Chapter 10
APPROXIMATIONS FOR ANALOG FILTERS
10.8 Analog-Filter Transformations

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Analog-Filter Transformations

- Given a normalized transfer function obtained by any one of the classical analog-filter approximations, a *denormalized* lowpass, highpass, bandpass, or bandstop transfer function can be obtained by applying a transformation of the form

$$H_X(\bar{s}) = H_N(s)\Big|_{s=f_X(\bar{s})}$$

where $f_X(\bar{s})$ is one of the four standard *analog-filter transformations*.

Standard forms of $f_X(\bar{s})$

Type	$f_X(\bar{s})$
LP to LP	$\lambda\bar{s}$
LP to HP	$\lambda/\bar{s}$
LP to BP	$\frac{1}{B}\left(\bar{s}+\frac{\omega_0^2}{\bar{s}}\right)$
LP to BS	$\frac{B\bar{s}}{\bar{s}^2+\omega_0^2}$

Lowpass-to-Lowpass Transformation

- Consider a normalized lowpass transfer function $H_N(s)$ with passband and stopband edges ω_p and ω_a.

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$$H_{LP}(\bar{s}) = H_N(s)\Big|_{s=\lambda\bar{s}}$$

- By letting $\bar{s} = j\bar{\omega}$ and $s = j\omega$, we get

$$|H_{LP}(j\bar{\omega})| = |H_N(j\omega)|\Big|_{j\omega\, =\, j\lambda\bar{\omega}}$$

Therefore, the gain (loss) of the denormalized lowpass filter is equal to the gain (loss) of the normalized lowpass filter provided that $\omega = \lambda\bar{\omega}$.

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$$|H_{LP}(j\bar{\omega})| = |H_N(j\omega)|\Big|_{j\omega = j\lambda\bar{\omega}}$$

- Thus points on the $j\omega$ axis of the s plane map onto points on the $j\bar{\omega}$ axis of the $\bar{s}$ plane.

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LP-to-LP Transformation *Cont'd*

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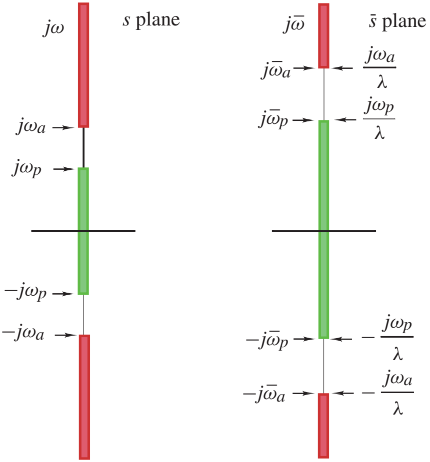
  - infinity of the s plane maps onto infinity of the $\bar{s}$ plane.


s plane
$j\omega$
$j\omega_a$
$j\omega_p$
$-j\omega_p$
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$\bar{s}$ plane
$j\bar{\omega}$
$j\bar{\omega}_a$
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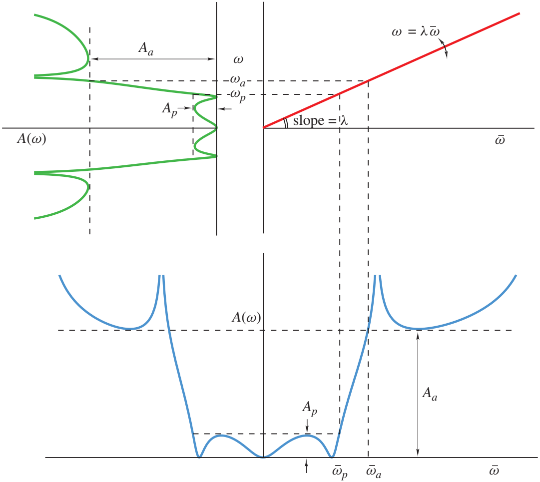
LP-to-LP Transformation – Graphical Illustration



Lowpass-to-Highpass Transformation

- The LP-to-HP transformation follows the pattern of the LP-to-LP transformation.

Lowpass-to-Bandpass Transformation

- A denormalized bandpass transfer function can be obtained from a normalized lowpass transfer function as follows:

$$H_{BP}(\bar{s}) = H_N(s)\Big|_{s=\frac{1}{B}\left(\bar{s}+\frac{\omega_0^2}{\bar{s}}\right)}$$

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- By letting $\bar{s} = j\bar{\omega}$ and $s = j\omega$, we get

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- Therefore, the gain (loss) of the denormalized bandpass filter is equal to the gain (loss) of the normalized lowpass filter provided that

$$\omega = \frac{1}{B}\left(\bar{\omega}-\frac{\omega_0^2}{\bar{\omega}}\right)$$

...

$$\omega = \frac{1}{B}\left(\bar{\omega} - \frac{\omega_0^2}{\bar{\omega}}\right)$$

- Solving for $\bar{\omega}$, we get

$$\bar{\omega} = \begin{cases} \omega_0 & \text{if } \omega = 0 \\ \pm\bar{\omega}_{p1}, \pm\bar{\omega}_{p2} & \text{if } \omega = \pm\omega_p \\ \pm\bar{\omega}_{a1}, \pm\bar{\omega}_{a2} & \text{if } \omega = \pm\omega_a \end{cases}$$

where

$$\bar{\omega}_{p1}, \bar{\omega}_{p2} = \mp\frac{\omega_p B}{2} + \sqrt{\omega_0^2 + \left(\frac{\omega_p B}{2}\right)^2}$$

$$\bar{\omega}_{a1}, \bar{\omega}_{a2} = \mp\frac{\omega_a B}{2} + \sqrt{\omega_0^2 + \left(\frac{\omega_a B}{2}\right)^2}$$

- Thus,

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LP-to-BP Transformation *Cont'd*

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- In effect,
  - the origin of the s plane maps onto point $j\omega_0$ of the $\bar{s}$ plane,

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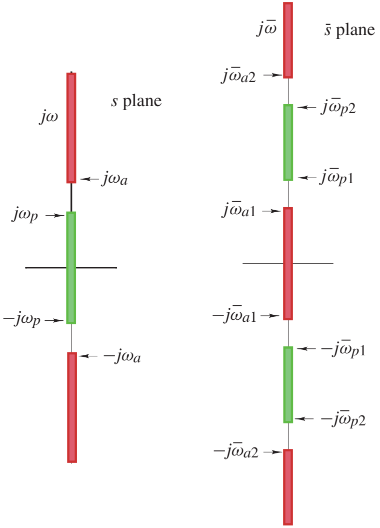
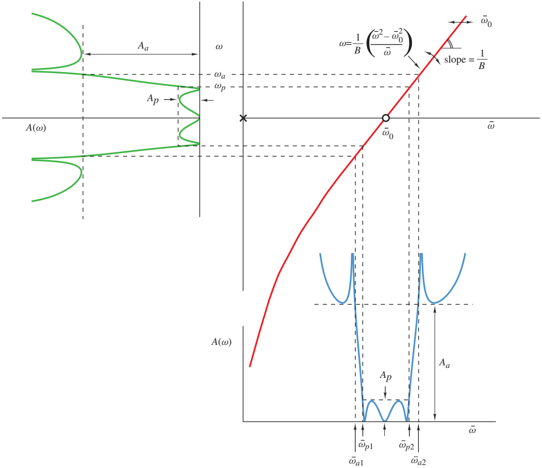
  - $\pm\infty$ of the s plane maps onto $\mp\infty$ of the $\bar{s}$ plane.

LP-to-BP Transformation – Mapping Properties



LP-to-BP Transformation – Graphical Illustration



Lowpass-to-Bandstop Transformation

- The LP-to-BS transformation follows the pattern of the LP-to-BP transformation.

Analog-Filter Transformations *Cont'd*

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  - Parameter λ scales the passband or stopband edge of a denormalized lowpass or highpass filter relative to the passband or stopband edge of the normalized filter.
  - Parameters ω_0 and B scale the location and passband or stopband width of a denormalized bandpass or bandstop filter.

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  - Parameter λ scales the passband or stopband edge of a denormalized lowpass or highpass filter relative to the passband or stopband edge of the normalized filter.
  - Parameters ω_0 and B scale the location and passband or stopband width of a denormalized bandpass or bandstop filter.
- The transformations are used in Chap. 12 to design recursive lowpass, highpass, bandpass, and bandstop filters that would satisfy arbitrary prescribed specifications.

This slide concludes the presentation.
Thank you for your attention.