Chapter 6
THE SAMPLING PROCESS
6.6 Sampling Theorem
6.7 Aliasing
6.8 Interrelations

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Introduction

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- Sampling obviously entails discarding part of the continuous-time signal and the question will immediately arise as to whether the sampling process will corrupt the signal.
- It turns out that under a certain condition that is part of the *sampling theorem*, the information content of the continuous-time signal can be fully preserved.

The Sampling Theorem

- The sampling theorem states:

  A bandlimited signal $x(t)$ for which

  $$X(j\omega) = 0 \quad \text{for} \quad |\omega| \geq \frac{\omega_s}{2}$$

  where $\omega_s = 2\pi/T$, can be uniquely determined from its values $x(nT)$.

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- Alternatively, in what amounts to the same thing, a continuous-time signal whose spectrum is zero outside the baseband (i.e., $-\omega_s/2$ to $\omega_s/2$) can, in theory, be recovered completely from an impulse-modulated version of the signal.

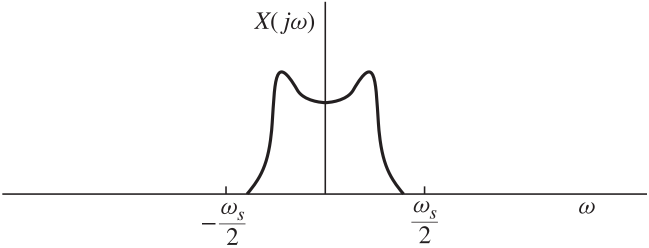
- Consider a two-sided bandlimited signal whose spectrum satisfies the condition of the sampling theorem.

The Sampling Theorem *Cont'd*

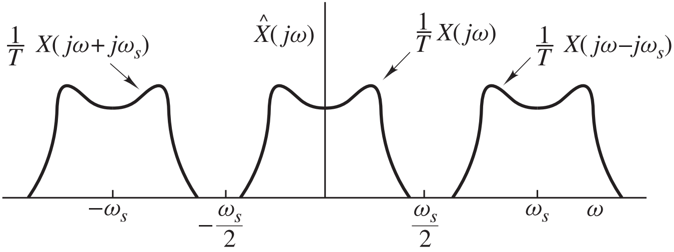
- Consider a two-sided bandlimited signal whose spectrum satisfies the condition of the sampling theorem.
- By virtue of Poisson's summation formula, i.e.,

$$\hat{X}(j\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j\omega + jn\omega_s)$$

impulse modulation will produce sidebands that are well separated from one another.


$X(j\omega)$
$-\frac{\omega_s}{2}$
$\frac{\omega_s}{2}$
ω

(*a*)


$\frac{1}{T}\, X(j\omega + j\omega_s)$
$\hat{X}(j\omega)$
$\frac{1}{T} X(j\omega)$
$\frac{1}{T}\, X(j\omega - j\omega_s)$
$-\omega_s$
$-\frac{\omega_s}{2}$
$\frac{\omega_s}{2}$
ω_s
ω

(*b*)

- Now if we pass the impulse-modulated signal through an ideal lowpass filter with a frequency response

$$H(j\omega) = \begin{cases} T & \text{for } \omega < \omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

then frequencies in the sidebands will be rejected and we will be left with the frequencies in the baseband, which constitute the original continuous-time signal.

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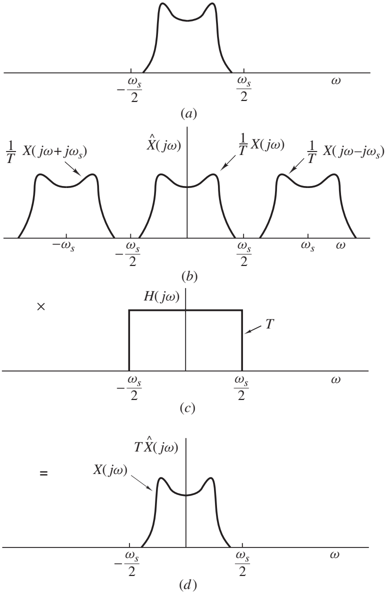
- A baseband gain of T is used to cancel out the scaling constant $1/T$ introduced by Poisson's summation formula.


$-\frac{\omega_s}{2}$
$\frac{\omega_s}{2}$
ω
(a)
$\frac{1}{T}X(j\omega+j\omega_s)$
$\hat{X}(j\omega)$
$\frac{1}{T}X(j\omega)$
$\frac{1}{T}X(j\omega-j\omega_s)$
$-\omega_s$
ω_s
(b)
×
$H(j\omega)$
T
(c)
=
$T\hat{X}(j\omega)$
$X(j\omega)$
(d)

- What has been done through a graphical illustration can now be repeated with mathematics.

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- If the impulse-modulated signal is passed through a lowpass filter with a frequency response $H(j\omega)$ as defined before, then the Fourier transform of the output of the filter will be

$$Y(j\omega) = H(j\omega)\hat{X}(j\omega)$$

where

$$H(j\omega) = \begin{cases} T & \text{for } \omega < \omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\hat{X}(j\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(j\omega + jn\omega_s)$$

...

$$Y(j\omega) = H(j\omega)\hat{X}(j\omega)$$

- If we apply the inverse Fourier transform, we get

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}\left[H(j\omega)\sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT}\right] \\ &= \sum_{n=-\infty}^{\infty} x(nT)\mathcal{F}^{-1}[H(j\omega)e^{-j\omega nT}] \qquad \text{(A)} \end{aligned}$$

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- The frequency response of a lowpass filter is actually a frequency-domain pulse of height T and base ω_s, i.e., $H(j\omega) = Tp_{\omega_s}(\omega)$ and hence from the table of Fourier transforms, we have

$$\frac{T\sin(\omega_s t/2)}{\pi t} \leftrightarrow H(j\omega) \qquad \text{(B)}$$

. . .

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT)\mathcal{F}^{-1}[H(j\omega)e^{-j\omega nT}] \quad \text{(A)}$$

$$\frac{T\sin(\omega_s t/2)}{\pi t} \leftrightarrow H(j\omega) \quad \text{(B)}$$

- From the time-shifting theorem of the Fourier transform

$$\frac{T\sin[\omega_s(t-nT)/2]}{\pi(t-nT)} \leftrightarrow H(j\omega)e^{-j\omega nT} \quad \text{(C)}$$

...

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT)\mathcal{F}^{-1}[H(j\omega)e^{-j\omega nT}] \qquad \text{(A)}$$

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- Therefore, from Eqs. (A) and (C), we conclude that

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT)\frac{\sin[\omega_s(t-nT)/2]}{\omega_s(t-nT)/2}$$

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- For $t = nT$, we have $y(nT) = x(nT)$ for $n = 0, 1, \ldots, kT$, and for all other values of t the output of the lowpass filter is an interpolated version of $x(t)$ according to the sampling theorem.

Aliasing

- If the spectrum of the continuous-time signal does *not* satisfy the condition imposed by the sampling theorem, i.e., if

$$X(j\omega) \neq 0 \quad \text{for} \quad |\omega| \geq \frac{\omega_s}{2}$$

then sideband frequencies will be aliased into baseband frequencies.

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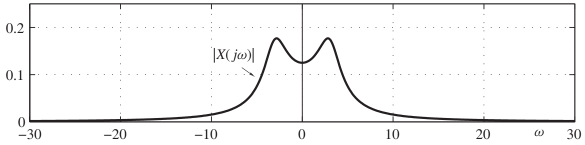
- As a result, $\hat{X}(j\omega)$ will not be equal to $X(j\omega)/T$ within the baseband.

- Under these circumstances, the use of an ideal lowpass filter will yield a distorted version of $x(t)$ at best.

Aliasing *Cont'd*

- Aliasing can be illustrated by examining an impulse-modulated signal generated by sampling the continuous-time signal

$$x(t) = u(t)e^{-at}\sin\omega_0 t$$

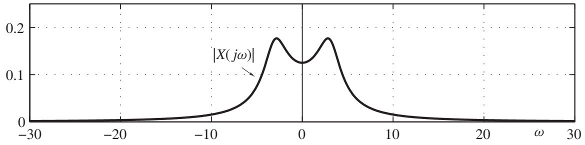


(a)

- Aliasing can be illustrated by examining an impulse-modulated signal generated by sampling the continuous-time signal

$$x(t) = u(t)e^{-at}\sin\omega_0 t$$

- The frequency spectrum of $x(t)$, $X(j\omega)$, extends over the infinite range $-\infty < \omega < \infty$.



(a)

- The frequency spectrum of impulse-modulated signal $\hat{x}(t)$ can be obtained as

$$\hat{X}(j\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j\omega + jn\omega_s)$$

by using Poisson's summation formula.

...

$$\hat{X}(j\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j\omega + jn\omega_s)$$

- The shifted copies of $X(j\omega)$ or sidebands, namely, ..., $X(j\omega - j2\omega_s)$, $X(j\omega - j\omega_s)$, $X(j\omega + j\omega_s)$, $X(j\omega + j2\omega_s)$, ... overlap with the baseband $-\omega_s/2 < \omega < \omega_s/2$ and, therefore, the above sum can be expressed as

$$\hat{X}(j\omega) = \frac{1}{T}[X(j\omega) + E(j\omega)]$$

where

$$E(j\omega) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} X(j\omega + jk\omega_s)$$

is the contribution of the sidebands to the baseband.

- Now if we filter the impulse-modulated signal, $\hat{x}(t)$, using an ideal lowpass filter with a frequency response

$$H(j\omega) = \begin{cases} T & \text{for } -\omega_s/2 < \omega < \omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

we will get a signal $y(t)$ whose frequency spectrum is given by

$$\begin{aligned} Y(j\omega) &= H(j\omega)\hat{X}(j\omega) \\ &= H(j\omega) \cdot \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j\omega + jn\omega_s) \\ &= X(j\omega) + E(j\omega) \end{aligned}$$

Aliasing *Cont'd*

. . .

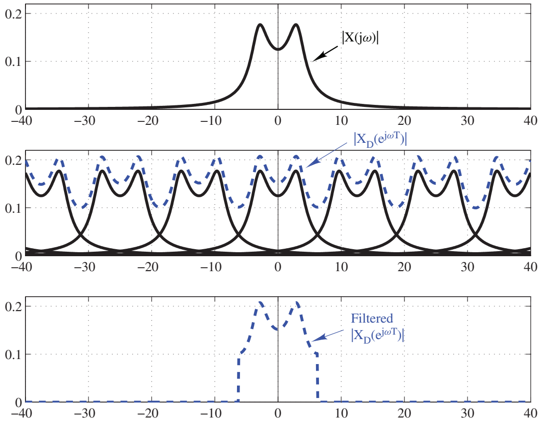
$$Y(j\omega) = X(j\omega) + E(j\omega)$$

- In other words, the output of the filter will be signal $x(t)$ plus an error

$$e(t) = \mathcal{F}^{-1}E(j\omega)$$

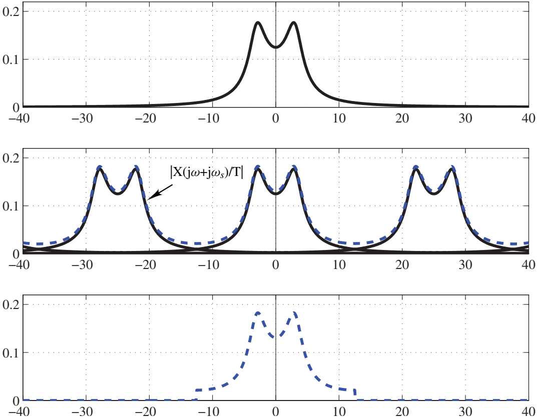
which is commonly referred to as the *aliasing error*.

- With a sampling frequency of 12.5 rad/s, $|E(j\omega)|$, i.e., the discrepancy between the solid and dashed curves in the figure is quite large.

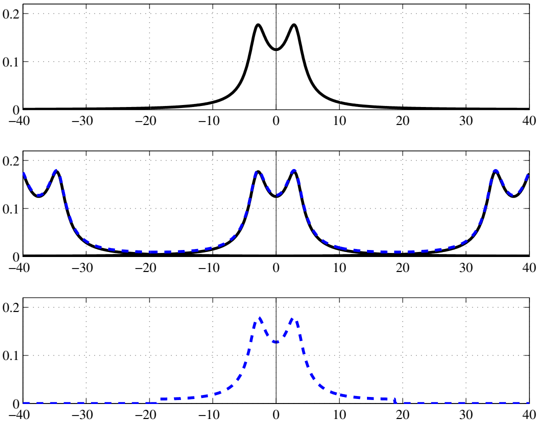


Aliasing *Cont'd*

- As the sampling frequency is increased to 25, the sidebands are spread out and $|E(j\omega)|$ will be decreased quite a bit as shown.



- A further increase to 40 rad/s will render $|E(j\omega)|$ for all practical purposes negligible as can be seen.



Summary of Interrelations

- Impulse-modulated signal:

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \tag{6.42c}$$

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- Impulse-modulated signal:

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \qquad (6.42c)$$

- Spectrum of impulse-modulated signal or discrete-time signal in terms of the spectrum of the original continuous-time signal:

$$\hat{X}(j\omega) = X_D(e^{j\omega T}) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(j\omega + jn\omega_s) \qquad (6.45a)$$

where

$$X_D(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT}$$

Summary of Interrelations *Cont'd*

- Spectrum of impulse-modulated signal (or discrete-time signal) in terms of the spectrum of the original continuous-time signal for a *right-sided signal*:

$$\hat{X}(j\omega) = X_D(e^{j\omega T}) = \frac{x(0+)}{2} + \frac{1}{T}\sum_{n=-\infty}^{\infty} X(j\omega + jn\omega_s) \quad (6.45b)$$

- Spectrum of impulse-modulated signal (or discrete-time signal) in terms of the spectrum of the original continuous-time signal for a *right-sided signal*:

$$\hat{X}(j\omega) = X_D(e^{j\omega T}) = \frac{x(0+)}{2} + \frac{1}{T}\sum_{n=-\infty}^{\infty} X(j\omega + jn\omega_s) \quad (6.45b)$$

- Laplace transform of impulse-modulated signal in terms of the Laplace transform of the original continuous-time signal for a *right-sided signal*:

$$\hat{X}(s) = X_D(z) = \frac{x(0+)}{2} + \frac{1}{T}\sum_{n=-\infty}^{\infty} X(s + jn\omega_s) \quad (6.46a)$$

where $z = e^{sT}$.

- Recovery of a continuous-time signal by lowpass filtering an impulse-modulated signal – *frequency domain*:

$$Y(j\omega) = H(j\omega)\hat{X}(j\omega) \tag{6.48}$$

where

$$H(j\omega) = \begin{cases} T & \text{for } |\omega| < \omega_s/2 \\ 0 & \text{for } |\omega| \geq \omega_s/2 \end{cases}$$

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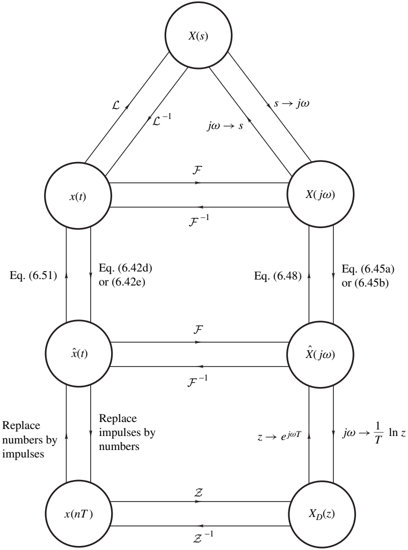
- Recovery of a continuous-time signal by lowpass filtering an impulse-modulated signal – *time-domain*:

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT)\frac{\sin[\omega_s(t-nT)/2]}{\omega_s(t-nT)/2} \tag{6.51}$$

Graphical Representation of Interrelations



This slide concludes the presentation.
Thank you for your attention.