feasible, an infinite-duration impulse response can only be achieved with a recursive system.
- To confuse the issue somewhat, it is possible to construct a recursive system that has a finite-duration impulse response!

# An FIR Recursive System

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- If we premultiply both sides of the equation by the operator $(1 + 4\mathcal{E}^{-1})$, we get

$$(1 + 4\mathcal{E}^{-1})y(nT) = (1 + 4\mathcal{E}^{-1})[x(nT) + 3x(nT - T)]$$

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- After simplification, we have

$$\begin{aligned} y(nT) + 4y(nT - T) = {} & x(nT) + 3x(nT - T) \\ & +4x(nT - T) + 12x(nT - 2T) \end{aligned}$$

# An FIR Recursive System *Cont'd*

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$$y(nT) + 4y(nT - T) = x(nT) + 3x(nT - T) + 4x(nT - T) + 12x(nT - 2T)$$

- Thus the FIR system can be represented by the recursive equation

$$y(nT) = x(nT) + 7x(nT - T) + 12x(nT - 2T) - 4y(nT - T)$$

...

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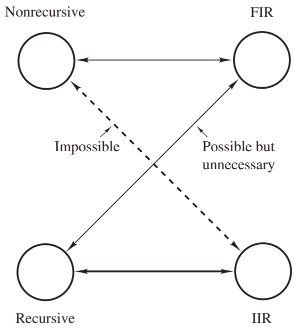
- For most practical purposes *nonrecursive systems are FIR systems and recursive systems are IIR systems*.

# Alternative Classification *Cont'd*




*Note:* An IIR system *cannot* be a nonrecursive system and vice-versa. However, a recursive system can be constructed that is also an FIR system but such a system would serve no useful purpose.

# Stability

- A discrete-time system is said to be *stable* if and only if any bounded excitation results in a bounded response, i.e.,

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- Hence

$$|y(nT)| = \left| \sum_{k=-\infty}^{\infty} h(kT)x(nT - kT) \right| \leq \sum_{k=-\infty}^{\infty} |h(kT) \cdot x(nT - kT)|$$

# Stability *Cont'd*

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- For example,

$$\left| \sum 2 \cdot 3 + (-1) \cdot 4 + 2 \cdot (-2) + (-3) \cdot (-3) \right| = 7$$

$$\leq \sum |2 \cdot 3| + |(-1) \cdot 4| + |2 \cdot (-2)| + |(-3) \cdot (-3)| = 23$$

...

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- If $\quad |x(nT)| \leq P < \infty \quad \text{for all } n$

  we have $\quad |y(nT)| \leq P \sum_{k=-\infty}^{\infty} |h(kT)|$

...

$$|y(nT)| \leq P \sum_{k=-\infty}^{\infty} |h(kT)|$$

- Clearly, if

$$\sum_{k=-\infty}^{\infty} |h(kT)| < \infty \qquad \text{(B)}$$

then

$$|y(nT)| < \infty \quad \text{for all } n$$

# Stability *Cont'd*

...

$$|y(nT)| \leq P \sum_{k=-\infty}^{\infty} |h(kT)|$$

- Clearly, if

$$\sum_{k=-\infty}^{\infty} |h(kT)| < \infty \qquad \text{(B)}$$

then $$|y(nT)| < \infty \quad \text{for all } n$$

- Therefore, Eq. (B) constitutes a *sufficient* condition for stability.

# Stability *Cont'd*

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where $P$ is a positive constant.

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- From the convolution summation, we get

$$|y(nT)| = \left| \sum_{k=-\infty}^{\infty} x(nT - kT)h(kT) \right|$$

- Hence

$$|y(nT)| = \sum_{k=-\infty}^{\infty} P \cdot |h(kT)| = P \sum_{k=-\infty}^{\infty} |h(kT)|$$

...

$$|y(nT)| = P \sum_{k=-\infty}^{\infty} |h(kT)|$$

- Evidently, at least for the type of signal under consideration, the response will be bounded if and only if

$$\sum_{k=-\infty}^{\infty} |h(kT)| < \infty$$

which implies that this condition is also a *necessary* condition for stability.

- Summarizing, the condition

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$$\sum_{k=-\infty}^{\infty} |h(kT)| < \infty$$

is both a *necessary and sufficient* condition for stability.

- *Note:* In nonrecursive systems, the impulse response is both finite in value and also of finite duration and hence the above condition is always satisfied, i.e., *nonrecursive systems are always stable*.

# Example

A first-order system is characterized by the equation

$$y(nT) = x(nT) + py(nT - T)$$

has an impulse response

$$h(nT) = u(nT)p^n$$

Check the stability of the system.

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This is a geometric series and has a sum

$$\sum_{k=-\infty}^{\infty} |h(kT)| = \lim_{n\to\infty} \frac{1 - |p|^{(n+1)}}{1 - |p|}$$

# Example *Cont'd*

If $p > 1$,

$$\sum_{k=-\infty}^{\infty} |h(kT)| = \lim_{n\to\infty} \frac{1 - |p|^{(n+1)}}{1 - |p|} \to \infty$$

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On the other hand, if $p < 1$,

$$\sum_{k=-\infty}^{\infty} |h(kT)| = \lim_{n\to\infty} \frac{1 - |p|^{(n+1)}}{1 - |p|} \to \frac{1}{1 - |p|} = K < \infty$$

where $K$ is a positive constant. Therefore, the system is stable if and only if

$$|p| < 1 \quad \blacksquare$$

# Example

A discrete-time system has an impulse response

$$h(nT) = u(nT)e^{0.1nT} \sin \frac{n\pi}{6}$$

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**Solution** We can write

$$\sum_{k=0}^{\infty} |h(nT)| = \sum_{k=0}^{\infty} \left| u(kT)e^{0.1kT} \sin \frac{k\pi}{6} \right|$$

$$= \sum_{k=3,9,15,\ldots}^{\infty} \left| e^{0.1kT} \right| + \sum_{k\neq 3,9,15,\ldots}^{\infty} \left| e^{0.1kT} \sin \frac{k\pi}{6} \right| \to \infty$$

Therefore, the system is *unstable*. ■

*This slide concludes the presentation.*
*Thank you for your attention.*