Chapter 12
DESIGN OF RECURSIVE FILTERS
12.6 Bilinear-Transformation Method
12.7 Digital-Filter Transformations

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Introduction

- In the invariant impulse-response approximation method and two other methods mentioned in Probs. 12.9 and 12.11, namely,
  - the invariant unit-step-response method, and
  - the invariant sinusoid-response method,

  the derived digital filter has *exactly* the same impulse, unit-step, or sinusoid response, as appropriate, as the original analog filter for $t = nT$.

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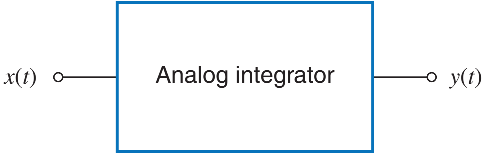
  the derived digital filter has *exactly* the same impulse, unit-step, or sinusoid response, as appropriate, as the original analog filter for $t = nT$.

- An approximation method will now be described whereby a digital filter is derived that has *approximately* the same time-domain response as the original analog filter *for any excitation*.

- Consider an *analog integrator* characterized by the transfer function

$$H_{AI}(s) = \frac{1}{s}$$

and assume that its response to an excitation $x(t)$ is $y(t)$ as shown in the figure.



Derivation *Cont'd*

- The impulse response of the integrator is given by

$$\mathcal{L}^{-1} H_I(s) = h_I(t) = \begin{cases} 1 & \text{for } t \geq 0+ \\ 0 & \text{for } t \leq 0- \end{cases}$$

  and its response at instant t to an arbitrary right-sided excitation $x(t)$, i.e., $x(t) = 0$ for $t < 0$, is given by the convolution integral

$$y(t) = \int_0^t x(\tau) h_I(t - \tau)\, d\tau$$

  (See Theorem 3.14 in textbook.)

Derivation *Cont'd*

...

$$y(t) = \int_0^t x(\tau) h_I(t-\tau)\, d\tau$$

- If $t_2 > t_1 > 0+$, we can write

$$y(t_2) - y(t_1) = \int_0^{t_2} x(\tau) h_I(t_2-\tau)\, d\tau - \int_0^{t_1} x(\tau) h_I(t_1-\tau)\, d\tau \qquad \text{(A)}$$

For $t_1, t_2 \geq \tau > 0+$

$$h_I(t_2-\tau) = h_I(t_1-\tau) = 1$$

and thus Eq. (A) simplifies to

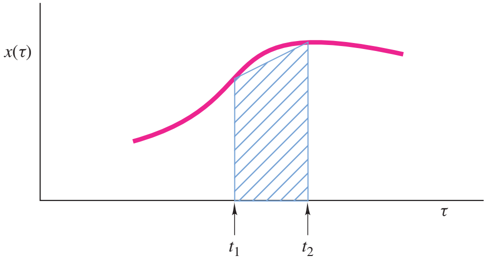
$$y(t_2) - y(t_1) = \int_{t_1}^{t_2} x(\tau)\, d\tau$$

Derivation *Cont'd*

$$y(t_2) - y(t_1) = \int_{t_1}^{t_2} x(\tau)\, d\tau$$

- As $t_1 \rightarrow t_2$, we note from the figure shown that

$$y(t_2) - y(t_1) \approx \frac{t_2 - t_1}{2}[x(t_1) + x(t_2)]$$



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$$y(t_2) - y(t_1) \approx \frac{t_2 - t_1}{2}[x(t_1) + x(t_2)]$$

- If we let $t_1 = nT - T$ and $t_2 = nT$, then the difference equation

$$y(nT) - y(nT - T) = \frac{T}{2}[x(nT - T) + x(nT)]$$

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- The above equation represents a *digital integrator* that has *approximately* the same time-domain response as the analog integrator *for any* excitation.

Derivation *Cont'd*

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- Hence the transfer function of the digital integrator can be derived as

$$H_{DI}(z) = \frac{Y(z)}{X(z)} = \frac{T}{2}\left(\frac{z+1}{z-1}\right)$$

Derivation *Cont'd*

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- The above equation can be expressed as

$$H_{DI}(z) = H_{AI}(s)\Bigg|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)}$$

where

$$H_{AI}(s) = \frac{1}{s}$$

is the transfer function of the analog integrator.

Derivation *Cont'd*

...

$$H_{DI}(z) = H_{AI}(s)\Bigg|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)}$$

- If we now apply the bilinear transformation to the transfer function of an arbitrary analog filter the discrete-time transfer function

$$H_D(z) = H_A(s)\Bigg|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)}$$

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will be obtained.

- The digital filter represented by $H_D(z)$ will produce *approximately* the same time-domain response as the analog filter from which it was derived for *any* excitation.

  Furthermore, the time-domain response of the digital filter would tend to approach that of the analog filter as $T \to 0$.

Mapping Properties

- The relation between the frequency response of the derived digital filter and that of the original analog filter can be established by examining the mapping properties of the bilinear transformation

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- With $s = \sigma + j\omega$, we obtain $z = re^{j\theta}$ where

$$r = \left[\frac{\left(\frac{2}{T}+\sigma\right)^2+\omega^2}{\left(\frac{2}{T}-\sigma\right)^2+\omega^2}\right]^{\frac{1}{2}} \quad \text{and} \quad \theta = \tan^{-1}\frac{\omega}{\frac{2}{T}+\sigma} + \tan^{-1}\frac{\omega}{\frac{2}{T}-\sigma}$$

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- Evidently,

$$\begin{array}{llll} \text{if} & \sigma > 0 & \text{then} & r > 1 \\ \text{if} & \sigma = 0 & \text{then} & r = 1 \\ \text{if} & \sigma < 0 & \text{then} & r < 1 \end{array}$$

  Therefore, the bilinear transformation maps

Mapping Properties *Cont'd*

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Mapping Properties *Cont'd*

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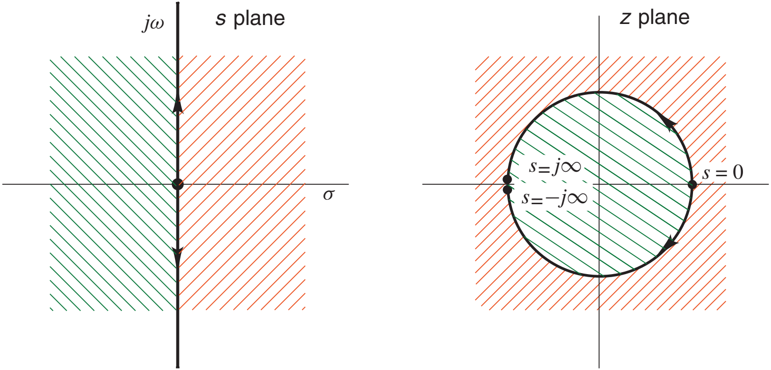
  - the $j\omega$ axis of the s plane onto the unit circle $|z| = 1$ of the z plane;
  - the open left-half s plane onto the region inside the unit circle $|z| = 1$ of the z plane.

Mapping Properties *Cont'd*



(a)

Mapping Properties *Cont'd*

. . .

$$\theta = \tan^{-1} \frac{\omega}{\frac{2}{T} + \sigma} + \tan^{-1} \frac{\omega}{\frac{2}{T} - \sigma}$$

- For the case where $\sigma = 0$, the $j\omega$ axis maps onto the unit circle $|z| = 1$ as was shown. For $\sigma = 0$, we have

$$\theta = 2 \tan^{-1} \frac{\omega T}{2}$$

Mapping Properties *Cont'd*

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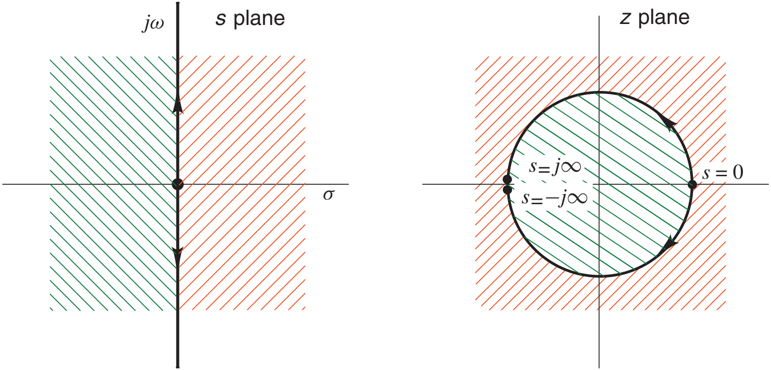
- Therefore, the origin of the s plane maps onto point $[1,\ 0]$ of the z plane, and the positive and negative $j\omega$ axes map onto the top and bottom semicircles $|z| = 1$, respectively.

Mapping Properties *Cont'd*



(*a*)

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- Similarly, the frequency response of a digital filter is the discrete-time transfer function evaluated on the unit circle $|z| = 1$ of the z plane.
- Since the $j\omega$ axis of the s plane maps onto the unit circle of the z plane, it follows that a given frequency in the analog filter ω must correspond to some frequency Ω in the baseband of the digital filter and vice-versa.

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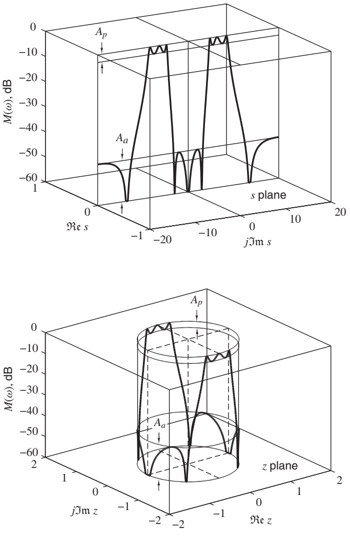
- Therefore, passbands and stopbands in the analog filter translate into passbands and stopbands in the digital filter, respectively.


0
−10
−20
−30
−40
−50
−60
$M(\omega)$, dB
A_p
A_a
1
0
−1
$\Re s$
−20
−10
0
10
20
$j\Im s$
s plane
0
−10
−20
−30
−40
−50
−60
$M(\omega)$, dB
A_p
A_a
2
1
0
−1
−2
$j\Im z$
−2
−1
0
1
2
$\Re z$
z plane

(*b*)

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- Since the numerator degree in $H_A(s)$ is equal to or less than the denominator degree, the numerator degree in $H_D(z)$ cannot be greater than the denominator degree.

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- Since the numerator degree in $H_A(s)$ is equal to or less than the denominator degree, the numerator degree in $H_D(z)$ cannot be greater than the denominator degree.

  This follows from the fact that the numerator degree in the bilinear transformation is equal to the denominator degree.

- The application of the bilinear transformation to a continuous-time transfer function $H_A(s)$ would give a discrete-time transfer function

$$H_D(z) = H_A(s)\Bigg|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)}$$

The Warping Effect

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$$H_D(z) = H_A(s) \Big|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)}$$

- Let ω and Ω be the frequency variables in the analog filter and the derived digital filter, respectively.

  From the above equation, we obtain

$$H_D(e^{j\Omega T}) = H_A(j\omega) \;\; \text{provided that} \;\; \omega = \frac{2}{T}\tan\frac{\Omega T}{2}$$

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$$\omega \approx \Omega$$

  and, as a result, the digital filter has the same frequency response as the analog filter.

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The Warping Effect *Cont'd*

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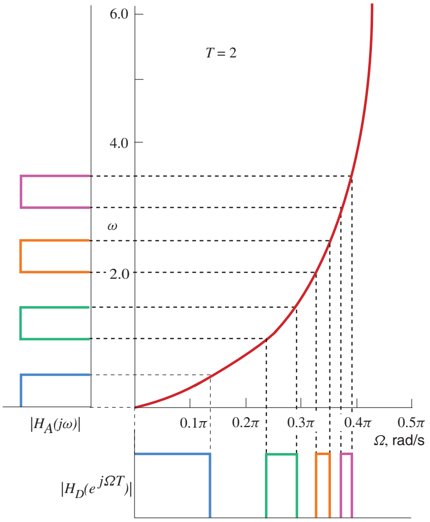
  and, as a result, the digital filter has the same frequency response as the analog filter.

- For higher frequencies, however, the relation between ω and Ω becomes nonlinear and *distortion* is introduced in the frequency scale of the digital filter relative to that of the analog filter.

- This is known as the *warping effect*.

- The influence of the warping effect on the amplitude response can be demonstrated by considering an analog filter with a number of uniformly spaced passbands centered at regular intervals as shown in the next slide.

The Warping Effect *Cont'd*



The Warping Effect *Cont'd*

- The derived digital filter has the same number of passbands, but the center frequencies and bandwidths of higher-frequency passbands tend to be reduced disproportionately.

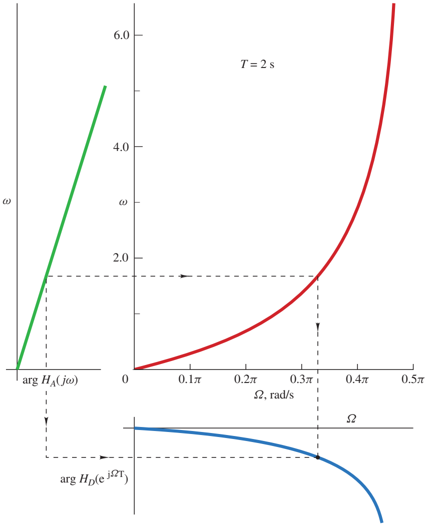
The Warping Effect *Cont'd*

- The effect of the bilinear transformation on the phase response can be examined by considering an analog filter with linear phase response.

The Warping Effect *Cont'd*

- The effect of the bilinear transformation on the phase response can be examined by considering an analog filter with linear phase response.
- Since the relation between the frequencies in the analog and digital filter is highly nonlinear, the digital filter obtained will have a nonlinear phase response as shown in the next slide.

The Warping Effect *Cont'd*



Example

The transfer function

$$H_A(s) = \prod_{j=1}^{3} \frac{a_{0j} + s^2}{b_{0j} + b_{1j}s + s^2}$$

where a_{0j} and b_{ij} are given in the table shown is an elliptic bandstop filter with a passband ripple of 1 dB and a minimum stopband loss of 34.45 dB.

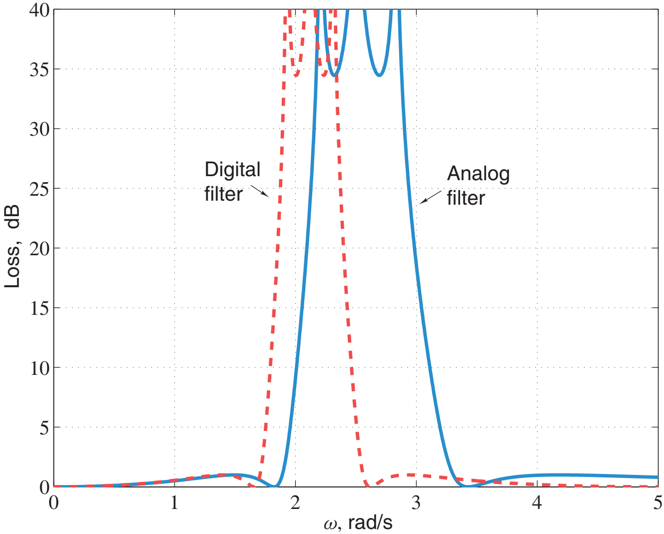
j	a_{0j}	b_{0j}	b_{1j}
1	6.250000	6.250000	2.618910
2	8.013554	1.076433E+1	3.843113E-1
3	4.874554	3.628885	2.231394E-1

Example *Cont'd*

Using the bilinear transformation method, obtain a corresponding discrete-time transfer function.

Assume a sampling frequency of 10 rad/s.

Example *Cont'd*



Prewarping

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- Let ω and Ω be the frequency variables in the analog and digital filter, respectively.

  Since

$$\omega = \frac{2}{T} \tan \frac{\Omega T}{2}$$

  we have

$$\Omega = \frac{2}{T} \tan^{-1} \frac{\omega T}{2}$$

Prewarping *Cont'd*

...

$$\omega = \frac{2}{T}\tan\frac{\Omega T}{2}, \quad \Omega = \frac{2}{T}\tan^{-1}\frac{\omega T}{2}$$

- If $\omega_1, \omega_2, \ldots, \omega_i, \ldots$ are the passband and stopband edges in the analog filter, then the corresponding passband and stopband edges in the digital filter are given by

$$\Omega_i = \frac{2}{T}\tan^{-1}\frac{\omega_i T}{2} \quad \text{for} \quad i = 1, 2, \ldots$$

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- Consequently, if prescribed passband and stopband edges $\tilde{\Omega}_1$, $\tilde{\Omega}_2, \ldots, \tilde{\Omega}_i, \ldots$ are to be achieved in the digital filter, the analog filter must be prewarped before application of the bilinear transformation to ensure that

$$\omega_i = \frac{2}{T}\tan\frac{\tilde{\Omega}_i T}{2}$$

Prewarping *Cont'd*

. . .

$$\omega_i = \frac{2}{T} \tan \frac{\tilde{\Omega}_i T}{2}, \quad \Omega_i = \frac{2}{T} \tan^{-1} \frac{\omega_i T}{2}$$

- Under these circumstances

$$\Omega_i = \frac{2}{T} \tan^{-1} \frac{\omega_i T}{2} = \frac{2}{T} \tan^{-1} \left[\frac{T}{2} \cdot \frac{2}{T} \tan \frac{\tilde{\Omega}_i T}{2} \right] = \tilde{\Omega}_i$$

as required.

The Warping Effect *Cont'd*

- The bilinear transformation method together with a prewarping technique can be used to design Butterworth, Chebyshev, inverse-Chebyshev, and elliptic filters that would satisfy *arbitrary prescribed* specifications as shown in Chap. 13.

Digital-Filter Transformations

- Given a lowpass digital filter, a corresponding highpass, bandpass, or bandstop filter can be deduced by using a family of digital-filter transformations known as the *Constantinides* transformations.

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- These transformations have certain other applications as well, for example, they can be used to construct filters with variable cutoff frequencies. (See textbook for details.)

Digital-Filter Transformations *Cont'd*

Type	Transformation	α, k
LP to LP	$z = \frac{\bar{z}-\alpha}{1-\alpha\bar{z}}$	$\alpha = \frac{\sin[(\Omega_p-\omega_p)T/2]}{\sin[(\Omega_p+\omega_p)T/2]}$
LP to HP	$z = -\frac{\bar{z}-\alpha}{1-\alpha\bar{z}}$	$\alpha = \frac{\cos[(\Omega_p-\omega_p)T/2]}{\cos[(\Omega_p+\omega_p)T/2]}$
LP to BP	$z = -\frac{\bar{z}^2-\frac{2\alpha k}{k+1}\bar{z}+\frac{k-1}{k+1}}{1-\frac{2\alpha k}{k+1}\bar{z}+\frac{k-1}{k+1}\bar{z}^2}$	$\alpha = \frac{\cos[(\omega_{p2}+\omega_{p1})T/2]}{\cos[(\omega_{p2}-\omega_{p1})T/2]}$ $k = \tan\frac{\Omega_p T}{2}\cot\frac{(\omega_{p2}-\omega_{p1})T}{2}$
LP to BS	$z = \frac{\bar{z}^2-\frac{2\alpha}{1+k}\bar{z}+\frac{1-k}{1+k}}{1-\frac{2\alpha}{1+k}\bar{z}+\frac{1-k}{1+k}\bar{z}^2}$	$\alpha = \frac{\cos[(\omega_{p2}+\omega_{p1})T/2]}{\cos[(\omega_{p2}-\omega_{p1})T/2]}$ $k = \tan\frac{\Omega_p T}{2}\tan\frac{(\omega_{p2}-\omega_{p1})T}{2}$

This slide concludes the presentation.
Thank you for your attention.