Chapter 6
THE SAMPLING PROCESS
6.2 Impulse-Modulated Signals

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Introduction

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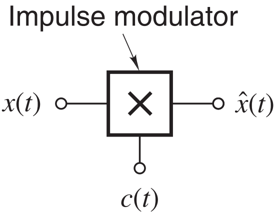
- These relationships are developed by defining a special class of signals known as *impulse-modulated* signals which comprise sequences of continuous-time impulse functions.

- Impulse-modulated signals are essentially continuous-time signals but simultaneously they are also sampled signals.

  Therefore, on the one hand, they have Fourier transforms and, on the other, they can be represented by z transforms.

  Consequently, impulse-modulated signals can serve as a mathematical *bridge* between continuous-time and discrete-time signals that facilitates the derivations of the various relationships between the two classes of signals.

- An impulse modulated-signal, denoted as $\hat{x}(t)$, can be generated by sampling a continuous-time signal $x(t)$ using an ideal impulse modulator.



(a)

- An impulse modulator is characterized by the equation

$$\hat{x}(t) = c(t)x(t)$$

where $c(t)$ is a carrier given by

$$c(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Impulse-Modulated Signals *Cont'd*

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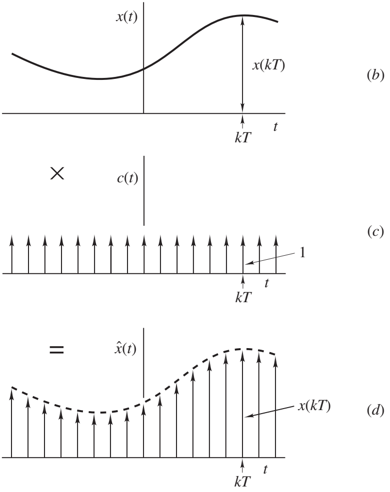
$$\hat{x}(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- From the properties of the unit impulse function, we get

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

The input and output of an impulse modulator are as follows:



Relationship between Impulse-Modulated and Discrete-Time Signals

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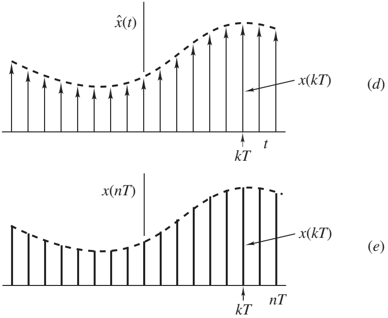
- They can be converted to discrete-time signals by replacing impulses by numbers.
- On the other hand, impulse-modulated signals can be obtained from discrete-time signals by replacing numbers by impulses.

Relationship between Impulse-Modulated and Discrete-Time Signals *Cont'd*



Relationship between Fourier Transform and Z Transform

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- As a continuous-time signal, an impulse-modulated signal has a Fourier transform given by

$$\hat{X}(j\omega) = \mathcal{F} \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT) = \sum_{n=-\infty}^{\infty} x(nT)\mathcal{F}\delta(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT}$$

. . .

$$\hat{X}(j\omega) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT}$$

- Therefore, from the definition of the z transform we note that

$$\hat{X}(j\omega) = X_D(z)\Big|_{z=e^{j\omega T}}$$

where

$$X_D(z) = \mathcal{Z}x(nT) \tag{A}$$

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- In effect, the *Fourier transform* of impulse-modulated signal $\hat{x}(t)$ *is numerically equal to the z transform* of the corresponding discrete-time signal $x(nT)$ evaluated on the unit circle $|z| = 1$.

- In other words, the frequency spectrum of $\hat{x}(t)$ is equal to that of $x(nT)$.

Example

The continuous-time signal

$$x(t) = \begin{cases} 0 & \text{for } t < -3.5 \text{ s} \\ 1 & \text{for } -3.5 \le t < -2.5 \\ 2 & \text{for } -2.5 \le t < 2.5 \\ 1 & \text{for } 2.5 \le t \le 3.5 \\ 0 & \text{for } t > 3.5 \end{cases}$$

is subjected to impulse modulation.

Find the frequency spectrum of $\hat{x}(t)$ in closed form assuming a sampling frequency of 2π rad/s.

Solution The frequency spectrum of an impulse-modulated signal, $\hat{x}(t)$, can be readily obtained by evaluating the z transform of $x(nT)$ on the unit circle of the z plane.

The impulse-modulated version of $x(t)$ can be expressed as

$$\begin{aligned}\hat{x}(t) = {} & \delta(t+3T) + 2\delta(t+2T) + 2\delta(t+T) + 2\delta(0) \\ & +2\delta(t-T) + 2\delta(t-2T) + \delta(t-3T)\end{aligned}$$

where $T = 1$ s.

A corresponding discrete-time signal can be obtained by replacing impulses by numbers as

$$\begin{aligned}x(nT) = {} & \delta(nT+3T) + 2\delta(nT+2T) + 2\delta(nT+T) + 2\delta(0) \\ & +2\delta(nT-T) + 2\delta(nT-2T) + \delta(nT-3T)\end{aligned}$$

Hence $\quad X_D(z) = \mathcal{Z}x(t) = z^3 + 2z^2 + 2z^1 + 2 + 2z^{-1} + 2z^{-2} + z^{-3}$

...

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Since the frequency spectrum of an impulse-modulated signal is given by

$$\hat{X}(j\omega) = X_D(e^{j\omega T})$$

we get

$$\begin{aligned}\hat{X}(j\omega) &= X_D(e^{j\omega T}) \\ &= (e^{j3\omega T} + e^{-j3\omega T}) + 2(e^{j2\omega T} + e^{-j2\omega T}) \\ &\quad + 2(e^{j\omega T} + e^{-j\omega T}) + 2 \\ &= 2\cos 3\omega T + 4\cos 2\omega T + 4\cos \omega T + 2 \quad \blacksquare\end{aligned}$$

Example

The continuous-time signal

$$x(t) = u(t)e^{-t}\sin 2t$$

is subjected to impulse modulation.

Find the frequency spectrum of $\hat{x}(t)$ in closed form assuming a sampling frequency of 2π rad/s.

Example *Cont'd*

Solution A discrete-time signal can be readily derived from $x(t)$ by replacing t by nT as

$$\begin{aligned} x(nT) &= u(nT)e^{-nT}\sin 2nT = u(nT)e^{-nT} \times \frac{1}{2j}\left(e^{j2nT} - e^{-j2nT}\right) \\ &= u(nT)\frac{1}{2j}\left(e^{nT(-1+j2)} - e^{nT(-1-j2)}\right) \end{aligned}$$

Since $T = 2\pi/\omega_s = 1$ s, the table of z transforms gives

$$X_D(z) = \frac{1}{2j}\left(\frac{z}{z - e^{-1+j2}} - \frac{z}{z - e^{-1-j2}}\right)$$

and after some manipulation

$$X_D(z) = \frac{ze^{-1}\sin 2}{z^2 - 2ze^{-1}\cos 2 + e^{-2}}$$

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Since the frequency spectrum of an impulse-modulated signal is given by

$$\hat{X}(j\omega) = X_D(e^{j\omega T})$$

we get

$$\hat{X}(j\omega) = X_D(e^{j\omega T}) = \frac{e^{j\omega - 1}\sin 2}{e^{2j\omega} - 2e^{j\omega - 1}\cos 2 + e^{-2}} \quad \blacksquare$$

Poisson's Summation Formula

- As may be expected, the spectrum of a discrete-time signal must be related to that spectrum of the continuous-time signal from which it was derived.

Poisson's Summation Formula

- As may be expected, the spectrum of a discrete-time signal must be related to that spectrum of the continuous-time signal from which it was derived.
- This relationship can be established by using Poisson's summation formula.

- Consider a signal $x(t)$ with a Fourier transform $X(j\omega)$.

  Poisson's summation formula states that

$$\sum_{n=-\infty}^{\infty} x(t+nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(jn\omega_s)e^{jn\omega_s t}$$

  where $\omega_s = 2\pi/T$.

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- If $t = 0$ and $x(t)$ is a two-sided signal, we have

$$\sum_{n=-\infty}^{\infty} x(nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(jn\omega_s) \qquad \text{(B)}$$

...

$$\sum_{n=-\infty}^{\infty} x(t+nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(jn\omega_s)e^{jn\omega_s t}$$

- If $t=0$ and $x(t)$ is a right-sided signal, i.e., $x(t)=0$ for $t<0$, then

$$\sum_{n=0}^{\infty} x(nT) = \frac{x(0+)}{2} + \frac{1}{T}\sum_{n=-\infty}^{\infty} X(jn\omega_s)$$

where

$$\lim_{t\to 0} x(t) = \frac{x(0-)+x(0+)}{2} = \frac{x(0+)}{2}$$

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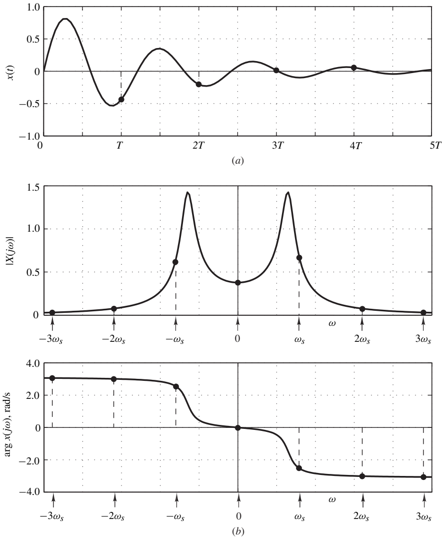
- *Note:* In Fourier analysis, the value of a time function at a discontinuity is always taken to be the average of the left and right limits (see textbook).

Poisson's Summation Formula *Cont'd*



Spectral Relationship between Discrete-Time and Continuous-Time Signals

- Given a continuous-time signal $x(t)$ with a Fourier transform $X(j\omega)$, then from the frequency-shifting theorem we have

$$x(t)e^{-j\omega_0 t} \leftrightarrow X(j\omega_0 + j\omega)$$

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$$x(t)e^{-j\omega_0 t} \leftrightarrow X(j\omega_0 + j\omega)$$

- From Poisson's summation formula, we get

$$\sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega_0 nT} = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j\omega_0 + jn\omega_s)$$

where $\omega_s = 2\pi/T$ and if we now replace ω_0 by ω, we deduce the important relationship

$$\sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT} = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j\omega + jn\omega_s)$$

...

$$\sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT} = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(j\omega + jn\omega_s)$$

- It was shown earlier that

$$\hat{X}(j\omega) = X_D(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT}$$

and hence $\hat{X}(j\omega) = X_D(e^{j\omega T}) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(j\omega + jn\omega_s)$

- Therefore, the frequency spectrum of the impulse-modulated signal $\hat{x}(t)$ is *numerically equal* to the frequency spectrum of discrete-time signal $x(nT)$ and the two can be *uniquely determined* from the frequency spectrum of the continuous-time signal $x(t)$, namely, $X(j\omega)$.

Spectral Relationship *Cont'd*

- As is to be expected, $\hat{X}(j\omega)$ is a *periodic function* of ω with period ω_s since the frequency spectrum of discrete-time signals is periodic.

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- As is to be expected, $\hat{X}(j\omega)$ is a *periodic function* of ω with period ω_s since the frequency spectrum of discrete-time signals is periodic.
- To check this out, we can replace $j\omega$ by $j\omega + jm\omega_s$ in

$$\hat{X}(j\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j\omega + jn\omega_s)$$

to obtain

$$\begin{aligned}\hat{X}(j\omega + jm\omega_s) &= \frac{1}{T} \sum_{n=-\infty}^{\infty} X[j\omega + j(m+n)\omega_s] \\ &= \frac{1}{T} \sum_{n'=-\infty}^{\infty} X(j\omega + jn'\omega_s) = \hat{X}(j\omega)\end{aligned}$$

- For a right-sided signal, $x(t) = 0$ for $t \leq 0-$, and hence the impulse-modulated signal assumes the form

$$\hat{x}(t) = \sum_{n=0}^{\infty} x(nT)\delta(t - nT)$$

where $x(0) \equiv x(0+)$.

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- The Fourier transform of the signal is given by

$$\hat{X}(j\omega) = \sum_{n=0}^{\infty} x(nT)e^{-j\omega nT} = X_D(e^{j\omega T})$$

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- Thus Poisson's summation formula gives

$$\hat{X}(j\omega) = X_D(e^{j\omega T}) = \frac{x(0+)}{2} + \frac{1}{T}\sum_{n=-\infty}^{\infty} X(j\omega + jn\omega_s) \qquad \text{(C)}$$

...

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- By letting $j\omega = s$ and $e^{sT} = z$, Eq. (C) can be expressed in the s domain as

$$\hat{X}(s) = X_D(z) = \frac{x(0+)}{2} + \frac{1}{T}\sum_{n=-\infty}^{\infty} X(s + jn\omega_s)$$

where $X(s)$ and $\hat{X}(s)$ are the Laplace transforms of $x(t)$ and $\hat{x}(t)$, respectively.

Spectral Relationship *Cont'd*

...

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  where $X(s)$ and $\hat{X}(s)$ are the Laplace transforms of $x(t)$ and $\hat{x}(t)$, respectively.

- This relationship will be used in Chap. 12 to design digital filters based on analog filters.

Example

Find $\hat{X}(j\omega)$ if $x(t) = \cos \omega_0 t$.

Solution From the table of Fourier transforms (Table 3.2), we have

$$X(j\omega) = \mathcal{F} \cos \omega_0 t = \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

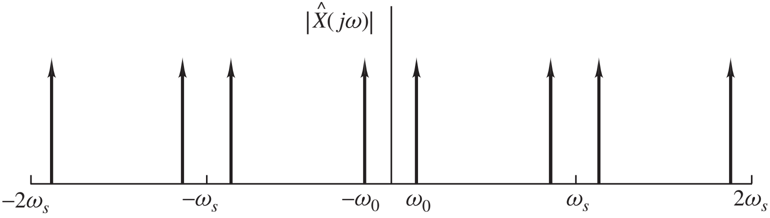
Hence Poisson's summation formula, i.e.,

$$\hat{X}(j\omega) = X_D(e^{j\omega T}) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j\omega + jn\omega_s)$$

gives

$$\hat{X}(j\omega) = \frac{\pi}{T} \sum_{n=-\infty}^{\infty} [\delta(\omega + n\omega_s + \omega_0) + \delta(\omega + n\omega_s - \omega_0)] \quad \blacksquare$$

Example *Cont'd*



(*a*)

Example

Find $\hat{X}(j\omega)$ if $x(t) = u(t)e^{-t}$.

Solution From the table of Fourier transforms (Table 3.2),

$$X(j\omega) = \mathcal{F}[u(t)e^{-t}] = \frac{1}{1 + j\omega}$$

Since $x(t) = 0$ for $t < 0$ in this case, we need to use the second form of Poisson's summation formula, i.e.,

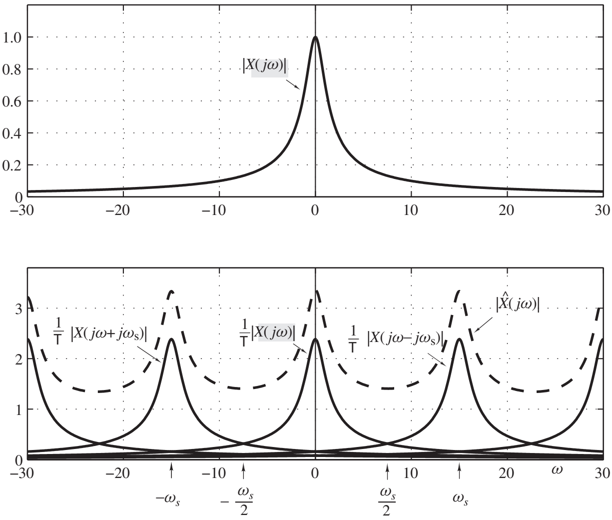
$$\hat{X}(j\omega) = X_D(e^{j\omega T}) = \frac{x(0+)}{2} + \frac{1}{T}\sum_{n=-\infty}^{\infty} X(j\omega + jn\omega_s)$$

The initial-value theorem of the Laplace transform gives

$$x(0+) = \lim_{s\to\infty}[sX(s)] = \lim_{s\to\infty}\frac{s}{1+s} = 1$$

and hence $$\hat{X}(j\omega) = \frac{1}{2} + \frac{1}{T}\sum_{n=-\infty}^{\infty}\frac{1}{1 + j(\omega + n\omega_s)}$$ ■

Example *Cont'd*



(b)

This slide concludes the presentation.
Thank you for your attention.