Chapter 3
THE FOURIER SERIES AND TRANSFORM
3.1. Introduction
3.3.1-3.3.3 Review of Fourier Transform
3.3.4 Impulse Functions
3.3.6 Fourier Transforms of Periodic Signals

Victoria, BC, Canada
Email: aantoniou@ieee.org

July 9, 2018

Introduction

- Digital filters are often used to process discrete-time signals that have been generated by sampling continuous-time signals.

Introduction

- Digital filters are often used to process discrete-time signals that have been generated by sampling continuous-time signals.
- Frequently digital filters are designed indirectly through the use of analog filters.

Introduction

- Digital filters are often used to process discrete-time signals that have been generated by sampling continuous-time signals.
- Frequently digital filters are designed indirectly through the use of analog filters.
- In order to understand the basis of these techniques, the spectral relationships among continuous-time, impulse-modulated, and discrete-time signals must be understood.

Introduction

- Digital filters are often used to process discrete-time signals that have been generated by sampling continuous-time signals.
- Frequently digital filters are designed indirectly through the use of analog filters.
- In order to understand the basis of these techniques, the spectral relationships among continuous-time, impulse-modulated, and discrete-time signals must be understood.
- These relationships are derived by using the Fourier transform, the Fourier series, the z transform, and Poisson's summation formula.

- Impulse-modulated signals comprise sequences of continuous-time impulse functions and to understand their significance, the properties of impulse functions must be understood.

  On the other hand, Poisson's summation formula is based on a relationship between the Fourier series and the Fourier transform of periodic signals.

Introduction *Cont'd*

- Impulse-modulated signals comprise sequences of continuous-time impulse functions and to understand their significance, the properties of impulse functions must be understood.

  On the other hand, Poisson's summation formula is based on a relationship between the Fourier series and the Fourier transform of periodic signals.

- This presentation begins with a review of the Fourier transform.

- Impulse-modulated signals comprise sequences of continuous-time impulse functions and to understand their significance, the properties of impulse functions must be understood.

  On the other hand, Poisson's summation formula is based on a relationship between the Fourier series and the Fourier transform of periodic signals.

- This presentation begins with a review of the Fourier transform.

- Then impulse functions are defined and their properties are examined.

Introduction *Cont'd*

- Impulse-modulated signals comprise sequences of continuous-time impulse functions and to understand their significance, the properties of impulse functions must be understood.

  On the other hand, Poisson's summation formula is based on a relationship between the Fourier series and the Fourier transform of periodic signals.

- This presentation begins with a review of the Fourier transform.

- Then impulse functions are defined and their properties are examined.

- Subsequently, the application of the Fourier transform to impulse functions and periodic signals is investigated.

Review of Fourier Transform

- The Fourier transform of a continuous-time signal $x(t)$ is defined as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt \qquad \text{(A)}$$

Review of Fourier Transform

- The Fourier transform of a continuous-time signal $x(t)$ is defined as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \qquad \text{(A)}$$

- In general, $X(j\omega)$ is complex and can be written as

$$X(j\omega) = A(\omega)e^{j\phi(\omega)}$$

where

$$A(\omega) = |X(j\omega)| \quad \text{and} \quad \phi(\omega) = \arg X(j\omega)$$

Review of Fourier Transform

- The Fourier transform of a continuous-time signal $x(t)$ is defined as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \tag{A}$$

- In general, $X(j\omega)$ is complex and can be written as

$$X(j\omega) = A(\omega)e^{j\phi(\omega)}$$

where

$$A(\omega) = |X(j\omega)| \quad \text{and} \quad \phi(\omega) = \arg X(j\omega)$$

- Functions $A(\omega)$ and $\phi(\omega)$ are the *amplitude spectrum* and *phase spectrum* of the signal, respectively.

Review of Fourier Transform

- The Fourier transform of a continuous-time signal $x(t)$ is defined as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt \qquad \text{(A)}$$

- In general, $X(j\omega)$ is complex and can be written as

$$X(j\omega) = A(\omega)e^{j\phi(\omega)}$$

where

$$A(\omega) = |X(j\omega)| \quad \text{and} \quad \phi(\omega) = \arg X(j\omega)$$

- Functions $A(\omega)$ and $\phi(\omega)$ are the *amplitude spectrum* and *phase spectrum* of the signal, respectively.

- Together, the amplitude and phase spectrums constitute the *frequency spectrum*.

Review of Fourier Transform *Cont'd*

...

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt \qquad \text{(A)}$$

- Function $x(t)$ is the *inverse Fourier transform* of $X(j\omega)$ and is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega \qquad \text{(B)}$$

. . .

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt \qquad \text{(A)}$$

- Function $x(t)$ is the *inverse Fourier transform* of $X(j\omega)$ and is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega \qquad \text{(B)}$$

- Eqs. (A) and (B) can be written in operator format as

$$X(j\omega) = \mathcal{F}x(t) \quad \text{and} \quad x(t) = \mathcal{F}^{-1}X(j\omega)$$

respectively.

...

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt \qquad \text{(A)}$$

- Function $x(t)$ is the *inverse Fourier transform* of $X(j\omega)$ and is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega \qquad \text{(B)}$$

- Eqs. (A) and (B) can be written in operator format as

$$X(j\omega) = \mathcal{F}x(t) \quad \text{and} \quad x(t) = \mathcal{F}^{-1}X(j\omega)$$

respectively.

- An alternative shorthand notation is

$$x(t) \leftrightarrow X(j\omega)$$

Convergence Theorem

- The convergence theorem of the Fourier transform states that if

$$\lim_{T\to\infty} \int_{-T}^{T} |x(t)|\, dt < \infty$$

then the Fourier transform of $x(t)$, $X(j\omega)$, exists and its inverse can be obtained by using the equation

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$$

Convergence Theorem

- The convergence theorem of the Fourier transform states that if

$$\lim_{T\to\infty} \int_{-T}^{T} |x(t)|\, dt < \infty$$

then the Fourier transform of $x(t)$, $X(j\omega)$, exists and its inverse can be obtained by using the equation

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$$

- Many signals that are of considerable interest in practice violate the above condition, for example, impulse functions, impulse-modulated signals, and periodic signals.

Convergence Theorem

- The convergence theorem of the Fourier transform states that if

$$\lim_{T\to\infty} \int_{-T}^{T} |x(t)|\, dt < \infty$$

then the Fourier transform of $x(t)$, $X(j\omega)$, exists and its inverse can be obtained by using the equation

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$$

- Many signals that are of considerable interest in practice violate the above condition, for example, impulse functions, impulse-modulated signals, and periodic signals.

- However, convergence problems can be circumvented by paying particular attention to the definition of impulse functions.

Impulse Functions

- The unit impulse function has been defined in the past as

$$\delta(t) = \lim_{\tau \to 0} \bar{p}_\tau(t) = \lim_{\tau \to 0} \begin{cases} \frac{1}{\tau} & \text{for } |t| \leq \tau/2 \\ 0 & \text{otherwise} \end{cases}$$

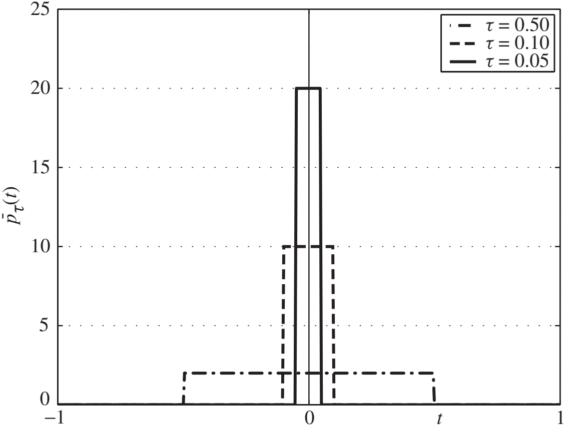
Impulse Functions

- The unit impulse function has been defined in the past as

$$\delta(t) = \lim_{\tau \to 0} \bar{p}_\tau(t) = \lim_{\tau \to 0} \begin{cases} \frac{1}{\tau} & \text{for } |t| \leq \tau/2 \\ 0 & \text{otherwise} \end{cases}$$

- Obviously, this is an infinitesimally thin, infinitely tall pulse whose area is equal to unity for any finite value of τ.

Pulse function $\bar{p}_\tau(t)$ for three values of τ:



(a)

Mathematical Problem

- The Fourier transform of the unit impulse function as defined in the past should be given by the integral

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$
$$= \int_{-\infty}^{\infty} \lim_{\tau\to 0}[\bar{p}_\tau(t)]e^{-j\omega t}\,dt$$

where

$$\bar{p}_\tau(t) = \begin{cases} \frac{1}{\tau} & \text{for } |t| \le \tau/2 \\ 0 & \text{otherwise} \end{cases}$$

Mathematical Problem

- The Fourier transform of the unit impulse function as defined in the past should be given by the integral

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} \lim_{\tau \to 0} [\bar{p}_\tau(t)] e^{-j\omega t}\, dt \end{aligned}$$

where

$$\bar{p}_\tau(t) = \begin{cases} \frac{1}{\tau} & \text{for } |t| \le \tau/2 \\ 0 & \text{otherwise} \end{cases}$$

- If we now attempt to evaluate the function $\bar{p}_\tau(t) e^{-j\omega t}$ at $\tau = 0$, we find that it becomes infinite and, therefore, the above integral cannot be evaluated.

Mathematical Problem *Cont'd*

- We can write

$$\mathcal{F} \lim_{\tau \to 0} \bar{p}_\tau(t) = \int_{-\infty}^{\infty} \lim_{\tau \to 0} [\bar{p}_\tau(t)] e^{-j\omega t}\, dt$$

$$\approx \int_{-\tau/2}^{\tau/2} \lim_{\tau \to 0} \left[\frac{1}{\tau} \right] dt$$

Mathematical Problem *Cont'd*

- We can write

$$\mathcal{F} \lim_{\tau \to 0} \bar{p}_\tau(t) = \int_{-\infty}^{\infty} \lim_{\tau \to 0} [\bar{p}_\tau(t)] e^{-j\omega t}\, dt$$
$$\approx \int_{-\tau/2}^{\tau/2} \lim_{\tau \to 0} \left[\frac{1}{\tau}\right] dt$$

- Since the area of the pulse function $\bar{p}_\tau(t)$ is unity for any finite value of τ, we might be tempted to assume that the area is equal to unity even for $\tau = 0$, i.e.,

$$\mathcal{F} \lim_{\tau \to 0} \bar{p}_\tau(t) = 1$$

Mathematical Problem *Cont'd*

- The Fourier transform of $\bar{p}_\tau(t)$ for a finite τ is given by

$$\mathcal{F}\bar{p}_\tau(t) = \frac{1}{\tau}\mathcal{F}p_\tau(t) = \frac{2\sin\omega\tau/2}{\omega\tau}$$

- The Fourier transform of $\bar{p}_\tau(t)$ for a finite τ is given by

$$\mathcal{F}\bar{p}_\tau(t) = \frac{1}{\tau}\mathcal{F}p_\tau(t) = \frac{2\sin\omega\tau/2}{\omega\tau}$$

- Obviously, this is well defined and, interestingly, it has the limit

$$\lim_{\tau\to 0}\mathcal{F}\bar{p}_\tau(t) = 1$$

Mathematical Problem *Cont'd*

- The Fourier transform of $\bar{p}_\tau(t)$ for a finite τ is given by

$$\mathcal{F}\bar{p}_\tau(t) = \frac{1}{\tau}\mathcal{F}p_\tau(t) = \frac{2\sin\omega\tau/2}{\omega\tau}$$

- Obviously, this is well defined and, interestingly, it has the limit

$$\lim_{\tau\to 0}\mathcal{F}\bar{p}_\tau(t) = 1$$

- So far so good!

- If we now attempt to find the inverse Fourier transform of 1, we run into certain mathematical difficulties.

Mathematical Problem *Cont'd*

- If we now attempt to find the inverse Fourier transform of 1, we run into certain mathematical difficulties.

- From the definition of the inverse Fourier transform, we have

$$\begin{aligned}\mathcal{F}^{-1}1 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\, d\omega \\ &= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} \cos\omega t\, d\omega + j\int_{-\infty}^{\infty} \sin\omega t\, d\omega\right]\end{aligned}$$

Mathematical Problem *Cont'd*

- If we now attempt to find the inverse Fourier transform of 1, we run into certain mathematical difficulties.

- From the definition of the inverse Fourier transform, we have

$$\begin{aligned}\mathcal{F}^{-1}1 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\,d\omega \\ &= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\cos\omega t\,d\omega + j\int_{-\infty}^{\infty}\sin\omega t\,d\omega\right]\end{aligned}$$

- However, mathematicians will tell us that these integrals do not converge or do not exist!

Mathematical Problem *Cont'd*

- Summarizing, by cheating a little bit we can get a more or less meaningful Fourier transform for the unit impulse function.

- Summarizing, by cheating a little bit we can get a more or less meaningful Fourier transform for the unit impulse function.
- Unfortunately, *it is impossible to recover the impulse function* from its Fourier transform by applying the inverse Fourier transform.

Mathematical Problem *Cont'd*

- The impulse-function problem can be circumvented in two ways, a *practical* and a *theoretical* one:

Mathematical Problem *Cont'd*

- The impulse-function problem can be circumvented in two ways, a *practical* and a *theoretical* one:
  - The practical approach is easy to understand and apply but it lacks rigor.

Mathematical Problem *Cont'd*

- The impulse-function problem can be circumvented in two ways, a *practical* and a *theoretical* one:
  - The practical approach is easy to understand and apply but it lacks rigor.
  - The theoretical approach is rigorous but it is rather abstract and more difficult to understand or apply in practical situations.

Practical Approach to Impulse Functions

- In the practical approach to impulse functions, a function $\gamma(t)$ is said to be a unit impulse function if, for any continuous function $x(t)$ over the range $-\epsilon < t < \epsilon$, the following relation is satisfied:

$$\int_{-\infty}^{\infty} \gamma(t)x(t)\,dt \doteq x(0) \qquad \text{(C)}$$

Practical Approach to Impulse Functions

- In the practical approach to impulse functions, a function $\gamma(t)$ is said to be a unit impulse function if, for any continuous function $x(t)$ over the range $-\epsilon < t < \epsilon$, the following relation is satisfied:

$$\int_{-\infty}^{\infty} \gamma(t)x(t)\,dt \doteq x(0) \qquad \text{(C)}$$

- The special symbol $\doteq$ is used to signify that the two sides can be made to approach one another to any desired degree of precision but *cannot be made exactly equal*.

Practical Approach to Impulse Functions

- In the practical approach to impulse functions, a function $\gamma(t)$ is said to be a unit impulse function if, for any continuous function $x(t)$ over the range $-\epsilon < t < \epsilon$, the following relation is satisfied:

$$\int_{-\infty}^{\infty} \gamma(t)x(t)\,dt \doteq x(0) \qquad \text{(C)}$$

- The special symbol $\doteq$ is used to signify that the two sides can be made to approach one another to any desired degree of precision but *cannot be made exactly equal*.

- Now consider the pulse function

$$\lim_{\tau \to \epsilon} \bar{p}_\tau(t) = \bar{p}_\epsilon(t) = \begin{cases} \frac{1}{\epsilon} & \text{for } |t| \leq \epsilon/2 \\ 0 & \text{otherwise} \end{cases}$$

where ϵ is a small but finite constant.

Practical Approach · · · *Cont'd*

· · ·

$$\int_{-\infty}^{\infty} \gamma(t)x(t)\,dt \doteq x(0) \qquad \text{(C)}$$

- If we let

$$\gamma(t) = \lim_{\tau \to \epsilon} \bar{p}_\tau(t)$$

in the left-hand side of Eq. (C), we obtain

$$\int_{-\infty}^{\infty} \lim_{\tau \to \epsilon} [\bar{p}_\tau(t)]x(t)\,dt = \int_{-\epsilon/2}^{\epsilon/2} \frac{1}{\epsilon} x(t)\,dt$$

$$\doteq \frac{1}{\epsilon} x(0) \int_{-\epsilon/2}^{\epsilon/2} dt \doteq x(0)$$

...

$$\int_{-\infty}^{\infty} \gamma(t)x(t)\,dt \doteq x(0) \qquad \text{(C)}$$

- If we let

$$\gamma(t) = \lim_{\tau \to \epsilon} \bar{p}_\tau(t)$$

in the left-hand side of Eq. (C), we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \lim_{\tau \to \epsilon} [\bar{p}_\tau(t)] x(t)\,dt &= \int_{-\epsilon/2}^{\epsilon/2} \frac{1}{\epsilon} x(t)\,dt \\ &\doteq \frac{1}{\epsilon} x(0) \int_{-\epsilon/2}^{\epsilon/2} dt \doteq x(0)\end{aligned}$$

- Thus we conclude that the very thin *pulse function* $\lim_{\tau \to \epsilon} \bar{p}_\tau(t)$ *behaves as an impulse function* and, therefore, we can write

$$\delta(t) = \lim_{\tau \to \epsilon} \bar{p}_\tau(t)$$

. . .

$$\delta(t) = \lim_{\tau \to \epsilon} \bar{p}_\tau(t)$$

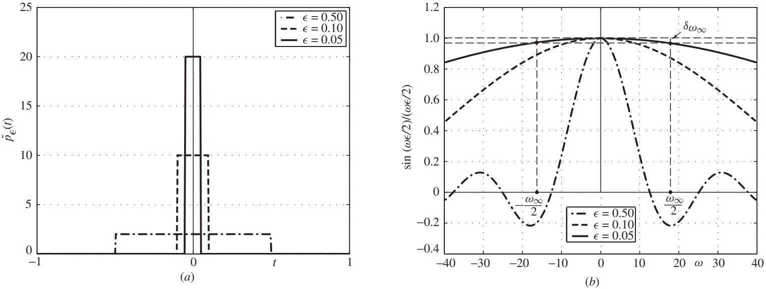
- Now if we apply the Fourier transform to the impulse function as defined, we get

$$\lim_{\tau \to \epsilon} \bar{p}_\tau(t) \leftrightarrow \lim_{\tau \to \epsilon} \frac{2 \sin \omega\tau/2}{\omega\tau}$$

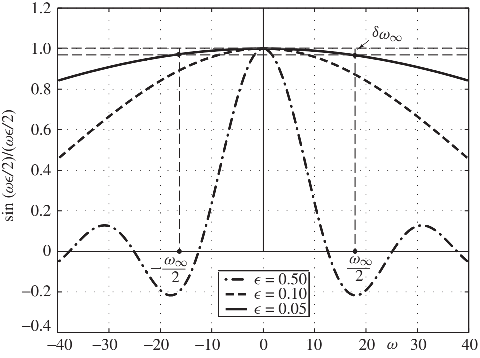
· · ·

$$\lim_{\tau \to \epsilon} \bar{p}_\tau(t) \leftrightarrow \lim_{\tau \to \epsilon} \frac{2 \sin \omega\tau/2}{\omega\tau}$$

- As τ is reduced, the *pulse function at the left tends to become thinner and taller* whereas the *sinc function at the right tends to be flattened out*.



- For some small but finite ϵ, the sinc function will be equal to *unity* to within an error δ_{ω_∞} over some frequency range $-\omega_\infty/2 < \omega < \omega_\infty/2$.



(*b*)

- Therefore, we can write

$$\delta(t) = \lim_{\tau \to \epsilon} \bar{p}_\tau(t) \leftrightarrow \lim_{\tau \to \epsilon} \frac{2 \sin \omega\tau/2}{\omega\tau} = i(\omega)$$

where $i(\omega)$ may be referred to as a *frequency-domain unity function*.

- Summarizing,
  - the Fourier transform of a time-domain impulse function is a frequency-domain unity function, and
  - the inverse Fourier transform of a frequency-domain unity function is a time-domain impulse function,

- Summarizing,
  - the Fourier transform of a time-domain impulse function is a frequency-domain unity function, and
  - the inverse Fourier transform of a frequency-domain unity function is a time-domain impulse function,

i.e.,
$$\delta(t) \leftrightarrow i(\omega)$$

- Summarizing,
  - the Fourier transform of a time-domain impulse function is a frequency-domain unity function, and
  - the inverse Fourier transform of a frequency-domain unity function is a time-domain impulse function,

i.e.,
$$\delta(t) \leftrightarrow i(\omega)$$

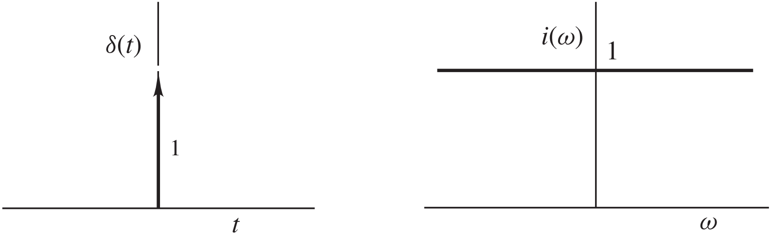
- Since $i(\omega) \doteq 1$ for the frequency range of interest, we can write

$$\delta(t) \leftrightsquigarrow 1$$

where the wavy double arrow $\leftrightsquigarrow$ signifies that the relation is *approximate* with the understanding that it can be made as exact as desired by making ϵ sufficiently small.

Practical Approach · · · *Cont'd*

The impulse and unity functions can be represented by the idealized graphs:



(a)

Properties of Impulse Functions

- Assuming that $x(t)$ is a continuous function of t over the range $-\epsilon < t < \epsilon$, the following relations apply:

(a) $$\int_{-\infty}^{\infty} \delta(t-\tau)x(t)\,dt = \int_{-\infty}^{\infty} \delta(-t+\tau)x(t)\,dt \doteq x(\tau)$$

(b) $$\delta(t-\tau)x(t) = \delta(-t+\tau)x(t) \doteq \delta(t-\tau)x(\tau)$$

(c) $$\delta(t)x(t) = \delta(-t)x(t) \doteq \delta(t)x(0)$$

(See textbook for proofs.)

Frequency-Domain Impulse Functions

- Given a transform pair

$$\delta(t) \leftrightarrow i(\omega)$$

where

$$\delta(t) = \lim_{\tau \to \epsilon} \bar{p}_\tau(t)$$
$$i(\omega) = \lim_{\tau \to \epsilon} \frac{2 \sin \omega\tau/2}{\omega\tau} \doteq 1 \quad \text{for} \quad |\omega| < \omega_\infty$$

the corresponding transform pair

$$i(t) \leftrightarrow 2\pi\delta(\omega)$$

where

$$i(t) = \frac{2 \sin t\epsilon/2}{t\epsilon} \doteq 1 \quad \text{for} \quad |t| < t_\infty$$
$$\delta(\omega) = \bar{p}_\epsilon(\omega)$$

can be generated by applying the *symmetry theorem* of the Fourier transform.

...

$$i(t) \leftrightarrow 2\pi\delta(\omega)$$

- Function $i(t)$ is a *time-domain unity function* whereas $\delta(\omega)$ is a *frequency-domain unit impulse function*.

. . .

$$i(t) \leftrightarrow 2\pi\delta(\omega)$$

- Function $i(t)$ is a *time-domain unity function* whereas $\delta(\omega)$ is a *frequency-domain unit impulse function*.

- Since $i(t) \doteq 1$, we have

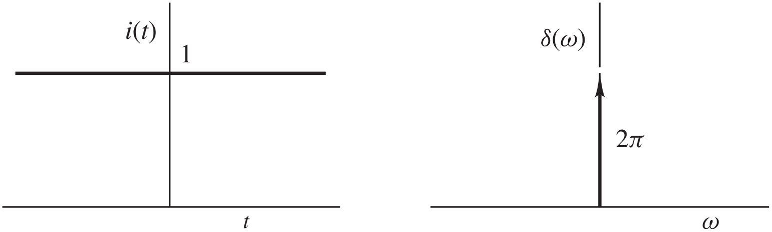
$$1 \leftrightsquigarrow 2\pi\delta(\omega)$$

Frequency-Domain Impulse Functions *Cont'd*

...

$$i(t) \leftrightarrow 2\pi\delta(\omega) \quad \text{or} \quad 1 \leftrightsquigarrow 2\pi\delta(\omega)$$

This transform pair can be represented by the idealized graphs shown.



(b)

Properties of Frequency-Domain Impulse Functions

- Assuming that $X(j\omega)$ is a continuous function of ω over the range $-\epsilon < \omega < \epsilon$, the following relations apply:

(a)
$$\int_{-\infty}^{\infty} \delta(\omega - \varpi) X(j\omega)\, dt$$
$$= \int_{-\infty}^{\infty} \delta(-\omega + \varpi) X(j\omega)\, dt \doteq X(j\varpi)$$

(b) $$\delta(\omega - \varpi) X(j\omega) = \delta(-\omega + \varpi) X(j\omega) \doteq \delta(t - \varpi) X(j\varpi)$$

(c) $$\delta(\omega) X(j\omega) = \delta(-\omega) X(j\omega) \doteq \delta(t) X(0)$$

(See textbook for details.)

Fourier Transforms of Exponentials

- Since

$$\delta(t) \leftrightarrow i(\omega)$$

the application of the time-shifting theorem gives

$$\delta(t - t_0) \leftrightarrow i(\omega) e^{-j\omega t_0}$$

and since $i(\omega) \doteq 1$, we get

$$\delta(t - t_0) \leftrightsquigarrow e^{-j\omega t_0}$$

Fourier Transforms of Exponentials

- Since

$$\delta(t) \leftrightarrow i(\omega)$$

the application of the time-shifting theorem gives

$$\delta(t - t_0) \leftrightarrow i(\omega)e^{-j\omega t_0}$$

and since $i(\omega) \doteq 1$, we get

$$\delta(t - t_0) \leftrightsquigarrow e^{-j\omega t_0}$$

- Now applying the frequency-shifting theorem to the frequency-domain impulse function, we obtain

$$i(t)e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$$

and since $i(t) \doteq 1$, we get

$$e^{j\omega_0 t} \leftrightsquigarrow 2\pi\delta(\omega - \omega_0)$$

Fourier Transforms of Sinusoidal Signals

- We know that

$$i(t)e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$$

and

$$i(t)e^{-j\omega_0 t} \leftrightarrow 2\pi\delta(\omega + \omega_0)$$

Fourier Transforms of Sinusoidal Signals

- We know that

$$i(t)e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$$

and

$$i(t)e^{-j\omega_0 t} \leftrightarrow 2\pi\delta(\omega + \omega_0)$$

- If we add the two equations, we get

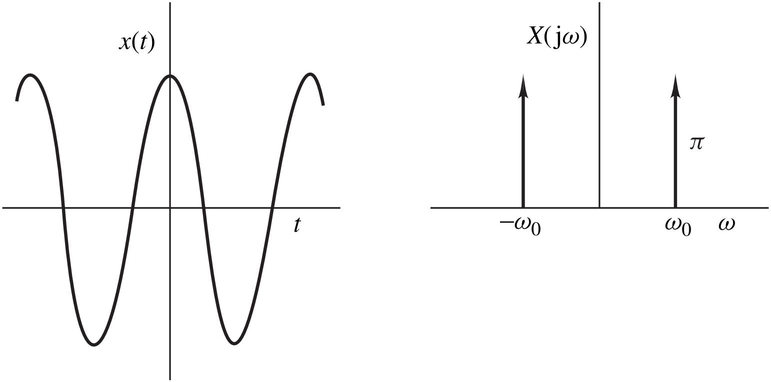
$$i(t)(e^{j\omega_0 t} + e^{-j\omega_0 t}) = 2i(t) \cdot \cos\omega_0 t \leftrightarrow 2\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

and since $i(t) \doteq 1$, we have

$$\cos\omega_0 t \leftrightsquigarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

...

$$\cos \omega_0 t \leftrightsquigarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$



- As before,

$$i(t)e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$$

and

$$i(t)e^{-j\omega_0 t} \leftrightarrow 2\pi\delta(\omega + \omega_0)$$

Fourier Transforms of Sinusoidal Signals *Cont'd*

- As before,

$$i(t)e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$$

and

$$i(t)e^{-j\omega_0 t} \leftrightarrow 2\pi\delta(\omega + \omega_0)$$

- If we subtract the top equation from the bottom one, we have

$$i(t)(e^{-j\omega_0 t} - e^{j\omega_0 t}) = -2ji(t) \cdot \sin\omega_0 t \leftrightarrow 2\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$$

and since $i(t) \risingdotseq 1$, we can write

$$\sin\omega_0 t \leftrightsquigarrow j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$

Fourier Transforms of Periodic Signals

- An arbitrary periodic signal can be represented by the Fourier series

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} X_k e^{-jk\omega_0 t}$$

Fourier Transforms of Periodic Signals

- An arbitrary periodic signal can be represented by the Fourier series

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} X_k e^{-jk\omega_0 t}$$

- Hence

$$\mathcal{F}\tilde{x}(t) = \sum_{k=-\infty}^{\infty} 2\pi X_k \mathcal{F} e^{-jk\omega_0 t} \doteq \sum_{k=-\infty}^{\infty} 2\pi X_k \delta(\omega - k\omega_0)$$

or

$$\tilde{x}(t) \leftrightsquigarrow 2\pi \sum_{k=-\infty}^{\infty} X_k \delta(\omega - k\omega_0)$$

Fourier Transforms of Periodic Signals *Cont'd*

. . .

$$\tilde{x}(t) \leftrightsquigarrow 2\pi \sum_{k=-\infty}^{\infty} X_k \delta(\omega - k\omega_0)$$

- Summarizing, the frequency spectrum of a periodic signal can be represented by *an infinite sequence of numbers* X_k for $-\infty < k < \infty$, i.e., the Fourier-series coefficients as shown in Chap. 3

. . .

$$\tilde{x}(t) \leftrightsquigarrow 2\pi \sum_{k=-\infty}^{\infty} X_k \delta(\omega - k\omega_0)$$

- Summarizing, the frequency spectrum of a periodic signal can be represented by *an infinite sequence of numbers* X_k for $-\infty < k < \infty$, i.e., the Fourier-series coefficients as shown in Chap. 3

  or

...

$$\tilde{x}(t) \leftrightsquigarrow 2\pi \sum_{k=-\infty}^{\infty} X_k \delta(\omega - k\omega_0)$$

- Summarizing, the frequency spectrum of a periodic signal can be represented by *an infinite sequence of numbers* X_k for $-\infty < k < \infty$, i.e., the Fourier-series coefficients as shown in Chap. 3

  or

- by *an infinite sequence of frequency-domain impulse functions* of strength $2\pi X_k$ for $-\infty < k < \infty$ as shown in the previous slide.

Theoretical Approach to Impulse Functions

- The Fourier transform pairs generated through the practical approach are approximate since the pulse width ϵ *cannot be reduced to absolute zero*.

Theoretical Approach to Impulse Functions

- The Fourier transform pairs generated through the practical approach are approximate since the pulse width ϵ *cannot be reduced to absolute zero*.

- However, by defining impulse functions in terms of *generalized functions*, analogous, but exact, Fourier transform pairs can be generated.

Theoretical Approach to Impulse Functions

- The Fourier transform pairs generated through the practical approach are approximate since the pulse width ϵ *cannot be reduced to absolute zero*.
- However, by defining impulse functions in terms of *generalized functions*, analogous, but exact, Fourier transform pairs can be generated.
- Unfortunately, impulse functions so defined are rather *impractical* and *difficult to implement* in a laboratory.

Theoretical Approach to Impulse Functions

- The Fourier transform pairs generated through the practical approach are approximate since the pulse width ϵ *cannot be reduced to absolute zero*.

- However, by defining impulse functions in terms of *generalized functions*, analogous, but exact, Fourier transform pairs can be generated.

- Unfortunately, impulse functions so defined are rather *impractical* and *difficult to implement* in a laboratory.

- See textbook for more details and references on generalized functions.

Summary of Fourier Transforms Derived

$x(t)$	$X(j\omega)$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\delta(t-t_0)$	$e^{-j\omega t_0}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$
$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$

This slide concludes the presentation.
Thank you for your attention.