Chapter 4
DISCRETE-TIME SYSTEMS
4.1 Introduction
4.2 Basic System Properties

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July 14, 2018

Introduction

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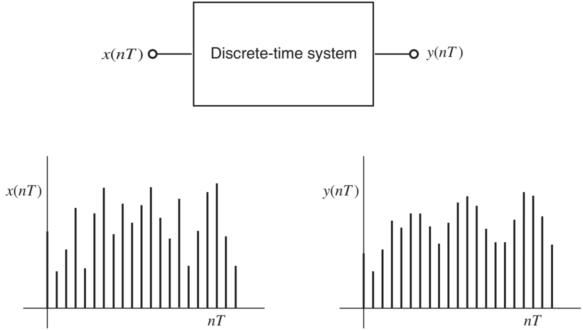
- Various types of discrete-time systems have emerged since the invention of the digital computer such as the systems used for digital control, robotics, data compression, and image-processing.
- This presentation will deal with the basic properties associated with discrete-time systems in general:
  - Linearity
  - Time invariance
  - Causality

The response (or output) $y(nT)$ of a discrete-time system is related to the excitation (or input) $x(nT)$ by some rule of correspondence, i.e.,

$$y(nT) = \mathcal{R}x(nT)$$

where $\mathcal{R}$ is an operator.



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- For a discrete-time system that can be used for the processing of signals, such as a digital filter, the rule of correspondence must of necessity involve some operation that changes the frequency spectrum of the input signal.

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- For a discrete-time system that can be used for the processing of signals, such as a digital filter, the rule of correspondence must of necessity involve some operation that changes the frequency spectrum of the input signal.
- For example, the operator $\mathcal{R}$ might transform an input signal $x(nT)$ into an output signal $y(nT)$ such that the high-frequency components in $x(nT)$ are removed.
  In such a case, the system would operate as a lowpass digital filter.

Basic System Properties *Cont'd*

Depending on the rule of correspondence, a discrete-time system can be classified as:

- Linear or nonlinear
- Time-invariant or time-dependent
- Causal or noncausal

Linearity

- A discrete-time system is *linear* if and only if it satisfies the conditions

$$\mathcal{R}\alpha x(nT) = \alpha\mathcal{R}x(nT) \qquad \text{(A)}$$

$$\mathcal{R}[x_1(nT) + x_2(nT)] = \mathcal{R}x_1(nT) + \mathcal{R}x_2(nT) \qquad \text{(B)}$$

for all possible values of α and all possible excitations $x_1(nT)$ and $x_2(nT)$.

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  for all possible values of α and all possible excitations $x_1(nT)$ and $x_2(nT)$.

  - The condition in Eq. (A) is referred to as the *proportionality* or *homogeneity* condition.
  - The condition in Eq. (B) is referred to as the *superposition* or *additivity* condition.

...

$$\mathcal{R}\alpha x(nT) = \alpha\mathcal{R}x(nT) \qquad \text{(A)}$$

$$\mathcal{R}\left[x_1(nT) + x_2(nT)\right] = \mathcal{R}x_1(nT) + \mathcal{R}x_2(nT) \qquad \text{(B)}$$

- From the superposition condition, i.e., Eq. (B), we get

$$y(nT) = \mathcal{R}\left[\alpha x_1(nT) + \beta x_2(nT)\right] = \mathcal{R}\left[\alpha x_1(nT)\right] + \mathcal{R}\left[\beta x_2(nT)\right]$$

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- Now from the proportionality condition, i.e., Eq. (A), we have

$$y(nT) = \mathcal{R}\left[\alpha x_1(nT) + \beta x_2(nT)\right] = \alpha \mathcal{R}x_1(nT) + \beta \mathcal{R}x_2(nT)$$

Thus, Eqs. (A) and (B) can be combined into one equation.

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  Thus, Eqs. (A) and (B) can be combined into one equation.

- If this condition is violated for any pair of excitations or any constant α or β, then the system is *nonlinear*.

- Linearity is a simple system property which is very familiar to the typical shopper.

Linearity *Cont'd*

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- If the price of apples were \$4.00 per kg and that of pears \$5.00 per kg, then 3 kgs of apples would cost \$12.00 and 5 kgs of pears would cost \$25.00 if the proportionality condition were satisfied.

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- If the price of apples were \$4.00 per kg and that of pears \$5.00 per kg, then 3 kgs of apples would cost \$12.00 and 5 kgs of pears would cost \$25.00 if the proportionality condition were satisfied.
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- If the price of apples were \$4.00 per kg and that of pears \$5.00 per kg, then 3 kgs of apples would cost \$12.00 and 5 kgs of pears would cost \$25.00 if the proportionality condition were satisfied.
- On the other hand, 1 kg of apples and 1 kg of pears would cost \$9.00 if the superposition condition were satisfied.
- Now if both conditions were satisfied, the situation at hand would be linear and 5 kgs of apples plus 3 kgs of pears would cost \$35.00.

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  In either case the situation would have become nonlinear.

- If the rule of correspondence of a discrete-time signal is known, then the system can be tested for linearity by checking whether the combined condition

$$\mathcal{R}\left[\alpha x_1(nT) + \beta x_2(nT)\right] = \alpha \mathcal{R} x_1(nT) + \beta \mathcal{R} x_2(nT)$$

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is satisfied. This tends to involve quite a bit of writing.

- A simpler approach that works well in the case where the system is nonlinear is to attempt to find a situation that would violate either the proportionality condition

$$\mathcal{R} \alpha x(nT) = \alpha \mathcal{R} x(nT) \qquad (A)$$

or the superposition condition

$$\mathcal{R}\left[x_1(nT) + x_2(nT)\right] = \mathcal{R} x_1(nT) + \mathcal{R} x_2(nT) \qquad (B)$$

- For example, if the rule of correspondence includes terms like $|x(nT)|$ or $x^k(nT)$ where $k \neq 1$, then the proportionality condition would most likely be violated and one would need to check only Eq. (A).

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  If it is violated, then the work is done and the system is classified as nonlinear.

  If it is not violated, then one must also check the superposition condition and if it is violated, the system is nonlinear.

  Otherwise, the system is linear.

Example

The response of a discrete-time system is given by

$$y(nT) = \mathcal{R}x(nT) = 7x^2(nT - T)$$

Check the system for linearity.

Solution A delayed version of the input signal appears squared in the characterization of the system and the proportionality condition is most likely violated.

For an arbitrary constant α, we have

$$\mathcal{R}\left[\alpha x(nT)\right] = 7\alpha^2 x^2(nT - T)$$

On the other hand,

$$\alpha \mathcal{R} x(nT) = 7\alpha x^2(nT - T)$$

Clearly if $\alpha \neq 1$, then

$$\mathcal{R}\left[\alpha x(nT)\right] \neq \alpha \mathcal{R} x(nT)$$

i.e., the proportionality condition is violated and, therefore, the system is *nonlinear*. ■

Example

The response of a discrete-time system is given by

$$y(nT) = \mathcal{R}x(nT) = (nT)^2 x(nT + 2T)$$

Check the system for linearity.

Solution For this case, the proportionality condition is not violated, as can be easily verified, and so we should check the combined equation

$$\mathcal{R}\left[\alpha x_1(nT) + \beta x_2(nT)\right] = \alpha \mathcal{R} x_1(nT) + \beta \mathcal{R} x_2(nT)$$

Example *Cont'd*

We can write

$$\begin{aligned}\mathcal{R}\left[\alpha x_1(nT) + \beta x_2(nT)\right] &= (nT)^2\left[\alpha x_1(nT+2T) + \beta x_2(nT+2T)\right] \\ &= \alpha(nT)^2 x_1(nT+2T) + \beta(nT)^2 x_2(nT+2T) \\ &= \alpha\mathcal{R}x_1(nT) + \beta\mathcal{R}x_2(nT)\end{aligned}$$

i.e., the system is *linear*. ■

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i.e., the system is *linear*. ■

Note: The squared term $(nT)^2$ may trick a few but it does not affect the linearity of the system since it is a time-dependent system parameter which is independent of the input signal.

Time Invariance

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- In practice, discrete-time systems utilize a certain type of digital element known as the *unit delay*.

  Unit delays are actually memory devices and their contents must be zero for the discrete-time system to be initially relaxed.

- A discrete-time system is said to be *time-invariant* if its response to an arbitrary excitation does not depend on the time of application of the excitation, i.e., its internal parameters do not change with time.

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- Formally, an initially relaxed discrete-time system is time-invariant if and only if

$$\mathcal{R}x(nT - kT) = y(nT - kT)$$

for all possible excitations $x(nT)$ and all integers k.

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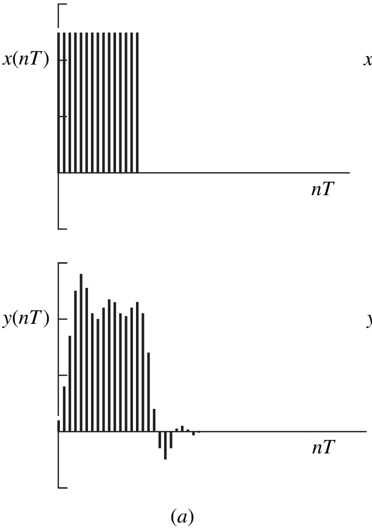
- A discrete-time system that does not satisfy the above test for at least one signal and any value of k other than 0 is *time-dependent*.

Time Invariance *Cont'd*



Example

A discrete-time system is characterized by the equation

$$y(nT) = \mathcal{R}x(nT) = 2nTx(nT)$$

Is the system time-invariant or time-dependent?

Solution The response to a delayed excitation is

$$\mathcal{R}x(nT - kT) = 2nTx(nT - kT)$$

The delayed response is: $y(nT - kT) = 2(nT - kT)x(nT - kT)$

For any $k \neq 0$, we have: $\mathcal{R}x(nT - kT) \neq y(nT - kT)$

Therefore, the system is *time-dependent*. ■

Example

A discrete-time system is characterized by the equation

$$y(nT) = \mathcal{R}x(nT) = 12x(nT - T) + 11x(nT - 2T)$$

Is the system time-invariant or time-dependent?

Solution The response to a delayed excitation is

$$\mathcal{R}x(nT-kT) = 12x(nT-T-kT) + 11x(nT-2T-kT)$$

The delayed response is

$$y(nT-kT) = 12x(nT-T-kT) + 11x(nT-2T-kT)$$

For any k, we have

$$\mathcal{R}x(nT-kT) = y(nT-kT)$$

Therefore, the system is *time-invariant*. ■

Causality

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- A more precise way of saying very much the same thing is as follows:

  An initially relaxed discrete-time system is causal if and only if

  $$\mathcal{R}x_1(nT) = \mathcal{R}x_2(nT) \quad \text{for} \quad n \leq k \tag{C}$$

  for all possible distinct excitations $x_1(nT)$ and $x_2(nT)$ such that

  $$x_1(nT) = x_2(nT) \quad \text{for} \quad n \leq k \tag{D}$$

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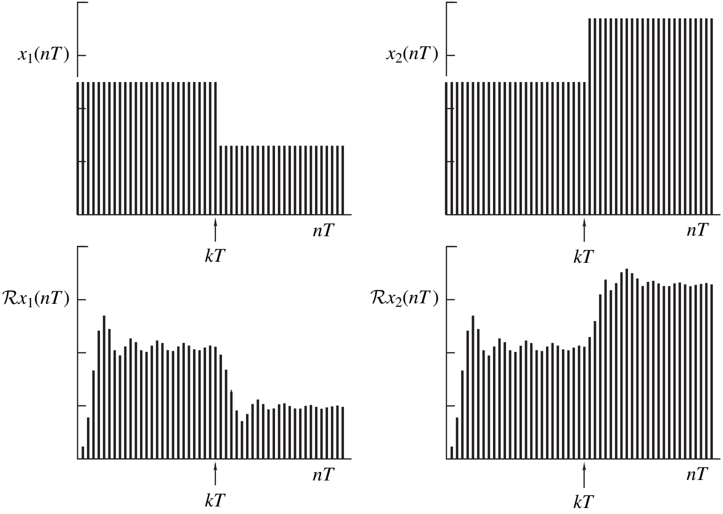
  $$x_1(nT) = x_2(nT) \quad \text{for} \quad n \le k \qquad \text{(D)}$$

- If Eq. (C) is not satisfied for at least one distinct pair of excitations that satisfy Eq. (D) and at least one value of k, then the system is *noncausal*.



(a) (b)

A discrete-time system is represented by the equation

$$y(nT) = \mathcal{R}x(nT) = 3x(nT - 2T) + 3x(nT + 2T)$$

Is the system causal or noncausal?

Solution Let $x_1(nT)$ and $x_2(nT)$ be distinct excitations such that

$$x_1(nT) = x_2(nT) \quad \text{for} \quad n \leq k \quad \text{and} \quad x_1(nT) \neq x_2(nT) \quad \text{for} \quad n > k \tag{E}$$

For $n = k$ we have

Example *Cont'd*

. . .

$$x_1(nT) = x_2(nT) \quad \text{for} \quad n \le k \quad \text{and} \quad x_1(nT) \ne x_2(nT) \quad \text{for} \quad n > k \qquad \text{(E)}$$

For $n = k$ we have

$$\mathcal{R}x_1(nT)|_{n=k} = 3x_1(kT - 2T) + 3x_1(kT + 2T)$$

$$\mathcal{R}x_2(nT)|_{n=k} = 3x_2(kT - 2T) + 3x_2(kT + 2T)$$

but since

$$3x_1(kT + 2T) \ne 3x_2(kT + 2T)$$

from our assumption in Eq. (E), we conclude that

$$\mathcal{R}x_1(nT) \ne \mathcal{R}x_2(nT) \quad \text{for} \quad n = k$$

Therefore, the system is *noncausal*. ■

Example

A discrete-time system is represented by the equation

$$y(nT) = \mathcal{R}x(nT) = 3x(nT - T) - 3x(nT - 2T)$$

Is the system causal or noncausal?

Solution Let $x_1(nT)$ and $x_2(nT)$ be distinct excitations such that

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...

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In this example, we have

$$\mathcal{R}x_1(nT) = 3x_1(nT - T) + 3x_1(nT - 2T)$$

$$\mathcal{R}x_2(nT) = 3x_2(nT - T) + 3x_2(nT - 2T)$$

If $n \leq k$, then $n-1,\ n-2 < k$ and so on the basis of our assumption in Eq. (F), we have

$$x_1(nT-T) = x_2(nT-T) \ \text{ and } \ x_1(nT-2T) = x_2(nT-2T) \ \text{ for } n \leq k$$

Hence we conclude that

$$\mathcal{R}x_1(nT) = \mathcal{R}x_2(nT) \quad \text{for} \quad n \leq k$$

Therefore, the system is *causal*. ■

- Analog systems such as analog filters are almost always *linear* and *time invariant*, and because they are real-time devices they have to be *causal*.

  Any nonlinearity or time-dependence is usually an imperfection.

System Properties *Cont'd*

- Analog systems such as analog filters are almost always *linear* and *time invariant*, and because they are real-time devices they have to be *causal*.

  Any nonlinearity or time-dependence is usually an imperfection.

- Discrete-time systems such as digital filters can be *nonlinear, time-dependent*, or *noncausal*, e.g., so-called median filters are nonlinear, adaptive filters are time-dependent, and nonrecursive filters are often noncausal.

This slide concludes the presentation.

Thank you for your attention.