

Chapter 5

APPLICATION OF TRANSFORM THEORY TO SYSTEMS

5.5.5 Frequency Response of Digital Filters

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Introduction

- In Sec. 5.5, it is shown that the steady-state response of a *stable* N th-order discrete-time system to a sinusoidal signal

$$x(nT) = u(nT) \sin \omega nT$$

is another sinusoidal signal of the form

$$\lim_{nT \rightarrow \infty} y(nT) = \tilde{y}(nT) = M(\omega) \sin[\omega nT + \theta(\omega)]$$

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- The quantities

$$M(\omega) = |H(e^{j\omega T})| \quad \text{and} \quad \theta(\omega) = \arg H(e^{j\omega T})$$

define the *amplitude response* and *phase response*, respectively, and

$$H(z) \Big|_{z=e^{j\omega T}} = H(e^{j\omega T}) = M(\omega) e^{j\theta(\omega)}$$

defines the *frequency response*.

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- It can be easily shown that the amplitude response is an *even* function and the phase response is an *odd* function of ω , i.e.,

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$$M(-\omega) = M(\omega) \quad \text{and} \quad \theta(-\omega) = -\theta(\omega)$$

- Therefore, the frequency response is completely specified if it is known over the positive half of the baseband, i.e., $0 \leq \omega \leq \omega_s/2$.

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- On the other hand, an analog highpass filter will pass high frequencies in the range $\omega_c \leq \omega < \infty$ and reject low frequencies in the range $0 < \omega \leq \omega_c$.

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- A digital *highpass* filter will pass high frequencies in the range $\omega_c \leq \omega < \omega_s/2$ and reject low frequencies in the range $0 < \omega \leq \omega_c$.

- A digital *bandpass* filter will pass midband frequencies in the range $\omega_{c1} \leq \omega < \omega_{c2}$ and reject low frequencies in the range $0 < \omega \leq \omega_{c1}$ and high frequencies in the range $\omega_{c2} \leq \omega < \omega_s/2$ where ω_{c1} and ω_{c2} are said to be the *lower and upper* cutoff frequencies, respectively.

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- On the other hand, a digital *bandstop* filter will reject midband frequencies in the range $\omega_{c1} \leq \omega < \omega_{c2}$ and pass low frequencies in the range $0 < \omega \leq \omega_{c1}$ and high frequencies in the range $\omega_{c2} \leq \omega < \omega_s/2$.

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- On the other hand, a digital *bandstop* filter will reject midband frequencies in the range $\omega_{c1} \leq \omega < \omega_{c2}$ and pass low frequencies in the range $0 < \omega \leq \omega_{c1}$ and high frequencies in the range $\omega_{c2} \leq \omega < \omega_s/2$.
- In other words, the upper edge of the baseband in digital systems is analogous to infinite frequency in analog systems.

- An arbitrary transfer function $H(z)$ can be expressed in terms of its magnitude and angle as

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$$|H(z)| = |\text{Re } H(z) + j\text{Im } H(z)|$$

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Note: The magnitude function $|H(z)|$ is, of course, a nonnegative quantity but $\arg H(z)$ can be positive or negative.

- If we let $z = e^{j\omega T}$, i.e., if z assumes values on the *unit circle* $|z| = 1$, then *3-D plots* of the form
 - $|H(e^{j\omega T})|$ versus $e^{j\omega T}$ and $\arg H(e^{j\omega T})$ versus $e^{j\omega T}$can be constructed which represent the amplitude and phase responses.

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- These 3-D plots are, of course, subsets of the plots
 - $|H(z)|$ versus z and $\arg(z)$ versus z .
- From these 3-D plots, *2-D plots* of the form
 - $M(\omega)$ versus ω and $\theta(\omega)$ versus ωcan be constructed, which represent the amplitude and phase responses.

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 - the transfer function,
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 - the phase response.
- The various representations are illustrated in terms of specific transfer functions for
 - a lowpass recursive filter,
 - a lowpass nonrecursive filter, and
 - a bandpass recursive filter.

Geometrical Representations

- If

$$H(z) = \frac{N(z)}{D(z)} = \frac{H_0 \prod_{i=1}^N (z - z_i)}{\prod_{i=1}^N (z - p_i)}$$

then the zeros z_1, z_2, \dots of $H(z)$ will show up as dimples in the surface $|H(z)|$ whereas the poles p_1, p_2, \dots will show up as spikes.

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- The slides that follow will illustrate the various geometrical representations that are associated with the transfer function, amplitude response and phase response, e.g.,
 - zero-pole plot
 - 3-D plots of $|H(z)|$ and $\arg H(z)$ versus $z = \operatorname{Re} z + j\operatorname{Im} z$
 - 3-D plots of $|H(e^{j\omega T})|$ and $\arg H(e^{j\omega T})$ versus $z = e^{j\omega T}$
 - 2-D plots of $M(\omega) = |H(e^{j\omega T})|$ and $\theta(\omega) = \arg H(e^{j\omega T})$ versus ω

Lowpass Filter

- Consider a fourth-order lowpass digital filter that has the following transfer function

$$H(z) = H_0 \prod_{i=1}^2 H_i(z) \quad \text{where} \quad H_i(z) = \frac{a_{0i} + a_{1i}z + z^2}{b_{0i} + b_{1i}z + z^2}$$

with

$$H_0 = 6.351486E - 02$$

$$a_{01} = 1.0, \quad a_{11} = 1.494070$$

$$b_{01} = 5.115041E - 01, \quad b_{11} = -1.015631$$

$$a_{02} = 1.0, \quad a_{12} = 4.188149E - 01$$

$$b_{02} = 8.839638E - 01 \quad b_{12} = -3.548538E - 01$$

- The transfer function can be expressed in terms of its zeros and poles as

$$H(z) = H_0 \prod_{i=1}^2 H_i(z) \quad \text{where} \quad H_i(z) = \frac{(z - z_i)(z - z_i^*)}{(z - p_i)(z - p_i^*)}$$

with

$$z_1, z_1^* = -0.7470 \pm j0.6648$$

$$z_2, z_2^* = -0.2094 \pm j0.9778$$

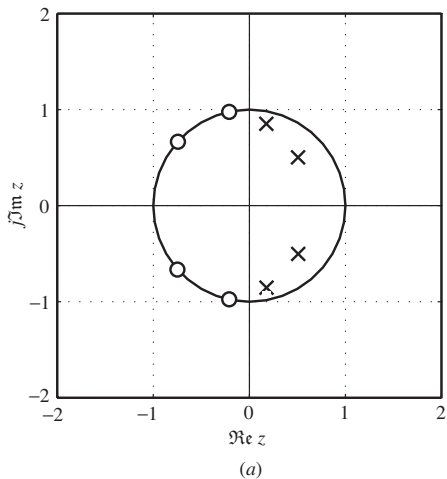
$$p_1, p_1^* = 0.5078 \pm j0.5036$$

$$p_2, p_2^* = 0.1774 \pm j0.9233$$

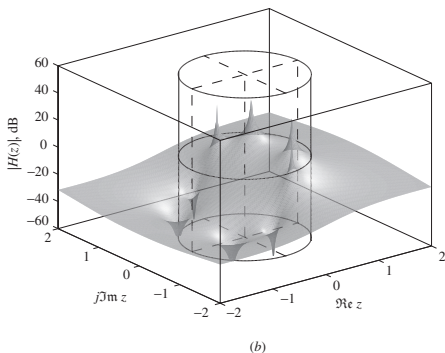
and

$$H_0 = 6.351486E - 02$$

- Zero-pole plot:



- Plot of $|H(z)|$ (in dB) versus $z = \text{Re } z + j\text{Im } z$:



The *dimples and spikes* are the *zeros and poles*, respectively.

- The amplitude response can be obtained as

$$M(\omega) = |H_0| \prod_{i=1}^2 |H_i(e^{j\omega T})| = |H_0| \prod_{i=1}^2 M_i(\omega) \quad \text{where}$$

$$\begin{aligned} M_i(\omega) &= |H_i(e^{j\omega T})| = \left| \frac{a_{0i} + a_{1i}e^{j\omega T} + e^{j2\omega T}}{b_{0i} + b_{1i}e^{j\omega T} + e^{j2\omega T}} \right| \\ &= \left| \frac{(a_{0i} + a_{1i} \cos \omega T + \cos 2\omega T) + j(a_{1i} \sin \omega T + \sin 2\omega T)}{(b_{0i} + b_{1i} \cos \omega T + \cos 2\omega T) + j(b_{1i} \sin \omega T + \sin 2\omega T)} \right| \\ &= \left[\frac{(a_{0i} + a_{1i} \cos \omega T + \cos 2\omega T)^2 + (a_{1i} \sin \omega T + \sin 2\omega T)^2}{(b_{0i} + b_{1i} \cos \omega T + \cos 2\omega T)^2 + (b_{1i} \sin \omega T + \sin 2\omega T)^2} \right]^{\frac{1}{2}} \\ &= \left[\frac{1 + a_{0i}^2 + a_{1i}^2 + 2(1 + a_{0i})a_{1i} \cos \omega T + 2a_{0i} \cos 2\omega T}{1 + b_{0i}^2 + b_{1i}^2 + 2(1 + b_{0i})b_{1i} \cos \omega T + 2b_{0i} \cos 2\omega T} \right]^{\frac{1}{2}} \end{aligned}$$

Lowpass Filter *Cont'd*

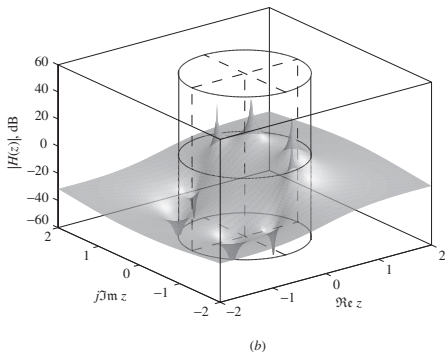
- Since $z = e^{j\omega nT}$ represents a circle of unit radius in the z plane, the amplitude response

$$M(\omega) = |H(e^{j\omega nT})|$$

can be represented geometrically by the intersection between the surface $|H(z)|$ and a cylinder of unit radius perpendicular to the z plane.

Lowpass Filter *Cont'd*

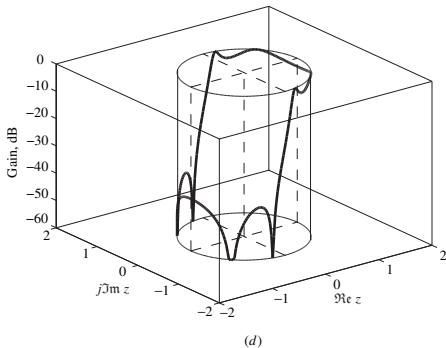
- Plot of $|H(z)|$ (in dB) versus $z = \text{Re } z + j\text{Im } z$:



The *intersection* between the surface $|H(z)|$ and the cylinder is the *amplitude response*.

Lowpass Filter *Cont'd*

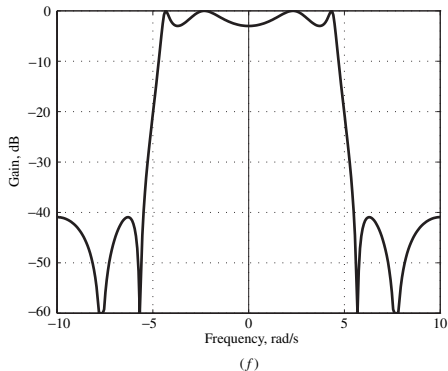
- Plot of $|H(z)|$ (in dB) versus $z = e^{j\omega T}$:



The *intersection* between surface $|H(z)|$ and the cylinder, i.e., the solid curve, is the *amplitude response*.

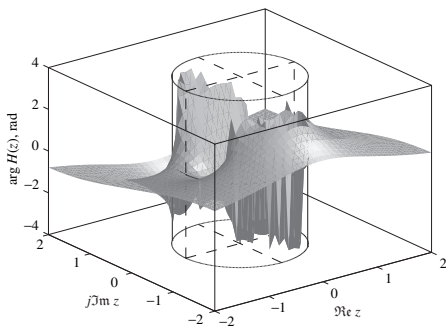
Lowpass Filter *Cont'd*

- Slicing the cylinder along the vertical line $z = -1$ and flattening it out will reveal the amplitude response, i.e., $M(\omega)$ versus ω , as a two-dimensional plot:



Lowpass Filter *Cont'd*

- Plot of $\arg H(z)$ (in rad) versus $z = \text{Re } z + j\text{Im } z$:



(c)

- The phase response can be obtained as

$$\theta(\omega) = \arg H_0 + \sum_{i=1}^2 \arg H_i(e^{j\omega T}) = \sum_{i=1}^2 \theta_i(\omega) \quad \text{where}$$

$$\begin{aligned}\theta_i(\omega) &= \arg H_i(e^{j\omega T}) \\ &= \arg \frac{a_{0i} + a_{1i}e^{j\omega T} + e^{j2\omega T}}{b_{0i} + b_{1i}e^{j\omega T} + e^{j2\omega T}} \\ &= \arg \frac{(a_{0i} + a_{1i} \cos \omega T + \cos 2\omega T) + j(a_{1i} \sin \omega T + \sin 2\omega T)}{(b_{0i} + b_{1i} \cos \omega T + \cos 2\omega T) + j(b_{1i} \sin \omega T + \sin 2\omega T)} \\ &= \tan^{-1} \frac{a_{1i} \sin \omega T + \sin 2\omega T}{a_{0i} + a_{1i} \cos \omega T + \cos 2\omega T} \\ &\quad - \tan^{-1} \frac{b_{1i} \sin \omega T + \sin 2\omega T}{b_{0i} + b_{1i} \cos \omega T + \cos 2\omega T}\end{aligned}$$

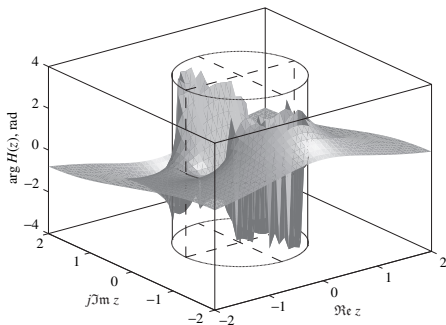
(See textbook for details.)

- Since $z = e^{j\omega T}$ represents a circle of unit radius in the z plane, the phase response

$$\theta(\omega) = \arg H(e^{j\omega T}) = \tan^{-1} \frac{\text{Im } H(e^{j\omega T})}{\text{Re } H(e^{j\omega T})}$$

can be represented geometrically by the intersection between the surface $\arg H(z)$ and a cylinder of unit radius perpendicular to the z plane.

- Plot of $\arg H(z)$ (in rad) versus $z = \text{Re } z + j\text{Im } z$:

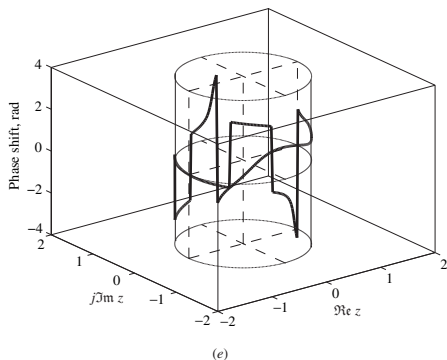


(c)

The *intersection* between surface $\arg H(z)$ and the cylinder is the *phase response*.

Lowpass Filter *Cont'd*

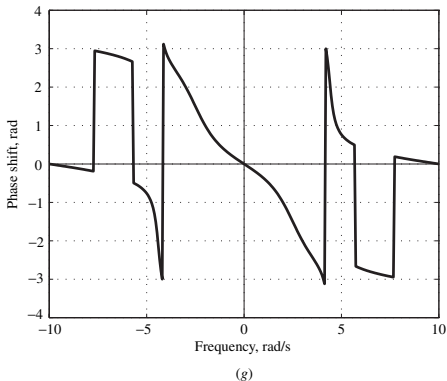
- Plot of $\arg H(z)$ (in rad) versus $z = e^{j\omega T}$:



The *intersection* between surface $\arg H(z)$ and the cylinder, i.e., the solid curve, is the *phase response*.

Lowpass Filter *Cont'd*

- Slicing the cylinder along the vertical line $z = -1$ and flattening it out will reveal the phase response, i.e., $\theta(\omega)$ versus ω , as a two-dimensional plot:



Pitfall

- The phase response shown in the previous slide is actually the phase response that would be computed by using MATLAB's function `atan2(y,x)` but *it is not correct!*

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$$\theta = \tan^{-1} \frac{x}{y}$$

is a multivalued function, and MATLAB's function `atan2(y,x)` would give a value for θ in the range $-2\pi \leq \theta \leq 2\pi$.

Computers in general would give a value in the range $-\pi \leq \theta \leq \pi$.

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- The problem can be corrected by noting that the phase response is a continuous function of ω .

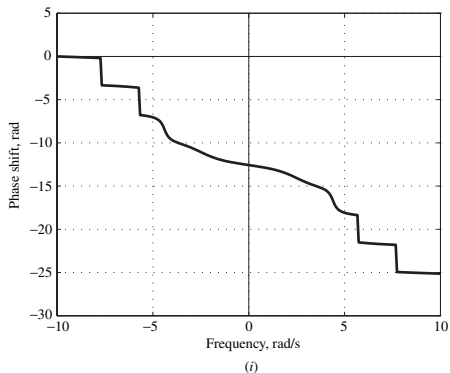
- For example, if function $\text{atan2}(y,x)$ gives a value of -179 followed by a value of $+179^\circ$ then, assuming a continuous phase response, an error of $+360^\circ$ has been committed and 360° should be subtracted from $+179^\circ$ to give the correct value of -181° .

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- Similarly, if function $\text{atan2}(y,x)$ gives a value of $+179$ followed by a value of -179° , then an error of -360° has been committed and 360° should be added to -179° to give the correct value $+181^\circ$.

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- Similarly, if function $\text{atan2}(y,x)$ gives a value of $+179$ followed by a value of -179° , then an error of -360° has been committed and 360° should be added to -179° to give the correct value $+181^\circ$.
- Alternatively, the correct value of the phase response can be obtained by using function $\text{unwrap}(p)$ of MATLAB, which will perform the necessary corrections.

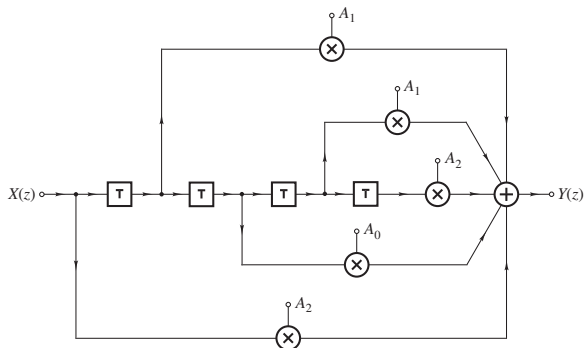
Lowpass Filter *Cont'd*

- Unwrapped phase response:



Example – Nonrecursive Lowpass Filter

The figure shows a nonrecursive filter:

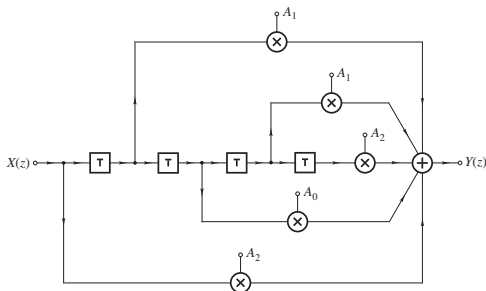


$$A_0 = 0.3352, \quad A_1 = 0.2540, \quad A_2 = 0.0784$$

Example *Cont'd*

- (a) Construct the zero-pole plot.
- (b) Plot the surface $|H(z)|$ as a function of $z = \text{Re } z + j\text{Im } z$.
- (c) Obtain expressions for the amplitude and phase responses.
- (d1) Plot the amplitude and phase responses in terms of 3-D plots.
- (d2) Plot the amplitude and phase responses in terms of 2-D plots.

Solution



Transfer function:

$$\begin{aligned}
 H(z) &= A_2 + A_1 z^{-1} + A_0 z^{-2} + A_1 z^{-3} + A_2 z^{-4} \\
 &= \frac{A_2 z^2 + A_1 z + A_0 + A_1 z^{-1} + A_2 z^{-2}}{z^2} \\
 &= \frac{A_2 z^4 + A_1 z^3 + A_0 z^2 + A_1 z + A_2}{z^4}
 \end{aligned}$$

Example *Cont'd*

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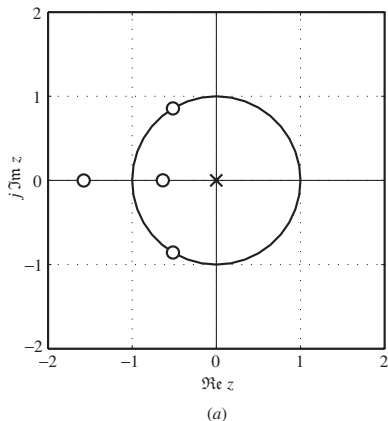
The zeros can be readily found by using D-Filter or MATLAB as

$$z_1 = -1.5756 \quad z_2 = -0.6347 \quad z_3, z_4 = -0.5148 \pm j0.8573$$

There is a 4th-order pole at the origin.

Example *Cont'd*

Zero-pole plot:

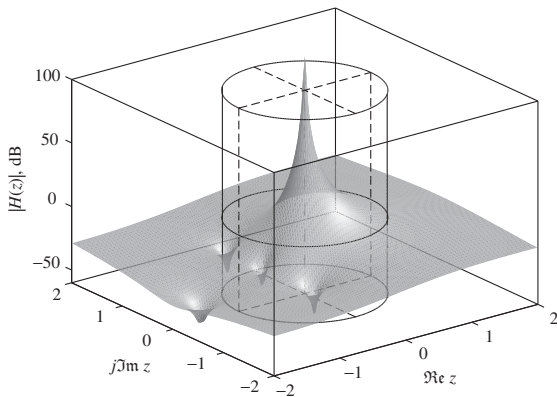


$$z_1 = -1.5756 \quad z_2 = -0.6347 \quad z_3, z_4 = -0.5148 \pm j0.8573$$

$$p_1 = p_2 = p_3 = p_4 = 0$$

Example *Cont'd*

$|H(z)|$ versus $z = \text{Re } z + j\text{Im } z$:



(b)

Dimples represent zeros, the huge spike represents the 4th-order pole at the origin.

Example *Cont'd*

Since

$$\begin{aligned} H(z) &= A_2 + A_1 z^{-1} + A_0 z^{-2} + A_1 z^{-3} + A_2 z^{-4} \\ &= \frac{A_2 z^2 + A_1 z + A_0 + A_1 z^{-1} + A_2 z^{-2}}{z^2} \\ &= \frac{A_2 z^4 + A_1 z^3 + A_0 z^2 + A_1 z + A_2}{z^4} \end{aligned} \quad (\text{A})$$

Eq. (A) gives the frequency response as

$$\begin{aligned} H(e^{j\omega T}) &= \frac{A_2(e^{j2\omega T} + e^{-j2\omega T}) + A_1(e^{j\omega T} + e^{-j\omega T}) + A_0}{e^{j2\omega T}} \\ &= \frac{2A_2 \cos 2\omega T + 2A_1 \cos \omega T + A_0}{e^{j2\omega T}} \end{aligned}$$

Example *Cont'd*

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$$H(e^{j\omega T}) = \frac{2A_2 \cos 2\omega T + 2A_1 \cos \omega T + A_0}{e^{j2\omega T}}$$

Therefore, the amplitude and phase responses are given by

$$M(\omega) = |2A_2 \cos 2\omega T + 2A_1 \cos \omega T + A_0| \quad \blacksquare$$

and

$$\theta(\omega) = \theta_N - 2\omega T \quad \blacksquare$$

respectively, where

$$\theta_N = \begin{cases} 0 & \text{if } 2A_2 \cos 2\omega T + 2A_1 \cos \omega T + A_0 \geq 0 \\ \pi & \text{otherwise} \end{cases}$$

Example *Cont'd*

• • •

$$H(e^{j\omega T}) = \frac{2A_2 \cos 2\omega T + 2A_1 \cos \omega T + A_0}{e^{j2\omega T}}$$

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and

$$\theta(\omega) = \theta_N - 2\omega T \quad \blacksquare$$

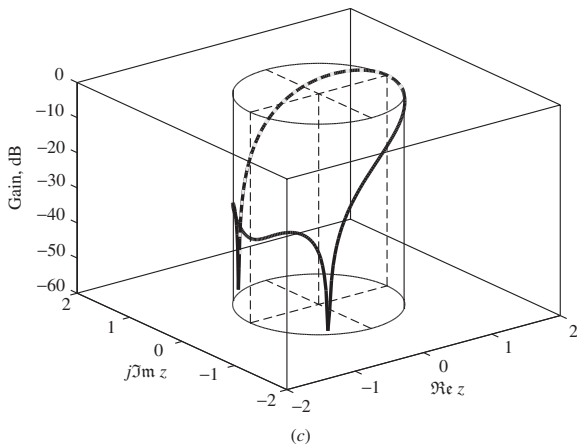
respectively, where

$$\theta_N = \begin{cases} 0 & \text{if } 2A_2 \cos 2\omega T + 2A_1 \cos \omega T + A_0 \geq 0 \\ \pi & \text{otherwise} \end{cases}$$

Note: The phase response is usually a linear function of ω in nonrecursive filters (see Chap. 10).

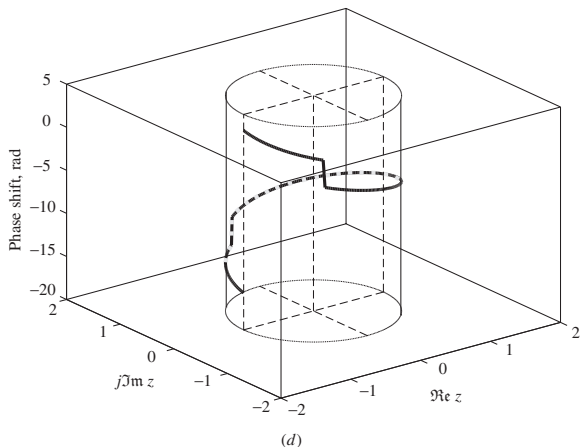
Example *Cont'd*

3-D plot of amplitude response, i.e., $\arg H(e^{j\omega T})$ versus $z = e^{j\omega T}$:



Example *Cont'd*

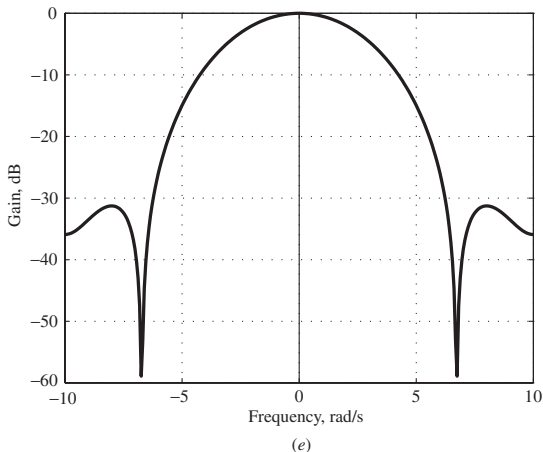
3-D plot of phase response, i.e., $\arg H(e^{j\omega T})$ versus $z = e^{j\omega T}$:



Note: The phase angle has been unwrapped.

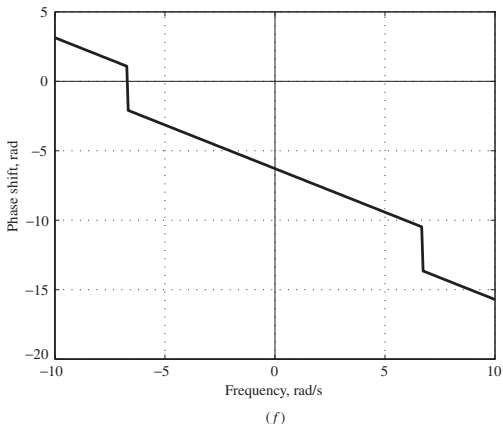
Example *Cont'd*

2-D plot of amplitude response, i.e., $M(\omega) = |H(e^{j\omega T})|$ versus ω :



Example *Cont'd*

2-D plot of phase response, i.e., $\arg H(e^{j\omega T})$ versus ω :



Note: The discontinuities are genuine: they are caused by zeros on the unit circle.

Example – Recursive Bandpass Filter

A recursive digital filter is characterized by the transfer function

$$H(z) = H_0 \prod_{i=1}^3 H_i(z)$$

where

$$H_i(z) = \frac{a_{0i} + a_{1i}z + z^2}{b_{0i} + b_{1i}z + z^2}$$

The sampling frequency is 20 rad/s.

Transfer-Function Coefficients

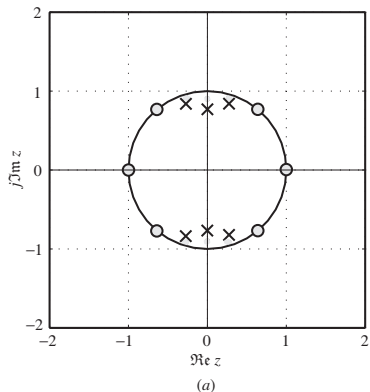
i	a_{0i}	a_{1i}	b_{0i}	b_{1i}
1	-1.0	0.0	8.131800E-1	7.870090E-8
2	1.0	-1.275258	9.211099E-1	5.484026E-1
3	1.0	1.275258	9.211097E-1	-5.484024E-1
$H_0 = 1.763161E - 2$				

Example *Cont'd*

- (a) Construct the zero-pole plot of the filter.
- (b) Plot the surface $|H(z)|$ as a function of $z = \text{Re } z + j\text{Im } z$.
- (c) Obtain expressions for the amplitude and phase responses.
- (d) Plot the amplitude and phase responses first in terms of 3-D plots and then in terms of 2-D plots.

Example *Cont'd*

Solution

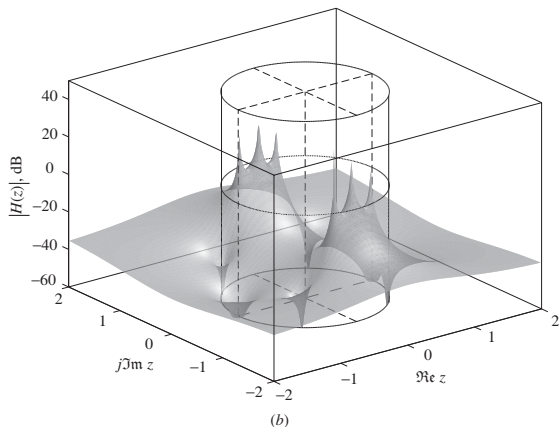


$$z_1, z_2 = \pm 1 \quad z_3, z_4 = 0.638 \pm j0.770 \quad z_5, z_6 = -0.638 \pm j0.770$$

$$p_1, p_2 = \pm j0.902 \quad p_3, p_4 = 0.274 \pm j0.770 \quad p_5, p_6 = -0.274 \pm j0.770$$

Example *Cont'd*

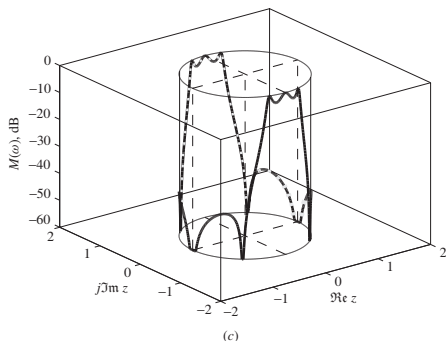
$|H(z)|$ versus $z = \text{Re } z + j\text{Im } z$:



Dimples represent zeros, the huge spike represents the 4th-order pole at the origin.

Example *Cont'd*

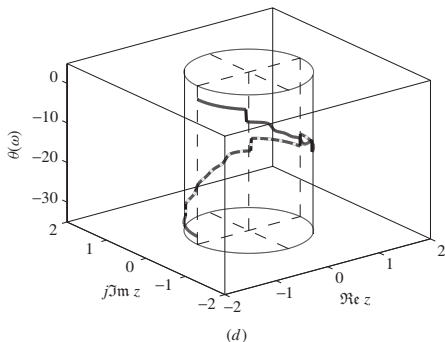
Plot of $|H(z)|$ (in dB) versus $z = e^{j\omega T}$:



The *intersection* between surface $|H(z)|$ and the cylinder, i.e., the solid curve, is the *amplitude response*.

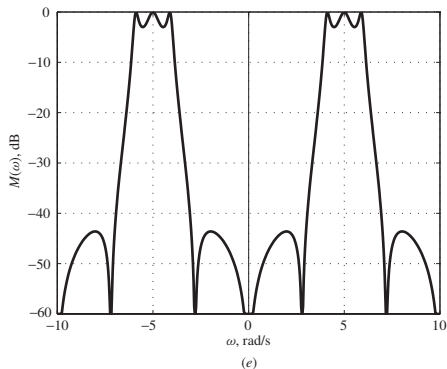
Example *Cont'd*

Plot of $\arg H(z)$ (in rad) versus $z = e^{j\omega T}$ with the phase angle unwrapped:



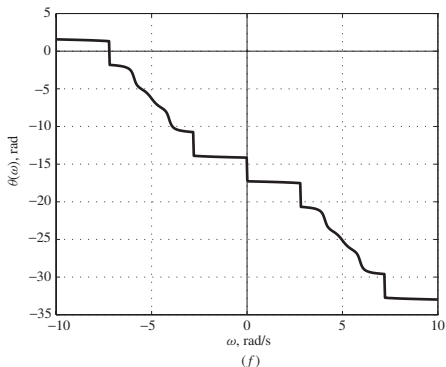
Example *Cont'd*

Slicing the cylinder along the vertical line $z = -1$ and flattening it out will reveal the amplitude response, i.e., $M(\omega)$ versus ω , as a two-dimensional plot:



Example *Cont'd*

Unwrapped phase response:



Note: The discontinuities shown are genuine. They are caused by the zeros on the unit circle.

*This slide concludes the presentation.
Thank you for your attention.*