Chapter 12
DESIGN OF RECURSIVE FILTERS
12.1 Introduction, 12.2 Realizability Constraints, 12.3 Invariant Impulse-Response Method, 12.4 Modified Invariant Impulse-Response Method, 12.5 Matched-Z Transformation Method

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- The basic reason is that in recursive filters the transfer function is a ratio of polynomials of z whereas in nonrecursive filters it is a polynomial of negative powers of z.
- In recursive filters, the approximation problem is usually solved through *indirect* or *direct* methods.

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- In direct methods, the design problem is formulated as an optimization problem which is then solved using standard optimization methods.
- This presentation will deal with some indirect methods for the design of recursive filters.

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  The bilinear transformation method will be discussed in the next presentation.

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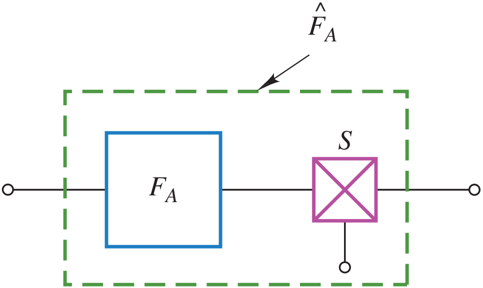
- These constraints will ensure that the transfer function is realizable in the form of a stable digital-filter network and are, therefore, said to be the *realizability* constraints.

Invariant Impulse-Response Method

- Given an analog filter, a corresponding digital filter can be obtained by constructing an impulse-modulated filter $\hat{F}_A$ as shown in the figure where S is an ideal impulse modulator and F_A is an analog filter characterized by a continuous-time transfer function $H_A(s)$.



- On the basis of the Poisson summation formula (see Chap. 6), the impulse-modulated filter can be represented by a continuous-time transfer function $\hat{H}_A(s)$ or, equivalently, by a discrete-time transfer function $H_D(z)$ as follows:

$$\hat{H}_A(j\omega) = H_D(e^{j\omega T}) = \frac{h_A(0+)}{2} + \frac{1}{T}\sum_{k=-\infty}^{\infty} H_A(j\omega + jk\omega_s)$$

where

$$h_A(t) = \mathcal{L}^{-1}H_A(s), \quad h_A(0+) = \lim_{s\to\infty}[sH_A(s)],$$

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- The method is referred to as the *invariant-impulse response* method because the impulse response of the digital filter is exactly equal to the impulse response of the analog filter at $t = nT$ for $n = 0, 1, \ldots, \infty$.

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  3. Would the digital filter obtained be causal?
  4. Would the discrete-time transfer function obtained have real coefficients?

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- If
$$H_A(j\omega) \approx 0 \quad \text{for} \quad |\omega| \geq \frac{\omega_s}{2}$$

  then

$$\sum_{k=-\infty, k\neq 0}^{\infty} H_A(j\omega + jk\omega_s) \approx 0 \quad \text{for} \quad |\omega| < \frac{\omega_s}{2}$$

  i.e., the side-bands contribute a negligible amount of *aliasing* error.

- If, in addition, $h_A(0+) = 0$ then

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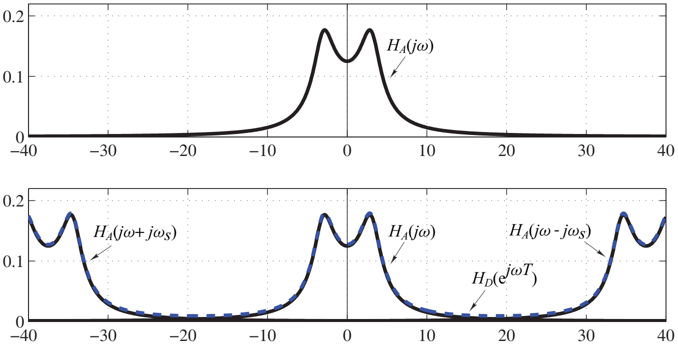
  the *condition* $h_A(0+) = 0$ *is satisfied* if the denominator degree in $H_A(s)$ exceeds the numerator degree by at least two.

- The *multiplier constant* $1/T$ *can be eliminated* by multiplying the discrete-time transfer function obtained, $H_D(z)$, by T.


0.2
0.1
0
–40
–30
–20
–10
0
10
20
30
40
HA(jω)
HA(jω+ jωS)
HA(jω)
HA(jω - jωS)
HD(ejωT)

Design Procedure

Given an analog filter characterized by a transfer function $H_A(s)$ *that satisfies the stated bandlimiting conditions*, a digital filter can be obtained by applying the following design procedure:

1. If the transfer function $H_A(s)$ is given in terms of its coefficients, i.e.,

$$H_A(s) = \frac{\sum_{i=0}^{M} a_i s^i}{\sum_{i=0}^{N} b_i s^i}$$

express it in terms of its *zeros and poles* as

$$H_A(s) = H_0 \frac{\prod_{i=1}^{M}(s - z_i)}{\prod_{i=1}^{N}(s - p_i)}$$

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3. Deduce the *impulse response* of the analog filter as follows:

$$h_A(t) = \mathcal{L}^{-1} H_A(s) = \sum_{i=1}^{N} A_i e^{p_i t}$$

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7. Combine partial fractions with complex conjugate poles to obtain the modified discrete-time transfer function

$$H'_D(z) = \sum_{i=1}^{N/2} \frac{a_{1i}z + a_{2i}z^2}{b_{0i} + b_{1i}z + b_{2i}z^2} \quad \text{for even } N$$

or

$$H'_D(z) = \frac{a_{11}z}{b_{01} + z} + \sum_{i=2}^{(N-1)/2} \frac{a_{1i}z + a_{2i}z^2}{b_{0i} + b_{1i}z + b_{2i}z^2} \quad \text{for odd } N$$

Notes

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  That is, *a stable analog filter will yield a stable digital filter*.

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  turn out to be *real* as required.

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  This is due to the fact that *complex conjugate poles give complex conjugate residues*.

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  Consequently, $H'_D(z)$ also represents a *causal filter*.
- A filter designed by using the invariant-impulse-response method can be conveniently realized by using the *parallel realization* since the overall transfer function is a sum of first- or second-order transfer functions.

Features

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- It *does not work* at all for highpass filters since these filters are not bandlimited by definition.

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  That is, the method is relatively *simple to apply*.

Example

Starting with a third-order normalized lowpass Chebyshev transfer function, obtain a discrete-time transfer function using the invariant-impulse response method.

Assume a maximum passband loss $A_p = 1.0$ dB and a sampling frequency $\omega_s = 10.0$ rad/s.

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Assume a maximum passband loss $A_p = 1.0$ dB and a sampling frequency $\omega_s = 10.0$ rad/s.

Solution The required Chebyshev transfer function can be readily obtained as

$$H_A(s) = \frac{H_0}{(s - p_1)(s - p_2)(s - p_2^*)}$$

where

$$H_0 = 0.4913, \; p_1 = -0.4942, \quad \text{and} \quad p_2, \; p_2^* = -0.2471 \pm j0.9660$$

(See Chap. 11.)

Example *Cont'd*

On expanding $H_A(s)$ into partial fractions as in Step 2 of the design procedure, we obtain

$$H_A(s) = \frac{A_1}{s - p_1} + \frac{A_2}{s - p_2} + \frac{A_3}{s - p_2^*}$$

where

$$\begin{aligned}
A_1 &= \left.\frac{H_0}{(s - p_2)(s - p_2^*)}\right|_{s=p_1} = \frac{H_0}{(p_1 - p_2)(p_1 - p_2^*)} = 0.4942 \\
A_2 &= \left.\frac{H_0}{(s - p_1)(s - p_2^*)}\right|_{s=p_2} = \frac{H_0}{(p_2 - p_1)(p_2 - p_2^*)} \\
&= -0.2471 - j0.0632 \\
A_3 &= A_2^* = -0.2471 + j0.0632
\end{aligned}$$

Example *Cont'd*

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From Step 3, the impulse response of the analog filter is obtained as

$$h_A(t) = \mathcal{L}^{-1} H_A(s) = A_1 e^{p_1 t} + A_2 e^{p_2 t} + A_2^* e^{p_2 * t}$$

...

$$H_A(s) = \frac{A_1}{s - p_1} + \frac{A_2}{s - p_2} + \frac{A_3}{s - p_2^*}$$

From Step 3, the impulse response of the analog filter is obtained as

$$h_A(t) = \mathcal{L}^{-1} H_A(s) = A_1 e^{p_1 t} + A_2 e^{p_2 t} + A_2^* e^{p_2 * t}$$

Replacing t by nT in $h_A(t)$, as in Step 4, gives

$$h_A(nT) = A_1 e^{p_1 nT} + A_2 e^{p_2 nT} + A_2^* e^{p_2^* nT}$$

where $T = 2\pi/\omega_s = 2\pi/10.0 = 0.6283$ s.

...

$$h_A(nT) = A_1 e^{p_1 nT} + A_2 e^{p_2 nT} + A_2^* e^{p_2^* nT}$$

On applying the z transform to $h_A(nT)$, as in Step 5, we get the discrete-time transfer function as

$$H_D(z) = \mathcal{Z} h_A(nT) = \frac{A_1 z}{z - e^{Tp_1}} + \frac{A_2 z}{z - e^{Tp_2}} + \frac{A_2^* z}{z - e^{Tp_2^*}}$$

Example *Cont'd*

Now on multiplying $H_D(z)$ by T (to adjust the gain of the filter) and then combining partial fractions with complex conjugate poles, the discrete-time transfer function can be expressed as

$$
\begin{aligned}
H'_D(z) &= \frac{A_1 z}{z - e^{Tp_1}} + \left(\frac{A_2 z}{z - e^{Tp_2}} + \frac{A_2^* z}{z - e^{Tp_3}} \right) \\
&= \frac{A_1 z}{z - e^{p_1 T}} + \frac{(A_2 + A_2^*)z^2 - (A_2 e^{p_2^* T} + A_2^* e^{p_2 T})z}{z^2 - (e^{p_2 T} + e^{p_2^* T})z + e^{p_2 T} \cdot e^{p_2^* T}} \\
&= \frac{A_1 z}{z - e^{p_1 T}} + \frac{2\text{Re}\,(A_2)z^2 - 2\text{Re}\,(A_2 e^{p_2^* T})z}{z^2 - 2\text{Re}\,(e^{p_2 T})z + |e^{p_2 T}|^2} \\
&= \frac{a_{11} z}{z + b_{01}} + \frac{a_{22} z^2 + a_{12} z}{z^2 + b_{12} z + b_{02}}
\end{aligned}
$$

...

$$H'_D(z) = \frac{a_{11}z}{z + b_{01}} + \frac{a_{22}z^2 + a_{12}z}{z^2 + b_{12}z + b_{02}} \quad \blacksquare$$

where

$$\begin{aligned}
a_{11} &= A_1 = 0.3105 \\
b_{01} &= -e^{Tp_1} = -0.7331 \\
a_{22} &= 2\text{Re}\,(A_2) = -0.4942 \\
a_{12} &= -2\text{Re}\,(A_2 e^{p_2^* T}) = 0.4093 \\
b_{12} &= -2\text{Re}\,(e^{p_2 T}) = -1.4065 \\
b_{02} &= |e^{p_2 T}|^2 = 0.7331
\end{aligned}$$

Example

Design a digital filter by applying the invariant impulse-response method to the Bessel-Thomson transfer function

$$H_A(s) = \frac{105}{105 + 105s + 45s^2 + 10s^3 + s^4}$$

Employ a sampling frequency $\omega_s = 8$ rad/s; repeat with $\omega_s = 16$ rad/s.

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Employ a sampling frequency $\omega_s = 8$ rad/s; repeat with $\omega_s = 16$ rad/s.

Solution The poles and residues of $H_A(s)$ are given by

$$\begin{aligned} p_1, p_1^* &= -2.896211 \pm j0.8672341 \\ p_2, p_2^* &= -2.103789 \pm j2.657418 \\ R_1, R_1^* &= 1.663392 \mp j8.396299 \\ R_2, R_2^* &= -1.663392 \pm j2.244076 \end{aligned}$$

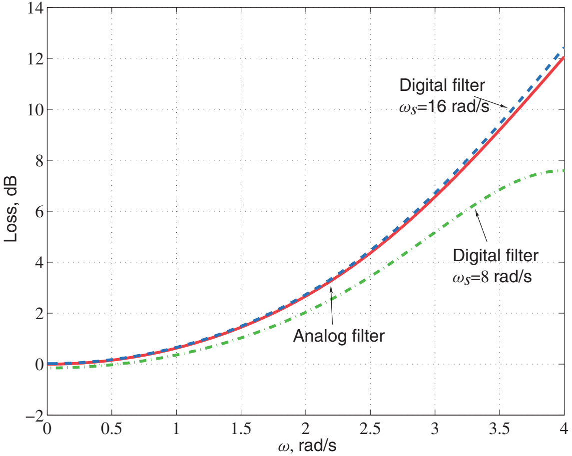
Example *Cont'd*

Steps 1 to 7 of the design procedure yield

$$TH_D(z) = \sum_{j=1}^{2} \frac{a_{1j}z + a_{2j}z^2}{b_{0j} + b_{1j}z + z^2}$$

ω_s	j	a_{1j}	a_{2j}	b_{0j}	b_{1j}
8	1	6.452333E-1	2.612851	1.057399E-2	-1.597700E-1
	2	-8.345233E-1	-2.612851	3.671301E-2	1.891907E-1
16	1	3.114550E-1	1.306425	1.028299E-1	-6.045080E-1
	2	-3.790011E-1	-1.306425	1.916064E-1	-4.404794E-1

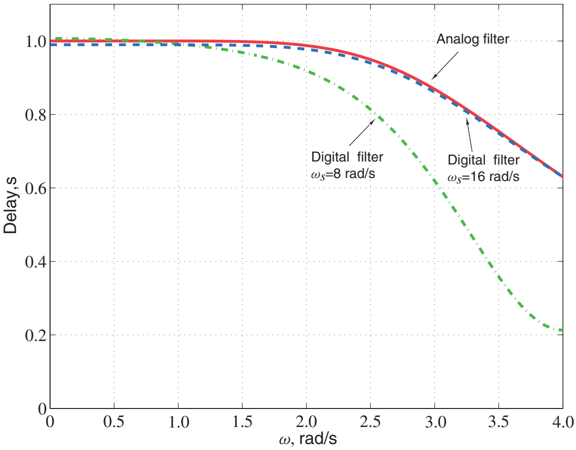
- Loss characteristic (i.e., $20\log[1/M(\omega)]$ versus ω):



(*a*)

Example *Cont'd*

- Delay characteristic (i.e., group delay versus ω):



(*b*)

- Aliasing errors tend to restrict the application of the invariant impulse-response method to the design of *allpole* filters, i.e., filters that have no zeros in the finite s plane.

Modified Invariant Impulse-Response Method

- Aliasing errors tend to restrict the application of the invariant impulse-response method to the design of *allpole* filters, i.e., filters that have no zeros in the finite s plane.
- However, a *modified* version of the method is available, which can be applied to filters that also have zeros in the finite s plane.

Modified Invariant Impulse-Response Method *Cont'd*

- Consider the transfer function

$$H_A(s) = \frac{H_0 N(s)}{D(s)} = \frac{H_0 \prod_{i=1}^{M}(s - z_i)}{\prod_{i=1}^{N}(s - p_i)}$$

where $N \geq M$.

Modified Invariant Impulse-Response Method *Cont'd*

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where $N \geq M$.

- We can write

$$H_A(s) = H_0 \frac{H_{A1}(s)}{H_{A2}(s)}$$

where $$H_{A1}(s) = \frac{1}{D(s)} \quad \text{and} \quad H_{A2}(s) = \frac{1}{N(s)}$$

...

$$H_{A1}(s) = \frac{1}{D(s)} \quad \text{and} \quad H_{A2}(s) = \frac{1}{N(s)}$$

- With M and $N \geq 2$, we have

$$h_A(0+) = \lim_{s\to\infty} [sH_A(s)] = 0$$

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- With M and $N \geq 2$, we have

$$h_A(0+) = \lim_{s\to\infty} [sH_A(s)] = 0$$

- Also

$$\lim_{s\to\infty} [H_{A1}(s)] \to 0 \quad \text{and} \quad \lim_{s\to\infty} [H_{A2}(s)] \to 0$$

and, consequently,

$$H_{A1}(j\omega) \approx 0 \quad \text{and} \quad H_{A2}(j\omega) \approx 0 \quad \text{for} \;\; |\omega| \geq \frac{\omega_s}{2}$$

for some sufficiently high value of ω_s.

- In effect, by using a sufficiently high sampling frequency, functions $H_{A1}(s)$ and $H_{A2}(s)$ can be considered to be bandlimited analog-filter transfer functions, and for each a discrete-time transfer function can be obtained, as follows, by using the invariant impulse-response method:

$$H_{D1}(z) = \frac{N_1(z)}{D_1(z)} = \sum_{i=1}^{N} \frac{A_i z}{z - e^{Tp_i}} \approx \frac{1}{T} H_{A1}(s) = \frac{1}{T} \frac{1}{D(s)}$$

$$H_{D2}(z) = \frac{N_2(z)}{D_2(z)} = \sum_{i=1}^{M} \frac{B_i z}{z - e^{Tz_i}} \approx \frac{1}{T} H_{A2}(s) = \frac{1}{T} \frac{1}{N(s)}$$

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- Therefore, a discrete-time transfer function can be formed as

$$H_D(z) = H_0 \frac{H_{D1}(z)}{H_{D2}(z)} = H_0 \frac{N_1(z) D_2(z)}{N_2(z) D_1(z)} \approx H_0 \frac{N(s)}{D(s)} = H_A(s)$$

...

$$H_D(z)H_0\frac{N_1(z)D_2(z)}{N_2(z)D_1(z)} \approx H_A(s)$$

- Evidently, given an arbitrary analog filter with frequency response $H_A(j\omega)$, a corresponding digital filter can be derived with a frequency response

$$H_D(e^{\omega T}) \approx H_A(j\omega) \quad \text{for} \quad |\omega| < \frac{\omega_s}{2}$$

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- Unfortunately, it also introduces two other problems.

...

$$H_D(z) = H_0 \frac{N_1(z) D_2(z)}{N_2(z) D_1(z)} \approx H_A(s)$$

- Polynomials $N_1(z)$ and $D_1(z)$ are of degree N which is the denominator degree in $H_A(s)$ and polynomials $N_2(z)$ and $D_2(z)$ are of degree M which is the numerator degree in $H_A(s)$.

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- The zeros of $D_1(z)$ are located in the unit circle $|z| = 1$.

  However, the zeros of $N_2(z)$ may be located outside the unit circle, which would render the derived digital filter *unstable*.

Stability Problem

- The stability problem mentioned in the previous slide can be easily circumvented without changing the amplitude response of the filter.

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- Consider a discrete-time transfer function

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and assume that it has poles $p_1, p_2, \ldots, p_K$ that are located outside the unit circle $|z| = 1$.

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$H_D(z)$ can be *stabilized* by simply replacing poles $p_1, p_2, \ldots, p_K$ by their reciprocals $1/p_1, 1/p_2, \ldots, 1/p_K$ and then replacing the multiplier constant H_0 by $H_0/\prod_{i=1}^{K} p_i$.

(See Chap. 12 for proof.)

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- The stabilization technique *preserves the amplitude response*, as stated.

  However, the *phase response is changed* which could be a problem in certain applications.
- The method *provides a theoretical basis* for the matched-z transformation method to be described later.

Example

The transfer function

$$H_A(s) = H_0 \prod_{j=1}^{3} \frac{a_{0j} + s^2}{b_{0j} + b_{1j}s + s^2}$$

where H_0, a_{0j}, and b_{1j} are given in the table, represents an analog lowpass elliptic filter.

j	a_{0j}	b_{0j}	b_{1j}
1	$1.199341E+1$	$3.581929E-1$	$9.508335E-1$
2	2.000130	$6.860742E-1$	$4.423164E-1$
3	1.302358	$8.633304E-1$	$1.088749E-1$
$H_0 = 6.713267E-3$			

Example *Cont'd*

The specifications of the filter are as follows:

- Passband ripple: 0.1 dB
- Minimum stopband loss: 43.46 dB
- Passband edge: $\sqrt{0.8}$ rad/s
- Stopband edge: $1/\sqrt{0.8}$ rad/s

Design a corresponding digital filter by employing the modified invariant impulse-response method.

Assume a sampling frequency $\omega_s = 7.5$ rad/s.

Example *Cont'd*

The design can be obtained through the following steps:

1. Let

$$H_{A1}(s) = \prod_{j=1}^{3} \frac{1}{b_{0j} + b_{1j}s + s^2} \quad \text{and} \quad H_{A2}(s) = \prod_{j=1}^{3} \frac{1}{a_{0j} + s^2}$$

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2. Find the poles and residues of $H_{A1}(s)$ and $H_{A2}(s)$.

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3. Form

$$H_{D1}(z) = \sum_{i=1}^{N} \frac{A_i z}{z - e^{Tp_i}} \quad \text{and} \quad H_{D2}(z) = \sum_{i=1}^{M} \frac{B_i z}{z - e^{Tz_i}}$$

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4. Form
$$H_D(z) = H_0 \frac{H_{D1}(z)}{H_{D2}(z)} = H_0 \frac{N_1(z)D_2(z)}{N_2(z)D_1(z)}$$

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$$H_D(z) = H_0 \frac{H_{D1}(z)}{H_{D2}(z)} = H_0 \frac{N_1(z)D_2(z)}{N_2(z)D_1(z)}$$

5. Replace poles $p_1, p_2, \ldots, p_K$ of $H_D(z)$ (zeros of $N_2(z)$) located outside the unit circle by their reciprocals and multiplier constant H_0 by $H_0 / \prod_{i=1}^{K} p_i$ in order to stabilize the transfer function.

Example *Cont'd*

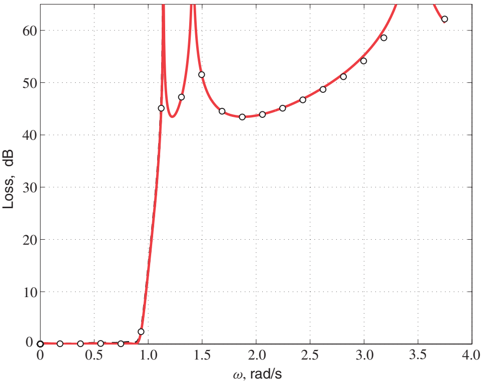
The design procedure gives a transfer function of the form

$$H_D(z) = H_0 \prod_{j=1}^{5} \frac{a_{0j} + a_{1j}z + z^2}{b_{0j} + b_{1j}z + z^2}$$

where H_0, a_{ij}, and b_{ij} are given in the table shown.

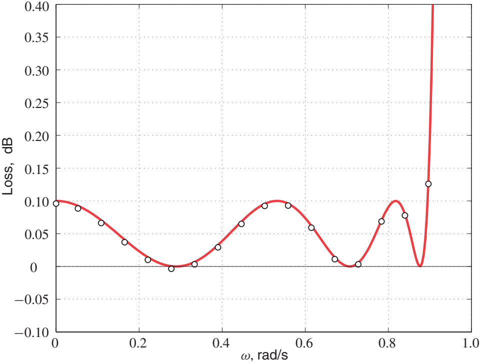
j	a_{0j}	a_{1j}	b_{0j}	b_{1j}
1	1.0	1.942528	4.508735E-1	$-$1.281134
2	1.0	$-$7.530225E-1	6.903732E-1	$-$1.303838
3	1.0	$-$1.153491	9.128252E-1	$-$1.362371
4	3.248990E+1	1.955491E+1	5.611278E-2	7.751650E-1
5	1.331746E-2	3.971465E-1	5.611278E-2	7.751650E-1
$H_0 = 3.847141E\text{-}4$				

- Loss characteristic with respect to the baseband: ——— Analog filter; ○ ○ ○ modified impulse-invariant response method.



(*a*)

- Loss characteristic with respect to the passband:
  ——— Analog filter; ∘ ∘ ∘ modified impulse-invariant response method.



(b)

- It was noted early in the history of digital-filter design that the invariant impulse-response method yields a discrete-time transfer function whose poles, $\bar{p}_i$, bear a one-to-one relation to the poles of the continuous-time transfer function, p_i, of the form

$$\bar{p}_i = e^{p_i T}$$

where T is the sampling period.

Matched-Z-Transformation Method

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$$\bar{p}_i = e^{p_i T}$$

where T is the sampling period.

- It did not take too long for someone to explore calculating the zeros of the discrete-time transfer function, $\bar{z}_i$, from the zeros of the continuous-time transfer function, z_i, using the same relation, i.e.,

$$\bar{z}_i = e^{z_i T}$$

- The technique seemed to work well for some types of analog filters but not in others, but it was soon discovered that improved results could be obtained by adding a number of zeros at the Nyquist point.

- The technique seemed to work well for some types of analog filters but not in others, but it was soon discovered that improved results could be obtained by adding a number of zeros at the Nyquist point.
- Further heuristic effort identified the number of Nyquist-point zeros needed for the various types of analog filters and the matched-z transformation method was formulated as detailed next.

- Given a continuous-time transfer function of the form

$$H_A(s) = H_0 \frac{\prod_{i=1}^{M}(s - z_i)}{\prod_{i=1}^{N}(s - p_i)}$$

a discrete-time transfer function can be obtained as

$$H_D(z) = H_0 (z+1)^L \frac{\prod_{i=1}^{M}(z - e^{z_i T})}{\prod_{i=1}^{N}(z - e^{p_i T})}$$

where L is an integer.

Matched-Z-Transformation Method *Cont'd*

- The value of L depends on the type of filter and it is given by the table shown.

Type of Filter	LP	HP	BP	BS
Butterworth	N	0	$N/2$	0
Chebyshev	N	0	$N/2$	0
Inverse-Chebyshev, N odd	1	0	n/a	n/a
N even	0	0	1 for odd $N/2$ 0 for even $N/2$	0
Elliptic, N odd	1	0	n/a	n/a
N even	0	0	1 for odd $N/2$ 0 for even $N/2$	0

- If we now compare the discrete-time transfer function given by the modified invariant impulse-response method, i.e.,

$$H_D(z) = H_0 \frac{N_1(z)}{N_2(z)} \cdot \frac{\prod_{i=1}^{M}(z - e^{z_i T})}{\prod_{i=1}^{N}(z - e^{p_i T})}$$

with that obtained by using the matched-z transformation method, i.e.,

$$H_D(z) = H_0(z+1)^L \cdot \frac{\prod_{i=1}^{M}(z - e^{z_i T})}{\prod_{i=1}^{N}(z - e^{p_i T})}$$

we note that the only difference is that the ratio of polynomial $N_1(z)/N_2(z)$ is replaced by the polynomial $(z+1)^L$.

- For the classical types of filters (elliptic and inverse-Chebyshev filters), it turns out that $N1(z)$ and $N_2(z)$ are mirror image polynomials with zeros on the negative real axis of the z plane clustered near the Nyquist point and, consequently,

$$\frac{N_1(z)}{N_2(z)} \approx (z+1)^L$$

- For the classical types of filters (elliptic and inverse-Chebyshev filters), it turns out that $N1(z)$ and $N_2(z)$ are mirror image polynomials with zeros on the negative real axis of the z plane clustered near the Nyquist point and, consequently,

$$\frac{N_1(z)}{N_2(z)} \approx (z+1)^L$$

- In effect, at least, for classical filters, the discrete-time transfer function obtained with the matched-z method is an *approximation* of that obtained with the modified invariant impulse-response method.

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- *Works moderately well* not only *for lowpass and bandpass filters* but also for *highpass and bandstop filters* including *elliptic filters*, i.e., no aliasing problems.
- The absence of $N_2(z)$ *eliminates the stability problem* associated with the modified invariant impulse-response method.

Disadvantages

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- A *high sampling frequency is usually necessary* to achieve good results which can introduce certain problems.

  A high sampling frequency corresponds to a reduced sampling period and, therefore, the amount of processing that can be done between samples is reduced.
- The *multiplier constant needs to be adjusted* at the end of the design (see Chap. 12) for details).

Example

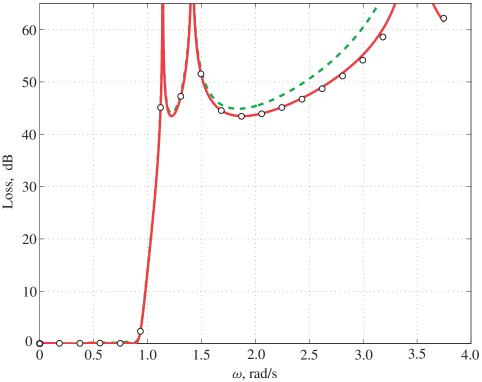
- The matched-z transformation method was used to redesign the elliptic filter considered earlier and the design obtained is as follows:

$$H_D(z) = H_0 \prod_{j=1}^{5} \frac{a_{0j} + a_{1j}z + z^2}{b_{0j} + b_{1j}z + z^2}$$

where H_0, a_{ij}, and b_{ij} are given in the table shown.

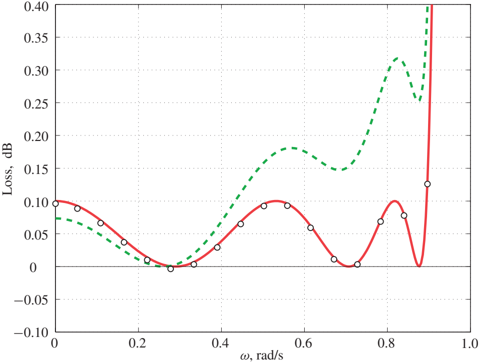
j	a_{0j}	a_{1j}	b_{0j}	b_{1j}
1	1.0	-1.153491	9.128252E-1	-1.362371
2	3.248990E+1	1.955491E+1	5.611278E-2	7.751650E-1
3	1.331746E-2	3.971465E-1	5.611278E-2	7.751650E-1
$H_0 = 3.847141E\text{-}4$				

- Loss characteristics with respect to the baseband:
  ——— Analog filter; ○ ○ ○ modified impulse-invariant response method; - - - - - matched-z transformation method.



(a)

- Loss characteristics with respect to the passband:
  ——— Analog filter; ∘ ∘ ∘ modified impulse-invariant response method; - - - - - matched-z transformation method.



(*b*)

This slide concludes the presentation.
Thank you for your attention.