Chapter 9
REALIZATION
9.2.3 State-Space Realization
9.2.4 Lattice Realization
9.2.5 Cascade Realization
9.2.6 Parallel Realization
9.2.7 Transposition

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State-Space Realization

- Another approach to the realization of digital filters is to start with the state-space characterization:

$$\begin{aligned} \mathbf{q}(nT+T) &= \mathbf{A}\mathbf{q}(nT) + \mathbf{b}x(nT) \\ y(nT) &= \mathbf{c}^T\mathbf{q}(nT) + dx(nT) \end{aligned}$$

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- The state-space equations can be written as

$$q_i(nT+T) = \sum_{j=1}^{N} a_{ij}q_j(nT) + b_i x(nT) \quad \text{for} \quad i = 1, 2, \ldots, N$$

$$y(nT) = \sum_{j=1}^{N} c_j q_j(nT) + d_0 x(nT)$$

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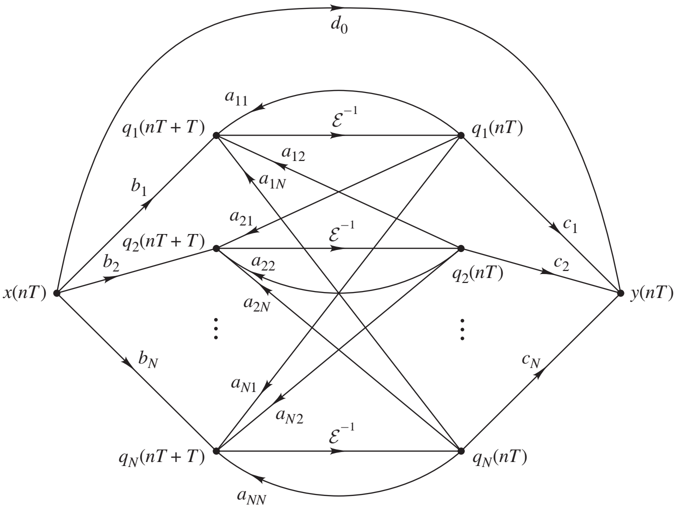
- A realization can now be obtained by converting the signal flow graph for the state-space equations into a network.



d_0
a_{11}
$\mathcal{E}^{-1}$
$q_1(nT+T)$
$q_1(nT)$
a_{12}
a_{1N}
b_1
c_1
a_{21}
$\mathcal{E}^{-1}$
$q_2(nT+T)$
a_{22}
b_2
c_2
$q_2(nT)$
$x(nT)$
$y(nT)$
a_{2N}
b_N
c_N
a_{N1}
a_{N2}
$\mathcal{E}^{-1}$
$q_N(nT+T)$
$q_N(nT)$
a_{NN}

Example

A discrete-time system can be represented by the state-space equations

$$
\begin{aligned}
\mathbf{q}(nT+T) &= \mathbf{A}\mathbf{q}(nT) + \mathbf{b}x(nT) \\
y(nT) &= \mathbf{c}^T\mathbf{q}(nT) + dx(nT)
\end{aligned}
$$

where

$$
\mathbf{A} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \ d = 2
$$

Obtain a state-space realization.

Example *Cont'd*

Solution For a general second-order system, we have

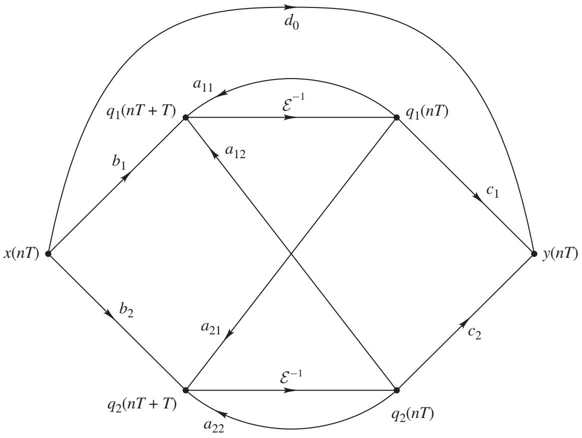
$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \; \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \; \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \; d = d_0$$

Hence the state-space equations can be expressed as

$$\begin{aligned} q_1(nT + T) &= a_{11}q_1(nT) + a_{12}q_2(nT) + b_1x(nT) \\ q_2(nT + T) &= a_{21}q_1(nT) + a_{22}q_2(nT) + b_2x(nT) \\ y(nT) &= c_1q_1(nT) + c_2(nT)q_2(nT) + dx(nT) \end{aligned}$$

Example *Cont'd*

Signal flow graph:



Example *Cont'd*

For the problem at hand, we have

$$a_{11} = m_1, \quad a_{12} = 0, \quad a_{21} = 0, \quad a_{22} = m_2$$
$$b_1 = 1, \quad b_2 = 1, \quad c_1 = m_1, \quad c_2 = m_2, \quad d_0 = 2$$

The required network can be obtained by replacing summing nodes by adders, distribution nodes by distribution nodes, and transmittances by multipliers and unit delays as appropriate. ■

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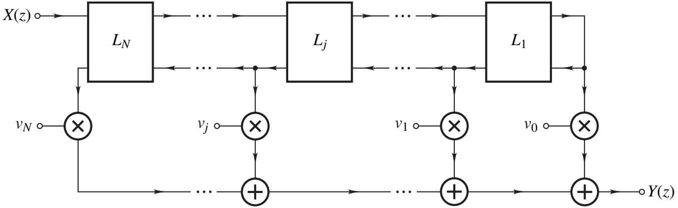
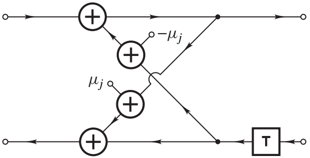
- However, they also offer certain advantages, as follows:
  - Reduced signal-to-noise ratios can be achieved.
  - A certain type of oscillations due to nonlinearities, known as *parasitic oscillations* can be eliminated in these structures (see Chap. 14).

- The lattice method was proposed by Gray and Markel and it is based on the configuration shown.



(a)



(b)

- A transfer function of the form

$$H(z) = \frac{N(z)}{D(z)} = \frac{\sum_{i=0}^{N} a_i z^{-i}}{1 + \sum_{i=1}^{N} b_i z^{-i}}$$

can be realized by applying a step-by-step recursive algorithm comprising N iterations to obtain a series of polynomials of the form

$$N_j(z) = \sum_{i=0}^{j} \alpha_{ji} z^{-i} \quad \text{and} \quad D_j(z) = \sum_{i=0}^{j} \beta_{ji} z^{-i}$$

for $j = N, N-1, \ldots, 0$.

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for $j = N,\ N-1,\ \ldots,\ 0$.

- Then for each value of j the multiplier constants ν_j and μ_j are evaluated using coefficients α_{jj} and β_{jj} in the above polynomials.

1. Let $N_j(z) = N(z)$ and $D_j(z) = D(z)$ and assume that $j = N$, that is

$$N_N(z) = \sum_{i=0}^{j} \alpha_{ji} z^{-i} = \sum_{i=0}^{N} a_i z^{-i}$$

$$D_N(z) = \sum_{i=0}^{j} \beta_{ji} z^{-i} = \sum_{i=0}^{N} b_i z^{-i} \quad \text{with } b_0 = 1$$

...

$$N_N(z) = \sum_{i=0}^{j} \alpha_{ji} z^{-i} = \sum_{i=0}^{N} a_i z^{-i} \quad \text{and} \quad D_N(z) = \sum_{i=0}^{j} \beta_{ji} z^{-i} = \sum_{i=0}^{N} b_i z^{-i}$$

2. Obtain ν_j, μ_j, $N_{j-1}(z)$, and $D_{j-1}(z)$ for $j = N, N-1, \ldots, 2$ using the following recursive relations:

$$\nu_j = \alpha_{jj}, \quad \mu_j = \beta_{jj}$$

$$P_j(z) = D_j\left(\frac{1}{z}\right) z^{-j} = \sum_{i=0}^{j} \beta_{ji} z^{i-j}$$

$$N_{j-1}(z) = N_j(z) - \nu_j P_j(z) = \sum_{i=0}^{j-1} \alpha_{ji} z^{-i}$$

$$D_{j-1}(z) = \frac{D_j(z) - \mu_j P_j(z)}{1 - \mu_j^2} = \sum_{i=0}^{j-1} \beta_{ji} z^{-i}$$

3. Obtain ν_1, μ_1, and $N_0(z)$ as follows:

$$\nu_1 = \alpha_{11}, \quad \mu_1 = \beta_{11}$$
$$P_1(z) = D_1\left(\frac{1}{z}\right)z^{-1} = \beta_{10}z^{-1} + \beta_{11}$$
$$N_0(z) = N_1(z) - \nu_1 P_1(z) = \alpha_{00}$$

3. Obtain ν_1, μ_1, and $N_0(z)$ as follows:

$$\begin{aligned}\nu_1 &= \alpha_{11}, \quad \mu_1 = \beta_{11} \\ P_1(z) &= D_1\left(\frac{1}{z}\right)z^{-1} = \beta_{10}z^{-1} + \beta_{11} \\ N_0(z) &= N_1(z) - \nu_1 P_1(z) = \alpha_{00}\end{aligned}$$

4. Complete the realization by letting

$$\nu_0 = \alpha_{00}$$

Example

Realize the transfer function

$$H(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

using the lattice method.

Solution

Step 1 We can write

$$N_2(z) = \alpha_{20} + \alpha_{21} z^{-1} + \alpha_{22} z^{-2} = a_0 + a_1 z^{-1} + a_2 z^{-2}$$
$$D_2(z) = \beta_{20} + \beta_{21} z^{-1} + \beta_{22} z^{-2} = 1 + b_1 z^{-1} + b_2 z^{-2}$$

Step 2: For $j = 2$, we get

$$
\begin{aligned}
\nu_2 &= \alpha_{22} = a_2 \quad \mu_2 = \beta_{22} = b_2 \\
P_2(z) &= D_2\left(\frac{1}{z}\right) z^{-2} = z^{-2} + b_1 z^{-1} + b_2 = \beta_{20} z^{-2} + \beta_{21} z^{-1} + \beta_{22} \\
N_1(z) &= N_2(z) - \nu_2 P_2(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} - \nu_2 (z^{-2} + b_1 z^{-1} + b_2) \\
&= \alpha_{10} + \alpha_{11} z^{-1} \\
D_1(z) &= \frac{D_2(z) - \mu_2 P_2(z)}{1 - \mu_2^2} = \frac{1 + b_1 z^{-1} + b_2 z^{-2} - \mu_2 (z^{-2} + b_1 z^{-1} + b_2)}{1 - \mu_2^2} \\
&= \beta_{10} + \beta_{11} z^{-1}
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_{10} &= a_0 - a_2 b_2 \quad \alpha_{11} = a_1 - a_2 b_1 \\
\beta_{10} &= 1, \quad \beta_{11} = \frac{b_1}{1 + b_2}
\end{aligned}
$$

Example *Cont'd*

Step 3 Similarly, for $j = 1$ we have

$$\nu_1 = \alpha_{11} = a_1 - a_2 b_1 \quad \mu_1 = \beta_{11} = \frac{b_1}{1 + b_2}$$

$$P_1(z) = D_1\left(\frac{1}{z}\right) z^{-1} = \beta_{10} z^{-1} + \beta_{11}$$

$$N_0(z) = N_1(z) - \nu_1 P_1(z) = \alpha_{10} + \alpha_{11} z^{-1} - \nu_1(\beta_{10} z^{-1} + \beta_{11}) = \alpha_{00}$$

where

$$\alpha_{00} = (a_0 - a_2 b_2) - \frac{(a_1 - a_2 b_1) b_1}{1 + b_2}$$

Example *Cont'd*

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where

$$\alpha_{00} = (a_0 - a_2 b_2) - \frac{(a_1 - a_2 b_1)b_1}{1 + b_2}$$

Step 4: Finally, step 4 gives

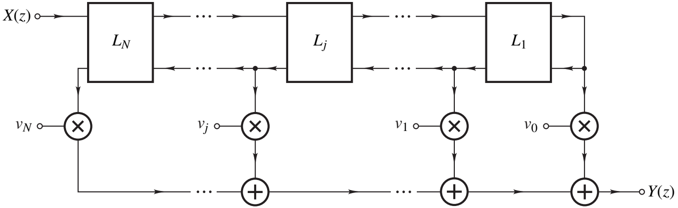
$$\nu_0 = \alpha_{00}$$

Example *Cont'd*

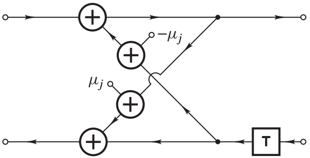
Summarizing, the multiplier constants for a general second-order lattice realization are as follows:

$$
\begin{aligned}
\nu_0 &= (a_0 - a_2 b_2) - \frac{(a_1 - a_2 b_1) b_1}{1 + b_2} \\
\nu_1 &= a_1 - a_2 b_1, \quad \nu_2 = a_2 \\
\mu_1 &= \frac{b_1}{1 + b_2}, \quad \mu_2 = b_2
\end{aligned}
$$

Example *Cont'd*



(*a*)



(*b*)

$$\nu_0 = (a_0 - a_2 b_2) - \frac{(a_1 - a_2 b_1) b_1}{1 + b_2}, \ \nu_1 = a_1 - a_2 b_1$$

$$\nu_2 = a_2, \ \mu_1 = \frac{b_1}{1 + b_2}, \ \mu_2 = b_2 \quad \blacksquare$$

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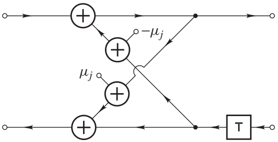
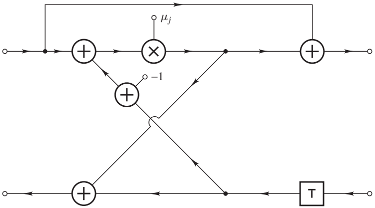
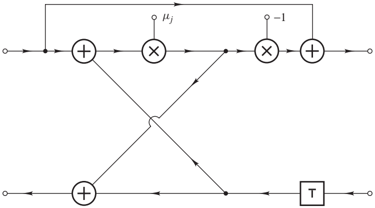
Lattice Realization *Cont'd*

- A problem associated with the lattice configuration presented is that it requires a large number of multipliers.
- Fortunately, a more economical lattice structure is possible.
- It turns out that the 2-multiplier lattice module shown earlier can be replaced by one of two 1-multiplier lattice modules as shown in the next slide.

Lattice Realization *Cont'd*







- Parameters μ_j for $j = 1, 2, \ldots, N$ stay the same as before.

Lattice Realization *Cont'd*

- Parameters μ_j for $j = 1, 2, \ldots, N$ stay the same as before.
- However, parameters ν_j need to be recalculated as

$$\tilde{\nu}_j = \frac{\nu_j}{\xi_j}$$

where

$$\xi_j = \begin{cases} 1 & \text{for } j = N \\ \prod_{i=j}^{N-1}(1 + \varepsilon_i \mu_{i+1}) & \text{for } j = 0, 1, \ldots, N-1 \end{cases}$$

Lattice Realization *Cont'd*

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where

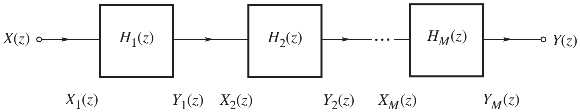
$$\xi_j = \begin{cases} 1 & \text{for } j = N \\ \prod_{i=j}^{N-1}(1 + \varepsilon_i \mu_{i+1}) & \text{for } j = 0, 1, \ldots, N-1 \end{cases}$$

- Parameter ε_i takes the value of $+1$ or -1 depending on which of the two 1-multiplier lattice modules is used.

Cascade Realization

- Consider an arbitrary number of filter sections connected in cascade as shown and assume that the ith section is characterized by

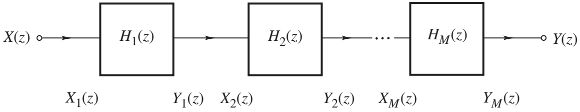
$$Y_i(z) = H_i(z)X_i(z)$$



(a)

- We can write

$$
\begin{aligned}
Y_1(z) &= H_1(z)X_1(z) = H_1(z)X(z) \\
Y_2(z) &= H_2(z)X_2(z) = H_2(z)Y_1(z) = H_1(z)H_2(z)X(z) \\
Y_3(z) &= H_3(z)X_3(z) = H_3(z)Y_2(z) = H_1(z)H_2(z)H_3(z)X(z) \\
\ldots\ldots &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
Y(z) &= Y_M(z) = H_M(z)Y_{M-1}(z) = H_1(z)H_2(z)\cdots H_M(z)X(z)
\end{aligned}
$$



(a)

- Therefore, the overall transfer function of a cascade arrangement of filter sections is equal to the *product* of the individual transfer functions, that is,

$$H(z) = \prod_{i=1}^{M} H_i(z)$$

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$$H(z) = \prod_{i=1}^{M} H_i(z)$$

- An Nth-order transfer function can be factorized into a product of first- and second-order transfer functions of the form

$$H_i(z) = \frac{a_{0i} + a_{1i}z^{-1}}{1 + b_{1i}z^{-1}} \quad \text{and} \quad H_i(z) = \frac{a_{0i} + a_{1i}z^{-1} + a_{2i}z^{-2}}{1 + b_{1i}z^{-1} + b_{2i}z^{-2}}$$

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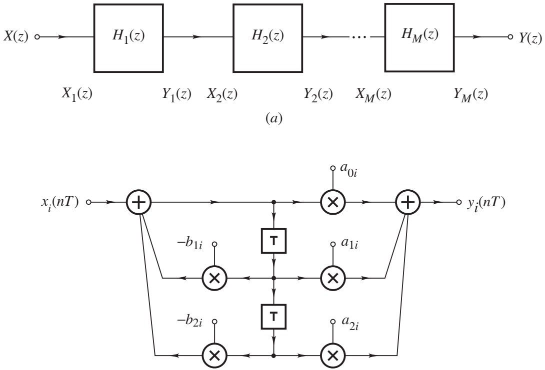
- Each of these low-order transfer functions can be realized using any one of the methods described.

- For example, an arbitrary transfer function can be realized by using a cascade arrangement of canonic sections as shown.

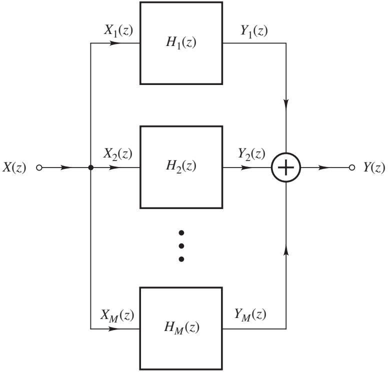


(a)

(b)

Parallel Realization

- Another realization comprising first- and second-order filter sections is based on the parallel configuration shown.



Parallel Realization *Cont'd*

- We note that all the parallel sections have a common input, i.e., $X_1(z) = X_2(z) = \cdots = X_M(z) = X(z)$.

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- Hence

$$\begin{aligned}
Y(z) &= Y_1(z) + Y_2(z) + \cdots + Y_M(z) \\
&= H_1(z)X_1(z) + H_2(z)X_2(z) + \cdots + H_M(z)X_M(z) \\
&= H_1(z)X(z) + H_2(z)X(z) + \cdots + H_M(z)X(z) \\
&= [H_1(z) + H_2(z) + \cdots + H_M(z)]X(z) \\
&= H(z)X(z)
\end{aligned}$$

where

$$H(z) = \sum_{i=1}^{M} H_i(z)$$

Example

Obtain a parallel realization of the transfer function

$$H(z) = \frac{10z^4 - 3.7z^3 - 1.28z^2 + 0.99z}{(z^2 - z + 0.34)(z^2 + 0.9z + 0.2)}$$

using canonic sections.

Solution The transfer function can be expressed as

$$H(z) = \frac{10z^4 - 3.7z^3 - 1.28z^2 + 0.99z}{(z - p_1)(z - p_2)(z - p_3)(z - p_4)}$$

where

$$\begin{aligned} p_1, p_2 &= 0.5 \mp j0.3 \\ p_3 &= -0.4 \\ p_4 &= -0.5 \end{aligned}$$

Example *Cont'd*

If we expand $H(z)/z$ into partial fractions, we get

$$\frac{H(z)}{z} = \frac{R_1}{z - 0.5 + j0.3} + \frac{R_2}{z - 0.5 - j0.3} + \frac{R_3}{z + 0.4} + \frac{R_4}{z + 0.5}$$

where

$$R_1 = 1, \quad R_2 = 1, \quad R_3 = 3, \quad R_4 = 5$$

Example *Cont'd*

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$$\frac{H(z)}{z} = \frac{R_1}{z - 0.5 + j0.3} + \frac{R_2}{z - 0.5 - j0.3} + \frac{R_3}{z + 0.4} + \frac{R_4}{z + 0.5}$$

where

$$R_1 = 1, \quad R_2 = 1, \quad R_3 = 3, \quad R_4 = 5$$

Thus

$$H(z) = \frac{z}{z - 0.5 + j0.3} + \frac{z}{z - 0.5 - j0.3} + \frac{3z}{z + 0.4} + \frac{5z}{z + 0.5}$$

...

$$H(z) = \frac{z}{z - 0.5 + j0.3} + \frac{z}{z - 0.5 - j0.3} + \frac{3z}{z + 0.4} + \frac{5z}{z + 0.5}$$

Combining the first two and the last two partial fractions into second-order transfer functions, we get

$$H(z) = H_1(z) + H_2(z)$$

where

$$H_1(z) = \frac{2 - z^{-1}}{1 - z^{-1} + 0.34z^{-2}} \quad \text{and} \quad H_2(z) = \frac{8 + 3.5z^{-1}}{1 + 0.9z^{-1} + 0.2z^{-2}}$$

...

$$H(z) = \frac{z}{z - 0.5 + j0.3} + \frac{z}{z - 0.5 - j0.3} + \frac{3z}{z + 0.4} + \frac{5z}{z + 0.5}$$

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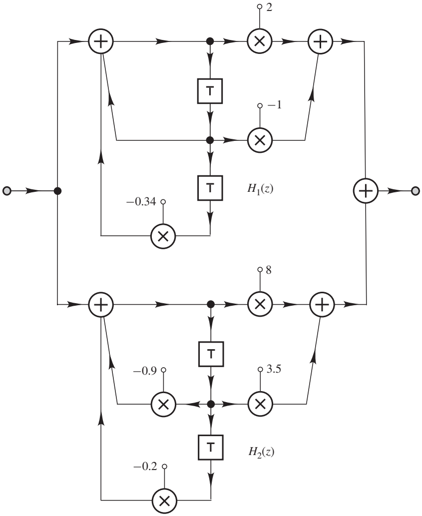
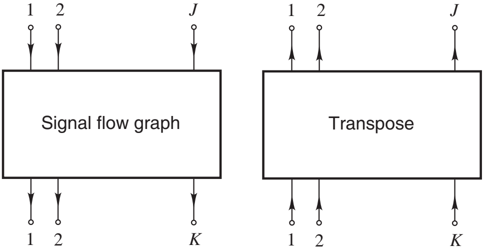
Using canonic structures for the two second-order transfer functions, the structure on the next slide is readily obtained.

Example *Cont'd*

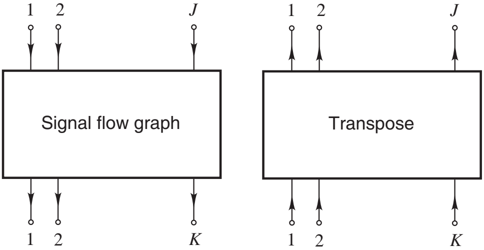


- Given a signal flow graph with inputs $j = 1, 2, \ldots, J$ and outputs $k = 1, 2, \ldots, K$, *a corresponding signal flow graph can be derived* by reversing the direction in each and every branch such that the J input nodes become output nodes and the K output nodes become input nodes.



Transpose

- Given a signal flow graph with inputs $j = 1, 2, \ldots, J$ and outputs $k = 1, 2, \ldots, K$, *a corresponding signal flow graph can be derived* by reversing the direction in each and every branch such that the J input nodes become output nodes and the K output nodes become input nodes.
- The signal flow graph so derived is said to be the *transpose* of the original signal flow graph.



- If a signal flow graph and its transpose are characterized by transfer functions $H_{jk}(z)$ and $H_{kj}(z)$, respectively, then

$$H_{jk}(z) = H_{kj}(z)$$

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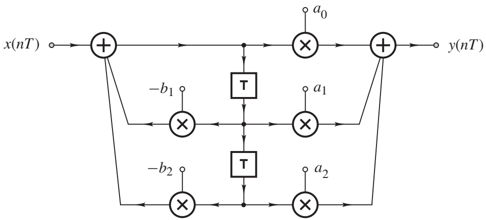
- In effect, given a digital-filter structure a corresponding transpose structure can be obtained that has the *same transfer function*.

- If a signal flow graph and its transpose are characterized by transfer functions $H_{jk}(z)$ and $H_{kj}(z)$, respectively, then

$$H_{jk}(z) = H_{kj}(z)$$

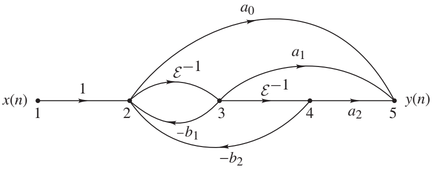
- In effect, given a digital-filter structure a corresponding transpose structure can be obtained that has the *same transfer function*.
- Sometimes, the derived transpose structure has improved features relative to the original structure.

Example



$x(nT)$
$y(nT)$
a_0
a_1
a_2
$-b_1$
$-b_2$
T
T

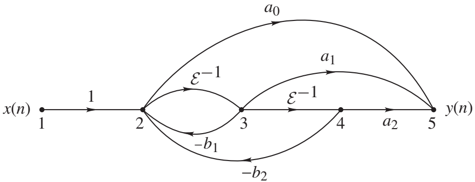
(a)



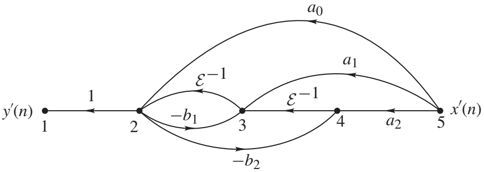
$x(n)$
$y(n)$
1
1
2
3
4
5
a_0
a_1
a_2
$\mathcal{E}^{-1}$
$\mathcal{E}^{-1}$
$-b_1$
$-b_2$

(b)

Example *Cont'd*

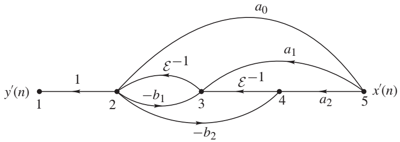


(*b*)

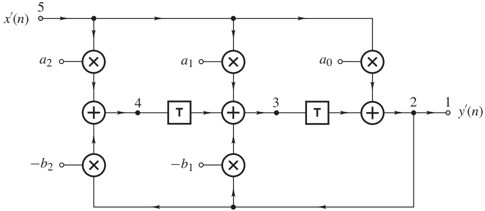


(*c*)

Example *Cont'd*



(c)



(d)

This slide concludes the presentation.
Thank you for your attention.