Chapter 5
APPLICATION OF TRANSFORM THEORY TO SYSTEMS
5.5.1 Steady-State Sinusoidal Response
5.5.2 Evaluation of Frequency Response
5.5.3 Periodicity of Frequency Response

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- If the discrete-time system is *stable*, then the transient component tends to zero and the sinusoidal component becomes the steady-state response of the system.
- The amplitude and phase angle of the steady-state sinusoidal response define the *frequency response* of the system.

- Consider a causal recursive system characterized by the Nth-order transfer function

$$H(z) = \frac{N(z)}{D(z)} = \frac{H_0 \prod_{i=1}^{N}(z - z_i)}{\prod_{i=1}^{N}(z - p_i)}$$

Steady-State Sinusoidal Response

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- The response of such a system to a sinusoidal signal of unit amplitude and zero phase angle that starts at time 0, i.e.,

$$x(nT) = u(nT) \sin \omega nT$$

is given by

$$y(nT) = \mathcal{Z}^{-1}[H(z)X(z)]$$

...

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- From the table of standard z transforms, we have

$$X(z) = \mathcal{Z}[u(nT)\sin \omega nT] = \frac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1}$$

and if we factorize the denominator, we get

$$X(z) = \frac{z \sin \omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})}$$

- Using the general inversion formula (see Chap. 4)

$$y(nT) = \frac{1}{2\pi j} \oint_{\Gamma} H(z)X(z)z^{n-1}\,dz$$

and applying the residue theorem, we have

$$y(nT) = u(nT) \sum_{i=1}^{N+2} \text{res}_{\,z=p_i}[H(z)X(z)z^{n-1}]$$

Sinusoidal Response *Cont'd*

- If we consider a transfer function with simple poles (for the sake of simplicity), we have

$$H(z) = \frac{N(z)}{D(z)} = \frac{H_0 \prod_{i=1}^{N}(z - z_i)}{\prod_{i=1}^{N}(z - p_i)}$$

and since

$$X(z) = \frac{z \sin \omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})}$$

the sinusoidal response is given by

$$y(nT) = u(nT) \sum_{i=1}^{N+2} \text{res}_{\,z=p_i} [H(z)X(z)z^{n-1}]$$

...

$$y(nT) = u(nT) \sum_{i=1}^{N+2} \text{res}_{\,z=p_i}[H(z)X(z)z^{n-1}]$$

or

$$y(nT) = u(nT)\Big\{ \sum_{i=1}^{N} X(p_i)p_i^{n-1}\text{res}_{\,z=p_i}H(z) + \frac{1}{2j}[H(e^{j\omega T})e^{j\omega nT} - H(e^{-j\omega T})e^{-j\omega nT}]\Big\}$$

where the first two terms are the residues of $H(z)X(z)z^{n-1}$ at the poles of $X(z)$ and the terms under the sum are its residues at the poles of $H(z)$.

- The sinusoidal response of a system can thus be expressed as a sum of two components, i.e.,

$$y(nT) = y_{\mathrm{TR}}(nT) + \tilde{y}(nT)$$

where

$$y_{\mathrm{TR}}(nT) = \sum_{i=1}^{N} X(p_i) p_i^{n-1} \operatorname{res}_{z=p_i} H(z) \qquad \text{(A)}$$

$$\tilde{y}(nT) = \frac{1}{2j}[H(e^{j\omega T})e^{j\omega nT} - H(e^{-j\omega T})e^{-j\omega nT}] \qquad \text{(B)}$$

- If we express pole p_i in the exponential form $p_i = r_i e^{j\psi_i}$, then

$$p_i^{n-1} = r_i^{n-1} e^{j(n-1)\psi_i}$$

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- If the system is stable, then $r_i < 1$ for $i = 1, 2, \ldots, N$ and hence

$$\lim_{n\to\infty} p_i^{n-1} = \lim_{n\to\infty} [r_i^{n-1} e^{j(n-1)\psi_i}] \to 0$$

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$$y_{\mathrm{TR}}(nT) = \sum_{i=1}^{N} X(p_i)p_i^{n-1}\text{res}_{\ z=p_i} H(z) \qquad \text{(A)}$$

$$\lim_{n\to\infty} p_i^{n-1} = \lim_{n\to\infty}[r_i^{n-1}e^{j(n-1)\psi_i}] \to 0$$

- Consequently, Eq. (A) gives

$$\lim_{n\to\infty} y_{\mathrm{TR}}(nT) = \lim_{n\to\infty} \sum_{i=1}^{N} X(p_i)p_i^{n-1}\text{res}_{\ z=p_i} H(z) \to 0$$

i.e., *$y_{TR}(nT)$ is a transient component* which tends to zero as $n \to \infty$ if the system is stable.

- Hence the steady-state response of the system can be obtained as

$$\tilde{y}(nT) = \lim_{n\to\infty} y(nT) = \frac{1}{2j}[H(e^{j\omega T})e^{j\omega nT} - H(e^{-j\omega T})e^{-j\omega nT}]$$

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- If we now let $H(e^{j\omega T}) = M(\omega)e^{j\theta(\omega)}$ where
$$M(\omega) = |H(e^{j\omega T})| \quad \text{and} \quad \theta(\omega) = \arg H(e^{j\omega T})$$
straightforward manipulation (see textbook) will show that $M(\omega)$ is an *even* function and $\theta(\omega)$ is an *odd* function of ω, i.e.,
$$M(-\omega) = M(\omega) \quad \text{and} \quad \theta(-\omega) = -\theta(\omega)$$

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where $\quad M(\omega) = |H(e^{j\omega T})|, \quad \theta(\omega) = \arg H(e^{j\omega T})$

and $\quad M(-\omega) = M(\omega), \quad \theta(-\omega) = -\theta(\omega)$

- Therefore, the steady-state sinusoidal response can be expressed as

$$\begin{aligned}\tilde{y}(nT) &= \frac{1}{2j}\left[M(\omega)e^{j\theta(\omega)}e^{j\omega nT} - M(\omega)e^{-j\theta(\omega)}e^{-j\omega nT}\right] \\ &= M(\omega)\frac{1}{2j}\left[e^{j[\omega nT+\theta(\omega)]} - e^{-j[\omega nT+\theta(\omega)]}\right] \\ &= M(\omega)\sin[\omega nT + \theta(\omega)]\end{aligned}$$

i.e., *$\tilde{y}(nT)$ is a sinusoidal component* with amplitude $M(\omega)$ and phase angle $\theta(\omega)$.

- Summarizing, the steady-state response of an N-order discrete-time system to a sinusoidal signal with unit amplitude and zero phase angle is another sinusoidal signal of the form

$$\lim_{nT\to\infty} y(nT) = \tilde{y}(nT) = M(\omega)\sin[\omega nT + \theta(\omega)]$$

which has an amplitude and phase angle

$$M(\omega) = |H(e^{j\omega T})| \quad \text{and} \quad \theta(\omega) = \arg H(e^{j\omega T})$$

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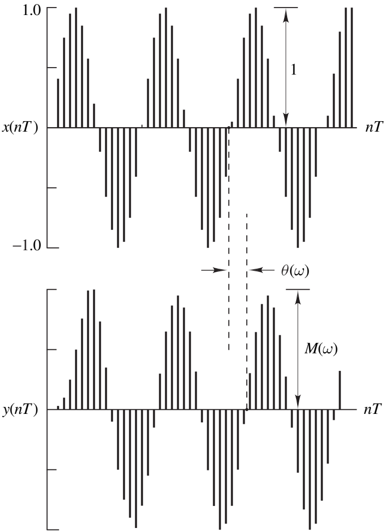
- In effect, *a discrete-time system will multiply the amplitude of a sinusoidal input by $M(\omega)$ and increase its phase angle by $\theta(\omega)$*.

Sinusoidal Response *Cont'd*



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$$M(\omega) = |H(e^{j\omega T})| \quad \text{and} \quad \theta(\omega) = \arg H(e^{j\omega T})$$

- $M(\omega)$ is said to be the *gain* of the system at frequency ω.

  In digital filters, $M(\omega)$ can vary over many orders of magnitude and is often expressed in *decibels (dB)* as $20 \log M(\omega)$.

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- As a function of ω, $M(\omega)$ is said to be the *amplitude response*.

- As a function of ω, $\theta(\omega)$ is said to be the *phase response*.

- The function

$$H(e^{j\omega T}) = M(\omega)e^{j\theta(\omega)}$$

which includes the amplitude response $M(\omega)$ and phase response $\theta(\omega)$ as components is said to be the *frequency response* of the system.

Physical Interpretation

- The frequency spectrum of a signal $x(nT)$ whose z transform is $X(z)$ is given by $X(e^{j\omega T})$ (see Chap. 4).

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- Since

$$Y(z) = H(z)X(z) \quad \text{or} \quad Y(e^{j\omega T}) = H(e^{j\omega T})X(e^{j\omega T})$$

we conclude that the *spectrum of the output* signal is equal to the *frequency response* (or the spectrum of the impulse response) of the system *times* the *spectrum of the input* signal.

Evaluation of Frequency Response

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- This amounts to evaluating the transfer function on the unit circle $|z| = 1$ of the z plane.
- If we let $z = e^{j\omega T}$ in the transfer function

$$H(z) = \frac{H_0 \prod_{i=1}^{N}(z - z_i)^{m_i}}{\prod_{i=1}^{N}(z - p_i)^{n_i}}$$

we obtain

$$H(e^{j\omega T}) = M(\omega)e^{j\theta(\omega)} = \frac{H_0 \prod_{i=1}^{N}(e^{j\omega T} - z_i)^{m_i}}{\prod_{i=1}^{N}(e^{j\omega T} - p_i)^{n_i}}$$

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- By letting $e^{j\omega T} - z_i = M_{z_i} e^{j\psi_{z_i}}$ and $e^{j\omega T} - p_i = M_{p_i} e^{j\psi_{p_i}}$

  we get
$$M(\omega) = \frac{|H_0| \prod_{i=1}^{N} M_{z_i}^{m_i}}{\prod_{i=1}^{N} M_{p_i}^{n_i}} \qquad \text{(B)}$$

$$\theta(\omega) = \arg H_0 + \sum_{i=1}^{N} m_i \psi_{z_i} - \sum_{i=1}^{N} n_i \psi_{p_i} \qquad \text{(C)}$$

  where $\arg H_0 = \pi$ if H_0 is negative and is zero otherwise.

Graphical Procedure

The gain and phase shift of a discrete-time system at a specified frequency ω can be determined *graphically* through the following procedure:

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3. Draw the complex number $e^{j\omega T}$.
4. Draw m_i complex numbers from each m_ith-order zero of $H(z)$ to meet complex number $e^{j\omega T}$ on the unit circle.
5. Draw n_i complex numbers from each n_ith-order pole to meet complex number $e^{j\omega T}$ on the unit circle.

6. Calculate the gain $M(\omega)$ using Eq. (C), i.e.,

$$M(\omega) = \frac{|H_0| \prod_{i=1}^{N} M_{z_i}^{m_i}}{\prod_{i=1}^{N} M_{p_i}^{n_i}} \qquad \text{(B)}$$

6. Calculate the gain $M(\omega)$ using Eq. (C), i.e.,

$$M(\omega) = \frac{|H_0| \prod_{i=1}^{N} M_{z_i}^{m_i}}{\prod_{i=1}^{N} M_{p_i}^{n_i}} \qquad \text{(B)}$$

7. Calculate phase shift $\theta(\omega)$ using Eq. (D), i.e.,

$$\theta(\omega) = \arg H_0 + \sum_{i=1}^{N} m_i \psi_{z_i} - \sum_{i=1}^{N} n_i \psi_{p_i} \qquad \text{(C)}$$

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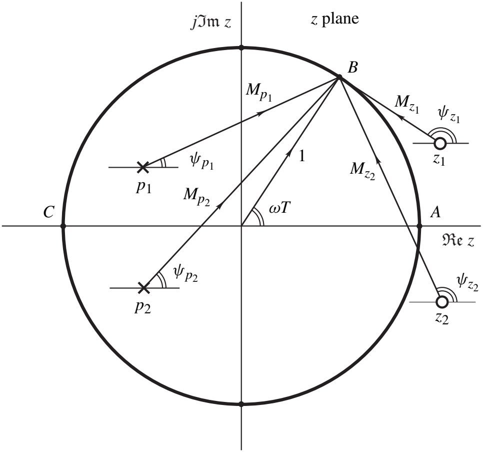
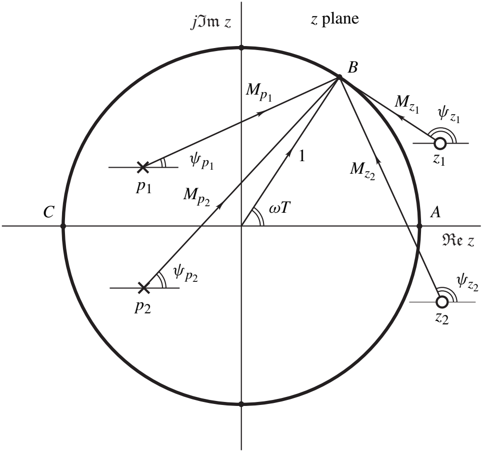
$$\theta(\omega) = \arg H_0 + \sum_{i=1}^{N} m_i \psi_{z_i} - \sum_{i=1}^{N} n_i \psi_{p_i} \qquad \text{(C)}$$

- The amplitude and phase responses of a system can be determined by repeating the above procedure for frequencies $\omega = \omega_1, \omega_2, \ldots$ in the range 0 to π/T.

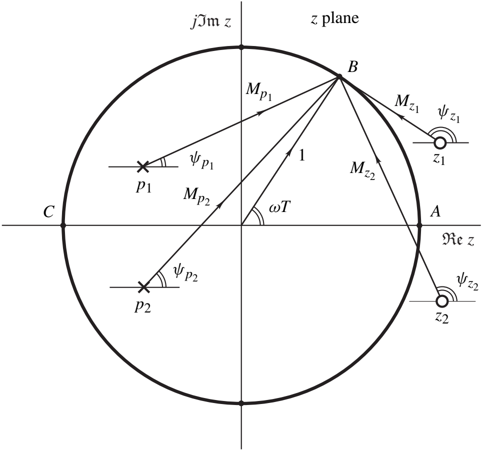
- Frequency response of second-order system:



- Point A corresponds to zero frequency.



- One complete revolution from point A in the counterclockwise sense back to point A corresponds to $\Delta\omega = \omega_s = 2\pi/T$.



Graphical Procedure *Cont'd*

...

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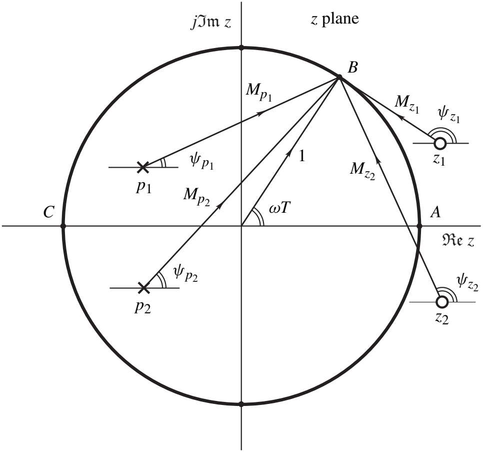
- One complete revolution from point A in the counterclockwise sense back to point A corresponds to $\Delta\omega = \omega_s = 2\pi/T$.
- Since T is the period between samples, ω_s is called the *sampling frequency* in rad/s.

Graphical Procedure *Cont'd*

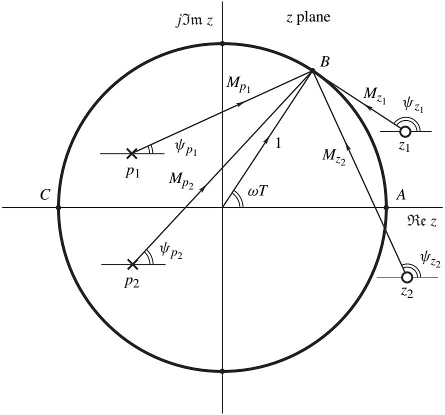
- Point C corresponds to frequency $\pi/T = \frac{1}{2}\omega_s$ which is commonly referred to as the *Nyquist frequency*.



Periodicity of Frequency Response

- If $e^{j\omega T}$ is rotated k complete revolutions, the values of $M(\omega)$ and $\theta(\omega)$ will obviously remain unchanged and so

$$H(e^{j(\omega+k\omega_s)T}) = H(e^{j\omega T})$$



Periodicity of Frequency Response *Cont'd*

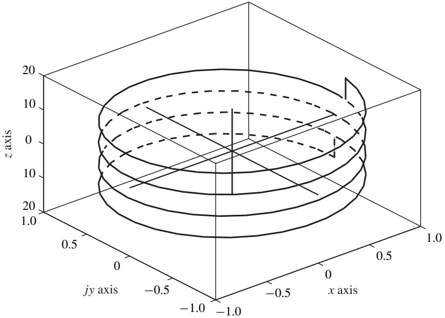
...

$$H(e^{j(\omega+k\omega_s)T}) = H(e^{j\omega T})$$

- We conclude that *the frequency response of a discrete-time system is periodic with period ω_s*.

Periodicity of Frequency Response *Cont'd*

- The periodicity of the frequency response can be visualized by considering the z plane as a Riemann surface of the form illustrated below. (See Appendix for details.)



Periodicity of Frequency Response *Cont'd*

- The periodicity of the frequency response can be viewed from a different perspective by examining the discrete-time sinusoidal signal

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$$x(nT) = \sin[(\omega + k\omega_s)nT]$$

- Using simple trigonometry, we can show that

$$\begin{aligned} x(nT) &= \sin \omega nT \cos k\omega_s nT + \cos \omega nT \sin k\omega_s nT \\ &= \sin \omega nT \cos \left(k \cdot \frac{2\pi}{T} \cdot nT\right) + \cos \omega nT \sin \left(k \cdot \frac{2\pi}{T} \cdot nT\right) \\ &= \sin \omega nT \cos 2kn\pi + \cos \omega nT \sin 2kn\pi \\ &= \sin \omega nT \end{aligned}$$

that is

$$x(nT) = \sin(\omega nT + k\omega_s nT) = \sin(\omega nT)$$

. . .

$$x(nT) = \sin(\omega nT + k\omega_s nT) = \sin(\omega nT)$$

- In effect, $\sin(\omega k + \omega_s)nT$ and $\sin \omega nT$ are *numerically identical* for any k, and if the two signals are applied at the input of a discrete-time system, they will produce the same response.

This slide concludes the presentation.
Thank you for your attention.