# Chapter 10
# DESIGN OF NONRECURSIVE FILTERS
# 10.1 Introduction
# 10.2 Properties of Constant-Delay Nonrecursive Filters


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- In certain filtering applications, delay distortion is highly undesirable and in such applications digital filters are required that have a *linear-phase response*.
- It turns out that linear-phase response can be easily achieved by designing the required filter as a nonrecursive filter.
- A linear-phase response is obtained by simply ensuring that the impulse response satisfies certain symmetry conditions.
- In this presentation, some basic properties of linear-phase nonrecursive filters are examined.

## Constant Delay in Nonrecursive Filters

- A causal nonrecursive filter can be represented by the transfer function

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- Its frequency response is given by

$$H(e^{j\omega T}) = M(\omega)e^{j\theta(\omega)} = \sum_{n=0}^{N-1} h(nT)e^{-j\omega nT}$$

where

$$M(\omega) = |H(e^{j\omega T})| \quad \text{and} \quad \theta(\omega) = \arg H(e^{j\omega T})$$

...

$$\theta(\omega) = \arg H(e^{j\omega T})$$

- The absolute delay, which is also known as the *phase delay*, and the *group delay* of a filter are given by

$$\tau_p = -\frac{\theta(\omega)}{\omega} \quad \text{and} \quad \tau_g = -\frac{d\theta(\omega)}{d\omega}$$

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$$\tau_p = -\frac{\theta(\omega)}{\omega} \quad \text{and} \quad \tau_g = -\frac{d\theta(\omega)}{d\omega}$$

- If both the phase and group delays are assumed to be constant, then the phase response must be linear, i.e.,

$$\theta(\omega) = -\tau\omega = \tan^{-1} \frac{-\sum_{n=0}^{N-1} h(nT) \sin \omega nT}{\sum_{n=0}^{N-1} h(nT) \cos \omega nT}$$

where $\tau$ is a constant.

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- Hence

$$\tan \tau\omega = \frac{\sum_{n=0}^{N-1} h(nT) \sin \omega nT}{\sum_{n=0}^{N-1} h(nT) \cos \omega nT}$$

or

$$\sum_{n=0}^{N-1} h(nT)(\cos \omega nT \sin \omega\tau - \sin \omega nT \cos \omega\tau) = 0$$

and so

$$\sum_{n=0}^{N-1} h(nT) \sin(\omega\tau - \omega nT) = 0$$

...

$$\sum_{n=0}^{N-1} h(nT)\sin(\omega\tau - \omega nT) = 0$$

- The solution of the above equation can be shown to be

$$\tau = \tfrac{1}{2}(N-1)T$$

$$h(nT) = h[(N-1-n)T] \quad \text{for} \quad 0 \le n \le N-1$$

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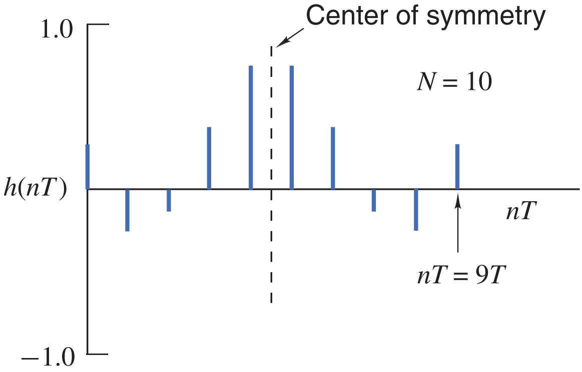
- Therefore, a nonrecursive filter can be designed to have *constant phase and group delays* over its entire baseband by simply ensuring that its impulse response is *symmetrical* about its center.

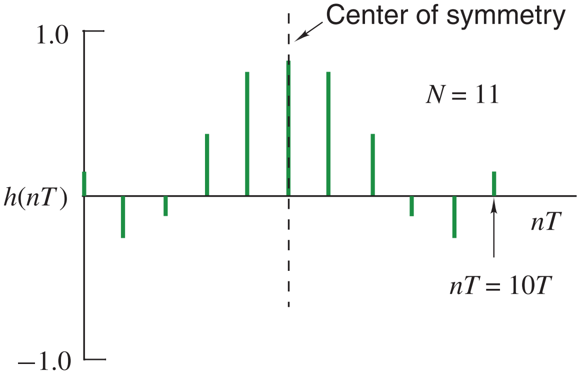
- For *even* $N$, the impulse response is symmetrical about the midpoint between samples $(N-2)/2$ and $N/2$ as shown:




(*a*)

# Constant Delay in Nonrecursive Filters *Cont'd*

- For *odd* $N$, the impulse response is symmetrical about sample $(N-1)/2$ as shown:




(b)

# Constant Delay in Nonrecursive Filters *Cont'd*

- In most applications only the group delay needs to be constant in which case the phase response can have the form

$$\theta(\omega) = \theta_0 - \tau\omega$$

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- If we assume that $\theta_0 = \pm\pi/2$, a second class of constant-delay nonrecursive filters is obtained where

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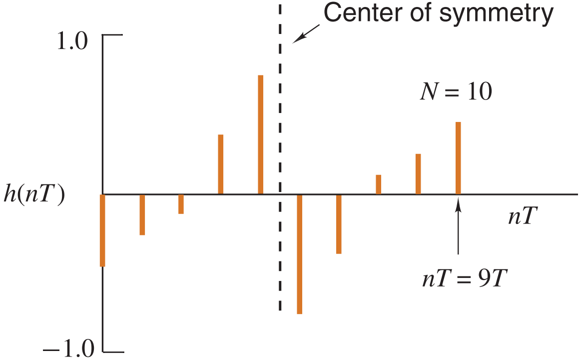
$$\begin{aligned}\tau &= \tfrac{1}{2}(N-1)T \\ h(nT) &= -h[(N-1-n)T]\end{aligned}$$

- In effect, a nonrecursive filter can be designed to have *constant group delay* over its entire baseband by simply ensuring that its impulse response is *antisymmetrical* about its center.

- For *even* $N$, the impulse response is antisymmetrical about the midpoint between samples $(N-2)/2$ and $N/2$ as shown:

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# Constant Delay in Nonrecursive Filters *Cont'd*

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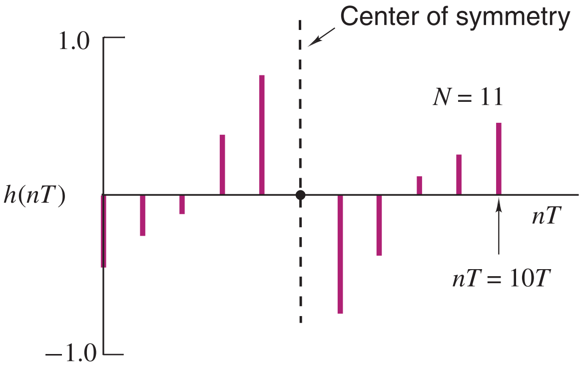


(b)

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- For the case of a *symmetrical* impulse response and odd $N$,

$$H(e^{j\omega T}) = \sum_{n=0}^{(N-3)/2} h(nT)e^{-j\omega nT} + h\left[\frac{(N-1)T}{2}\right]e^{-j\omega(N-1)T/2} + \sum_{n=(N+1)/2}^{N-1} h(nT)e^{-j\omega nT} \qquad \text{(A)}$$

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- If we first let $N-1-n=m$ and then let $m=n$, we get

$$\sum_{n=(N+1)/2}^{N-1} h(nT)e^{-j\omega nT} = \sum_{n=(N+1)/2}^{N-1} h[(N-1-n)T]e^{-j\omega nT} = \sum_{n=0}^{(N-3)/2} h(nT)e^{-j\omega(N-1-n)T} \qquad \text{(B)}$$

- From Eqs. (A) and (B)

$$H(e^{j\omega T}) = e^{-j\omega(N-1)T/2}\left\{h\left[\frac{(N-1)T}{2}\right] + \sum_{n=0}^{(N-3)/2} 2h(nT)\cos\left[\omega\left(\frac{N-1}{2}-n\right)T\right]\right\}$$

and with $(N-1)/2 - n = k$, we have

$$H(e^{j\omega T}) = e^{-j\omega(N-1)T/2}\sum_{k=0}^{(N-1)/2} a_k \cos\omega kT$$

where $a_0 = h\left[\frac{(N-1)T}{2}\right]$ and $a_k = 2h\left[\left(\frac{N-1}{2}-k\right)T\right]$

| $h(nT)$ | $N$ | $H(e^{j\omega T})$ |
|---|---|---|
| Symmetrical | Odd | $e^{-j\omega(N-1)T/2} \sum_{k=0}^{(N-1)/2} a_k \cos \omega kT$ |
| | Even | $e^{-j\omega(N-1)T/2} \sum_{k=1}^{N/2} b_k \cos[\omega(k - \frac{1}{2})T]$ |
| Antisymmetrical | Odd | $e^{-j[\omega(N-1)T/2-\pi/2]} \sum_{k=1}^{(N-1)/2} a_k \sin \omega kT$ |
| | Even | $e^{-j[\omega(N-1)T/2-\pi/2]} \sum_{k=1}^{N/2} b_k \sin[\omega(k - \frac{1}{2})T]$ |

where $a_0 = h\left[\frac{(N-1)T}{2}\right], \ a_k = 2h\left[\left(\frac{N-1}{2} - k\right)T\right], \ b_k = 2h\left[\left(\frac{N}{2} - k\right)T\right]$

# Location of Zeros

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- For *odd* $N$, we can write

$$H(z) = \frac{1}{z^{(N-1)/2}} \left\{ \sum_{n=0}^{(N-3)/2} h(nT)\left(z^{(N-1)/2-n} \pm z^{-[(N-1)/2-n]}\right) \right.$$
$$\left. + \frac{1}{2} h\left[\frac{(N-1)T}{2}\right](z^0 \pm z^0) \right\} \qquad \text{(C)}$$

where the negative sign applies to the case of antisymmetrical impulse response.

...

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- With $(N-1)/2 - n = k$, Eq. (C) can be expressed as

$$H(z) = \frac{N(z)}{D(z)} = \frac{1}{z^{(N-1)/2}} \sum_{k=0}^{(N-1)/2} \frac{a_k}{2}(z^k \pm z^{-k})$$

where $a_0$ and $a_k$ are given in the table of frequency responses shown earlier.

# Location of Zeros *Cont'd*

...

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- If we replace $z$ by $z^{-1}$ in $N(z)$, we get

$$N(z^{-1}) = \sum_{k=0}^{(N-1)/2} a_k(z^{-k} \pm z^k)$$

$$= \pm \sum_{k=0}^{(N-1)/2} a_k(z^k \pm z^{-k}) = \pm N(z)$$

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- The same relation holds for *even* $N$, as can be easily shown.
- Therefore, if $z_i = r_i e^{j\psi_i}$ is a zero of $H(z)$, then its *reciprocal* $z_i^{-1} = e^{-j\psi_i}/r_i$ must also be a zero of $H(z)$.

# Location of Zeros *Cont'd*

The property $N(z^{-1}) = \pm N(z)$ imposes the following constraints on the zeros of the transfer function:

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1. An arbitrary number of zeros can be located at $z_i = \pm 1$ since $z_i^{-1} = \pm 1$.
2. An arbitrary number of complex-conjugate pairs of zeros can be located on the unit circle since

$$(z-z_i)(z-z_i^*) = (z-e^{j\psi_i})(z-e^{-j\psi_i}) = \left(z - \frac{1}{z_i^*}\right)\left(z - \frac{1}{z_i}\right)$$

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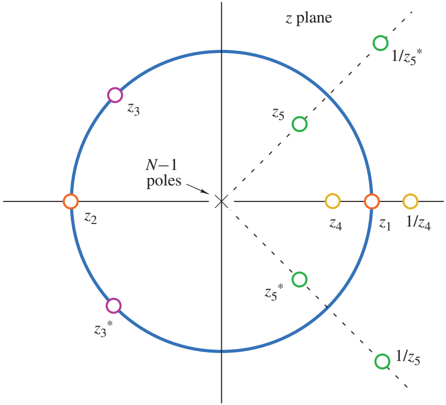
4. Complex zeros off the unit circle must occur in groups of four, namely, $z_i$, $z_i^*$, and their reciprocals.




*Note:* Polynomials with these properties are called *mirror-image polynomials*.

*This slide concludes the presentation.*
*Thank you for your attention.*