Chapter 10
DESIGN OF NONRECURSIVE FILTERS
10.6 Design Based on Numerical-Analysis Formulas

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Introduction

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- Such mathematical operations can be performed by using many classical numerical-analysis formulas.
- Formulas of this type can be readily derived from the Taylor series.
- This presentation will show that numerical-analysis formulas can be used to design nonrecursive filters that can be used to perform interpolation, differentiation, and integration.

Interpolation Formulas

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- The most fundamental numerical analysis formulas are the formulas for *interpolation*.
- The value of $x(t)$ at $t = nT + pT$, where $0 \le p < 1$, is given by the *forward* Gregory-Newton interpolation formula as

$$\begin{aligned} x(nT + pT) &= (1 + \Delta)^p x(nT) \\ &= \left[1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \cdots \right] x(nT) \end{aligned}$$

where

$$\Delta x(nT) = x(nT + T) - x(nT)$$

is commonly referred to as the *forward* difference.

- Similarly, the *backward* Gregory-Newton interpolation formula gives

$$\begin{aligned} x(nT + pT) &= (1 - \nabla)^{-p} x(nT) \\ &= \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \cdots\right] x(nT) \end{aligned}$$

where

$$\nabla x(nT) = x(nT) - x(nT - T)$$

is known as the *backward difference*.

- Another interpolation formula known as the *Stirling formula* assumes the form

$$
\begin{aligned}
x(nT+pT) = {} & \left[1+\frac{p^2}{2!}\delta^2+\frac{p^2(p^2-1)}{4!}\delta^4+\cdots\right]x(nT) \\
& +\frac{p}{2}\left[\delta x\left(nT-\tfrac{1}{2}T\right)+\delta x\left(nT+\tfrac{1}{2}T\right)\right] \\
& +\frac{p(p^2-1)}{2(3!)}\left[\delta^3 x\left(nT-\tfrac{1}{2}T\right)+\delta^3 x\left(nT+\tfrac{1}{2}T\right)\right] \\
& +\frac{p(p^2-1)(p^2-2^2)}{2(5!)}\left[\delta^5 x\left(nT-\tfrac{1}{2}T\right)+\delta^5 x\left(nT+\tfrac{1}{2}T\right)\right] \\
& +\cdots
\end{aligned}
$$

where

$$\delta x\left(nT+\tfrac{1}{2}T\right)=x(nT+T)-x(nT)$$

is known as the *central difference*.

- The forward, backward, and central differences are linear operators.

  Hence higher-order differences can be readily obtained, e.g.,

$$\begin{aligned}
\delta^3 x\left(nT + \tfrac{1}{2}T\right) &= \delta^2\left[\delta x\left(nT + \tfrac{1}{2}T\right)\right] = \delta^2[x(nT + T) - x(nT)] \\
&= \delta[\delta x(nT + T) - \delta x(nT)] \\
&= \delta\left\{x\left(nT + \tfrac{3}{2}T\right) - x\left(nT + \tfrac{1}{2}T\right)\right. \\
&\quad \left. - \left[x\left(nT + \tfrac{1}{2}T\right) - x\left(nT - \tfrac{1}{2}T\right)\right]\right\} \\
&= \delta x\left(nT + \tfrac{3}{2}T\right) - 2\delta x\left(nT + \tfrac{1}{2}T\right) + \delta x\left(nT - \tfrac{1}{2}T\right) \\
&= [x(nT + 2T) - x(nT + T)] - 2[x(nT + T) - x(nT)] \\
&\quad + [x(nT) - x(nT - T)] \\
&= x(nT + 2T) - 3x(nT + T) + 3x(nT) - x(nT - T)
\end{aligned}$$

Differentiation Formulas

- The first derivative of $x(t)$ with respect to time at instant $t = nT + pT$ can be expressed as

$$\left.\frac{dx(t)}{dt}\right|_{t=nT+pT} = \frac{dx(nT+pT)}{dp} \times \frac{dp}{dt}$$
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- By differentiating each of the interpolation formulas considered with respect to p, corresponding differentiation formulas can be obtained.

Integration Formulas

- Integration formulas can be derived by writing

$$\int_{nT}^{t_2} x(t)\,dt = T\int_0^{p_2} x(nT + pT)\,dp$$

where

$$nT < t_2 \le nT + T$$

and

$$t_2 = nT + Tp_2 \quad \text{or} \quad p_2 = \frac{t_2 - nT}{T}$$

with $0 < p_2 \le 1$.

- Nonrecursive filters that can perform *interpolation*, *differentiation*, or *integration* can be obtained by expressing one of the available numerical formulas for these operations in the form of a difference equation.

Digital Interpolators, Differentiators, Integrators

- Nonrecursive filters that can perform *interpolation*, *differentiation*, or *integration* can be obtained by expressing one of the available numerical formulas for these operations in the form of a difference equation.
- Let $x(nT)$ and $y(nT)$ be the input and output of a nonrecursive filter and assume that $y(nT)$ is equal to the required function of $x(t)$, i.e.,

$$y(nT) = f[x(t)]\Big|_{t=nT+pT}$$

- For interpolation, differentiation, or integration, we would have

$$y(nT) = x(t)\Big|_{t=nT+pT}$$

$$y(nT) = \frac{dx(t)}{dt}\Big|_{t=nT+pT}$$

or

$$y(nT) = \int_{nT}^{nT+pT} x(t)\,dt$$

as appropriate.

- By choosing an appropriate numerical formula for the operation of interest and then eliminating all the difference operators using their definitions, we can obtain a difference equation of the form

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- Now by applying the z transform, a transfer function

$$H(z) = \sum_{n=-K}^{M} h(nT) z^{-n}$$

can be deduced.

. . .

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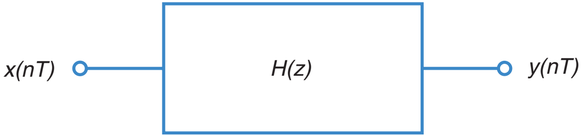
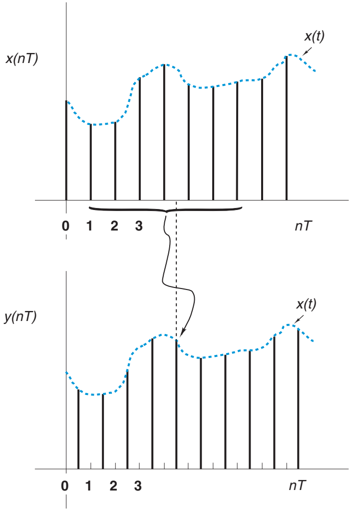


Interpolation:



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- For real-time applications it is necessary to convert a noncausal into a causal design.
- This is done by multiplying the transfer function by an appropriate negative power of z, which corresponds to delaying the impulse response of the filter to ensure that $h(nT) = 0$ for $n < 0$.

Example

- A signal $x(t)$ is sampled at a rate of $1/T$ Hz.

  Design a sixth-order differentiator with a time-domain response

  $$y(nT) = \left.\frac{dx(t)}{dt}\right|_{t=nT}$$

  Use the Stirling formula.

- **Solution** From Stirling's formula for interpolation

$$
\begin{aligned}
y(nT) &= \left.\frac{dx(t)}{dt}\right|_{t=nT+pT} = \frac{1}{T}\left.\frac{dx(nT+pT)}{dp}\right|_{p=0} \\
&= \frac{1}{2T}\left[\delta x\left(nT-\tfrac{1}{2}T\right)+\delta x\left(nT+\tfrac{1}{2}T\right)\right] \\
&\quad - \frac{1}{12T}\left[\delta^3 x\left(nT-\tfrac{1}{2}T\right)+\delta^3 x\left(nT+\tfrac{1}{2}T\right)\right] \\
&\quad + \frac{1}{60T}\left[\delta^5 x\left(nT-\tfrac{1}{2}T\right)+\delta^5 x\left(nT+\tfrac{1}{2}T\right)\right]+\cdots
\end{aligned}
$$

Example *Cont'd*

- From the definition of the central difference, we get

$$\begin{aligned}
\delta x\left(nT - \tfrac{1}{2}T\right) + \delta x\left(nT + \tfrac{1}{2}T\right) &= x(nT + T) - x(nT - T)\\
\delta^3 x\left(nT - \tfrac{1}{2}T\right) + \delta^3 x\left(nT + \tfrac{1}{2}T\right) &= x(nT + 2T) - 2x(nT + T)\\
&\quad + 2x(nT - T) - x(nT - 2T)\\
\delta^5 x\left(nT - \tfrac{1}{2}T\right) + \delta^5 x\left(nT + \tfrac{1}{2}T\right) &= x(nT + 3T) - 4x(nT + 2T)\\
&\quad + 5x(nT + T) - 5x(nT - T)\\
&\quad + 4x(nT - 2T) - x(nT - 3T)
\end{aligned}$$

- Hence

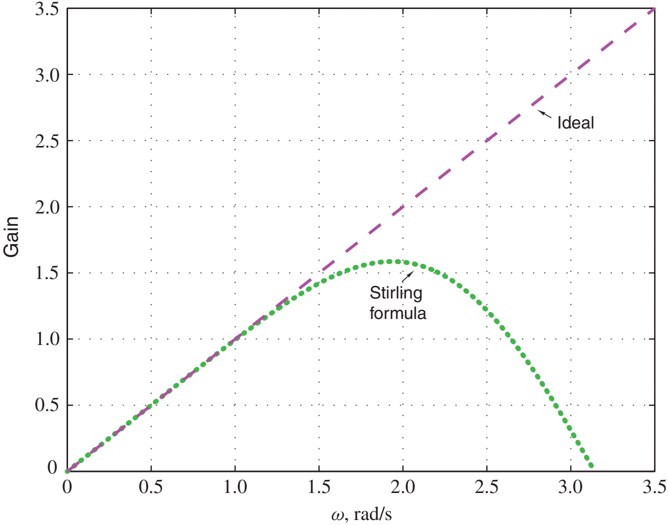
$$y(nT) = \frac{1}{60T}[x(nT+3T) - 9x(nT+2T) + 45x(nT+T) \\ -45x(nT-T) + 9x(nT-2T) - x(nT-3T)]$$

and, therefore

$$H(z) = \frac{1}{60T}(z^3 - 9z^2 + 45z - 45z^{-1} + 9z^{-2} - z^{-3})$$

- Note that the differentiator has an antisymmetrical impulse response, i.e., it has a constant group delay, and it is also noncausal.
- A causal filter can be obtained by multiplying $H(z)$ by z^{-3}.

Example *Cont'd*



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$$H(e^{j\omega T}) = j\omega \quad \text{for} \;\; 0 \le |\omega| < \omega_s/2$$
- Then on assuming a periodic frequency response, the appropriate impulse response can be determined.
- Gibbs' oscillations due to the transition in $H(e^{j\omega T})$ at $\omega = \omega_s/2$ can be reduced, as before, by using the window technique.

Example

- Design a sixth-order differentiator by employing the Fourier-series method.

  Use (a) a rectangular window and (b) the Kaiser window with $\alpha = 3.0$.

Example *Cont'd*

- **Solution** Using the Fourier-series method, the impulse response of the differentiator can be obtained as

$$h(nT) = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} j\omega e^{j\omega nT}\, d\omega = -\frac{1}{\omega_s} \int_{0}^{\omega_s/2} 2\omega \sin(\omega nT)\, d\omega$$

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- On integrating by parts, we get

$$h(nT) = \frac{1}{nT}\cos \pi n - \frac{1}{n^2 \pi T}\sin \pi n$$

or

$$h(nT) = \begin{cases} 0 & \text{for } n = 0 \\ \dfrac{1}{nT}\cos \pi n & \text{otherwise} \end{cases}$$

- If we now use the rectangular window with $N = 7$, we deduce

$$H_w(z) = \frac{1}{6T}(2z^3 - 3z^2 + 6z - 6z^{-1} + 3z^{-2} - 2z^{-3})$$

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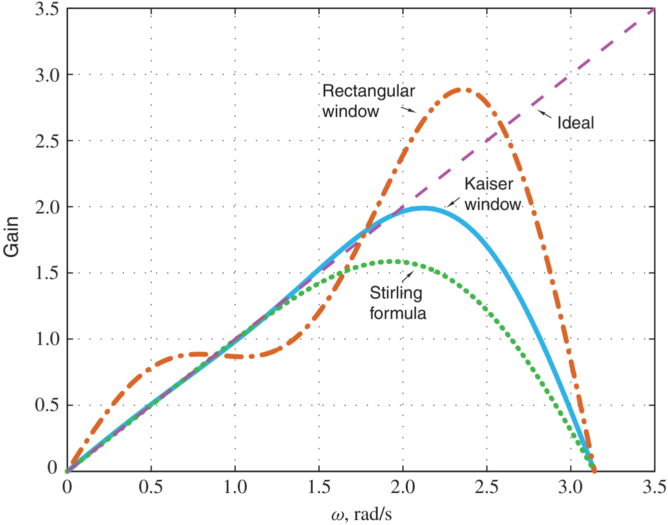
- The design of digital differentiators that would satisfy prescribed specifications is considered in Chap. 15.


3.5
3.0
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3.0
3.5
ω, rad/s
Rectangular window
Ideal
Kaiser window
Stirling formula

This slide concludes the presentation.
Thank you for your attention.