Chapter 15
DESIGN OF NONRECURSIVE FILTERS USING OPTIMIZATION

15.1 Introduction 15.2 Problem Formulation
15.3 Remez Exchange Algorithm 15.4 Improved Search Methods 15.5 Efficient Remez Exchange Algorithm 15.7 Prescribed Specifications

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The weighted-Chebyshev method for the design of nonrecursive filters is an iterative multivariable optimization method based on the *Remez Exchange Algorithm*. 
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Enhancements to the weighted-Chebyshev method were proposed by Antoniou during the early eighties.
Consider a nonrecursive filter characterized by the transfer function

\[ H(z) = \sum_{n=0}^{N-1} h(nT)z^{-n} \]

and assume that

- the filter length \( N \) is odd (the filter order \( N - 1 \) is even),
- the impulse response is symmetrical, and
- the sampling frequency is \( \omega_s = 2\pi \) rad/s (the Nyquist frequency is \( \pi \) rad/s) and the sampling period is \( T = 1 \) s.
The frequency response of the filter can be expressed as

\[ H(e^{j\omega}) = e^{-j\omega} P_c(\omega) \]

where

\[ P_c(\omega) = \sum_{k=0}^{c} a_k \cos k\omega \]

is the frequency response of a noncausal version of the required filter and

\[ a_0 = h(c) \]
\[ a_k = 2h(c - k) \quad \text{for} \quad k = 1, 2, \ldots, c \]
\[ c = (N - 1)/2 \]
An error function $E(\omega)$ can be constructed as

$$E(\omega) = W(\omega)[D(\omega) - P_c(\omega)]$$

where $e^{-j\omega}D(\omega)$ is the idealized frequency response of the desired filter, $W(\omega)$ is a weighting function, and

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If $|E(\omega)|$ is minimized such that

$$|E(\omega)| = |W(\omega)[D(\omega) - P_c(\omega)]| \leq \delta_p \quad \text{for } \omega \in \Omega \quad (B)$$

with respect to a set of frequencies in the interval $[0, \pi]$, say $\Omega$, a filter can be obtained in which

$$|E_0(\omega)| = |D(\omega) - P_c(\omega)| \leq \frac{\delta_p}{|W(\omega)|} \quad \text{for } \omega \in \Omega \quad (C)$$
In the case of a lowpass filter, the minimization of $|E(\omega)|$ will force the inequality

$$|E_0(\omega)| = |D(\omega) - P_c(\omega)| \leq \frac{\delta_p}{|W(\omega)|} \quad \text{for} \quad \omega \in \Omega \quad \text{(C)}$$

where

$$D(\omega) = \begin{cases} 
1 & \text{for } 0 \leq \omega \leq \omega_p \\
0 & \text{for } \omega_a \leq \omega \leq \pi
\end{cases}$$
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where

$$D(\omega) = \begin{cases} 
1 & \text{for } 0 \leq \omega \leq \omega_p \\
0 & \text{for } \omega_a \leq \omega \leq \pi 
\end{cases}$$

In effect, a minimization algorithm will force the actual gain function $P_c(\omega)$ to approach the ideal gain function $D(\omega)$. 
Lowpass Filters *Cont’d*

![Diagram showing gain response of lowpass filters with labels $D(\omega)$, $E_0(\omega)$, $P_c(\omega)$, $\omega_p$, and $\omega_d$.](image)
If we choose the weighting function

$$W(\omega) = \begin{cases} 
1 & \text{for } 0 \leq \omega \leq \omega_p \\
\frac{\delta_p}{\delta_a} & \text{for } \omega_a \leq \omega \leq \pi 
\end{cases}$$

then from Eq. (C), i.e.,

$$|E_0(\omega)| = |D(\omega) - P_c(\omega)| \leq \frac{\delta_p}{|W(\omega)|} \quad \text{for } \omega \in \Omega \quad \text{(C)}$$

we get

$$|E_0(\omega)| \leq \begin{cases} 
\delta_p & \text{for } 0 \leq \omega \leq \omega_p \\
\delta_a & \text{for } \omega_a \leq \omega \leq \pi 
\end{cases}$$
The most appropriate approach for the solution of the optimization problem just described is to solve the *minimax* problem

\[
\text{minimize } \{ \max_\omega |E(\omega)| \}
\]

where

\[
x = [a_0 \ a_1 \ \cdots \ a_c]^T
\]
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\minimize_x \{ \max_\omega |E(\omega)| \}
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By virtue of the so-called \textit{alternation theorem}, there is a \textit{unique equiripple} solution of the above minimax problem.
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$$\minimize_x \{ \max_\omega |E(\omega)| \}$$

where

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By virtue of the so-called \textit{alternation theorem}, there is a \textit{unique equiripple} solution of the above minimax problem.

\textit{Note} that weighted-Chebyshev filters are so called because they have an \textit{equiripple} amplitude response just like Chebyshev filters but are not related to Chebyshev filters in any other way.
Minimax Problem \textit{Cont’d}

\[ D(\omega) \quad P_c(\omega) \]

\[ 1 + \delta_p \]

\[ 1 - \delta_p \]

\[ \delta_a \]

\[ \omega_p \quad \omega_a \quad \omega \]
Alternation Theorem

- If \( P_c(\omega) \) is a linear combination of \( r = c + 1 \) cosine functions of the form

\[
P_c(\omega) = \sum_{k=0}^{c} a_k \cos k\omega
\]

then a necessary and sufficient condition that \( P_c(\omega) \) be the unique, best, weighted-Chebyshev approximation to a continuous function \( D(\omega) \) on \( \Omega \), where \( \Omega \) is a dense and compact subset of the frequency interval \([0, \pi]\), is that the weighted error function \( E(\omega) \) exhibit at least \( r + 1 \) extremal frequencies \( \hat{\omega}_i \) in \( \Omega \) such that

\[
\hat{\omega}_0 < \hat{\omega}_1 < \cdots < \hat{\omega}_r
\]

\[
E(\hat{\omega}_{i+1}) = -E(\hat{\omega}_i) \quad \text{for } i = 0, 1, \ldots, r - 1
\]

and

\[
|E(\hat{\omega}_i)| = \max_{\omega \in \Omega} |E(\omega)| \quad \text{for } i = 0, 1, \ldots, r
\]
Notes:

– A subset Ω is *dense* if it has a sufficiently large number of members for the application at hand.
– A subset Ω is *compact* if it is closed and bounded.
– A subset is *closed* if all its limits are members of the set.
– A subset is *bounded* if all its members are bounded.
From the alternation theorem and Eq. (B), i.e.,

\[ E(\omega) = W(\omega)[D(\omega) - P_c(\omega)] \]  \hspace{1cm} (B)

we can write

\[ E(\hat{\omega}_i) = W(\hat{\omega}_i)[D(\hat{\omega}_i) - P_c(\hat{\omega}_i)] = (-1)^i \delta \]

for \( i = 0, 1, \ldots, r \), where \( \delta \) is a constant.
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for \( i = 0, 1, \ldots, r \), where \( \delta \) is a constant.

The above system of equations can be put in matrix form as

\[
\begin{bmatrix}
1 & \cos \hat{\omega}_0 & \cos 2\hat{\omega}_0 & \cdots & \cos c\hat{\omega}_0 & \frac{1}{W(\hat{\omega}_0)} \\
1 & \cos \hat{\omega}_1 & \cos 2\hat{\omega}_1 & \cdots & \cos c\hat{\omega}_1 & \frac{-1}{W(\hat{\omega}_1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \cos \hat{\omega}_r & \cos 2\hat{\omega}_r & \cdots & \cos c\hat{\omega}_r & \frac{(-1)^r}{W(\hat{\omega}_r)}
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_c \\
\delta
\end{bmatrix}
= \begin{bmatrix}
D(\hat{\omega}_0) \\
D(\hat{\omega}_1) \\
\vdots \\
D(\hat{\omega}_{r-1}) \\
D(\hat{\omega}_r)
\end{bmatrix}
\]
Alternation Theorem  *Cont’d*

- If the extremal frequencies (or extremals for short) were known, coefficients $a_k$ and, in turn, the frequency response of the filter could be computed using Eq. (A), i.e.,

$$P_c(\omega) = \sum_{k=0}^{c} a_k \cos k\omega$$

(A)
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$$P_c(\omega) = \sum_{k=0}^{c} a_k \cos k\omega$$  \hspace{1cm} (A)

The solution of this system exists since the above $(r + 1) \times (r + 1)$ matrix is known to be nonsingular.
The Remez exchange algorithm is an *iterative multivariable algorithm* that is naturally suited for the solution of the minimax problem just described.

It is based on the *second optimization method of Remez*. 
1. Initialize extremal frequencies $\hat{\omega}_0, \hat{\omega}_1, \ldots, \hat{\omega}_r$ and ensure that an extremal is assigned at each band edge.
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2. Locate the frequencies $\tilde{\omega}_0, \tilde{\omega}_1, \ldots, \tilde{\omega}_\rho$ at which the magnitude of the error

$$|E(\omega)| = |W(\omega)[D(\omega) - P_c(\omega)]|$$

is maximum and $|E(\tilde{\omega}_i)| \geq \delta$ (these frequencies are potential extremals for the next iteration).
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is maximum and $|E(\tilde{\omega}_i)| \geq \delta$ (these frequencies are potential extremals for the next iteration).

3. Compute the convergence parameter

$$Q = \frac{\max |E(\tilde{\omega}_i)| - \min |E(\tilde{\omega}_i)|}{\max |E(\tilde{\omega}_i)|}$$

where $i = 0, 1, \ldots, \rho$. 
4. Reject $\rho - r$ superfluous potential extremals $\widehat{\omega}_i$ according to an appropriate rejection criterion and renumber the remaining $\widehat{\omega}_i$ by setting $\hat{\omega}_i = \widehat{\omega}_i$ for $i = 0, 1, \ldots, r$. 
4. Reject $\rho - r$ superfluous potential extremals $\hat{\omega}_i$ according to an appropriate rejection criterion and renumber the remaining $\hat{\omega}_i$ by setting $\hat{\omega}_i = \hat{\omega}_i$ for $i = 0, 1, \ldots, r$.

5. If $Q > \varepsilon$, where $\varepsilon$ is a convergence tolerance (say $\varepsilon = 0.01$), repeat from step 2; otherwise continue to step 6.
4. Reject $\rho - r$ superfluous potential extremals $\hat{\omega}_i$ according to an appropriate rejection criterion and renumber the remaining $\hat{\omega}_i$ by setting $\hat{\omega}_i = \bar{\omega}_i$ for $i = 0, 1, \ldots, r$.

5. If $Q > \varepsilon$, where $\varepsilon$ is a convergence tolerance (say $\varepsilon = 0.01$), repeat from step 2; otherwise continue to step 6.

6. Compute $P_c(\omega)$ using the last set of extremal frequencies; then deduce $h(n)$, the impulse response of the required filter, and stop.
Initialization of Extremal Frequencies

The implementation of the basic Remez algorithm can be accomplished as follows:

**Step 1:**
- A simple initialization scheme is to distribute the extremals uniformly in each passband and stopband such that
  - the total number of extremals is exactly equal to \( r + 1 = \frac{N + 3}{2} \),
  - the number of extremals in each passband or stopband is proportional to the bandwidth of the passband or stopband,
  - there is an extremal at each band edge.

Such a scheme is illustrated in the next slide.
The implementation of the basic Remez algorithm can be accomplished as follows:

**Step 1:**
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Such a scheme is illustrated in the next slide.
Initialization of Extremal Frequencies

**Bands: 1**
- Extremals: $r+1$ (12)
- Intervals: $r$ (11)

\[ W_0 = \frac{B_1}{r} \]

**Bands: 2**
- Extremals: $r+1$ (13)
- Intervals: $r-1$ (11)

\[ W_1 = \frac{B_1}{m_1}, \quad W_2 = \frac{B_2}{m_2} \]

**Bands: 3**
- Extremals: $r+1$ (14)
- Intervals: $r-2$ (11)

\[ W_1 = \frac{B_1}{m_1}, \quad W_3 = \frac{B_3}{m_3}, \quad W_2 = \frac{B_2}{m_2} \]
For a filter with $J$ bands with bandwidths $B_1, B_2, \ldots, B_J$, the number of extremals and intervals between extremals for each band can be calculated by using the formulas

$$W_0 = \frac{1}{r + 1 - J} \sum_{j=1}^{J} B_j$$

$$m_j = \text{int} \left( \frac{B_j}{W_0} + 0.5 \right) \quad \text{for } j = 1, 2, \ldots, J - 1$$

and

$$m_J = r - \sum_{j=1}^{J-1} (m_j + 1)$$

$$W_j = \frac{B_j}{m_j} \quad \text{for } j = 1, 2, \ldots, J$$

where $r = (N + 1)/2$ and $N$ is the filter length.
**Step 2:**

- In order to locate the frequencies $\hat{\omega}_0, \hat{\omega}_1, \ldots, \hat{\omega}_\rho$ at which $|E(\omega)|$ is maximum such that $|E(\hat{\omega}_i)| \geq \delta$, we calculate coefficients $a_0, a_1, \ldots, a_c$ and parameter $\delta$ by solving the system

\[
\begin{bmatrix}
1 & \cos \hat{\omega}_0 & \cos 2\hat{\omega}_0 & \cdots & \cos c\hat{\omega}_0 & \frac{1}{W(\hat{\omega}_0)} \\
1 & \cos \hat{\omega}_1 & \cos 2\hat{\omega}_1 & \cdots & \cos c\hat{\omega}_1 & \frac{1}{W(\hat{\omega}_1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \cos \hat{\omega}_r & \cos 2\hat{\omega}_r & \cdots & \cos c\hat{\omega}_r & \frac{(-1)^r}{W(\hat{\omega}_r)}
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_c \\
\delta
\end{bmatrix}
= 
\begin{bmatrix}
D(\hat{\omega}_0) \\
D(\hat{\omega}_1) \\
\vdots \\
D(\hat{\omega}_{r-1}) \\
D(\hat{\omega}_r)
\end{bmatrix}
\]
• With coefficients $a_0, a_1, \ldots, a_c$ known, polynomial

$$P_c(\omega) = \sum_{k=0}^{c} a_k \cos k\omega$$

can be calculated.
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$$P_c(\omega) = \sum_{k=0}^{c} a_k \cos k\omega$$

can be calculated.

With $P_c(\omega)$ known, the error function

$$|E(\omega)| = |W(\omega)[D(\omega) - P_c(\omega)]|$$

can be calculated.
Updating of Extremals  Cont’d

- The maxima of the error function can be obtained by evaluating $|E(\omega)|$ over a dense set of frequencies in the passband(s) and stopband(s) of the required filter.
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A sufficient number of frequency points for most applications is around 16 sample points per ripple in $|E(\omega)|$, i.e., $8(N + 1)$. 
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A sufficient number of frequency points for most applications is around 16 sample points per ripple in $|E(\omega)|$, i.e., $8(N + 1)$.

An actual plot of $|E(\omega)|$ versus $\omega$ is shown in the next slide.
Updating of Extremals Cont’d

Filter length: 27
Iteration no: 1

Error at Sample Points

<table>
<thead>
<tr>
<th>Frequency, rad/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>0.01</td>
</tr>
<tr>
<td>0.02</td>
</tr>
<tr>
<td>0.03</td>
</tr>
<tr>
<td>0.04</td>
</tr>
<tr>
<td>0.05</td>
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<tr>
<td>0.06</td>
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<tr>
<td>0.07</td>
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<td>0.08</td>
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<tr>
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<tr>
<td>0.12</td>
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<tr>
<td>0.13</td>
</tr>
<tr>
<td>0.14</td>
</tr>
<tr>
<td>0.15</td>
</tr>
</tbody>
</table>

Function Evals: 0
The approach just described is easy to apply. However, it is *inefficient* and may be subject to numerical ill-conditioning in particular if $\delta$ is small and $N$ is large.

*Note* that a $50 \times 50$ matrix is quite typical and a $100 \times 100$ matrix is not unusual.
An alternative and *more efficient* approach is to deduce \( \delta \) analytically (by using Cramer’s rule) and then interpolate \( P_c(\omega) \) on the \( r \) frequency points using the *barycentric* form of the Lagrange interpolation formula, as follows:
An alternative and more efficient approach is to deduce $\delta$ analytically (by using Cramer’s rule) and then interpolate $P_c(\omega)$ on the $r$ frequency points using the barycentric form of the Lagrange interpolation formula, as follows:

Calculate parameter $\delta$ as

$$\delta = \sum_{k=0}^{r} \frac{\alpha_k D(\hat{\omega}_k)}{\sum_{k=0}^{r} (-1)^k \alpha_k W(\hat{\omega})}$$
With $\delta$ and $P_{c}(\hat{\omega}_{k}) = C_{k} = D(\hat{\omega}_{k}) - (-1)^{k} \frac{\delta}{W(\hat{\omega}_{k})}$ known, the following interpolation formula can be constructed:

$$P_{c}(\omega) = \begin{cases} 
C_{k} & \text{for } \omega = \hat{\omega}_{0}, \hat{\omega}_{1}, \ldots, \hat{\omega}_{r-1} \\
\sum_{k=0}^{r-1} \frac{\beta_{k} C_{k}}{x - x_{k}} & \text{otherwise}
\end{cases}$$

where $\alpha_{k} = \prod_{i=0, i \neq k}^{r} \frac{1}{x_{k} - x_{i}}$, $\beta_{k} = \prod_{i=0, i \neq k}^{r-1} \frac{1}{x_{k} - x_{i}}$

and $x = \cos \omega$ and $x_{i} = \cos \hat{\omega}_{i}$ for $i = 0, 1, \ldots, r$
Using the interpolation formula, the value of $P_c(\omega)$ for any frequency $\omega$ can be computed.
Using the interpolation formula, the value of $P_c(\omega)$ for any frequency $\omega$ can be computed.

Since $W(\omega)$ and $D(\omega)$ are known, the error function

$$|E(\omega)| = |W(\omega)[D(\omega) - P_c(\omega)]|$$

and, in turn, the frequencies $\tilde{\omega}_0, \tilde{\omega}_1, \ldots, \tilde{\omega}_\rho$ at which $|E(\omega)|$ is maximum can be deduced.
Updating of Extremals \( Cont'd \)

\[\begin{align*}
\omega_4 j^+ &+ \omega_3 j^- - \omega_5 j^- - \omega_6 j^+ + \omega_2 j^+ - \omega_7 j^- \\
\end{align*}\]

\( |P_c(\omega)| \)

\( |\delta| \)

\( \omega_L j \quad \omega_2 j \quad \omega_3 j \quad \omega_4 j \quad \omega_5 j \quad \omega_6 j \quad \omega_R j \)

\( \hat{\omega}_1 j \quad \hat{\omega}_2 j \quad \hat{\omega}_3 j \quad \hat{\omega}_4 j \quad \hat{\omega}_5 j \quad \hat{\omega}_6 j \quad \hat{\omega}_7 j \)
Step 3:
- Compute the convergence parameter

\[ Q = \frac{\max |E(\hat{\omega}_i)| - \min |E(\hat{\omega}_i)|}{\max |E(\hat{\omega}_i)|} \]
\[ |\omega_4j^\ast + \omega_3j^\ast - \omega_5j^\ast - \omega_6j^\ast + \omega_2j^\ast - \omega_7j^\ast| \]

\[ \omega_Rj \]

\[ \omega_1j^\ast \]

\[ \omega_Lj \]

\[ \omega_{1j} \]

\[ \omega_{2j} \]

\[ \omega_{3j} \]

\[ \omega_{4j} \]

\[ \omega_{5j} \]

\[ \omega_{6j} \]

\[ \omega_{7j} \]

\[ |\delta| \]

\[ E(\omega) \]

\[ E(\bar{\omega}_i) \]

\[ E(\bar{\omega}_i) \max \]

\[ E(\bar{\omega}_i) \min \]
Step 4:

- The problem formulation is such that there must be exactly $r + 1$ extremals in each iteration.
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- Analysis will show that \( |E(\omega)| \) can have as many as \( r + 2J - 1 \) maxima where \( J \) is the number of bands:
  - For a 1-band filter (differentiators): \( r + 1 \) (no extra maxima)
  - For a 2-band filter (lowpass or highpass filter): \( r + 3 \) (2 extra maxima)
  - For a 3-band filter (bandpass or bandstop filter): \( r + 5 \) (4 extra maxima)

If in any iteration the number of maxima exceeds \( r + 1 \), then the iteration is said to have generated superfluous potential extremals.
Rejection of Superfluous Potential Extremals

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Step 4:

- The problem formulation is such that there must be exactly \( r + 1 \) extremals in each iteration.

- Analysis will show that \(|E(\omega)|\) can have as many as \( r + 2J - 1 \) maxima where \( J \) is the number of bands:
  - For a 1-band filter (differentiators): \( r+1 \) (no extra maxima)
  - For a 2-band filter (lowpass or highpass filter): \( r+3 \) (2 extra maxima)
  - For a 3-band filter (bandpass or bandstop filter): \( r+5 \) (4 extra maxima)

- If in any iteration the number of maxima exceeds \( r + 1 \), then the iteration is said to have generated superfluous potential extremals.
In the standard McClellan, Rabiner, and Parks algorithm, this difficulty is circumvented by rejecting the $\rho - r$ potential extremals $\hat{\omega}_i$ that yield the lowest error $|E(\omega)|$. 
Rejection of Superfluous Potential Extremals Cont’d

\[ |E(\omega)| \]

\[ |\delta| \]

\[ \omega_1 \]
\[ \omega_2 \]
\[ \omega_3 \]
\[ \omega_4 \]
\[ \omega_5 \]
\[ \omega_6 \]
\[ \omega_R \]
\[ \omega_L \]
**Step 5:**

- If the convergence parameter is not small enough, i.e., if the ripples have not equalized sufficiently, repeat from Step 2.
**Step 6:**

- The impulse response can be determined by recalling that function $P_c(\omega)$ is the frequency response of a noncausal version of the required filter.
Step 6:

- The impulse response can be determined by recalling that function $P_c(\omega)$ is the frequency response of a noncausal version of the required filter.

- The impulse response of the noncausal filter, denoted as $h_0(n)$ for $-c \leq n \leq c$, can be determined by computing $P_c(k\Omega)$ for $k = 0, 1, \ldots, c$ where $\Omega = 2\pi/N$, and then using the inverse discrete Fourier transform.
It can be shown that

\[ h_0(n) = h_0(-n) = \frac{1}{N} \left\{ P_c(0) + \sum_{k=1}^{c} 2P_c(k\Omega) \cos \left( \frac{2\pi kn}{N} \right) \right\} \]

for \( n = 0, 1, \ldots, c \).
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for \( n = 0, 1, \ldots, c \).

The impulse response of the required causal filter is given by

\[ h(n) = h_0(n - c) \]

for \( n = 0, 1, \ldots, c \).
### Example

<table>
<thead>
<tr>
<th>Band</th>
<th>$D(\omega)$</th>
<th>$W(\omega)$</th>
<th>Left band edge</th>
<th>Right band edge</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1.0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.4</td>
<td>1.25</td>
<td>$\pi$</td>
</tr>
</tbody>
</table>

Sampling frequency: $2\pi$
Filter length: 27
Iteration no: 1

Error at Sample Points

|\|E(\omega)| | |\|E(\omega)| |

Function Evals: 0
Example Cont’d

Filter length: 27
Iteration no: 2

Function Evals: 199

Error at Sample Points

Frequency, rad/s

Filter length: 27
Iteration no: 2

Function Evals: 199

Error at Sample Points

Frequency, rad/s
Example Cont’d

Filter length: 27
Iteration no: 3

Error at Sample Points

Function Evals: 398
Filter length: 27
Iteration no: 4

Function Evals: 597
Filter length: 27
Iteration no: 5

Function Evals: 796

Error at Sample Points

Frequency, rad/s
Filter length: 27
Iteration no: 6  

Function Evals: 995
The Remez exchange described is using an *exhaustive search* to identify the maxima of $|E(\omega)|$. Consider a filter of length $N$ and assume that $|E(\omega)|$ is evaluated at $S$ sample points per ripple. The algorithm presented would require $S \times \frac{(N+1)}{2}$ function evaluations. One function evaluation requires:

- $N - 1$ additions
- $\frac{(N+1)}{2}$ multiplications
- $\frac{(N+1)}{2}$ divisions
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- $(N + 1)/2$ multiplications
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A Remez optimization usually requires
- 4 to 8 iterations for lowpass or highpass filters,
- 6 to 10 iterations for bandpass filters, and
- 8 to 12 iterations for bandstop filters.
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- 4 to 8 iterations for lowpass or highpass filters,
- 6 to 10 iterations for bandpass filters, and
- 8 to 12 iterations for bandstop filters.

If prescribed specifications are to be achieved and the appropriate value of $N$ is unknown, typically two to four Remez optimizations have to be performed.
For example, if
- \( N = 101 \),
- \( S = 16 \),
- number of Remez optimizations = 4,
- iterations per optimization = 6,
the design would entail 24 iterations, 19,200 function evaluations, \( 1.92 \times 10^6 \) additions, \( 0.979 \times 10^6 \) multiplications, and \( 0.979 \times 10^6 \) divisions.
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This is in addition to the computation required for the evaluation of \( \delta \) and coefficients \( \alpha_k, C_k, \) and \( \beta_k \) once per iteration.
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- $S = 16$, 
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This is in addition to the computation required for the evaluation of $\delta$ and coefficients $\alpha_k$, $C_k$, and $\beta_k$ once per iteration.

In effect, the amount of computation required to complete a design is quite substantial.
Selective Step-by-Step Search

- When the system of equations

\[
\begin{bmatrix}
1 & \cos \hat{\omega}_0 & \cos 2\hat{\omega}_0 & \cdots & \cos c\hat{\omega}_0 & \frac{1}{W(\hat{\omega}_0)} \\
1 & \cos \hat{\omega}_1 & \cos 2\hat{\omega}_1 & \cdots & \cos c\hat{\omega}_1 & \frac{-1}{W(\hat{\omega}_1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \cos \hat{\omega}_r & \cos 2\hat{\omega}_r & \cdots & \cos c\hat{\omega}_r & \frac{(-1)^r}{W(\hat{\omega}_r)}
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_c \\
\delta
\end{bmatrix}
= \begin{bmatrix}
D(\hat{\omega}_0) \\
D(\hat{\omega}_1) \\
\vdots \\
D(\hat{\omega}_{r-1}) \\
D(\hat{\omega}_r)
\end{bmatrix}
\]

is solved, the error function $|E(\omega)|$ is forced to satisfy the relation

\[
|E(\hat{\omega}_i)| = |W(\hat{\omega}_i)[D(\hat{\omega}_i) - P_c(\hat{\omega}_i)]| = |\delta|
\]
Selective Step-by-Step Search

When the system of equations

\[
\begin{bmatrix}
1 & \cos \hat{\omega}_0 & \cos 2\hat{\omega}_0 & \cdots & \cos c\hat{\omega}_0 & \frac{1}{W(\hat{\omega}_0)} \\
1 & \cos \hat{\omega}_1 & \cos 2\hat{\omega}_1 & \cdots & \cos c\hat{\omega}_1 & \frac{-1}{W(\hat{\omega}_1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
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\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_c \\
\delta
\end{bmatrix}
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D(\hat{\omega}_0) \\
D(\hat{\omega}_1) \\
\vdots \\
D(\hat{\omega}_{r-1}) \\
D(\hat{\omega}_r)
\end{bmatrix}
\]

is solved, the error function \( |E(\omega)| \) is forced to satisfy the relation

\[
|E(\hat{\omega}_i)| = |W(\hat{\omega}_i)[D(\hat{\omega}_i) - P_c(\hat{\omega}_i)]| = |\delta|
\]

This relation can be satisfied in a number of ways but the most likely possibility for the \( j \)th band is illustrated in the next slide where \( \omega_{Lj} \) and \( \omega_{Rj} \) are the left-hand and right-hand edges, respectively.
Selective Step-by-Step Search  Cont’d

\[ |E(\omega)| \]

Frame # 50  Slide # 90  A. Antoniou  Digital Signal Processing – Secs. 15.1-15.5, 15.7
Because of the special nature of the error function

(a) the maxima of $|E(\omega)|$ can be easily found by searching in the vicinity of the extremals;

(b) gradient information can be used to expedite the search for the maxima of $|E(\omega)|$; and

(c) the closer we get to the solution, the closer are the maxima of the error function to the extremals.
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(a) the maxima of $|E(\omega)|$ can be easily found by searching in the vicinity of the extremals;

(b) gradient information can be used to expedite the search for the maxima of $|E(\omega)|$; and

(c) the closer we get to the solution, the closer are the maxima of the error function to the extremals.

By using a selective step-by-step search, a large amount of computation can be eliminated.
Extra ripples can arise in the first and last bands:
● Also in interior bands:

\[ |E(\omega)| \]

\[ |\delta| \]

\[ \omega_{L_j} \quad \hat{\omega}_{1j} \quad \hat{\omega}_{2j} \quad \hat{\omega}_{3j} \]

\[ (f) \]

\[ \omega_{R_j} \quad \hat{\omega}_{(\mu_j-1)j} \quad \hat{\omega}_{\mu_j} \]

\[ (g) \]
Increased computational efficiency can be achieved by using a search based on *cubic interpolation*.
Cubic Interpolation Search

- Increased computational efficiency can be achieved by using a search based on cubic interpolation.

- Assuming that the magnitude of the error can be represented by the third-order polynomial

\[ |E(\omega)| = M = a + b\omega + c\omega^2 + d\omega^3 \]

where \( a, b, c, \) and \( d \) are constants then

\[ \frac{dM}{d\omega} = G = b + 2c\omega + 3d\omega^2 \]

Hence, the frequencies at which \( M \) has stationary points are given by

\[ \bar{\omega} = \frac{1}{3d} \left[ -c \pm \sqrt{c^2 - 3bd} \right] \]
Increased computational efficiency can be achieved by using a search based on cubic interpolation.

Assuming that the magnitude of the error can be represented by the third-order polynomial

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Hence, the frequencies at which $M$ has stationary points are given by

$$\bar{\omega} = \frac{1}{3d} \left[-c \pm \sqrt{(c^2 - 3bd)}\right]$$

Therefore, $|E(\omega)|$ has a maximum if

$$\frac{d^2M}{d\omega^2} = 2c + 6d\bar{\omega} < 0 \text{ or } \bar{\omega} < -\frac{c}{3d}$$
Cubic Interpolation Search \textit{Cont'd}

\[ |E(\omega)| \]

\[ \tilde{\omega}_1 \quad \tilde{\omega}_2 \quad \tilde{\omega}_3 \quad \omega \]
The cubic interpolation method requires four function evaluations per potential extremal consistently. The selective step-by-step search may require as many as eight function evaluations per potential extremal in the first two or three iterations but as the solution is approached only two or three function evaluations are required. By using the cubic interpolation to start with and then switching over to the step-by-step search, a very efficient algorithm can be constructed. The decision to switch from cubic to selective can be based on the value of the convergence parameter $Q$ (see Step 5). Switching from the cubic to the selective when $Q$ is reduced below 0.65 works well.
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The decision to switch from cubic to selective can be based on the value of the convergence parameter $Q$ (see Step 5). Switching from the cubic to the selective when $Q$ is reduced below 0.65 works well.
If an extremal does not move from one iteration to the next, then the minimum value of $E(\hat{\omega}_i)$ is simply $\delta$, as can be easily shown, and this happens quite often even in the first or second iteration of the Remez algorithm.
Improved Rejection Scheme \textit{Cont’d}

Filter length: 27  
Iteration no: 1

Function Evals: 0

![Error at Sample Points](image-url)

Frame # 58  Slide # 104  A. Antoniou  Digital Signal Processing – Secs. 15.1-15.5, 15.7
As a consequence, rejecting potential extremals on the basis of the individual values of $E(\omega_i)$ tends to become random and this can slow the Remez algorithm quite significantly particularly for multiband filters.
As a consequence, rejecting potential extremals on the basis of the individual values of $E(\tilde{\omega};)$ tends to become random and this can slow the Remez algorithm quite significantly particularly for multiband filters.

An improved scheme for the rejection of superfluous extremals based the rejection on the lowest average band error as well as the individual values of $E(\tilde{\omega};)$ is described in the next slide.
Compute the average band errors

\[ E_j = \frac{1}{\nu_j} \sum_{\omega_i \in \Omega_j} |E(\omega_i)| \quad \text{for } j = 1, 2, \ldots, J \]

where \( \Omega_j \) is the set of extremals in band \( j \) given by

\[ \Omega_j = \{ \omega_i : \omega_{Lj} \leq \omega_i \leq \omega_{Rj} \} \]

\( \nu_j \) is the number of potential extremals in band \( j \), and \( J \) is the number of bands.
- Compute the average band errors

\[ E_j = \frac{1}{\nu_j} \sum_{\omega_i \in \Omega_j} |E(\omega_i)| \quad \text{for} \quad j = 1, 2, \ldots, J \]

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\( \nu_j \) is the number of potential extremals in band \( j \), and \( J \) is the number of bands.

- Rank the \( J \) bands in the order of lowest average error and let \( l_1, l_2, \ldots, l_J \) be the ranked list obtained, i.e., \( l_1 \) and \( l_J \) are the bands with the lowest and highest average error, respectively.
● Reject one \( \tilde{\omega}_i \) in each of bands \( l_1, l_2, \ldots, l_{J-1}, l_1, l_2, \ldots \) until \( \rho - r \) superfluous \( \tilde{\omega}_i \) are rejected. In each case, reject the \( \tilde{\omega}_i \), other than a band edge, that yields the lowest \( |E(\tilde{\omega}_i)| \) in the band.

**Example:**

If \( J = 3, \rho - r = 3, \) and the average errors for bands 1, 2, and 3 are 0.05, 0.08, and 0.02, then \( \tilde{\omega}_i \) are rejected in bands 3, 1, and 3.

**Note:** The potential extremals are not rejected in band 2 which is the band of highest average error.
### Example

<table>
<thead>
<tr>
<th>Band</th>
<th>$D(\omega)$</th>
<th>$W(\omega)$</th>
<th>Left band edge</th>
<th>Right band edge</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1.0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.4</td>
<td>1.25</td>
<td>$\pi$</td>
</tr>
</tbody>
</table>

**Sampling frequency:** $2\pi$
Example Cont’d

Filter length: 27
Iteration no: 1

Function Evals: 0
Filter length: 27
Iteration no: 2

Error at Sample Points

Function Evals: 87
Example Cont’d

Filter length: 27
Iteration no: 3

Function Evals: 134
Filter length: 27
Iteration no: 4

Function Evals: 171
Filter length: 27
Iteration no: 5

Function Evals: 208
Filter length: 27
Iteration no: 6

Function Evals: 250
ExampleCont’d

Filter length: 27
Iteration no: 7

Error at Sample Points

Function Evals: 278
### Comparisons — Amount of Computation

<table>
<thead>
<tr>
<th>Type of Filter</th>
<th>No. of Examples</th>
<th>Range of $N$</th>
<th>Ave. Funct. Evals.</th>
<th>Saving, %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>LP</td>
<td>45</td>
<td>9-101</td>
<td>2691</td>
<td>722</td>
</tr>
<tr>
<td>HP</td>
<td>42</td>
<td>9-101</td>
<td>2774</td>
<td>710</td>
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<tr>
<td>BS</td>
<td>35</td>
<td>21-91</td>
<td>2720</td>
<td>639</td>
</tr>
</tbody>
</table>

**A:** Exhaustive search  
**B:** Selective search  
**C:** Selective plus cubic search
## Comparisons — Robustness

<table>
<thead>
<tr>
<th>Type of Filter</th>
<th>No. of Examples</th>
<th>No. Failures</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>A</td>
</tr>
<tr>
<td>LP</td>
<td>46</td>
<td>1</td>
</tr>
<tr>
<td>HP</td>
<td>43</td>
<td>1</td>
</tr>
<tr>
<td>BP</td>
<td>50</td>
<td>3</td>
</tr>
<tr>
<td>BS</td>
<td>45</td>
<td>6</td>
</tr>
</tbody>
</table>

A: Exhaustive search  
B: Selective search  
C: Selective plus cubic search
A nonrecursive filter of length $N$, passband and stopband weights of 1 and $\delta_p/\delta_a$, respectively, and specified passband and stopband edges can be readily designed.
Prescribed Specifications

- A nonrecursive filter of length $N$, passband and stopband weights of 1 and $\delta_p/\delta_a$, respectively, and specified passband and stopband edges can be readily designed.

- While the filter obtained will have passband and stopband edges at the correct locations and the ratio $\delta_p/\delta_a$ will be exactly as required, the amplitudes of the passband and stopband ripples are highly unlikely to have the specified values.
Prescribed Specifications

- A nonrecursive filter of length $N$, passband and stopband weights of 1 and $\delta_p/\delta_a$, respectively, and specified passband and stopband edges can be readily designed.

- While the filter obtained will have passband and stopband edges at the correct locations and the ratio $\delta_p/\delta_a$ will be exactly as required, the amplitudes of the passband and stopband ripples are highly unlikely to have the specified values.

- An acceptable design can be obtained by predicting the value of $N$ on the basis of the required specifications and then designing filters for increasing or decreasing values of $N$ until the lowest value of $N$ that satisfies the specifications is found.
A reasonably accurate empirical formula for the prediction of
the required filter length, \( N \), for the case of lowpass and
highpass filters, due to Herrmann, Rabiner, and Chan, is

\[
N = \text{int} \left[ \frac{(D - FB^2)}{B} + 1.5 \right]
\]

where

\[
B = \frac{|\omega_a - \omega_p|}{2\pi}
\]

\[
D = [0.005309(\log_{10} \delta_p)^2 + 0.07114 \log_{10} \delta_p - 0.4761] \log_{10} \delta_a
\]
\[-[0.00266(\log_{10} \delta_p)^2 + 0.5941 \log_{10} \delta_p + 0.4278]
\]

\[
F = 0.51244(\log_{10} \delta_p - \log_{10} \delta_a) + 11.012
\]
The formula of Herrmann et al. can also be used to predict the filter length in the design of bandpass, bandstop, and multiband filters in general.
The formula of Herrmann et al. can also be used to predict the filter length in the design of bandpass, bandstop, and multiband filters in general.

In these filters, a value of $N$ is computed for each transition band between a passband and stopband or a stopband and passband and the largest value of $N$ so obtained is taken to be the predicted filter length.
Algorithm

1. Compute $N$ using the prediction formula of Herrmann et al.; if $N$ is even, set $N = N + 1$.

2. Design a filter of length $N$ using the Remez algorithm and determine the minimum value of $\delta$, say $\tilde{\delta}$.
   
   (A) If $\tilde{\delta} > \delta_p$, then do:
   
   (a) Set $N = N + 2$, design a filter of length $N$ using the Remez algorithm, and find $\delta$;
   
   (b) If $\delta \leq \delta_p$, then go to step 3; else, go to step 2(A)(a).

(B) If $\tilde{\delta} < \delta_p$, then do:

(a) Set $N = N - 2$, design a filter of length $N$ using the Remez algorithm, and find $\delta$;

(b) If $\delta > \delta_p$, then go to step 4; else, go to step 2(B)(a).
3. If part A of the algorithm was executed, use the last set of extremals and the corresponding value of \( N \) to obtain the impulse response of the required filter and stop.

4. If part B of the algorithm was executed, use the last but one set of extremals and the corresponding value of \( N \) to obtain the impulse response of the required filter and stop.
Example

In an application, a nonrecursive equiripple bandstop filter is required which should satisfy the following specifications:

- Odd filter length
- Passband ripple $A_p : 0.5$ dB
- Minimum stopband attenuation $A_a : 50.0$ dB
- Lower passband edge $\omega_{p1} : 0.8$ rad/s
- Upper passband edge $\omega_{p2} : 2.2$ rad/s
- Lower stopband edge $\omega_{a1} : 1.2$ rad/s
- Upper stopband edge $\omega_{a2} : 1.8$ rad/s
- Sampling frequency $\omega_s : 2\pi$ rad/s

Design the lowest-order filter that will satisfy the specifications.
The design algorithm gave a filter with the following specifications:

- Passband ripple: 0.4342 dB
- Minimum stopband attenuation: 51.23 dB

<table>
<thead>
<tr>
<th>$N$</th>
<th>Iters.</th>
<th>FE’s</th>
<th>$A_p$, dB</th>
<th>$A_a$, dB</th>
</tr>
</thead>
<tbody>
<tr>
<td>31</td>
<td>10</td>
<td>582</td>
<td>0.5055</td>
<td>49.91</td>
</tr>
<tr>
<td>33</td>
<td>7</td>
<td>376</td>
<td>0.5037</td>
<td>49.94</td>
</tr>
<tr>
<td>35</td>
<td>9</td>
<td>545</td>
<td>0.4342</td>
<td>51.23</td>
</tr>
</tbody>
</table>
Note: Passband errors multiplied by a factor of 40.
Advantages of Weighted-Chebyshev Method

- Designs are optimal, i.e., the required filter order for a set of prescribed specifications is the lowest that can be achieved.
- The minimum filter order to satisfy certain prescribed specifications can be predicted by using certain empirical formulas.
- Minimum filter order implies a more efficient and faster filter implementation for real-time applications.
- The method is very flexible in that it can be used to design filters, differentiators, Hilbert transformers, etc.
- The solutions achieved are equiripple.
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Disadvantages of Weighted-Chebyshev Method

- The design requires a very large amount of computation.
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- The design requires a very large amount of computation.
- Not suitable for applications where the design has to be carried out in real- or quasi-real time, for example, in programmable or adaptable filters.
A DSP software package that incorporates the design techniques described in this presentation is \textit{D-Filter}.

For more information about D-Filter or to download a \textit{free} copy, click the following link:

\url{http://ece.uvic.ca/~dsp/Software-ne.html}
Three design techniques that bring about substantial improvements in the efficiency of the Remez algorithm have been described:

- A step-by-step exhaustive search
- A cubic interpolation search
- An improved scheme for the rejection of superfluous potential extremals

These techniques are implemented in a DSP software package known as D-Filter.

Extensive experimentation has shown that the selective and cubic interpolation searches reduce the amount of computation required by the Remez algorithm by almost 90% without degrading its robustness.
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Summary

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For off-line applications, the Remez algorithm continues to be the method of choice for the design of linear-phase filters, multiband filters, differentiators, Hilbert transformers.
Despite the improvements described, the Remez algorithm continues to require a large amount of computation.

For applications that need the filter to be designed on-line in real or quasi-real time, *the window method is preferred* although the filters obtained are suboptimal.
This slide concludes the presentation. Thank you for your attention.