SAMPLE SOLUTIONS

DIGITAL FILTERS: ***Analysis, Design, and Signal Processing Applications***

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SA.1 A periodic signal can be represented by the equation

$$\tilde{x}(t) = \sum_{k=1}^{10} A_k \sin(\omega_k t + \phi_k)$$

where the frequencies ω_k, amplitudes A_k, and phase angles ϕ_k are given in Table SA.1.

The signal is to be processed first by an ideal bandpass filter and then by a differentiator, as shown in Fig. SA.1. The bandpass filter will pass frequencies in the range $4 \leq \omega \leq 6$ rad/s and reject all other frequencies and the differentiator will differentiate the signal with respect to time.

(a) Obtain a time-domain representation for the signal at the outputs of the bandpass filter and differentiator, i.e., at nodes B and C, respectively, in Fig. SA.1.

(b) Obtain a frequency-domain representation for the signal at the output of the bandpass filter.

(c) Obtain a frequency-domain representation for the signal at the output of the differentiator.

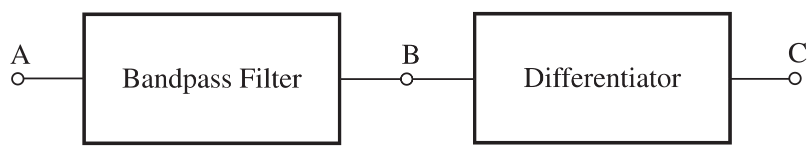

Figure SA.1

Table SA.1 Frequency Spectrum

k	ω_k, rad/s	A_k	ϕ_k
1	1	0.3819	−0.3478
2	2	0.3614	0.8222
3	3	0.8575	2.3502
4	4	0.0629	−0.3292
5	5	0.1342	−0.1693
6	6	0.8648	0.6648
7	7	0.5155	−2.4473
8	8	0.6797	1.7780
9	9	0.7001	−1.5824
10	10	0.3	1.1

Solution

(*a*) Node B:

$$x_B(t) = \sum_{k=4}^{6} A_k \sin(\omega_k t + \phi_k) \quad \blacksquare$$

Node C:

$$x_C(t) = \frac{d}{dt}\left[\sum_{k=4}^{6} A_k \sin(\omega_k t + \phi_k)\right] = \sum_{k=4}^{6} \frac{d}{dt}\left[A_k \sin(\omega_k t + \phi_k)\right]$$

$$= \sum_{k=4}^{6} \omega_k A_k \cos(\omega_k t + \phi_k) = \sum_{k=4}^{6} \omega_k A_k \sin(\omega_k t + \phi_k + \pi/2) \quad \blacksquare$$

(*b*) The amplitude and phase spectrums at Node B are given in Table SA.2 and are plotted in Fig. SA.2. ■

Table SA.2 Frequency Spectrum at Node B

k	ω_k, rad/s	A_k	ϕ_k
4	4	0.0629	−0.3292
5	5	0.1342	−0.1693
6	6	0.8648	0.6648

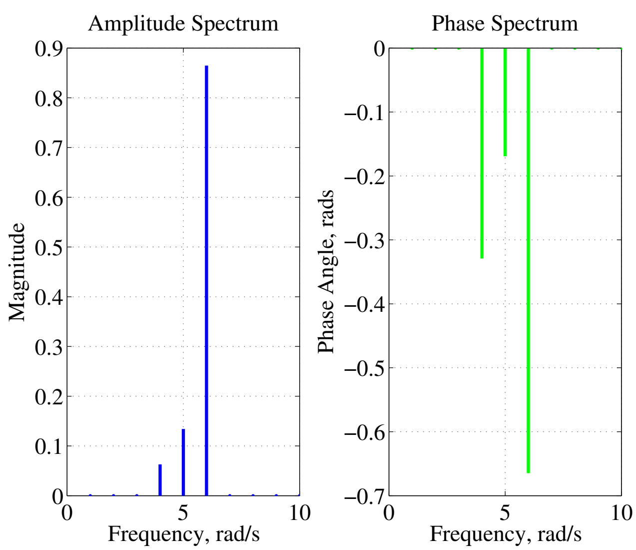

Figure SA.2

(*c*) Similarly, the amplitude and phase spectrums for Node C are given in Table SA.3 and are plotted in Fig. SA.3. ■

Table SA.3 Frequency Spectrum at Node C

k	ω_k, rad/s	A_k	ϕ_k
4	4	0.2516	1.2416
5	5	0.6710	1.4015
6	6	5.1888	2.2356

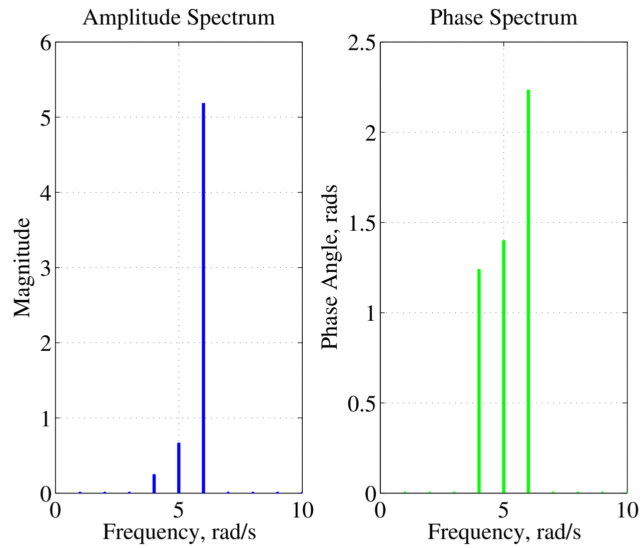

Figure SA.3

SA.2 A periodic signal defined by

$$\tilde{x}(t) = \sum_{r=-\infty}^{\infty} x(t + r\tau_0)$$

where $x(t)$ is zero outside the range $-\tau_0/2 \leq t \leq \tau_0/2$ has a Fourier series of the form

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t} \qquad \text{for } -\tau_0/2 \leq t \leq \tau_0/2$$

where

$$X_k = \frac{1}{\tau_0} \int_{-\tau_0/2}^{\tau_0/2} x(t) e^{-jk\omega_0 t}\, dt$$

(a) Assuming that $x(t)$ is an even function, show that

$$X_k = \frac{2}{\tau_0} \int_{0}^{\tau_0/2} x(t) \cos k\omega_0 t\, dt$$

Justify your steps.

(b) The periodic signal of Fig. SA.4 is described by the equation

$$x(t) = \begin{cases} \cos \omega_0 t/2 & \text{for } -\tau_0/4 \leq t \leq \tau_0/4 \\ 0 & \text{otherwise} \end{cases}$$

where $\omega_0 = 2\pi/\tau_0$. Using the formula in part (a), obtain an expression for the Fourier series coefficients X_k.

(c) Give expressions for the amplitude and phase spectrums of $\tilde{x}(t)$.

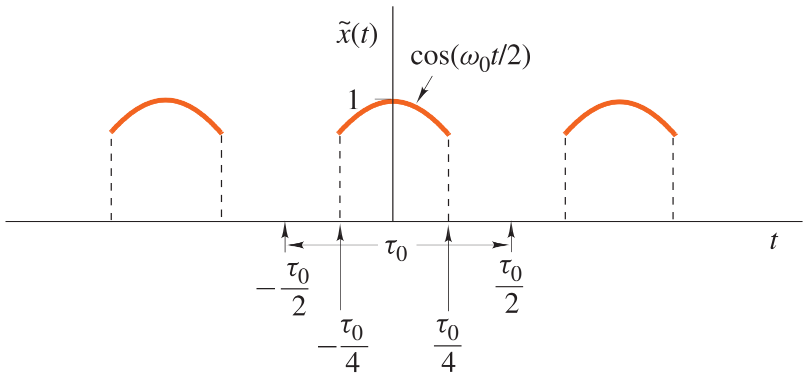

Figure SA.4

Solution

(a) From the definition of the Fourier series

$$
\begin{aligned}
X_k &= \frac{1}{\tau_0}\int_{-\tau_0/2}^{\tau_0/2} x(t)e^{-jk\omega_0 t}\,dt \\
&= \frac{1}{\tau_0}\int_{-\tau_0/2}^{\tau_0/2} x(t)(\cos k\omega_0 t - j\sin k\omega_0 t)\,dt \\
&= \frac{1}{\tau_0}\int_{-\tau_0/2}^{\tau_0/2} x(t)\cos k\omega_0 t\,dt - j\frac{1}{\tau_0}\int_{-\tau_0/2}^{\tau_0/2} x(t)\sin k\omega_0 t\,dt \\
&= \frac{1}{\tau_0}\left[\int_{-\tau_0/2}^{0} x(t)\cos k\omega_0 t\,dt + \int_{0}^{\tau_0/2} x(t)\cos k\omega_0 t\,dt\right] \\
&\quad -\frac{1}{\tau_0}\left[\int_{-\tau_0/2}^{0} x(t)\sin k\omega_0 t\,dt + \int_{0}^{\tau_0/2} x(t)\sin k\omega_0 t\,dt\right]
\end{aligned}
$$

If $x(t)$ is even, then $x(t)\cos k\omega_0 t$ is an even function and $x(t)\sin k\omega_0 t$ is an odd function. Hence

$$\int_{-\tau_0/2}^{0} x(t)\cos k\omega_0 t\,dt = \int_{0}^{\tau_0/2} x(t)\cos k\omega_0 t\,dt$$

and

$$\int_{-\tau_0/2}^{0} x(t)\sin k\omega_0 t\,dt = -\int_{0}^{\tau_0/2} x(t)\sin k\omega_0 t\,dt$$

Therefore

$$X_k = \frac{2}{\tau_0}\int_{0}^{\tau_0/2} x(t)\cos k\omega_0 t\,dt \quad \blacksquare$$

(b) The given signal is symmetrical about the y axis and, therefore, it is an even function. Hence we have

$$
\begin{aligned}
X_k &= \frac{2}{\tau_0}\int_{0}^{\tau_0/2} x(t)\cos k\omega_0 t\,dt \\
&= \frac{2}{\tau_0}\int_{0}^{\tau_0/4} \cos(\omega_0 t/2)\cos(k\omega_0 t)\,dt = \frac{2}{\tau_0}\int_{0}^{\tau_0/4} \cos(k\omega_0 t)\cos(\omega_0 t/2)\,dt
\end{aligned}
$$

From trigonometry, we have

$$\cos A\cos B = \tfrac{1}{2}[\cos(A+B) + \cos(A-B)]$$

and, therefore,

$$\begin{aligned}
X_k &= \frac{2}{\tau_0}\int_0^{\tau_0/4} \tfrac{1}{2}[\cos(k\omega_0 t+\omega_0 t/2)+\cos(k\omega_0 t-\omega_0 t/2)]\,dt \\
&= \frac{1}{\tau_0}\int_0^{\tau_0/4} \cos\left[\left(k+\tfrac{1}{2}\right)\omega_0 t\right]+\cos\left[\left(k-\tfrac{1}{2}\right)\omega_0 t\right]\,dt \\
&= \frac{1}{\tau_0}\left[\frac{\sin\left[\left(k+\frac{1}{2}\right)\omega_0 t\right]}{\left(k+\frac{1}{2}\right)\omega_0}+\frac{\sin\left[\left(k-\frac{1}{2}\right)\omega_0 t\right]}{\left(k-\frac{1}{2}\right)\omega_0}\right]_0^{\tau_0/4} \\
&= \frac{1}{\tau_0}\left[\frac{\sin\left[\left(k+\frac{1}{2}\right)\frac{2\pi}{\tau_0}\cdot\frac{\tau_0}{4}\right]}{\left(k+\frac{1}{2}\right)\frac{2\pi}{\tau_0}}+\frac{\sin\left[\left(k-\frac{1}{2}\right)\frac{2\pi}{\tau_0}\cdot\frac{\tau_0}{4}\right]}{\left(k-\frac{1}{2}\right)\frac{2\pi}{\tau_0}}\right] \\
&= \frac{1}{2\pi}\left[\frac{\sin\left[\left(k+\frac{1}{2}\right)\frac{\pi}{2}\right]}{\left(k+\frac{1}{2}\right)}+\frac{\sin\left[\left(k-\frac{1}{2}\right)\frac{\pi}{2}\right]}{\left(k-\frac{1}{2}\right)}\right]
\end{aligned}$$

(*c*) The amplitude and phase spectrums of $\tilde{x}(t)$ are the magnitude and angle of X_k, i.e., $|X_k|$ and $\arg X_k$. Since X_k is real, the angle of X_k is 0 or π depending on whether X_k is positive or negative. ■

SA.3 A z transform is given by

$$X(z)=\frac{(z^2+1)(z+1)}{(z^2+z-2)(z-3)}$$

(*a*) Construct the zero-pole plot of $X(z)$.

(*b*) Function $X(z)$ is known to have as many Laurent series as there are annuli of convergence but only one of these series is a z transform that satisfies the absolute convergence theorem (Theorem 3.1). Identify the annulus of convergence of that series on the zero-pole plot obtained in part (*a*).

(*c*) Through the use of partial fractions obtain a closed-form expression for $x(nT)=\mathcal{Z}^{-1}X(z)$.

Solution

(*a*) $X(z)$ can be expressed as

$$\begin{aligned}
X(z) &= \frac{(z^2+1)(z+1)}{(z^2+z-2)(z-3)} \\
&= \frac{(z+j)(z-j)(z+1)}{(z-1)(z+2)(z-3)}
\end{aligned}$$

Hence $X(z)$ has zeros at $z=\pm j,\ -1$ and poles at $z=1,\ -2,\ 3$. The zero-pole plot is depicted in Fig. SA.5. ■

(*b*) The correct annulus is the outer annulus which can be represented by

$$3\le|z|<R \quad \text{where} \quad R\to\infty \quad ■$$

See Fig. SA.5.

(*c*) Using Technique I, we can write

$$\begin{aligned}
\frac{X(z)}{z} &= \frac{(z^2+1)(z+1)}{z(z-1)(z+2)(z-3)} \\
&= \frac{R_0}{z}+\frac{R_1}{z-1}+\frac{R_2}{z+2}+\frac{R_3}{z-3}
\end{aligned} \qquad \text{(SA.1)}$$

Since the poles are simple, we have

$$R_0=\lim_{z\to 0}\frac{(z^2+1)(z+1)}{(z-1)(z+2)(z-3)}=\frac{1}{(-1)\times 2\times(-3)}=\frac{1}{6}$$

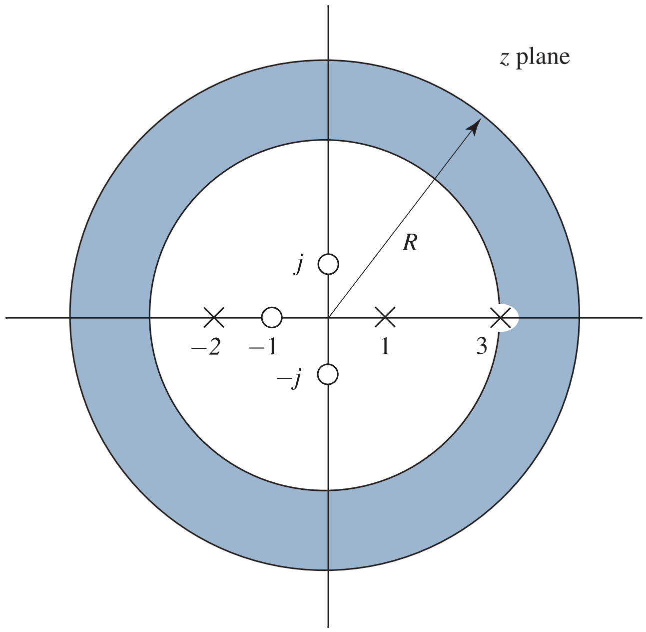

Figure SA.5

$$R_1 = \lim_{z\to 1} \frac{(z^2+1)(z+1)}{z(z+2)(z-3)} = \frac{2\times 2}{1\times 3\times(-2)} = -\frac{2}{3}$$

$$R_2 = \lim_{z\to -2} \frac{(z^2+1)(z+1)}{z(z-1)(z-3)} = \frac{5\times(-1)}{(-2)\times(-3)\times(-5)} = \frac{1}{6}$$

$$R_3 = \lim_{z\to 3} \frac{(z^2+1)(z+1)}{z(z-1)(z+2)} = \frac{10\times 4}{3\times 2\times 5} = \frac{4}{3}$$

From Eq. (SA.1), we can write

$$\begin{aligned} X(z) &= R_0 + \frac{R_1 z}{z-1} + \frac{R_2 z}{z+2} + \frac{R_3 z}{z-3} \\ &= \frac{1}{6} - \frac{\frac{2}{3}z}{z-1} + \frac{\frac{1}{6}z}{z+2} + \frac{\frac{4}{3}z}{z-3} \end{aligned}$$

Therefore, for $n \geq 0$, the use of Table 3.2 gives

$$\begin{aligned} x(nT) &= \tfrac{1}{6}\delta(nT) - \tfrac{2}{3}u(nT) + \tfrac{1}{6}u(nT)(-2)^n + \tfrac{4}{3}u(nT)(3)^n \\ &= \tfrac{1}{6}\delta(nT) + u(nT)[-\tfrac{2}{3} + \tfrac{1}{6}(-2)^n + \tfrac{4}{3}(3)^n] \quad \blacksquare \end{aligned}$$

Since the numerator degree of $X(z)$ is equal to the denominator degree, it follows from the corollary of the initial-value theorem (Theorem 3.8) that $x(nT) = 0$ for $n < 0$, i.e., the above solution applies for all values of n.

An alternative but equivalent solution can be readily obtained by using Technique II (see p. 115) whereby we expand $X(z)$ instead of $X(z)/z$ into partial fractions. We can write

$$\begin{aligned} X(z) &= \frac{(z^2+1)(z+1)}{(z-1)(z+2)(z-3)} \\ &= R_0 + \frac{R_1}{z-1} + \frac{R_2}{z+2} + \frac{R_3}{z-3} \end{aligned}$$

where

$$R_0 = \lim_{z\to\infty} \frac{(z^2+1)(z+1)}{(z-1)(z+2)(z-3)} = \lim_{z\to\infty} \frac{z^3}{z^3} = 1$$

$$R_1 = \lim_{z\to 1} \frac{(z^2+1)(z+1)}{(z+2)(z-3)} = \frac{2\times 2}{3\times(-2)} = -\frac{2}{3}$$

Thus

$$R_2 = \lim_{z\to -2} \frac{(z^2+1)(z+1)}{(z-1)(z-3)} = \frac{5\times(-1)}{(-3)\times(-5)} = -\frac{1}{3}$$

$$R_3 = \lim_{z\to 3} \frac{(z^2+1)(z+1)}{(z-1)(z+2)} = \frac{10\times 4}{2\times 5} = 4$$

Thus

$$X(z) = 1 - \frac{\frac{2}{3}}{z-1} - \frac{\frac{1}{3}}{z+2} + \frac{4}{z-3}$$

and for $n \geq 0$, we have

$$\begin{aligned} x(nT) &= \delta(nT) - \tfrac{2}{3}u(nT-T) - \tfrac{1}{3}u(nT-T)(-2)^{n-1} + 4u(nT-T)(3)^{n-1} \\ &= \delta(nT) + u(nT-T)[-\tfrac{2}{3} - \tfrac{1}{3}(-2)^{n-1} + 4(3)^{n-1}] \quad \blacksquare \end{aligned}$$

SA.4 (*a*) Find the z transform of the following discrete-time signal

$$x(nT) = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } 0 \leq n \leq 5 \\ 1 + (n-5)T & \text{for } n > 5 \end{cases}$$

(*b*) The z transform of a discrete-time signal $x(nT)$ is given by

$$X(z) = \frac{z(3z^2 - 2z + 1)}{(z^2+1)(z-1)}$$

Using the initial-value theorem (Theorem 3.8), show that $x(nT) = 0$ for $n < 0$. Then find $x(nT)$ for $n \geq 0$ using the general inversion formula

$$x(nT) = \frac{1}{2\pi j} \oint_\Gamma X(z) z^{n-1}\, dz.$$

Solution

(*a*) The signal can be expressed as

$$x(nT) = u(nT) + r(nT - 5T)$$

Hence

$$\begin{aligned} \mathcal{Z}x(nT) &= \mathcal{Z}u(nT) + \mathcal{Z}r(nT-5T) \\ &= \mathcal{Z}u(nT) + z^{-5}\mathcal{Z}r(nT) \\ &= \frac{z}{z-1} - \frac{Tz^{-4}}{(z-1)^2} \end{aligned}$$

(*b*) The first nonzero value of $x(nT)$ occurs at $KT = (N - M)T$ where N is the denominator degree and M in the numerator degree in $X(z)$. Since $N = M = 3$, we have $K = 0$, i.e., the signal starts at $KT = 0$. Hence for $n < 0$, we have

$$x(nT) = 0 \quad \blacksquare$$

For $n \geq 0$, we can write

$$\begin{aligned} x(nT) &= \sum_{i=1}^{M} \underset{z=p_i}{\mathfrak{Res}} \left[X(z)z^{n-1}\right] \\ &= \sum_{i=1}^{M} \underset{z=p_i}{\mathfrak{Res}} \frac{z(3z^2 - 2z + 1)z^{n-1}}{(z^2+1)(z-1)} \\ &= \sum_{i=1}^{M} \underset{z=p_i}{\mathfrak{Res}} \frac{(3z^2 - 2z + 1)z^n}{(z-j)(z+j)(z-1)} \\ &= \frac{R_1}{z-j} + \frac{R_1^*}{z+j} + \frac{R_2}{z-1} \end{aligned} \tag{SA.2}$$

where M is the number of poles and R_1^* is the complex conjugate of R_1 since the pole at $z = j = e^{j\pi/2}$ is the complex conjugate of the pole at $z = -j = e^{-j\pi/2}$. Since the poles are simple, we have

$$\begin{aligned} R_1 &= \lim_{z \to j} \left[(z-j)\frac{(3z^2 - 2z + 1)z^n}{(z-j)(z+j)(z-1)}\right] \\ &= \lim_{z \to j} \frac{(3z^2 - 2z + 1)z^n}{(z+j)(z-1)} = \frac{(-3 - 2j + 1)j^n}{2j(j-1)} \\ &= \frac{-2 - 2j}{-2 - 2j} = j^n = e^{jn\pi/2} \end{aligned}$$

and

$$R_2 = \lim_{z \to 1} \frac{(3z^2 - 2z + 1)z^n}{(z^2+1)} = \frac{2}{2} = 1$$

Therefore, from Eq. (SA.2), we can write

$$\begin{aligned} x(nT) &= u(nT)e^{jn\pi/2} + u(nT)e^{-jn\pi/2} + u(nT) \\ &= u(nT)(2\cos n\pi/2 + 1) \quad \blacksquare \end{aligned}$$

SA.5 An initially relaxed discrete-time system can be represented by the equation

$$y(nT) = \mathcal{R}x(nT) = 2.5x(nT) + |e^{0.1(nT+2T)}|x(nT-T) + x(nT-2T)$$

By using *appropriate tests*, check the system for

(*a*) linearity,

(*b*) time invariance, and

(*c*) causality.

Solution

(*a*) **Linearity**

$$\begin{aligned} \mathcal{R}[\alpha x_1(nT) + \beta x_2(nT)] &= 2.5[\alpha x_1(nT) + \beta x_2(nT)] + |e^{0.1(nT+2T}|[\alpha x_1(nT-T) \\ &\quad +\beta x_2(nT-T)] + [\alpha x_1(nT-2T) + \beta x_2(nT-2T)] \\ &= \alpha[2.5x_1(nT) + |e^{0.1(nT+2T)}|x_1(nT-T) + x_1(nT-2T)] \\ &\quad +\beta[2.5x_2(nT) + |e^{0.1(nT+2T)}|x_2(nT-T) + x_2(nT-2T)] \\ &= \alpha\mathcal{R}x_1(nT) + \beta\mathcal{R}x_2(nT) \end{aligned}$$

Therefore, the system is linear. $\blacksquare$

(*b*) **Time invariance**
The response to a delayed excitation is

$$\mathcal{R}x(nT - kT) = 2.5x(nT - kT) + |e^{0.1(nT+2T)}|x(nT - kT - T) + x(nT - kT - 2T)$$

The delayed response is

$$y(nT - kT) = 2.5x(nT - kT) + |e^{0.1(nT-kT+2T)}|x(nT - kT - T) + x(nT - kT - 2T)$$

For any $k \neq 0$, we have

$$|e^{0.1nT+2T}| \neq |e^{0.1(nT-kT+2T)}|$$

Thus

$$y(nT - kT) \neq \mathcal{R}x(nT - kT)$$

and, therefore, the system is time dependent. ■

(*c*) Let $x_1(nT)$ and $x_2(nT)$ be arbitrary discrete-time signals such that

$$\begin{aligned} x_1(nT) &= x_2(nT) \qquad \text{for } n \leq k \\ x_1(nT) &\neq x_2(nT) \qquad \text{for } n > k \end{aligned}$$

We have

$$\mathcal{R}x_1(nT) = 2.5x_1(nT) + |e^{0.1(nT+2T)}|x_1(nT - T) + x_1(nT - 2T) \qquad \text{(SA.3)}$$

and

$$\mathcal{R}x_2(nT) = 2.5x_2(nT) + |e^{0.1(nT+2T)}|x_2(nT - T) + x_2(nT - 2T) \qquad \text{(SA.4)}$$

Since

$$x_1(nT) = x_2(nT) \qquad \text{for } n \leq k$$

then

$$x_1(nT - T) = x_2(nT - T) \qquad \text{for } n \leq k$$

and

$$x_1(nT - 2T) = x_2(nT - 2T) \qquad \text{for } n \leq k$$

Hence the right-hand side in Eq. (SA.3) is equal to the right-hand side in Eq. (SA.4) for $n \leq k$ and thus

$$\mathcal{R}x_1(nT) = \mathcal{R}x_2nT) \qquad \text{for } n \leq k$$

Therefore, the filter is causal. ■

SA.6 (*a*) Derive a state-space representation for the filter shown in Fig. SA.6.

(*b*) Using the state-space representation obtained in part (*a*), compute the impulse response of the filter at $nT = 5T$.

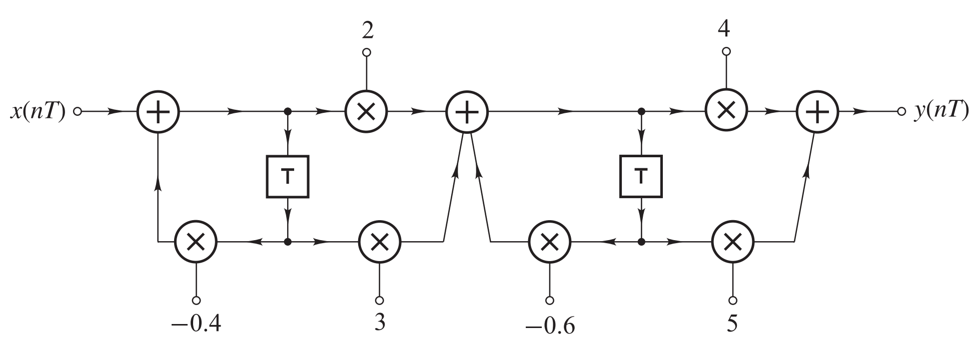

Figure SA.6

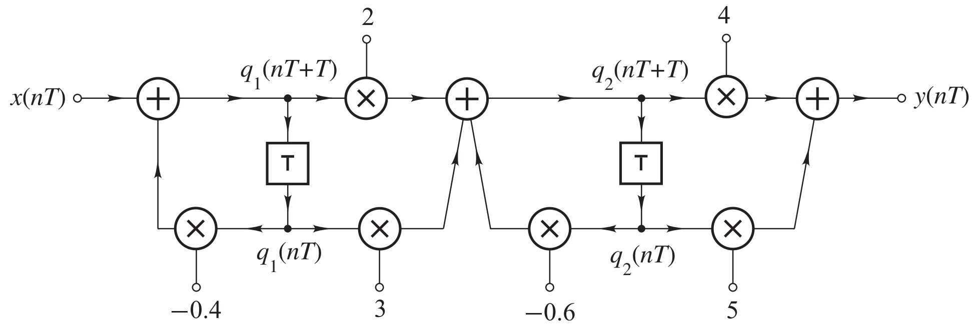

Figure SA.7

Solution

(*a*) State variables can be assigned as shown in Fig. SA.7. Hence we can write

$$q_1(nT+T) = x(nT) - 0.4q_1(nT) \qquad \text{(SA.5)}$$
$$q_2(nT+T) = 2q_1(nT+T) + 3q_1(nT) - 0.6q_2(nT) \qquad \text{(SA.6)}$$

On eliminating $q_1(nT+T)$ in Eq. (SA.6) using Eq. (SA.5), we get

$$\begin{aligned} q_2(nT+T) &= 2x(nT) - 0.8q_1(nT) + 3q_1(nT) - 0.6q_2(nT) \\ &= 2.2q_1(nT) - 0.6q_2(nT) + 2x(nT) \qquad \text{(SA.7)} \end{aligned}$$

From Eqs. (SA.5) and (SA.7)

$$\begin{bmatrix} q_1(nT+T) \\ q_2(nT+T) \end{bmatrix} = \begin{bmatrix} -0.4 & 0 \\ 2.2 & -0.6 \end{bmatrix} \begin{bmatrix} q_1(nT) \\ q_2(nT) \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} x(nT) \qquad \text{(SA.8)}$$

The output is given by

$$y(nT) = 4q_2(nT+T) + 5q_2(nT) \qquad \text{(SA.9)}$$

and from Eqs. (SA.6) and (SA.9), we have

$$\begin{aligned} y(nT) &= 4 \times 2.2q_1(nT) - 4 \times 0.6q_2(nT) + 4 \times 2x(nT) + 5q_2(nT) \\ &= 8.8q_1(nT) + 2.6q_2(nT) + 8x(nT) \\ &= [8.8 \;\; 2.6] \begin{bmatrix} q_1(nT) \\ q_2(nT) \end{bmatrix} + 8x(nT) \qquad \text{(SA.10)} \end{aligned}$$

Therefore, Eqs. (SA.8) and (SA.10) can be written as

$$\begin{aligned} \mathbf{q}(nT+T) &= \mathbf{A}\mathbf{q}(nT) + \mathbf{b}x(nT) \\ y(nT) &= \mathbf{c}^T\mathbf{q}(nT) = dx(nT) \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} -0.4 & 0 \\ 2.2 & -0.6 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad ■$$
$$\mathbf{c}^T = [8.8 \;\; 2.6] \qquad d = 8 \quad ■$$

(*b*) The impulse response is given by

$$h(nT) = \begin{cases} a_1 & \text{for } n = 0 \\ \mathbf{c}^T\mathbf{A}^{n-1}\mathbf{b} & \text{otherwise} \end{cases}$$

For $n = 5$

$$h(5T) = \mathbf{c}^T\mathbf{A}^4\mathbf{b} = [8.8 \;\; 2.6] \begin{bmatrix} -0.4 & 0 \\ 2.2 & -0.6 \end{bmatrix}^4 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Since

$$\begin{bmatrix} -0.4 & 0 \\ 2.2 & -0.6 \end{bmatrix}^2 = \begin{bmatrix} -0.4 & 0 \\ 2.2 & -0.6 \end{bmatrix} \begin{bmatrix} -0.4 & 0 \\ 2.2 & -0.6 \end{bmatrix} = \begin{bmatrix} 0.16 & 0.0 \\ -2.20 & 0.36 \end{bmatrix}$$

$$\begin{bmatrix} -0.4 & 0 \\ 2.2 & -0.6 \end{bmatrix}^4 = \begin{bmatrix} 0.16 & 0.0 \\ -2.20 & 0.36 \end{bmatrix} \begin{bmatrix} 0.16 & 0.0 \\ -2.20 & 0.36 \end{bmatrix} = \begin{bmatrix} 0.0256 & 0.0 \\ -1.144 & 0.1296 \end{bmatrix}$$

we get

$$h(5T) = [8.8 \;\; 2.6] \begin{bmatrix} 0.0256 & 0.0 \\ -1.144 & 0.1296 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = [8.8 \;\; 2.6] \begin{bmatrix} 0.0256 \\ -0.8848 \end{bmatrix} = -2.0752 \quad \blacksquare$$

SA.7 Fig. SA.8 shows a recursive digital filter.

(*a*) Find its transfer function.

(*b*) By using the Jury-Marden stability criterion, determine whether the filter is stable or unstable.

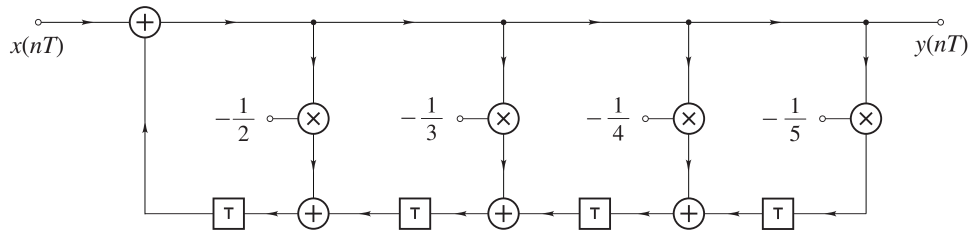

Figure SA.8

Solution

(a) From Fig. SA.8, we get

$$y(nT) = x(nT) - \tfrac{1}{2}y(nT-T) - \tfrac{1}{3}y(nT-2T) - \tfrac{1}{4}y(nT-3T) - \tfrac{1}{5}y(nT-4T)$$

Hence the z transform gives

$$Y(z) = X(z) - \tfrac{1}{2}z^{-1}Y(z) - \tfrac{1}{3}z^{-2}Y(z) - \tfrac{1}{4}z^{-3}Y(z) - \tfrac{1}{5}z^{-4}Y(z)$$

or

$$Y(z) + \tfrac{1}{2}z^{-1}Y(z) + \tfrac{1}{3}z^{-2}Y(z) + \tfrac{1}{4}z^{-3}Y(z) + \tfrac{1}{5}z^{-4}Y(z) = X(z)$$

and so

$$\frac{Y(z)}{X(z)} = \frac{1}{1 + \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2} + \frac{1}{4}z^{-3} + \frac{1}{5}z^{-4}}$$

In effect,

$$H(z) = \frac{N(z)}{D(z)} = \frac{z^4}{z^4 + \frac{1}{2}z^3 + \frac{1}{3}z^2 + \frac{1}{4}z + \frac{1}{5}}$$

where

$$D(z) = z^4 + \tfrac{1}{2}z^3 + \tfrac{1}{3}z^2 + \tfrac{1}{4}z + \tfrac{1}{5}$$

We note that

$$D(1) = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} = 2.283 > 0 \qquad \text{(SA.11a)}$$

$$(-1)^4 D(-1) = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} = 0.783 > 0 \qquad \text{(SA.11b)}$$

(b) The Jury-Marden array can be constructed as shown in Table SA.4.

Table SA.4 The Jury-Marden array

	b_0	b_1	b_2	b_3	b_4
1	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
2	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1
3	$\frac{24}{25}$	$\frac{9}{20}$	$\frac{4}{15}$	$\frac{3}{20}$	
4	$\frac{3}{20}$	$\frac{4}{15}$	$\frac{9}{20}$	$\frac{24}{25}$	
5	0.8991	0.392	0.1885		

From Eqs. (SA.11a) and (SA.11b) and Table SA.4, we have

$$\begin{aligned} D(1) &> 0 \qquad (-1)^4 D(-1) > 0 \\ b_0 &= 1 > \frac{1}{5} = |b_4| \\ |c_0| &= \frac{24}{25} > \frac{3}{20} = |c_3| \\ |d_0| &= 0.8991 > 0.1885 = |d_2| \end{aligned}$$

Therefore, conditions (i) to (iii) of the Jury-Marden stability criterion (see p. 220) are satisfied and the filter is stable. ■

SA.8 The filter of Fig. SA.9 is subjected to an input

$$x(nT) = \begin{cases} 1 & \text{for } n = 0 \text{ and } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the time-domain response in *closed* form if $m_1 = -\frac{3}{4}$ and $m_2 = -\frac{1}{8}$. The filter is linear and time-invariant.

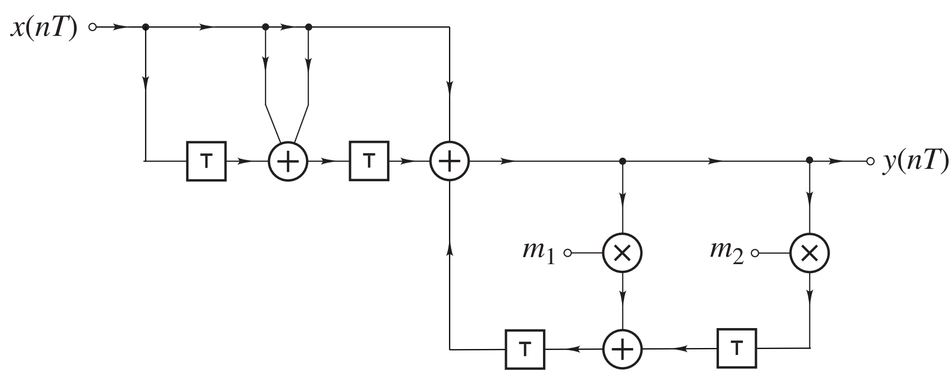

Figure SA.9

Solution

From Fig. SA.9

$$Y(z) = X(z) + 2z^{-1}X(z) + z^{-2}X(z) + m_1 z^{-1}Y(z) + m_2 z^{-2}Y(z)$$

Hence

$$Y(z)(1 - m_1 z^{-1} - m_2 z^{-2}) = (1 + 2z^{-1} + z^{-2})X(z)$$

or

$$Y(z) = \frac{(1 + 2z^{-1} + z^{-2})}{(1 - m_1 z^{-1} - m_2 z^{-2})} X(z)$$

Therefore,

$$\frac{Y(z)}{X(z)} = H(z) = \frac{z^2 + 2z + 1}{z^2 - m_1 z - m_2} = \frac{z^2 + 2z + 1}{z^2 + \frac{3}{4}z + \frac{1}{8}} = \frac{z^2 + 2z + 1}{\left(z + \frac{1}{2}\right)\left(z + \frac{1}{4}\right)}$$

The time-domain response can be obtained in a number of ways, as detailed below.

Method 1:

The input is the sum of two impulse functions, i.e.,

$$x(nT) = \delta(nT) + \delta(nT - T)$$

Hence

$$\mathcal{R}x(nT) = \mathcal{R}[\delta(nT) + \delta(nT - T)]$$

Since the filter is linear, we have

$$\mathcal{R}x(nT) = \mathcal{R}\delta(nT) + \mathcal{R}\delta(nT - T)$$

and if $h(nT)$ is the impulse response, i.e.,

$$h(nT) = \mathcal{R}\delta(nT)$$

then by virtue of the fact that the filter is time-invariant, we get

$$y(nT) = h(nT) + h(nT - T)$$

In effect, all we have to do is find the impulse response. Expanding $H(z)/z$ into partial fractions gives

$$\frac{H(z)}{z} = \frac{z^2 + 2z + 1}{z\left(z + \frac{1}{2}\right)\left(z + \frac{1}{4}\right)} = \frac{R_1}{z} + \frac{R_2}{z + \frac{1}{2}} + \frac{R_2}{z + \frac{1}{4}}$$

where

$$\begin{aligned}
R_1 &= \left.\frac{z^2 + 2z + 1}{\left(z + \frac{1}{2}\right)\left(z + \frac{1}{4}\right)}\right|_{z=0} = \frac{1}{\frac{1}{2} \times \frac{1}{4}} = 8 \\
R_2 &= \left.\frac{z^2 + 2z + 1}{z\left(z + \frac{1}{4}\right)}\right|_{z=-\frac{1}{2}} = \frac{\frac{1}{4} - \frac{2}{2} + 1}{\left(-\frac{1}{2}\right)\left(-\frac{1}{2} + \frac{1}{4}\right)} = \frac{\frac{1}{4}}{\left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)} = 2 \\
R_3 &= \left.\frac{z^2 + 2z + 1}{z\left(z + \frac{1}{2}\right)}\right|_{z=-\frac{1}{4}} = \frac{\frac{1}{16} - \frac{2}{4} + 1}{\left(-\frac{1}{4}\right)\left(-\frac{1}{4} + \frac{1}{2}\right)} = \frac{\frac{1}{16} + \frac{8}{16}}{-\frac{1}{16}} = -9
\end{aligned}$$

i.e.,

$$H(z) = R_1 + \frac{R_2 z}{z + \frac{1}{2}} + \frac{R_3 z}{z + \frac{1}{4}}$$

and

$$\begin{aligned}
h(nT) &= R_1\delta(nT) + \left[R_2\left(-\tfrac{1}{2}\right)^n + R_3\left(-\tfrac{1}{4}\right)^n\right]u(nT) \\
&= 8\delta(nT) + \left[2\left(-\tfrac{1}{2}\right)^n - 9\left(-\tfrac{1}{4}\right)^n\right]u(nT)
\end{aligned}$$

Now

$$y(nT) = h(nT) + h(nT - T)$$

Hence

$$\begin{aligned}
y(nT) &= 8[\delta(nT) + \delta(nT - T)] + \left[2\left(-\tfrac{1}{2}\right)^n - 9\left(-\tfrac{1}{4}\right)^n\right]u(nT) \\
&\quad + \left[2\left(-\tfrac{1}{2}\right)^{n-1} - 9\left(-\tfrac{1}{4}\right)^{n-1}\right]u(nT - T) \quad \blacksquare
\end{aligned}$$

Method 2

Since

$$x(nT) = \delta(nT) + \delta(nT - T)$$

the z transform gives

$$X(z) = 1 + z^{-1} = \frac{z+1}{z}$$

Hence

$$Y(z) = H(z)X(z) = \frac{(z^2 + 2z + 1)}{\left(z + \frac{1}{2}\right)\left(z + \frac{1}{4}\right)} \times \frac{(z+1)}{z}$$

Expanding $H(z)X(z)$ into partial fractions, gives

$$H(z)X(z) = R_1 + \frac{R_2}{z} + \frac{R_3}{z + \frac{1}{2}} + \frac{R_4}{z + \frac{1}{4}}$$

where

$$\begin{aligned}
R_1 &= \lim_{z \to \infty} H(z)X(z) = 1 \\
R_2 &= \left.\frac{(z^2 + 2z + 1)(z+1)}{\left(z + \frac{1}{2}\right)\left(z + \frac{1}{4}\right)}\right|_{z=0} = \frac{1}{\frac{1}{2} \times \frac{1}{4}} = 8 \\
R_3 &= \left.\frac{(z^2 + 2z + 1)(z+1)}{\left(z + \frac{1}{4}\right)z}\right|_{z=-\frac{1}{2}} = \frac{\left(\frac{1}{4} - 1 + 1\right)\left(-\frac{1}{2} + 1\right)}{\left(-\frac{1}{2} + \frac{1}{4}\right)\left(-\frac{1}{2}\right)} = \frac{\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)}{\left(-\frac{1}{4}\right)\left(-\frac{1}{2}\right)} = 1 \\
R_4 &= \left.\frac{(z^2 + 2z + 1)(z+1)}{\left(z + \frac{1}{2}\right)z}\right|_{z=-\frac{1}{4}} = \frac{\left(\frac{1}{16} - \frac{2}{4} + 1\right)\left(-\frac{1}{4} + 1\right)}{\left(-\frac{1}{4} + \frac{1}{2}\right)\left(-\frac{1}{4}\right)} = \frac{\left(\frac{1}{16} + \frac{8}{16}\right)\left(\frac{3}{4}\right)}{\left(\frac{1}{4}\right)\left(-\frac{1}{4}\right)} \\
&= \frac{\frac{9}{16} \times \frac{3}{4}}{-\frac{1}{16}} = -\frac{27}{4}
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(nT) &= R_1\delta(nT) + R_2\delta(nT - T) + R_3 u(nT - T)\left(-\tfrac{1}{2}\right)^{n-1} + R_4 u(nT - T)\left(-\tfrac{1}{4}\right)^{n-1} \\
&= \delta(nT) + 8\delta(nT - T) + u(nT - T)\left(-\tfrac{1}{2}\right)^{n-1} - \tfrac{27}{4} u(nT - T)\left(-\tfrac{1}{4}\right)^{n-1} \quad \blacksquare
\end{aligned}$$

Method 3

The inverse of $Y(z)$ is obtained from first principles as

$$y(nT) = \sum_{\text{res}} Y_0(z)$$

where

$$Y_0(z) = Y(z)z^{n-1} = H(z)X(z)z^{n-1} = \frac{(z^2 + 2z + 1)(z+1)}{z\left(z + \frac{1}{2}\right)\left(z + \frac{1}{4}\right)} z^{n-1}$$

However, watch out for pitfalls at the origin. In this case, we have a second-order pole at the origin if $n = 0$, a first-order pole at the origin if $n = 1$, and no poles at the origin if $n \geq 2$. Hence, we have to find $y(0)$ and $y(T)$ individually and then proceed to $y(nT)$ for $n \geq 2$. This would make this method quite long.

Method 4

We can express $Y(z)$ into partial fractions as

$$Y(z) = \frac{(z^2 + 2z + 1)(z+1)}{z\left(z + \frac{1}{2}\right)\left(z + \frac{1}{4}\right)} = R_1 + \frac{R_2}{z} + \frac{R_3 z}{z + \frac{1}{2}} + \frac{R_4 z}{z + \frac{1}{4}}$$

where

$$\begin{aligned}
R_2 &= \lim_{z=0} zY(z) = \left.\frac{z(z^2+2z+1)(z+1)}{z\left(z+\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}\right|_{z=0} = \frac{1\times 1}{\frac{1}{2}\times\frac{1}{4}} = 8 \\
R_3 &= \lim_{z=-\frac{1}{2}} \frac{z+\frac{1}{2}}{z}\times Y(z) = \left.\frac{(z^2+2z+1)(z+1)}{z^2\left(z+\frac{1}{4}\right)}\right|_{z=-\frac{1}{2}} \\
&= \frac{\left[\left(-\frac{1}{2}\right)^2+2\left(-\frac{1}{2}\right)+1\right]\left(-\frac{1}{2}+1\right)}{\left(-\frac{1}{2}\right)^2\left(-\frac{1}{2}+\frac{1}{4}\right)} = \frac{\left(\frac{1}{4}-1+1\right)\left(\frac{1}{2}\right)}{\left(\frac{1}{4}\right)\left(-\frac{1}{4}\right)} = \frac{\frac{1}{8}}{-\frac{1}{16}} = -2 \\
R_4 &= \lim_{z=-\frac{1}{4}} \frac{z+\frac{1}{4}}{z}\times Y(z) = \left.\frac{(z^2+2z+1)(z+1)}{z^2\left(z+\frac{1}{2}\right)}\right|_{z=-\frac{1}{4}} \\
&= \frac{\left[\left(-\frac{1}{4}\right)^2+2\left(-\frac{1}{4}\right)+1\right]\left(-\frac{1}{4}+1\right)}{\left(-\frac{1}{4}\right)^2\left(-\frac{1}{4}+\frac{1}{2}\right)} = \frac{\left(\frac{1}{16}-\frac{2}{4}+1\right)\left(\frac{3}{4}\right)}{\frac{1}{16}\times\frac{1}{4}} = \frac{\frac{9}{16}\times\frac{3}{4}}{\frac{1}{64}} = 27
\end{aligned}$$

Constant A can be obtained by noting that

$$\lim_{z=\infty} Y(z) = R_1 + R_3 + R_4 = \frac{z^3}{z^3} = 1$$

Thus

$$R_1 = 1 - R_3 - R_4 = 1 - (-2) - 27 = 3 - 27 = -24$$

Hence

$$Y(z) = -24 + \frac{8}{z} - \frac{2z}{z+\frac{1}{2}} + \frac{27z}{z+\frac{1}{4}}$$

Therefore,

$$y(nT) = -24\delta(nT) + 8\delta(nT-T) + \left[27\left(-\tfrac{1}{4}\right)^n - 2\left(-\tfrac{1}{2}\right)^n\right]u(nT) \quad \blacksquare$$

Method 5

One could expand $Y(z)/z$ into partial fractions as

$$\frac{Y(z)}{z} = \frac{R_1}{z} + \frac{R_2}{z^2} + \frac{R_3}{z+\frac{1}{2}} + \frac{R_4}{z+\frac{1}{4}}$$

However, this is essentially the same as method 4.

SA.9 A discrete-time system has a transfer function

$$H(z) = \frac{z^2+2}{z^2-(2r\cos\theta)z+r^2}$$

(*a*) Find the unit-step response in closed form.

(*b*) Using MATLAB, D-Filter, or similar software, plot the unit-step response for $r = 0.3$ and $\theta = \pi/4$.

(*c*) Repeat part (*b*) for $r = 0.6$ and $\theta = \pi/4$.

(*d*) Repeat part (*b*) for $r = 0.9$ and $\theta = \pi/4$.

(*e*) Compare the unit-step responses in parts (*b*) to (*d*).

Solution

(*a*) The z transform of the output of the system is given by

$$Y(z) = H(z)X(z) \tag{SA.12}$$

where $H(z)$ is the transfer function and $X(z)$ is the z transform of the input. Since the input is a unit step, we have

$$X(z) = \mathcal{Z}u(nT) = \frac{z}{z-1} \tag{SA.13}$$

Thus Eqs. (SA.12) and (SA.13) give

$$Y(z) = \frac{z^2+2}{z^2-(2r\cos\theta)z+r^2} \cdot \frac{z}{z-1} = \frac{z^3+2z}{(z-1)[z^2-(2r\cos\theta)z+r^2]}$$

The general inversion formula gives

$$y(nT) = \frac{1}{2\pi j}\oint_\Gamma Y(z)z^{n-1}dz = \sum_{i=1}^{M} \underset{z=p_i}{\mathfrak{R}es}\; Y_0(z) \tag{SA.14}$$

where

$$\begin{aligned} Y_0(z) &= Y(z)z^{n-1} = \frac{(z^3+2z)z^{n-1}}{(z-1)[z^2-(2r\cos\theta)z+r^2]} \\ &= \frac{(z^3+2z)z^{n-1}}{(z-1)[z^2-r(e^{j\theta}+e^{-j\theta})z+r^2]} \\ &= \frac{(z^2+2)z^n}{(z-1)(z-re^{j\theta})(z-re^{-j\theta})} \end{aligned}$$

and $M = 3$. The residues of $Y_0(z)$ can be obtained as follows:

$$\begin{aligned} R_1 &= \lim_{z=1}(z-1)Y_0(z) = \left.\frac{(z^2+2)z^n}{z^2-(2r\cos\theta)z+r^2}\right|_{z=1} \\ &= \frac{3}{1-(2r\cos\theta)+r^2} \end{aligned} \tag{SA.15}$$

$$\begin{aligned} R_2 &= \lim_{z=re^{j\theta}}(z-re^{j\theta})Y_0(z) = \left.\frac{(z^2+2)z^n}{(z-1)(z-re^{-j\theta})}\right|_{z=re^{j\theta}} \\ &= \frac{(r^2e^{j2\theta}+2)r^ne^{jn\theta}}{(re^{j\theta}-1)(re^{j\theta}-re^{-j\theta})} = \frac{(r^2e^{j2\theta}+2)r^{n-1}e^{jn\theta}}{(re^{j\theta}-1)2j\sin\theta} \\ &= \frac{(r^2\cos 2\theta+2+jr^2\sin 2\theta)r^{n-1}e^{jn\theta}}{(r\cos\theta-1+jr\sin\theta)2j\sin\theta} \\ &= \frac{(r^2\cos 2\theta+2+jr^2\sin 2\theta)r^{n-1}e^{jn\theta}}{[(r\cos\theta-1)+jr\sin\theta]2e^{j\pi/2}\sin\theta} \\ &= Me^{j\phi}r^{n-1}e^{jn\theta} = Mr^{n-1}e^{j(n\theta+\phi)} \end{aligned} \tag{SA.16}$$

where

$$\begin{aligned} M &= \left|\frac{r^2\cos 2\theta+2+jr^2\sin 2\theta}{[j(r\cos\theta-1)-r\sin\theta]2\sin\theta}\right| \\ &= \sqrt{\frac{(r^2\cos 2\theta+2)^2+(r^2\sin 2\theta)^2}{4[(r\cos\theta-1)^2+(r\sin\theta)^2]\sin^2\theta}} \end{aligned}$$

and

$$\begin{aligned} \phi &= \tan^{-1}\frac{r^2\sin 2\theta}{r^2\cos 2\theta+2} - \tan^{-1}\frac{(r\cos\theta-1)}{-r\sin\theta} \\ &= \tan^{-1}\frac{r^2\sin 2\theta}{r^2\cos 2\theta+2} - \tan^{-1}\frac{r\sin\theta}{(r\cos\theta-1)} - \frac{\pi}{2} \end{aligned}$$

Similarly,

$$R_3 = R_2^* \tag{SA.17}$$

since complex conjugate poles give complex conjugate residues. We note that there is no additional pole at the origin when $n = 0$ and hence for $n \geq 0$ Eq. (SA.14) gives

$$y(nT) = R_1 + R_2 + R_2^* \tag{SA.18}$$

Since the numerator degree in $Y(z)$ does not exceed the denominator degree, we have

$$y(nT) = 0 \qquad \text{for} \;\; n < 0$$

Therefore, for any n, Eqs. (SA.15)–(SA.18) give

$$\begin{aligned} y(nT) &= u(nT)\left[R_1 + Mr^{n-1}e^{j(n\theta+\phi)} + Mr^{n-1}e^{-j(n\theta+\phi)}\right] \\ &= u(nT)\left[R_1 + 2Mr^{n-1}\cos(n\theta+\phi)\right] \quad \blacksquare \end{aligned}$$

(*b*–*d*) The step responses for the three cases are illustrated in Fig. SA.10*a*–*c*. ■

(*e*) On the basis of the step responses obtained, the system in part (*b*) is what they call *overdamped*, the one in part (*c*) is *critically damped*, and the one in part (*d*) is referred to as *underdamped*. ■

SA.10 A second-order digital filter has zeros $z_1 = e^{j\pi/3}$ and $z_2 = e^{-j\pi/3}$ and poles $p_1 = 0.5e^{j\pi/4}$ and $p_2 = 0.5e^{-j\pi/4}$ and its multiplier constant is 2.

(*a*) Obtain the transfer function of the filter.

(*b*) Obtain an expression for the gain.

(*c*) Assuming that the sampling frequency is 2π, calculate the gain at $\omega = 0,\ \pi/4,\ \pi/3$, and π.

Solution

(*a*) Since we have the zeros, poles, and multiplier constant of the filter, the transfer function can be readily constructed as

$$\begin{aligned} H(z) &= \frac{2(z - e^{j\pi/3})(z - e^{-j\pi/3})}{(z - \frac{1}{2}e^{j\pi/4})(z - \frac{1}{2}e^{-j\pi/4})} = \frac{2[z^2 - (e^{j\pi/3} + e^{-j\pi/3})z + 1]}{z^2 - \frac{1}{2}(e^{j\pi/4} + e^{-j\pi/4}) + \frac{1}{4}} \\ &= \frac{2[z^2 - 2(\cos\pi/3)z + 1]}{z^2 - (\cos\pi/4)z + \frac{1}{4}} = \frac{2[z^2 - z + 1]}{z^2 - \frac{\sqrt{2}}{2}z + \frac{1}{4}} \quad \blacksquare \end{aligned}$$

(*b*) The gain is given by

$$\begin{aligned} M(\omega) &= |H(e^{j\omega T})| = \left|\frac{2[e^{2j\omega T} - e^{j\omega T} + 1]}{e^{j2\omega T} - \frac{\sqrt{2}}{2}e^{j\omega T} + \frac{1}{4}}\right| \\ &= 2\left|\frac{\cos 2\omega T + j\sin 2\omega T - \cos\omega T - j\sin\omega T + 1}{\cos 2\omega T + j\sin 2\omega T - \frac{\sqrt{2}}{2}(\cos\omega T + j\sin\omega T) + \frac{1}{4}}\right| \\ &= 2\sqrt{\frac{(\cos 2\omega T - \cos\omega T + 1)^2 + (\sin 2\omega T - \sin\omega T)^2}{(\cos 2\omega T - \frac{\sqrt{2}}{2}\cos\omega T + \frac{1}{4})^2 + (\sin 2\omega T - \frac{\sqrt{2}}{2}\sin\omega T)^2}} \quad \blacksquare \end{aligned}$$

(*c*) Since $\omega_s = 2\pi$, we have $2\pi f_s = 2\pi/T = 2\pi$. Hence $T = 1$ s. For the frequencies given, the numerical values in Table SA.5 can be readily calculated.

Table SA.5

ω	$\cos\omega T$	$\sin\omega T$	$\cos 2\omega T$	$\sin 2\omega T$
0	1	0	1	0
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	0	1
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
π	-1	0	1	0

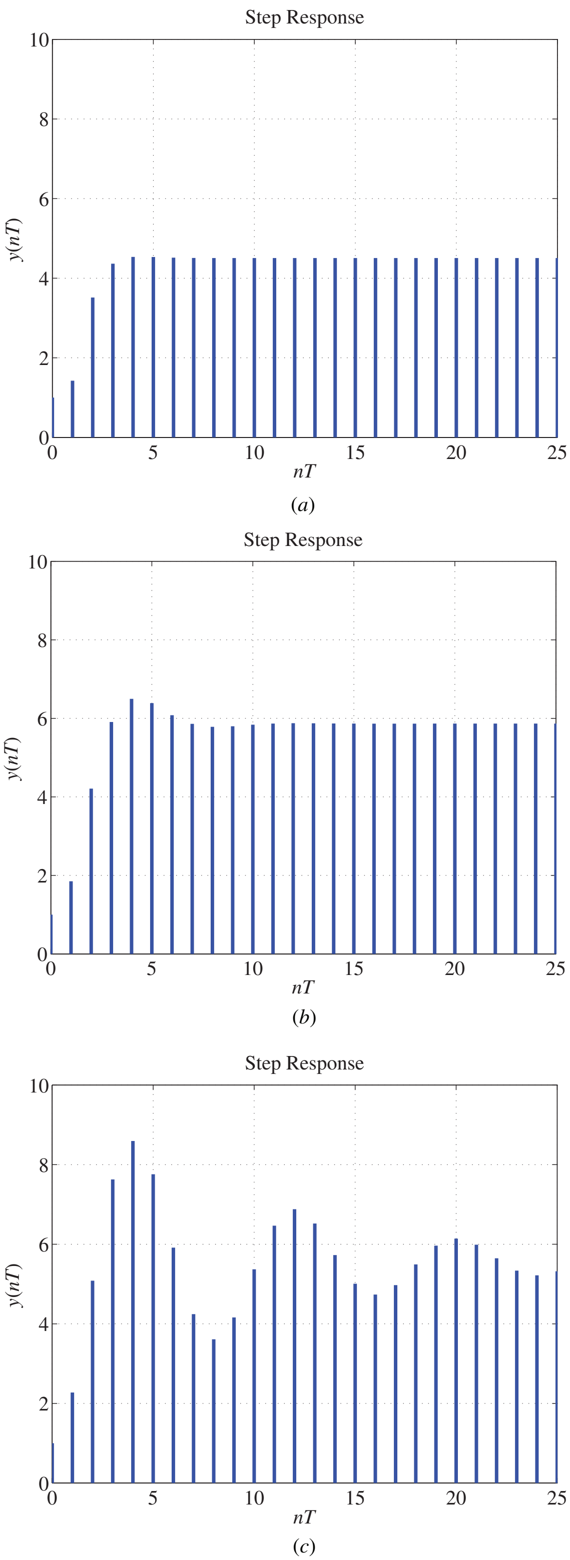
Step Response
y(nT)
nT
(a)
Step Response
y(nT)
nT
(b)
Step Response
y(nT)
nT
(c)

Figure SA.10

Thus

$$M(0) = 2\sqrt{\frac{(1-1+1)^2+(0-0)^2}{(1-\frac{\sqrt{2}}{2}+\frac{1}{4})^2+(0-0)^2}} = \frac{2}{\frac{5-2\sqrt{2}}{4}} = 3.6840 \quad \blacksquare$$

$$\begin{aligned} M\left(\frac{\pi}{4}\right) &= 2\sqrt{\frac{(0-\frac{\sqrt{2}}{2}+1)^2+(1-\frac{\sqrt{2}}{2})^2}{(0-\frac{\sqrt{2}}{2}\times\frac{\sqrt{2}}{2}+\frac{1}{4})^2+(1-\frac{\sqrt{2}}{2}\times\frac{\sqrt{2}}{2})^2}} \\ &= 2\sqrt{\frac{(2-\sqrt{2})^2+(2-\sqrt{2})^2}{(\frac{1}{2}-1)^2+1}} = 2\sqrt{\frac{2(2-\sqrt{2})^2}{(-\frac{1}{2})^2+1}} = 1.489 \quad \blacksquare \end{aligned}$$

$$\begin{aligned} M\left(\frac{\pi}{3}\right) &= 2\sqrt{\frac{(-\frac{1}{2}-\frac{1}{2}+1)^2+(\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2})^2}{(-\frac{1}{2}-\frac{\sqrt{2}}{2}\times\frac{1}{2}+\frac{1}{4})^2+(\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2})}} \\ &= 2\sqrt{\frac{0+0}{\left(\frac{1-2-\sqrt{2}}{4}\right)^2+\cdots}} = 0 \quad \blacksquare \end{aligned}$$

$$\begin{aligned} M(\pi) &= 2\sqrt{\frac{(1+1+1)^2+(0-0)}{(1+\frac{\sqrt{2}}{2}+\frac{1}{4})^2+(0-0)}} = 2\sqrt{\frac{3^2\times 16}{(4+2\sqrt{2}+1)^2}} \\ &= \frac{2\times 3\times 4}{4+2\sqrt{2}+1} = 3.0658 \quad \blacksquare \end{aligned}$$

SA.11 A digital filter characterized by the transfer function

$$H(z) = \frac{2(z^2-z+1)}{z^2-\frac{\sqrt{2}}{2}z+\frac{1}{4}}$$

and a practical D/A converter are connected in cascade as shown in Fig. SA.11*a*. The output waveform of the D/A converter is of the form illustrated in Fig. S11*b* where $\tau = 0.01T$ s and T is the sampling period. The sampling frequency is 2π rad/s.

(*a*) Obtain an expression for the gain of just the digital filter.

(*b*) Obtain an expression for the overall gain of the digital filter in cascade with the D/A converter.

(*c*) Calculate the gain of just the digital filter at $\omega = 0,\ \pi/4,\ \pi/3$, and π.

(*d*) Calculate the overall gain of the digital filter in cascade with the D/A converter at $\omega = 0,\ \pi/4,\ \pi/3$, and π.

(*e*) Sketch (i) the gain of just the digital filter and (ii) the overall gain of the digital filter in cascade with the D/A converter, and explain the effect of the D/A converter on the amplitude response.

Solution

(*a*) The gain is given by

$$\begin{aligned} M(\omega) &= |H(e^{j\omega T})| = \left|\frac{2[e^{2j\omega T}-e^{j\omega T}+1]}{e^{j2\omega T}-\frac{\sqrt{2}}{2}e^{j\omega T}+\frac{1}{4}}\right| \\ &= 2\left|\frac{\cos 2\omega T + j\sin 2\omega T - \cos\omega T - j\sin\omega T + 1}{\cos 2\omega T + j\sin 2\omega T - \frac{\sqrt{2}}{2}(\cos\omega T + j\sin\omega T)+\frac{1}{4}}\right| \\ &= 2\sqrt{\frac{(\cos 2\omega T - \cos\omega T + 1)^2 + (\sin 2\omega T - \sin\omega T)^2}{(\cos 2\omega T - \frac{\sqrt{2}}{2}\cos\omega T + \frac{1}{4})^2 + (\sin 2\omega T - \frac{\sqrt{2}}{2}\sin\omega T)^2}} \quad \blacksquare \end{aligned}$$

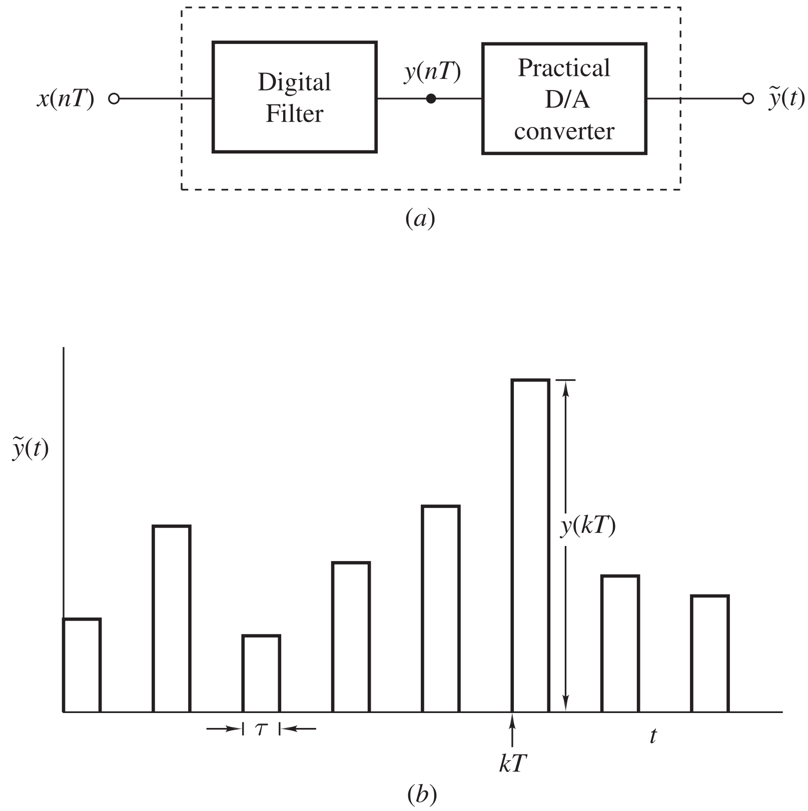

Figure SA.11

(*b*) The practical D/A converter has a gain

$$|H_p(j\omega)| = \tau \left| \frac{\sin(\omega\tau/2)}{\omega\tau/2} \right| \tag{SA.19}$$

where $\tau = 0.01T$ and $T = 2\pi/\omega_s = 2\pi/2\pi = 1$ s (see Eq. (6.60) in textbook). Hence the overall gain of the digital filter in cascade with the D/A converter is given by

$$M_T(\omega) = 2\sqrt{\frac{(\cos 2\omega T - \cos\omega T + 1)^2 + (\sin 2\omega T - \sin\omega T)^2}{(\cos 2\omega T - \frac{\sqrt{2}}{2}\cos\omega T + \frac{1}{4})^2 + (\sin 2\omega T - \frac{\sqrt{2}}{2}\sin\omega T)^2}} \cdot \tau \left| \frac{\sin\omega\tau/2}{\omega\tau/2} \right|$$

(*c*) Since $\omega_s = 2\pi$, we have $2\pi f_s = 2\pi/T = 2\pi$. Hence $T = 1$ s. For the frequencies given, the numerical values in Table SA.6 can be readily calculated.

Table SA.6

ω	$\cos\omega T$	$\sin\omega T$	$\cos 2\omega T$	$\sin 2\omega T$
0	1	0	1	0
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	0	1
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
π	-1	0	1	0

Thus

$$M(0) = 2\sqrt{\frac{(1-1+1)^2+(0-0)^2}{(1-\frac{\sqrt{2}}{2}+\frac{1}{4})^2+(0-0)^2}} = \frac{2}{\frac{5-2\sqrt{2}}{4}} = 3.6840 \quad \blacksquare$$

$$M\left(\frac{\pi}{4}\right) = 2\sqrt{\frac{(0-\frac{\sqrt{2}}{2}+1)^2+(1-\frac{\sqrt{2}}{2})^2}{(0-\frac{\sqrt{2}}{2}\times\frac{\sqrt{2}}{2}+\frac{1}{4})^2+(1-\frac{\sqrt{2}}{2}\times\frac{\sqrt{2}}{2})^2}}$$

$$= 2\sqrt{\frac{(2-\sqrt{2})^2+(2-\sqrt{2})^2}{(\frac{1}{2}-1)^2+1}} = 2\sqrt{\frac{2(2-\sqrt{2})^2}{(-\frac{1}{2})^2+1}} = 1.4890 \quad \blacksquare$$

$$M\left(\frac{\pi}{3}\right) = 2\sqrt{\frac{(-\frac{1}{2}-\frac{1}{2}+1)^2+(\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2})^2}{(-\frac{1}{2}-\frac{\sqrt{2}}{2}\times\frac{1}{2}+\frac{1}{4})^2+(\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2})}}$$

$$= 2\sqrt{\frac{0+0}{\left(\frac{1-2-\sqrt{2}}{4}\right)^2+\cdots}} = 0 \quad \blacksquare$$

$$M(\pi) = 2\sqrt{\frac{(1+1+1)^2+(0-0)}{(1+\frac{\sqrt{2}}{2}+\frac{1}{4})^2+(0-0)}} = 2\sqrt{\frac{3^2\times 16}{(4+2\sqrt{2}+1)^2}}$$

$$= \frac{2\times 3\times 4}{4+2\sqrt{2}+1} = 3.0657 \quad \blacksquare$$

(*d*) Since $T = 1$s, Eq. (SA.19) gives

$$|H_P(j\omega)| = 0.01T\left|\frac{\sin(0.01T\omega/2)}{0.01T\omega/2}\right| = 0.01\left|\frac{\sin(0.01\omega/2)}{0.01\omega/2}\right|$$
$$= p0.01$$

Hence the overall gain of the digital filter in cascade with the D/A converter is obtained from the above numerical values as follows:

$$M_T(0) = 3.6840\times 0.01 = 3.6840\times 10^{-2} \quad \blacksquare$$

$$M_T\left(\frac{\pi}{4}\right) = 1.4890\times 0.01 = 1.4890\times 10^{-2} \quad \blacksquare$$

$$M_T\left(\frac{\pi}{3}\right) = 0\times 0.0100 = 0.0 \quad \blacksquare$$

$$M_T(\pi) = 3.0557\times 0.01 = 3.0557\times 10^{-2} \quad \blacksquare$$

SA.12 Realize the transfer function

$$H(z) = \left(\frac{z}{z-0.5+j0.3}+\frac{z}{z-0.5-j0.3}\right)\cdot\left(\frac{3z}{z+0.4}+\frac{5z}{z+0.5}\right)$$

using two second-order canonic filter sections in cascade.

Solution

The transfer function can be expressed as

$$H(z) = \frac{2z^2-0.5z-0.5z}{z^2-z+0.5^2+0.3^2}\cdot\frac{3z^2+1.5z+5z^2+2z}{z^2+0.9z+0.2}$$
$$= \frac{2z^2-z}{z^2-z+0.34}\cdot\frac{8z^2+3.5z}{z^2+0.9z+0.2}$$
$$= \frac{2-z^{-1}}{1-z^{-1}+0.34z^{-2}}\cdot\frac{8+3.5z^{-1}}{1+0.9z^{-1}+0.2z^{-2}}$$
$$= H_1(z)\cdot H_2(z)$$

where

$$H_1(z) = \frac{2 - z^{-1}}{1 - z^{-1} + 0.34z^{-2}} \qquad \text{and} \qquad H_2(z) = \frac{8 + 3.5z^{-1}}{1 + 0.9z^{-1} + 0.2z^{-2}}$$

Now if we realize $H_1(z)$ and $H_2(z)$ in terms of direct canonic sections, the cascade realization shown in Fig. SA.12 can be readily obtained. ■

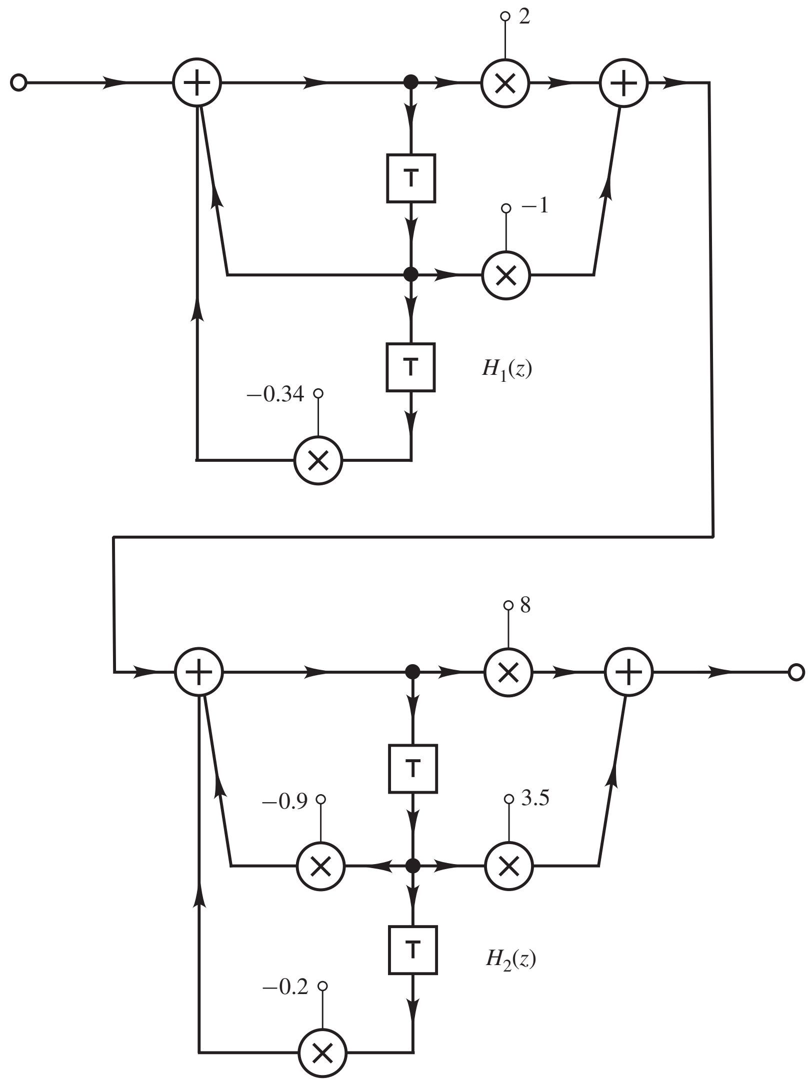

Figure SA.12

SA.13 An analog elliptic lowpass filter with a cutoff frequency of 1 rad/s has a transfer function of the form

$$H(s) = \frac{0.075(s^2 + 2.6)}{(s + 0.38)(s^2 + 0.31s + 0.51)}$$

(*a*) By applying the lowpass-to-highpass transformation

$$s = \frac{1}{\bar{s}}$$

get a continuous-time highpass transfer function.

(*b*) Construct the zero-pole plot of the continuous-time highpass transfer function.

(*c*) Using the zeros and poles obtained in part (*b*), get a corresponding discrete-time highpass transfer function using the matched-z-transformation method. The sampling frequency is $\omega_s = 20$ rad/s.

(d) How does the matched-z-transformation method compare with the invariant-impulse-response method?

Solution

The transfer function has zeros at

$$z = z_1,\ z_1^*$$

where

$$z_1 = 0 + j1.6125$$

and poles at

$$z = p_0,\ p_1,\ p_1^*$$

where

$$\begin{aligned} p_0 &= -0.3800 + j0.0000 \\ p_1 &= -0.1550 + j0.6971 \end{aligned}$$

(a) The highpass transfer function is obtained as

$$\begin{aligned} H_{HP}(\bar{s}) &= H(s)\Big|_{\bar{s}\to\frac{1}{s}} = \frac{0.075(\bar{s}^2 + 2.6)}{(s+0.38)(s^2+0.31s+0.51)}\Big|_{s=1/\bar{s}} \\ &= \frac{0.075(\frac{1}{\bar{s}^2} + 2.6)}{(\frac{1}{\bar{s}}+0.38)(\frac{1}{\bar{s}^2}+0.31\frac{1}{\bar{s}}+0.51)} \\ &= \frac{0.075\bar{s}(1+2.6\bar{s}^2)}{(1+0.38\bar{s})(0.51\bar{s}^2+0.31\bar{s}+1)} \\ &= \frac{0.075\times 2.6}{0.38\times 0.51} \times \frac{\bar{s}(\bar{s}^2+\frac{1}{2.6})}{(\bar{s}+\frac{1}{0.38})(\bar{s}^2+\frac{0.31}{0.51}\bar{s}+\frac{1}{0.51})} \\ &= 1.0063 \times \frac{\bar{s}(\bar{s}^2+0.3846)}{(\bar{s}+2.6316)(\bar{s}^2+0.6079\bar{s}+1.9609)} \quad \blacksquare \end{aligned}$$

(b) Therefore, the highpass filter has zeros at

$$\bar{s} = \bar{s}_0,\ \bar{s}_1,\ \bar{s}_2$$

where

$$\bar{s}_0 = 0, \quad \bar{s}_1,\ \bar{s}_2 = 0 \pm j0.6202$$

and poles at

$$\bar{s} = \bar{p}_0,\ \bar{p}_1,\ \bar{p}_2$$

where

$$\bar{p}_0 = -2.6316, \quad \bar{p}_1,\ \bar{p}_2 = -0.3040 \pm j1.3669$$

(c) The transfer function of the digital filter is given by

$$H_D(z) = (z+1)^L \frac{H_0 \prod_{i=1}^{M} (z - e^{\bar{s}_i T})}{\prod_{i=1}^{N} (z - e^{\bar{p}_i T})}$$

where $L = 0$ for a highpass filter. Hence

$$H_D(z) = \frac{\prod_{i=1}^{3} (z - \tilde{z}_i)}{\prod_{i=1}^{3} (z - \tilde{p}_i)} \quad \blacksquare$$

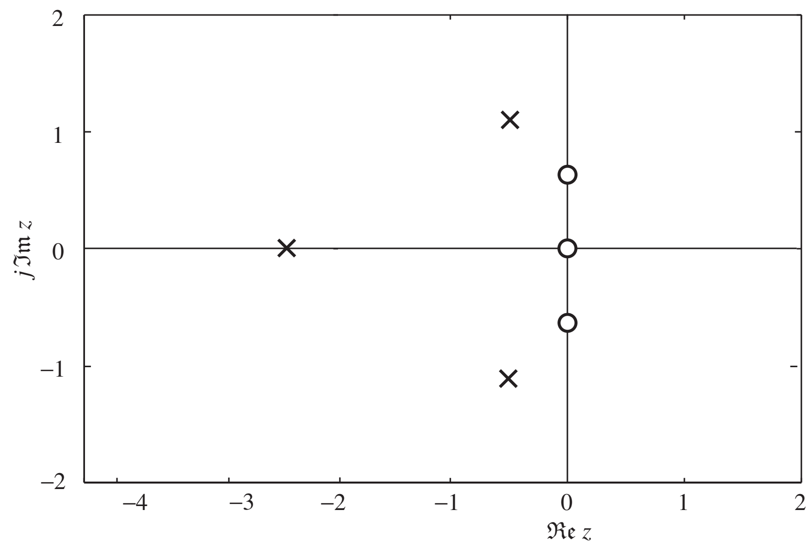

Figure SA.13

where

$$\tilde{z}_0 = e^{0T} = 1, \quad \tilde{z}_1,\ \tilde{z}_2 = e^{\pm j0.6202T}$$
$$\tilde{p}_0 = e^{-2.6316T}, \quad \tilde{p}_1,\ \tilde{p}_2 = e^{(-0.3040\pm j1.3669)T}$$

with

$$T = \frac{2\pi}{\omega_s} = \frac{2\pi}{20} = \frac{\pi}{10}$$

(d) The method works with all types of filters, i.e., LP, HP, BP, and BS, and is easy to apply. However, it tends to increase the passband ripple. ■

SA.14 A lowpass analog filter has a transfer function

$$H_A(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

(a) Assuming a sampling frequency of 10π, design a digital filter using the bilinear transformation method.

(b) Find the 1-dB and 30-dB frequencies of the analog filter.

(c) Find the 1-dB and 30-dB frequencies of the digital filter.

(d) What should be the 3-dB frequency of the analog filter to get a 3-dB frequency in the digital filter at 1 rad/s?

Solution

(a) The sampling period is given by

$$T = \frac{1}{f_s} = \frac{2\pi}{\omega_s} = \frac{2\pi}{10\pi} = \frac{1}{5}$$

The discrete-time transfer function is given by

$$\begin{aligned}
H_D(z) &= H_A(s)\,\Big|_{s=\frac{2}{T}\frac{z-1}{z+1}} \\
&= \left.\frac{1}{s^2+\sqrt{2}s+1}\right|_{s=\frac{10(z-1)}{z+1}} \\
&= \frac{1}{\left[\frac{10(z-1)}{z+1}\right]^2+\sqrt{2}\left[\frac{10(z-1)}{z+1}\right]+1} \\
&= \frac{z^2+2z+1}{100(z^2-2z+1)+10\sqrt{2}(z^2-1)+(z^2+2z+1)} \\
&= \frac{z^2+2z+1}{b_2z^2+b_1z+b_0} \quad \blacksquare
\end{aligned}$$

where

$$\begin{aligned}
b_0 &= 100-10\sqrt{2}+1 = 86.86 \\
b_1 &= -200+2 = -198.00 \\
b_2 &= 100+10\sqrt{2}+1 = 115.14
\end{aligned}$$

Alternatively, by factorizing constant b_2 in the denominator, the transfer function can be expressed as

$$H_D(z) = H_0\frac{z^2+2z+1}{z^2+b_1'z+b_0'} \quad \blacksquare$$

where

$$H_0 = 8.685\times 10^{-3}, \quad b_0' = 0.7544, \quad b_1' = -1.720$$

(*b*) The frequency response of the analog filter

$$H_A(j\omega) = \frac{1}{-\omega^2+j\sqrt{2}\omega+1}$$

Hence the loss is given by

$$\begin{aligned}
A(\omega) &= 20\log\sqrt{(1-\omega^2)^2+2\omega^2} \\
&= 10\log(1-2\omega^2+\omega^4+2\omega^2)
\end{aligned}$$

Thus

$$1+\omega^4 = 10^{0.1\times A(\omega)}$$

By letting $A(\omega) = 1$ dB, the 1-dB frequency is obtained as

$$\omega_1 = (10^{0.1}-1)^{1/4} = 0.7133 \text{ rad/s} \quad \blacksquare$$

By letting $A(\omega) = 30$ dB, the 30-dB frequency is obtained as

$$\omega_2 = (10^{0.1\times 30}-1)^{1/4} = 5.622 \text{ rad/s} \quad \blacksquare$$

(*c*) A frequency ω_i in the analog filter transforms into a frequency Ω_i given by

$$\Omega_i = \frac{2}{T}\tan^{-1}\frac{\omega_i T}{2} \tag{SA.20}$$

Hence the 1- and 30-dB frequencies in the digital filter are obtained as

$$\Omega_1 = 10\times\tan^{-1}\frac{0.7133}{10} = 0.7121 \text{ rad/s} \quad \blacksquare$$

and

$$\Omega_2 = 10\times\tan^{-1}\frac{5.622}{10} = 5.122 \text{ rad/s} \quad \blacksquare$$

respectively.

(*d*) Now if the 3-dB frequency in the digital filter is required to be 1 rad/s, then according to Eq. (SA.20) the 3-dB frequency in the analog filter should be

$$\omega_3 = \frac{2}{T} \tan \frac{\Omega_3 T}{2} = 10 \times \tan \frac{1}{10} = 1.003 \text{ rad/s} \quad \blacksquare$$