Chapter 5
THE APPLICATION OF THE Z TRANSFORM
5.4 Time-Domain Analysis

Victoria, BC, Canada
Email: aantoniou@ieee.org

July 14, 2018

Introduction

★ Previous presentations dealt with time-domain analysis through the use of mathematical induction or on the basis of the state-space representation.

Introduction

- ★ Previous presentations dealt with time-domain analysis through the use of mathematical induction or on the basis of the state-space representation.
- ★ Although the induction method is rather intuitive, it runs into serious difficulties when the system order is increased to two or higher.

Introduction

- ★ Previous presentations dealt with time-domain analysis through the use of mathematical induction or on the basis of the state-space representation.
- ★ Although the induction method is rather intuitive, it runs into serious difficulties when the system order is increased to two or higher.
- ★ The state-space approach, on the other hand, yields solutions in the form of infinite summations rather than in terms of closed-form solutions.

Introduction

- Previous presentations dealt with time-domain analysis through the use of mathematical induction or on the basis of the state-space representation.
- Although the induction method is rather intuitive, it runs into serious difficulties when the system order is increased to two or higher.
- The state-space approach, on the other hand, yields solutions in the form of infinite summations rather than in terms of closed-form solutions.
- The z transform approach overcomes these difficulties and it is, therefore, the preferred approach.

Time-Domain Analysis

★ As is shown earlier, a discrete-time system with excitation $x(nT)$, response $y(nT)$, and impulse response $h(nT)$ is characterized by the equation

$$Y(z) = H(z)X(z)$$

Time-Domain Analysis

- ★ As is shown earlier, a discrete-time system with excitation $x(nT)$, response $y(nT)$, and impulse response $h(nT)$ is characterized by the equation

$$Y(z) = H(z)X(z)$$

- ★ Therefore, the response produced by an arbitrary excitation can be readily obtained as

$$y(nT) = \mathcal{Z}^{-1}[H(z)X(z)]$$

Time-Domain Analysis

- ★ As is shown earlier, a discrete-time system with excitation $x(nT)$, response $y(nT)$, and impulse response $h(nT)$ is characterized by the equation

$$Y(z) = H(z)X(z)$$

- ★ Therefore, the response produced by an arbitrary excitation can be readily obtained as

$$y(nT) = \mathcal{Z}^{-1}[H(z)X(z)]$$

- ★ The inverse z transform can be obtained by using any one of the standard inversion techniques described in Chap. 3.

Example

A discrete-time system is characterized by the transfer function

$$H(z) = \frac{z^2 - z + 1}{(z - p_1)(z - p_2)}$$

where

$$p_1,\ p_2 = \tfrac{1}{2} \pm j\tfrac{1}{2} = \frac{1}{\sqrt{2}} e^{\pm j\pi/4}$$

Find the unit-step response.

Solution The response of the system is given by

$$y(nT) = \mathcal{Z}^{-1}[H(z)X(z)]$$

The z transform of the input is given by

$$X(z) = \mathcal{Z}u(nT) = \frac{z}{z-1}$$

Expanding $H(z)X(z)/z$ into partial fractions gives

$$H(z)X(z) = \frac{R_0 z}{z-1} + \frac{R_1 z}{(z-p_1)} + \frac{R_2 z}{(z-p_2)}$$

where $R_0 = 2$, $\quad R_1 = \frac{1}{\sqrt{2}}e^{-j5\pi/4}$, $\quad$ and $\quad R_2 = R_1^* = \frac{1}{\sqrt{2}}e^{j5\pi/4}$.

Solution The response of the system is given by

$$y(nT) = \mathcal{Z}^{-1}[H(z)X(z)]$$

The z transform of the input is given by

$$X(z) = \mathcal{Z}u(nT) = \frac{z}{z-1}$$

Expanding $H(z)X(z)/z$ into partial fractions gives

$$H(z)X(z) = \frac{R_0 z}{z-1} + \frac{R_1 z}{(z-p_1)} + \frac{R_2 z}{(z-p_2)}$$

where $R_0 = 2, \quad R_1 = \frac{1}{\sqrt{2}}e^{-j5\pi/4}, \quad$ and $\quad R_2 = R_1^* = \frac{1}{\sqrt{2}}e^{j5\pi/4}$.

From the table of standard z transforms, we have

$$y(nT) = 2u(nT) + u(nT)\left(\frac{1}{\sqrt{2}}e^{j\pi/4}\right)^n \cdot \frac{1}{\sqrt{2}}e^{-j5\pi/4}$$

...

$$y(nT) = 2u(nT) + u(nT)\left(\frac{1}{\sqrt{2}}e^{j\pi/4}\right)^n \cdot \frac{1}{\sqrt{2}}e^{-j5\pi/4} + u(nT)\left(\frac{1}{\sqrt{2}}e^{-j\pi/4}\right)^n \cdot \frac{1}{\sqrt{2}}e^{j5\pi/4}$$

Example *Cont'd*

...

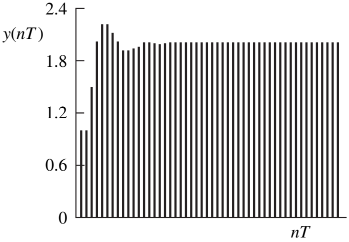
$$
\begin{aligned}
y(nT) &= 2u(nT) + u(nT)\left(\frac{1}{\sqrt{2}}e^{j\pi/4}\right)^n \cdot \frac{1}{\sqrt{2}}e^{-j5\pi/4} \\
&\quad + u(nT)\left(\frac{1}{\sqrt{2}}e^{-j\pi/4}\right)^n \cdot \frac{1}{\sqrt{2}}e^{j5\pi/4} \\
&= 2u(nT) + \frac{1}{\left(\sqrt{2}\right)^{n+1}}u(nT)(e^{j(n-5)\pi/4} + e^{-j(n-5)\pi/4})
\end{aligned}
$$

...

$$
\begin{aligned}
y(nT) &= 2u(nT) + u(nT)\left(\frac{1}{\sqrt{2}}e^{j\pi/4}\right)^n \cdot \frac{1}{\sqrt{2}}e^{-j5\pi/4} \\
&\quad + u(nT)\left(\frac{1}{\sqrt{2}}e^{-j\pi/4}\right)^n \cdot \frac{1}{\sqrt{2}}e^{j5\pi/4} \\
&= 2u(nT) + \frac{1}{(\sqrt{2})^{n+1}}u(nT)(e^{j(n-5)\pi/4} + e^{-j(n-5)\pi/4}) \\
&= 2u(nT) + \frac{1}{(\sqrt{2})^{n-1}}u(nT)\cos\left[(n-5)\frac{\pi}{4}\right] \qquad \blacksquare
\end{aligned}
$$

...

$$y(nT) = 2u(nT) + \frac{1}{\left(\sqrt{2}\right)^{n-1}} u(nT) \cos\left[(n-5)\frac{\pi}{4}\right] \quad \blacksquare$$



Unit-step response

Example

A discrete-time system is characterized by the transfer function

$$H(z) = \frac{z^2 - z + 1}{(z - p_1)(z - p_2)}$$

where

$$p_1,\ p_2 = \tfrac{1}{2} \pm j\tfrac{1}{2} = \frac{1}{\sqrt{2}} e^{\pm j\pi/4}$$

Find the response of the system to a sinusoidal excitation

$$x(nT) = u(nT) \sin \omega nT$$

Solution The response of the system is given by

$$y(nT) = \mathcal{Z}^{-1}[H(z)X(z)]$$

The z transform of the input is given by

$$\begin{aligned} X(z) &= \mathcal{Z}[u(nT)\sin\omega nT] = \frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1} \\ &= \frac{z\sin\omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})} \end{aligned}$$

and hence

$$\begin{aligned} H(z)X(z)z^{n-1} &= \frac{z^2 - z + 1}{(z - p_1)(z - p_2)} \cdot \frac{z\sin\omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})} \cdot z^{n-1} \\ &= \frac{z^2 - z + 1}{(z - p_1)(z - p_2)} \cdot \frac{\sin\omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})} \cdot z^n \end{aligned}$$

. . .

$$H(z)X(z)z^{n-1} = \frac{z^2 - z + 1}{(z - p_1)(z - p_2)} \cdot \frac{\sin \omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})} \cdot z^n$$

Since the system is causal $y(nT) = 0$ for $n < 0$ and hence the general inversion formula gives

$$y(nT) = u(nT)[R_1 + R_2 + R_3 + R_4]$$

where R_1, R_2, R_3, and R_4 are the residues of $H(z)X(z)z^{n-1}$ at poles p_1, p_2, $p_3 = e^{j\omega T}$, and $p_4 = e^{j\omega T}$, respectively.

The residues can be evaluated as shown in the next three slides.

Example *Cont'd*

...

$$H(z)X(z)z^{n-1} = \frac{z^2 - z + 1}{(z-p_1)(z-p_2)} \cdot \frac{\sin\omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})} \cdot z^n$$

$$\begin{aligned}
R_1 &= \lim_{z=p_1} \left[\frac{z^2 - z + 1}{(z-p_2)} \cdot \frac{\sin\omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})} \cdot z^n \right] \\
&= \left[\frac{p_1^2 - p_1 + 1}{(p_1 - p_2)} \cdot \frac{\sin\omega T}{(p_1 - e^{j\omega T})(p_1 - e^{-j\omega T})} \cdot p_1^n \right] \\
&= \rho(\omega) e^{j\psi(\omega)} \left(\frac{1}{\sqrt{2}} \right)^n e^{jn\pi/4} = \rho(\omega) \left(\frac{1}{\sqrt{2}} \right)^n e^{j[n\pi/4 + \psi(\omega)]}
\end{aligned}$$

where
$$\begin{aligned}
\rho(\omega) &= \left| \frac{p_1^2 - p_1 + 1}{(p_1 - p_2)} \cdot \frac{\sin\omega T}{(p_1 - e^{j\omega T})(p_1 - e^{-j\omega T})} \right| \\
\psi(\omega) &= \arg \left[\frac{p_1^2 - p_1 + 1}{(p_1 - p_2)} \cdot \frac{\sin\omega T}{(p_1 - e^{j\omega T})(p_1 - e^{-j\omega T})} \right]
\end{aligned}$$

Example *Cont'd*

...

$$H(z)X(z)z^{n-1} = \frac{z^2 - z + 1}{(z - p_1)(z - p_2)} \cdot \frac{\sin \omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})} \cdot z^n$$

$$\begin{aligned}
R_2 &= R_1^* = \rho(\omega)\left(\tfrac{1}{\sqrt{2}}\right)^n e^{-j[n\pi/4+\psi(\omega)]} \\
R_3 &= \lim_{z=e^{j\omega T}}[H(z)X(z)z^{n-1}] \\
&= H(e^{j\omega T}) \cdot \frac{\sin \omega T}{(e^{j\omega T} - e^{-j\omega T})} \cdot e^{jn\omega T} \\
&= \frac{1}{2j} H(e^{j\omega T})e^{jn\omega T} \\
R_4 &= R_3^* = -\frac{1}{2j} H(e^{-j\omega T})e^{-jn\omega T}
\end{aligned}$$

...

$$R_1 = \rho(\omega)\left(\tfrac{1}{\sqrt{2}}\right)^n e^{j[n\pi/4+\psi(\omega)]}, \quad R_2 = \rho(\omega)\left(\tfrac{1}{\sqrt{2}}\right)^n e^{-j[n\pi/4+\psi(\omega)]}$$
$$R_3 = \frac{1}{2j}H(e^{j\omega T})e^{jn\omega T}, \quad R_4 = -\frac{1}{2j}H(e^{-j\omega T})e^{-jn\omega T}$$

If we now let

$$H(e^{j\omega T}) = M(\omega)e^{j\theta(\omega)} \quad \text{then} \quad H(e^{-j\omega T}) = M(\omega)e^{-j\theta(\omega)}$$

and so

$$y(nT) = u(nT)\Big[\rho(\omega)\left(\tfrac{1}{\sqrt{2}}\right)^n e^{j[n\pi/4+\psi(\omega)]} + \rho(\omega)\left(\tfrac{1}{\sqrt{2}}\right)^n e^{-j[n\pi/4+\psi(\omega)]}$$
$$+\frac{1}{2j}M(\omega)e^{j\theta(\omega)}e^{jn\omega T} - \frac{1}{2j}M(\omega)e^{-j\theta(\omega)}e^{-jn\omega T}\Big]$$

...

$$
\begin{aligned}
y(nT) &= u(nT)\Big[\rho(\omega)\left(\tfrac{1}{\sqrt{2}}\right)^{n} e^{j[n\pi/4+\psi(\omega)]} + \rho(\omega)\left(\tfrac{1}{\sqrt{2}}\right)^{n} e^{-j[n\pi/4+\psi(\omega)]} \\
&\quad + \frac{1}{2j}M(\omega)e^{j\theta(\omega)}e^{jn\omega T} - \frac{1}{2j}M(\omega)e^{-j\theta(\omega)}e^{-jn\omega T}\Big] \\
&= u(nT)\Big\{\rho(\omega)\left(\tfrac{1}{\sqrt{2}}\right)^{n}\left[e^{j[n\pi/4+\psi(\omega)]} + e^{-j[n\pi/4+\psi(\omega)]}\right] \\
&\quad + M(\omega)\frac{1}{2j}\left[e^{j[n\omega T+\theta(\omega)]} - e^{-j[n\omega T+\theta(\omega)]}\right]\Big\} \\
&= u(nT)\Big\{\rho(\omega)\left(\tfrac{1}{\sqrt{2}}\right)^{n-2}\cos[\tfrac{n\pi}{4} + \psi(\omega)] \\
&\quad + M(\omega)\sin[n\omega T + \theta(\omega)]\Big\} \quad \blacksquare
\end{aligned}
$$

The cosine term is a *transient* component that tends to zero as $n \to \infty$ whereas the sine term represents the *steady-state* response of the system.

This slide concludes the presentation.
Thank you for your attention.