

Chapter 5

APPLICATION OF TRANSFORM THEORY TO SYSTEMS

5.6 Transfer Functions for Digital Filters

5.7 Amplitude and Delay Distortion

Copyright © 2018 Andreas Antoniou
Victoria, BC, Canada
Email: aantoniou@ieee.org

July 10, 2018

Introduction

- ⇒ Previous presentations dealt with the frequency response of discrete-time systems, which is obtained by using the transfer function.

Introduction

- ⇒ Previous presentations dealt with the frequency response of discrete-time systems, which is obtained by using the transfer function.
- ⇒ In this presentation, we examine some of the basic types of transfer functions that characterize some typical first- and second-order filter types known as *biquads*.

Introduction

- ⇒ Previous presentations dealt with the frequency response of discrete-time systems, which is obtained by using the transfer function.
- ⇒ In this presentation, we examine some of the basic types of transfer functions that characterize some typical first- and second-order filter types known as *biquads*.
- ⇒ Biquads are often used as basic digital-filter blocks to construct high-order filters.

First-Order Transfer Functions

⇒ A first-order transfer function can have only a real zero and a real pole, i.e.,

$$H(z) = \frac{z - z_0}{z - p_0}$$

First-Order Transfer Functions

- ⇒ A first-order transfer function can have only a real zero and a real pole, i.e.,

$$H(z) = \frac{z - z_0}{z - p_0}$$

- ⇒ To ensure that the system is stable, the pole must satisfy the condition $-1 < p_0 < 1$.

First-Order Transfer Functions

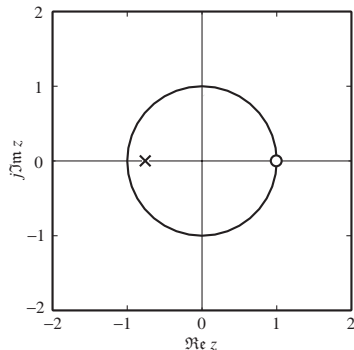
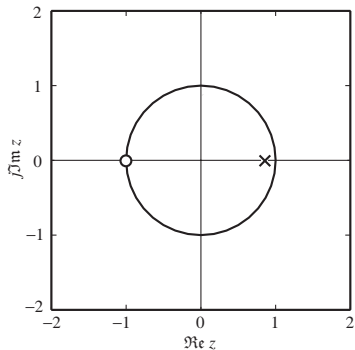
- ⇒ A first-order transfer function can have only a real zero and a real pole, i.e.,

$$H(z) = \frac{z - z_0}{z - p_0}$$

- ⇒ To ensure that the system is stable, the pole must satisfy the condition $-1 < p_0 < 1$.
- ⇒ The zero can be anywhere on the real axis of the z plane.

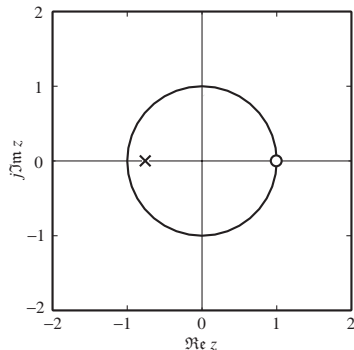
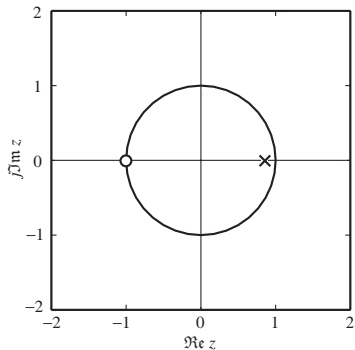
First-Order Transfer Functions *Cont'd*

- ⇒ If the pole is close to point $(1, 0)$ and the zero is close to or at point $(-1, 0)$, then we have a *lowpass* filter.



First-Order Transfer Functions *Cont'd*

- ⇒ If the pole is close to point $(1, 0)$ and the zero is close to or at point $(-1, 0)$, then we have a *lowpass* filter.
- ⇒ If the zero and pole positions are interchanged, then we get a *highpass* filter.



First-Order Transfer Functions *Cont'd*

- ⇒ Certain applications require discrete-time systems that have a constant amplitude response and a varying phase response. Such systems can be constructed by using *allpass* transfer functions.

First-Order Transfer Functions *Cont'd*

⇒ Certain applications require discrete-time systems that have a constant amplitude response and a varying phase response.

Such systems can be constructed by using *allpass* transfer functions.

⇒ A first-order allpass transfer function is of the form

$$H(z) = \frac{p_0 z - 1}{z - p_0} = p_0 \frac{z - 1/p_0}{z - p_0}$$

where the zero is the reciprocal of the pole.

First-Order Transfer Functions *Cont'd*

⇒ Certain applications require discrete-time systems that have a constant amplitude response and a varying phase response.

Such systems can be constructed by using *allpass* transfer functions.

⇒ A first-order allpass transfer function is of the form

$$H(z) = \frac{p_0 z - 1}{z - p_0} = p_0 \frac{z - 1/p_0}{z - p_0}$$

where the zero is the reciprocal of the pole.

⇒ The frequency response of a system characterized by $H(z)$ is given by

$$H(e^{j\omega T}) = \frac{p_0 e^{j\omega T} - 1}{e^{j\omega T} - p_0} = \frac{p_0 \cos \omega T + jp_0 \sin \omega T - 1}{\cos \omega T + j \sin \omega T - p_0}$$

First-Order Transfer Functions *Cont'd*

...

$$H(e^{j\omega T}) = \frac{p_0 e^{j\omega T} - 1}{e^{j\omega T} - p_0} = \frac{p_0 \cos \omega T + jp_0 \sin \omega T - 1}{\cos \omega T + j \sin \omega T - p_0}$$

⇒ The amplitude and phase responses are given by

$$\begin{aligned} M(\omega) &= \left| \frac{p_0 \cos \omega T - 1 + jp_0 \sin \omega T}{\cos \omega T - p_0 + j \sin \omega T} \right| \\ &= \left[\frac{(p_0 \cos \omega T - 1)^2 + (p_0 \sin \omega T)^2}{(\cos \omega T - p_0)^2 + (\sin \omega T)^2} \right]^{\frac{1}{2}} = 1 \end{aligned}$$

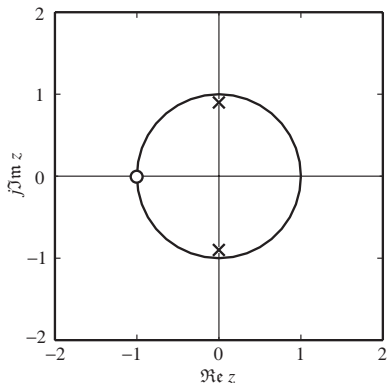
and

$$\theta(\omega) = \tan^{-1} \frac{p_0 \sin \omega T}{p_0 \cos \omega T - 1} - \tan^{-1} \frac{\sin \omega T}{\cos \omega T - p_0}$$

respectively.

Second-Order Lowpass Biquad

- ⇒ A *lowpass* second-order transfer function can be constructed by placing a complex-conjugate pair of poles anywhere inside the unit circle and a pair of zeros at the Nyquist point:



Second-Order Lowpass Biquad *Cont'd*

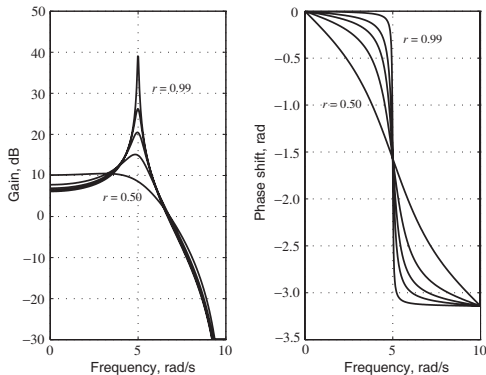
⇒ The transfer function of the lowpass biquad assumes the form:

$$H_{LP}(z) = \frac{(z + 1)^2}{(z - re^{j\phi})(z - re^{-j\phi})} = \frac{z^2 + 2z + 1}{z^2 - 2r(\cos \phi)z + r^2}$$

where $0 < r < 1$.

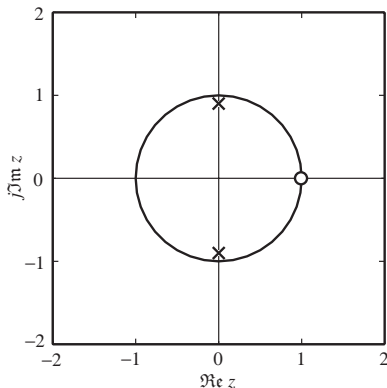
Second-Order Lowpass Biquad *Cont'd*

- ⇒ As the poles move closer to the unit circle, the amplitude response develops a peak at frequency $\omega = \phi/T$ while the slope of the phase response tends to become steeper and steeper at that frequency.



Second-Order Highpass Biquad

- ⇒ A *highpass* second-order transfer function can be constructed by placing a complex-conjugate pair of poles anywhere inside the unit circle and a pair of zeros at point $(1, 0)$:



Second-Order Highpass Biquad *Cont'd*

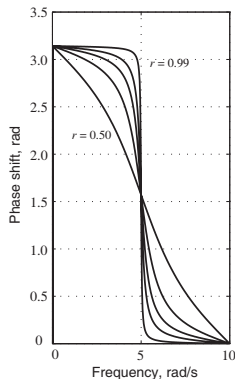
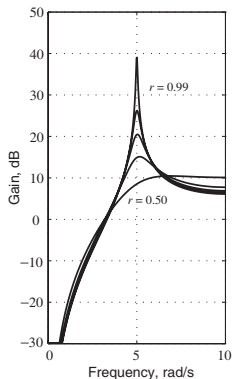
⇒ The transfer function of the highpass biquad assumes the form:

$$H_{HP}(z) = \frac{(z - 1)^2}{z^2 - 2r(\cos \phi)z + r^2} = \frac{(z^2 - 2z + 1)}{z^2 - 2r(\cos \phi)z + r^2}$$

where $0 < r < 1$.

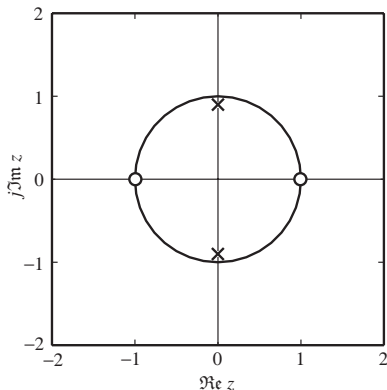
Second-Order Highpass Biquad *Cont'd*

- ⇒ As the poles move closer to the unit circle, the amplitude response develops a peak at frequency $\omega = \phi/T$ while the slope of the phase response tends to become steeper and steeper at that frequency.



Second-Order Bandpass Biquad

- ⇒ A *bandpass* second-order transfer function can be constructed by placing a complex-conjugate pair of poles anywhere inside the unit circle, zeros at points $(-1, 0)$ and $(1, 0)$:



Second-Order Bandpass Biquad *Cont'd*

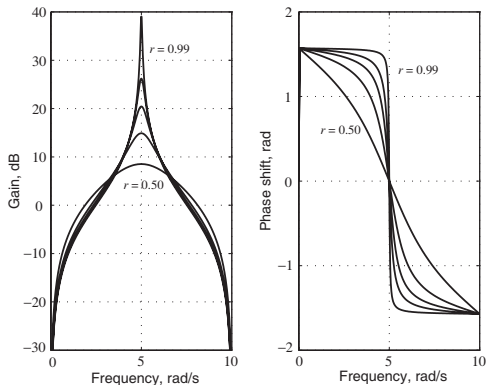
⇒ The transfer function of the bandpass biquad assumes the form:

$$H_{BP}(z) = \frac{(z + 1)(z - 1)}{z^2 - 2r(\cos \phi)z + r^2}$$

where $0 < r < 1$.

Second-Order Bandpass Biquad *Cont'd*

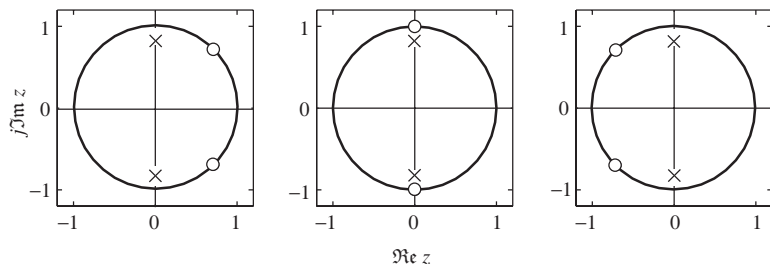
- ⇒ As the poles move closer to the unit circle, the amplitude response develops a peak at frequency $\omega = \phi/T$ while the slope of the phase response tends to become steeper and steeper at that frequency.



Second-Order Notch Biquad

- ⇒ A *notch* second-order transfer function can be constructed by placing a complex-conjugate pair of poles anywhere inside the unit circle, and a complex-conjugate pair of zeros on the unit circle.

There are three possibilities:



Second-Order Notch Biquad *Cont'd*

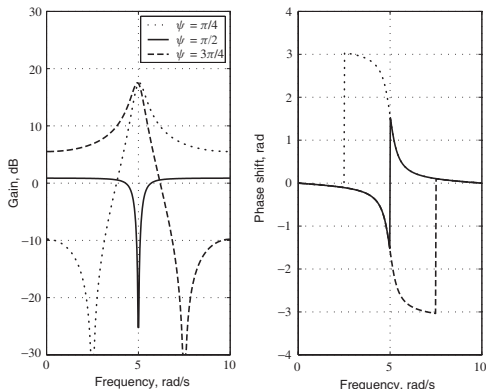
⇒ The transfer function of the bandpass biquad assumes the form:

$$H_N(z) = \frac{z^2 - 2(\cos \psi)z + 1}{z^2 - 2r(\cos \phi)z + r^2}$$

where $0 < r < 1$.

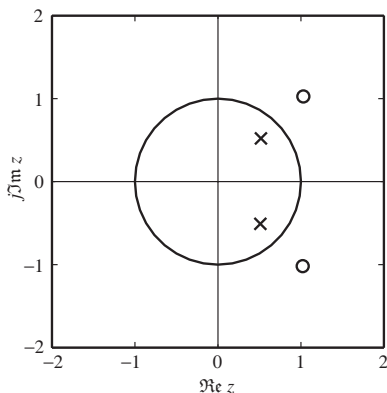
Second-Order Notch Biquad *Cont'd*

⇒ If $\psi = \pi/4$, $\psi = \pi/2$, or $\psi = 3\pi/4$, the notch filter behaves as a highpass, bandstop, or lowpass filter.



Second-Order Allpass Biquad

- ⇒ An *allpass* second-order transfer function can be constructed by placing a complex-conjugate pair of poles anywhere inside the unit circle and a complex-conjugate pair of zeros that are the reciprocals of the poles outside the unit circle.



Second-Order Allpass Biquad *Cont'd*

⇒ The transfer function of the bandpass biquad assumes the form:

$$H_{AP}(z) = \frac{r^2 z^2 - 2r(\cos \phi)z + 1}{z^2 - 2r(\cos \phi)z + r^2}$$

where $0 < r < 1$.

Second-Order Allpass Biquad *Cont'd*

- ⇒ The transfer function of the bandpass biquad assumes the form:

$$H_{AP}(z) = \frac{r^2 z^2 - 2r(\cos \phi)z + 1}{z^2 - 2r(\cos \phi)z + r^2}$$

where $0 < r < 1$.

- ⇒ We note that the numerator coefficients are the same as the denominator coefficients but in the reverse order.

Second-Order Allpass Biquad *Cont'd*

- ⇒ The transfer function of the bandpass biquad assumes the form:

$$H_{AP}(z) = \frac{r^2 z^2 - 2r(\cos \phi)z + 1}{z^2 - 2r(\cos \phi)z + r^2}$$

where $0 < r < 1$.

- ⇒ We note that the numerator coefficients are the same as the denominator coefficients but in the reverse order.
- ⇒ The above is a general property, that is, an arbitrary transfer function with the above coefficient symmetry is an allpass transfer function independently of the order.

Second-Order Allpass Biquad *Cont'd*

$$\begin{aligned}M_{AP}(\omega) &= |H_{AP}(e^{j\omega T})| = [H_{AP}(e^{j\omega T}) \cdot H_{AP}^*(e^{j\omega T})]^{\frac{1}{2}} \\&= [H_{AP}(e^{j\omega T}) \cdot H_{AP}(e^{-j\omega T})]^{\frac{1}{2}} \\&= \left\{ [H_{AP}(z) \cdot H_{AP}(z^{-1})]_{z=e^{j\omega T}} \right\}^{\frac{1}{2}} \\&= \left\{ \left[\frac{r^2 z^2 + 2r(\cos \phi)z + 1}{z^2 + 2r(\cos \phi)z + r^2} \cdot \frac{r^2 z^{-2} + 2r(\cos \phi)z^{-1} + 1}{z^{-2} + 2r(\cos \phi)z^{-1} + r^2} \right]_{z=e^{j\omega T}} \right\}^{\frac{1}{2}} \\&= \left\{ \left[\frac{r^2 z^2 + 2r(\cos \phi)z + 1}{z^2 + 2r(\cos \phi)z + r^2} \cdot \frac{r^2 + 2r(\cos \phi)z + z^2}{1 + 2r(\cos \phi)z + z^2 r^2} \right]_{z=e^{j\omega T}} \right\}^{\frac{1}{2}} = 1\end{aligned}$$

High-Order Filters

- ⇒ Higher-order transfer functions can be obtained by forming products or sums of first- and/or second-order transfer functions.

High-Order Filters

- ⇒ Higher-order transfer functions can be obtained by forming products or sums of first- and/or second-order transfer functions.
- ⇒ Corresponding high-order filters can be constructed by connecting several biquads in cascade or in parallel.

High-Order Filters

- ⇒ Higher-order transfer functions can be obtained by forming products or sums of first- and/or second-order transfer functions.
- ⇒ Corresponding high-order filters can be constructed by connecting several biquads in cascade or in parallel.
- ⇒ Methods for obtaining transfer functions that will yield specified frequency responses will be explored in later chapters.

Amplitude and Delay Distortion

- ⇒ In practice, a discrete-time system can distort the information content of a signal to be processed.

Amplitude and Delay Distortion

- ⇒ In practice, a discrete-time system can distort the information content of a signal to be processed.
- ⇒ Two types of distortion can be introduced as follows:

Amplitude and Delay Distortion

- ⇒ In practice, a discrete-time system can distort the information content of a signal to be processed.
- ⇒ Two types of distortion can be introduced as follows:
 - Amplitude distortion

Amplitude and Delay Distortion

- ⇒ In practice, a discrete-time system can distort the information content of a signal to be processed.
- ⇒ Two types of distortion can be introduced as follows:
 - Amplitude distortion
 - Delay (or phase) distortion

Amplitude and Delay Distortion *Cont'd*

- ⇒ Consider an application where a digital filter characterized by a transfer function $H(z)$ is to be used to select a specific signal $x_k(nT)$ from a sum of signals

$$x(nT) = \sum_{i=1}^m x_i(nT)$$

Amplitude and Delay Distortion *Cont'd*

- ⇒ Consider an application where a digital filter characterized by a transfer function $H(z)$ is to be used to select a specific signal $x_k(nT)$ from a sum of signals

$$x(nT) = \sum_{i=1}^m x_i(nT)$$

- ⇒ Let the amplitude and phase responses of the filter be $M(\omega)$ and $\theta(\omega)$, respectively.

Amplitude and Delay Distortion *Cont'd*

- ⇒ Consider an application where a digital filter characterized by a transfer function $H(z)$ is to be used to select a specific signal $x_k(nT)$ from a sum of signals

$$x(nT) = \sum_{i=1}^m x_i(nT)$$

- ⇒ Let the amplitude and phase responses of the filter be $M(\omega)$ and $\theta(\omega)$, respectively.
- ⇒ Two parameters associated with the phase response are the *absolute delay* $\tau_a(\omega)$ and the *group delay* $\tau_g(\omega)$ which are defined as

$$\tau_a(\omega) = -\frac{\theta(\omega)}{\omega} \quad \text{and} \quad \tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega}$$

Amplitude and Delay Distortion *Cont'd*

- ⇒ Consider an application where a digital filter characterized by a transfer function $H(z)$ is to be used to select a specific signal $x_k(nT)$ from a sum of signals

$$x(nT) = \sum_{i=1}^m x_i(nT)$$

- ⇒ Let the amplitude and phase responses of the filter be $M(\omega)$ and $\theta(\omega)$, respectively.
- ⇒ Two parameters associated with the phase response are the *absolute delay* $\tau_a(\omega)$ and the *group delay* $\tau_g(\omega)$ which are defined as

$$\tau_a(\omega) = -\frac{\theta(\omega)}{\omega} \quad \text{and} \quad \tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega}$$

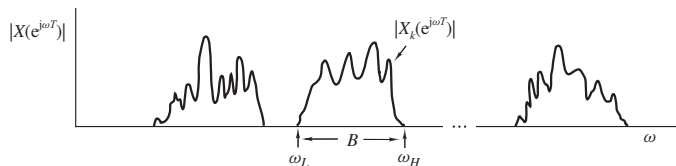
- ⇒ As functions of frequency, $\tau_a(\omega)$ and $\tau_g(\omega)$ are known as the *absolute-delay and group-delay characteristics*.

Amplitude and Delay Distortion *Cont'd*

- ⇒ Now assume that the amplitude spectrum of signal $x_k(nT)$ is concentrated in frequency band B given by

$$B = \{\omega : \omega_L \leq \omega \leq \omega_H\}$$

as shown.



Amplitude and Delay Distortion *Cont'd*

- ⇒ Now assume that the amplitude spectrum of signal $x_k(nT)$ is concentrated in frequency band B given by

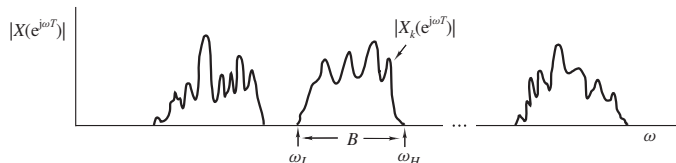
$$B = \{\omega : \omega_L \leq \omega \leq \omega_H\}$$

as shown.

- ⇒ Also assume that the filter has amplitude and phase responses

$$M(\omega) = \begin{cases} G_0 & \text{for } \omega \in B \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \theta(\omega) = -\tau_g \omega + \theta_0 \quad \text{for } \omega \in B$$

respectively, where G_0 and τ_g are constants.



⇒ The z transform of the output of the filter is given by

$$Y(z) = H(z)X(z) = H(z) \sum_{i=1}^m X_i(z) = \sum_{i=1}^m H(z)X_i(z)$$

⇒ The z transform of the output of the filter is given by

$$Y(z) = H(z)X(z) = H(z) \sum_{i=1}^m X_i(z) = \sum_{i=1}^m H(z)X_i(z)$$

⇒ Thus the frequency spectrum of the output signal is obtained as

$$\begin{aligned} Y(e^{j\omega T}) &= \sum_{i=1}^m H(e^{j\omega T})X_i(e^{j\omega T}) \\ &= \sum_{i=1}^m M(\omega)e^{j\theta(\omega)}X_i(e^{j\omega T}) \end{aligned}$$

...

$$Y(e^{j\omega T}) = \sum_{i=1}^m M(\omega) e^{j\theta(\omega)} X_i(e^{j\omega T})$$

⇒ We have assumed that

$$M(\omega) = \begin{cases} G_0 & \text{for } \omega \in B \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \theta(\omega) = -\tau_g \omega + \theta_0 \quad \text{for } \omega \in B$$

and hence we get

$$Y(e^{j\omega T}) = G_0 e^{-j\omega \tau_g + j\theta_0} X_k(e^{j\omega T})$$

since all signal spectrums except $X_k(e^{j\omega T})$ will be multiplied by zero.

Amplitude and Delay Distortion *Cont'd*

• • •

$$Y(e^{j\omega T}) = G_0 e^{-j\omega\tau_g + j\theta_0} X_k(e^{j\omega T})$$

⇒ If we now let $\tau_g = mT$ where m is a constant, we can write

$$Y(z) = G_0 e^{j\theta_0} z^{-m} X_k(z)$$

Amplitude and Delay Distortion *Cont'd*

• • •

$$Y(e^{j\omega T}) = G_0 e^{-j\omega\tau_g + j\theta_0} X_k(e^{j\omega T})$$

⇒ If we now let $\tau_g = mT$ where m is a constant, we can write

$$Y(z) = G_0 e^{j\theta_0} z^{-m} X_k(z)$$

⇒ Therefore, from the time-shifting theorem of the z transform, we deduce the output of the filter as

$$y(nT) = G_0 e^{j\theta_0} x_k(nT - mT)$$

• • •

$$Y(e^{j\omega T}) = G_0 e^{-j\omega\tau_g + j\theta_0} X_k(e^{j\omega T})$$

⇒ If we now let $\tau_g = mT$ where m is a constant, we can write

$$Y(z) = G_0 e^{j\theta_0} z^{-m} X_k(z)$$

⇒ Therefore, from the time-shifting theorem of the z transform, we deduce the output of the filter as

$$y(nT) = G_0 e^{j\theta_0} x_k(nT - mT)$$

⇒ In effect, if the *amplitude response* of the filter is *constant* with respect to frequency band B and zero elsewhere and its *phase response* is a *linear* function of ω , that is, the group delay is constant in frequency band B , then the output signal is a *delayed replica* of signal $x_k(nT)$ except that a constant multiplier $G_0 e^{j\theta_0}$ is introduced.

Amplitude and Delay Distortion *Cont'd*

- ⇒ If the amplitude response of the system is not constant in frequency band B , then so-called *amplitude distortion* will be introduced since different frequency components of the signal will be amplified by different amounts.

Amplitude and Delay Distortion *Cont'd*

- ⇒ If the amplitude response of the system is not constant in frequency band B , then so-called *amplitude distortion* will be introduced since different frequency components of the signal will be amplified by different amounts.
- ⇒ If the group delay is not constant in band B , different frequency components will be delayed by different amounts, and *delay (or phase) distortion* will be introduced.

Amplitude and Delay Distortion *Cont'd*

- ⇒ Amplitude distortion can be quite objectionable in practice. Consequently, the amplitude response is required to be flat to within a prescribed tolerance in each frequency band that carries information.
- ⇒ If the ultimate receiver of the signal is the human ear, e.g., when a speech or music signal is to be processed, delay distortion turns out to be quite tolerable.
- ⇒ In other applications where images are involved, e.g., transmission of video signals, delay distortion can be as objectionable as amplitude distortion, and the delay characteristic is required to be fairly flat.

*This slide concludes the presentation.
Thank you for your attention.*