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## SAMPLE SOLUTIONS

### DIGITAL SIGNAL PROCESSING: *Signals, Systems, and Filters*

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SA.1 A periodic signal can be represented by the equation

$$\tilde{x}(t) = \sum_{k=1}^{10} A_k \sin(\omega_k t + \phi_k)$$

where the frequencies  $\omega_k$ , amplitudes  $A_k$ , and phase angles  $\phi_k$  are given in Table SA.1.

The signal is to be processed first by an ideal bandpass filter and then by a differentiator, as shown in Fig. SA.1. The bandpass filter will pass frequencies in the range  $4 \leq \omega \leq 6$  rad/s and reject all other frequencies and the differentiator will differentiate the signal with respect to time.

- Obtain a time-domain representation for the signal at the outputs of the bandpass filter and differentiator, i.e., at nodes B and C, respectively, in Fig. SA.1.
- Obtain a frequency-domain representation for the signal at the output of the bandpass filter.
- Obtain a frequency-domain representation for the signal at the output of the differentiator.

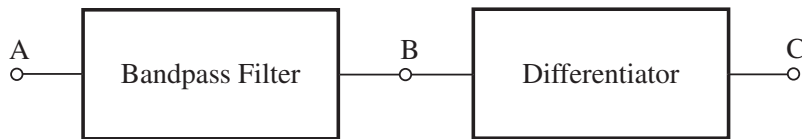


Figure SA.1

Table SA.1 Frequency Spectrum

$k$	$\omega_k$ , rad/s	$A_k$	$\phi_k$
1	1	0.3819	-0.3478
2	2	0.3614	0.8222
3	3	0.8575	2.3502
4	4	0.0629	-0.3292
5	5	0.1342	-0.1693
6	6	0.8648	0.6648
7	7	0.5155	-2.4473
8	8	0.6797	1.7780
9	9	0.7001	-1.5824
10	10	0.3	1.1

**Solution**

(a) Node B:

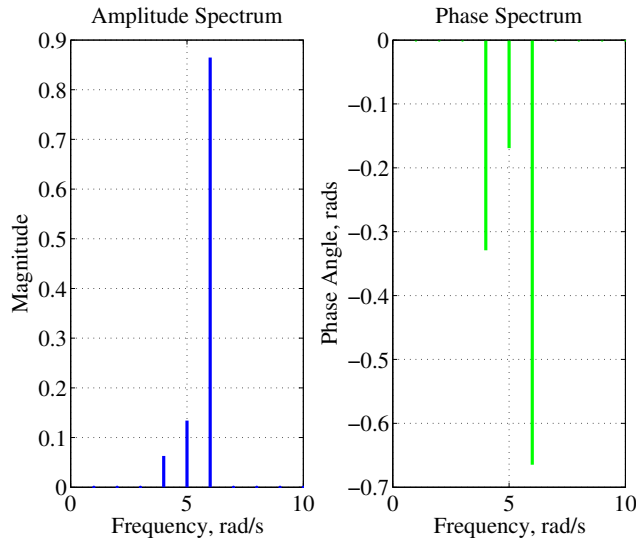
$$x_B(t) = \sum_{k=4}^6 A_k \sin(\omega_k t + \phi_k) \quad \blacksquare$$

Node C:

$$\begin{aligned} x_C(t) &= \frac{d}{dt} \left[ \sum_{k=4}^6 A_k \sin(\omega_k t + \phi_k) \right] = \sum_{k=4}^6 \frac{d}{dt} [A_k \sin(\omega_k t + \phi_k)] \\ &= \sum_{k=4}^6 \omega_k A_k \cos(\omega_k t + \phi_k) = \sum_{k=4}^6 \omega_k A_k \sin(\omega_k t + \phi_k + \pi/2) \quad \blacksquare \end{aligned}$$

(b) The amplitude and phase spectrums at Node B are given in Table SA.2 and are plotted in Fig. SA.2.  $\blacksquare$ **Table SA.2** Frequency Spectrum at Node B

$k$	$\omega_k$ , rad/s	$A_k$	$\phi_k$
4	4	0.0629	-0.3292
5	5	0.1342	-0.1693
6	6	0.8648	0.6648

**Figure SA.2**(c) Similarly, the amplitude and phase spectrums for Node C are given in Table SA.3 and are plotted in Fig. SA.3.  $\blacksquare$ **Table SA.3** Frequency Spectrum at Node C

$k$	$\omega_k$ , rad/s	$A_k$	$\phi_k$
4	4	0.2516	1.2416
5	5	0.6710	1.4015
6	6	5.1888	2.2356

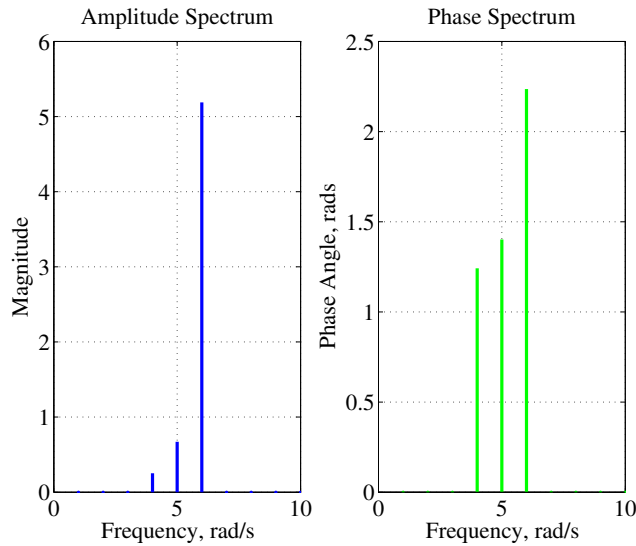


Figure SA.3

SA.2 A periodic signal defined by

$$\tilde{x}(t) = \sum_{r=-\infty}^{\infty} x(t + r\tau_0)$$

where  $x(t)$  is zero outside the range  $-\tau_0/2 \leq t \leq \tau_0/2$  has a Fourier series of the form

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t} \quad \text{for } -\tau_0/2 \leq t \leq \tau_0/2$$

where

$$X_k = \frac{1}{\tau_0} \int_{-\tau_0/2}^{\tau_0/2} x(t) e^{-jk\omega_0 t} dt$$

(a) Assuming that  $x(t)$  is an even function, show that

$$X_k = \frac{2}{\tau_0} \int_0^{\tau_0/2} x(t) \cos k\omega_0 t dt$$

Justify your steps.

(b) The periodic signal of Fig. SA.4 is described by the equation

$$x(t) = \begin{cases} \cos \omega_0 t / 2 & \text{for } -\tau_0/4 \leq t \leq \tau_0/4 \\ 0 & \text{otherwise} \end{cases}$$

where  $\omega_0 = 2\pi/\tau_0$ . Using the formula in part (a), obtain an expression for the Fourier series coefficients  $X_k$ .

(c) Give expressions for the amplitude and phase spectrums of  $\tilde{x}(t)$ .

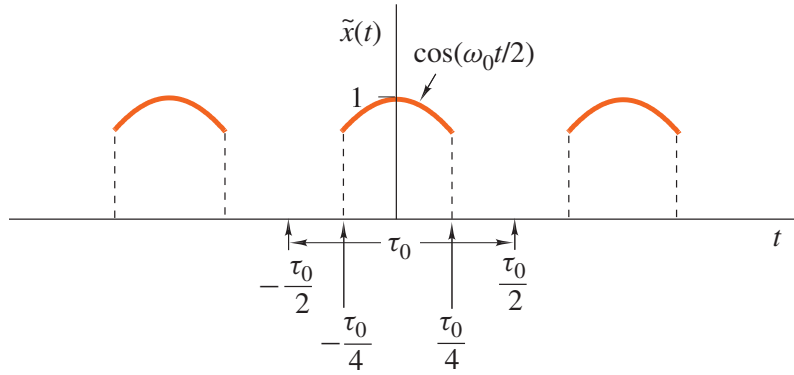


Figure SA.4

**Solution**

(a) From the definition of the Fourier series

$$\begin{aligned}
 X_k &= \frac{1}{\tau_0} \int_{-\tau_0/2}^{\tau_0/2} x(t) e^{-jk\omega_0 t} dt \\
 &= \frac{1}{\tau_0} \int_{-\tau_0/2}^{\tau_0/2} x(t) (\cos k\omega_0 t - j \sin k\omega_0 t) dt \\
 &= \frac{1}{\tau_0} \int_{-\tau_0/2}^{\tau_0/2} x(t) \cos k\omega_0 t dt - j \frac{1}{\tau_0} \int_{-\tau_0/2}^{\tau_0/2} x(t) \sin k\omega_0 t dt \\
 &= \frac{1}{\tau_0} \left[ \int_{-\tau_0/2}^0 x(t) \cos k\omega_0 t dt + \int_0^{\tau_0/2} x(t) \cos k\omega_0 t dt \right] \\
 &\quad - \frac{1}{\tau_0} \left[ \int_{-\tau_0/2}^0 x(t) \sin k\omega_0 t dt + \int_0^{\tau_0/2} x(t) \sin k\omega_0 t dt \right]
 \end{aligned}$$

If  $x(t)$  is even, then  $x(t) \cos k\omega_0 t$  is an even function and  $x(t) \sin k\omega_0 t$  is an odd function. Hence

$$\int_{-\tau_0/2}^0 x(t) \cos k\omega_0 t dt = \int_0^{\tau_0/2} x(t) \cos k\omega_0 t dt$$

and

$$\int_{-\tau_0/2}^0 x(t) \sin k\omega_0 t dt = - \int_0^{\tau_0/2} x(t) \sin k\omega_0 t dt$$

Therefore

$$X_k = \frac{2}{\tau_0} \int_0^{\tau_0/2} x(t) \cos k\omega_0 t dt \quad \blacksquare$$

(b) The given signal is symmetrical about the  $y$  axis and, therefore, it is an even function. Hence we have

$$\begin{aligned}
 X_k &= \frac{2}{\tau_0} \int_0^{\tau_0/2} x(t) \cos k\omega_0 t dt \\
 &= \frac{2}{\tau_0} \int_0^{\tau_0/4} \cos(\omega_0 t/2) \cos(k\omega_0 t) dt = \frac{2}{\tau_0} \int_0^{\tau_0/4} \cos(k\omega_0 t) \cos(\omega_0 t/2) dt
 \end{aligned}$$

From trigonometry, we have

$$\cos A \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)]$$

and, therefore,

$$\begin{aligned}
 X_k &= \frac{2}{\tau_0} \int_0^{\tau_0/4} \frac{1}{2} [\cos(k\omega_0 t + \omega_0 t/2) + \cos(k\omega_0 t - \omega_0 t/2)] dt \\
 &= \frac{1}{\tau_0} \int_0^{\tau_0/4} \cos \left[ \left(k + \frac{1}{2}\right) \omega_0 t \right] + \cos \left[ \left(k - \frac{1}{2}\right) \omega_0 t \right] dt \\
 &= \frac{1}{\tau_0} \left[ \frac{\sin \left[ \left(k + \frac{1}{2}\right) \omega_0 t \right]}{\left(k + \frac{1}{2}\right) \omega_0} + \frac{\sin \left[ \left(k - \frac{1}{2}\right) \omega_0 t \right]}{\left(k - \frac{1}{2}\right) \omega_0} \right]_0^{\tau_0/4} \\
 &= \frac{1}{\tau_0} \left[ \frac{\sin \left[ \left(k + \frac{1}{2}\right) \frac{2\pi}{\tau_0} \cdot \frac{\tau_0}{4} \right]}{\left(k + \frac{1}{2}\right) \frac{2\pi}{\tau_0}} + \frac{\sin \left[ \left(k - \frac{1}{2}\right) \frac{2\pi}{\tau_0} \cdot \frac{\tau_0}{4} \right]}{\left(k - \frac{1}{2}\right) \frac{2\pi}{\tau_0}} \right] \\
 &= \frac{1}{2\pi} \left[ \frac{\sin \left[ \left(k + \frac{1}{2}\right) \frac{\pi}{2} \right]}{\left(k + \frac{1}{2}\right)} + \frac{\sin \left[ \left(k - \frac{1}{2}\right) \frac{\pi}{2} \right]}{\left(k - \frac{1}{2}\right)} \right]
 \end{aligned}$$

- (c) The amplitude and phase spectrums of  $\tilde{x}(t)$  are the magnitude and angle of  $X_k$ , i.e.,  $|X_k|$  and  $\arg X_k$ . Since  $X_k$  is real, the angle of  $X_k$  is 0 or  $\pi$  depending on whether  $X_k$  is positive or negative. ■

SA.3 A  $z$  transform is given by

$$X(z) = \frac{(z^2 + 1)(z + 1)}{(z^2 + z - 2)(z - 3)}$$

- (a) Construct the zero-pole plot of  $X(z)$ .  
 (b) Function  $X(z)$  is known to have as many Laurent series as there are annuli of convergence but only one of these series is a  $z$  transform that satisfies the absolute convergence theorem (Theorem 3.1). Identify the annulus of convergence of that series on the zero-pole plot obtained in part (a).  
 (c) Through the use of partial fractions obtain a closed-form expression for  $x(nT) = \mathcal{Z}^{-1}X(z)$ .

**Solution**

- (a)  $X(z)$  can be expressed as

$$\begin{aligned}
 X(z) &= \frac{(z^2 + 1)(z + 1)}{(z^2 + z - 2)(z - 3)} \\
 &= \frac{(z + j)(z - j)(z + 1)}{(z - 1)(z + 2)(z - 3)}
 \end{aligned}$$

Hence  $X(z)$  has zeros at  $z = \pm j$ ,  $-1$  and poles at  $z = 1$ ,  $-2$ ,  $3$ . The zero-pole plot is depicted in Fig. SA.5. ■

- (b) The correct annulus is the outer annulus which can be represented by

$$3 \leq |z| < R \quad \text{where } R \rightarrow \infty \quad \blacksquare$$

See Fig. SA.5.

- (c) Using Technique I, we can write

$$\begin{aligned}
 \frac{X(z)}{z} &= \frac{(z^2 + 1)(z + 1)}{z(z - 1)(z + 2)(z - 3)} \\
 &= \frac{R_0}{z} + \frac{R_1}{z - 1} + \frac{R_2}{z + 2} + \frac{R_3}{z - 3}
 \end{aligned} \tag{A}$$

Since the poles are simple, we have

$$R_0 = \lim_{z \rightarrow 0} \frac{(z^2 + 1)(z + 1)}{(z - 1)(z + 2)(z - 3)} = \frac{1}{(-1) \times 2 \times (-3)} = \frac{1}{6}$$

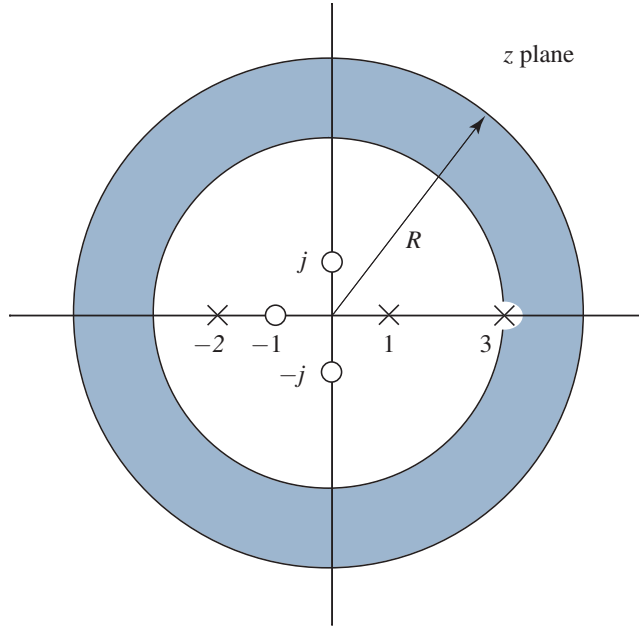


Figure SA.5

$$R_1 = \lim_{z \rightarrow 1} \frac{(z^2 + 1)(z + 1)}{z(z + 2)(z - 3)} = \frac{2 \times 2}{1 \times 3 \times (-2)} = -\frac{2}{3}$$

$$R_2 = \lim_{z \rightarrow -2} \frac{(z^2 + 1)(z + 1)}{z(z - 1)(z - 3)} = \frac{5 \times (-1)}{(-2) \times (-3) \times (-5)} = \frac{1}{6}$$

$$R_3 = \lim_{z \rightarrow 3} \frac{(z^2 + 1)(z + 1)}{z(z - 1)(z + 2)} = \frac{10 \times 4}{3 \times 2 \times 5} = \frac{4}{3}$$

From Eq. (SA.1), we can write

$$\begin{aligned} X(z) &= R_0 + \frac{R_1 z}{z - 1} + \frac{R_2 z}{z + 2} + \frac{R_3 z}{z - 3} \\ &= \frac{1}{6} - \frac{\frac{2}{3}z}{z - 1} + \frac{\frac{1}{6}z}{z + 2} + \frac{\frac{4}{3}z}{z - 3} \end{aligned}$$

Therefore, for  $n \geq 0$ , the use of Table 3.2 gives

$$\begin{aligned} x(nT) &= \frac{1}{6}\delta(nT) - \frac{2}{3}u(nT) + \frac{1}{6}u(nT)(-2)^n + \frac{4}{3}u(nT)(3)^n \\ &= \frac{1}{6}\delta(nT) + u(nT)\left[-\frac{2}{3} + \frac{1}{6}(-2)^n + \frac{4}{3}(3)^n\right] \quad \blacksquare \end{aligned}$$

Since the numerator degree of  $X(z)$  is equal to the denominator degree, it follows from the corollary of the initial-value theorem (Theorem 3.8) that  $x(nT) = 0$  for  $n < 0$ , i.e., the above solution applies for all values of  $n$ .

An alternative but equivalent solution can be readily obtained by using Technique II (see p. 115) whereby we expand  $X(z)$  instead of  $X(z)/z$  into partial fractions. We can write

$$\begin{aligned} X(z) &= \frac{(z^2 + 1)(z + 1)}{(z - 1)(z + 2)(z - 3)} \\ &= R_0 + \frac{R_1}{z - 1} + \frac{R_2}{z + 2} + \frac{R_3}{z - 3} \end{aligned}$$

where

$$R_0 = \lim_{z \rightarrow \infty} \frac{(z^2 + 1)(z + 1)}{(z - 1)(z + 2)(z - 3)} = \lim_{z \rightarrow \infty} \frac{z^3}{z^3} = 1$$

$$R_1 = \lim_{z \rightarrow 1} \frac{(z^2 + 1)(z + 1)}{(z + 2)(z - 3)} = \frac{2 \times 2}{3 \times (-2)} = -\frac{2}{3}$$

Thus

$$R_2 = \lim_{z \rightarrow -2} \frac{(z^2 + 1)(z + 1)}{(z - 1)(z - 3)} = \frac{5 \times (-1)}{(-3) \times (-5)} = -\frac{1}{3}$$

$$R_3 = \lim_{z \rightarrow 3} \frac{(z^2 + 1)(z + 1)}{(z - 1)(z + 2)} = \frac{10 \times 4}{2 \times 5} = 4$$

Thus

$$X(z) = 1 - \frac{\frac{2}{3}}{z - 1} - \frac{\frac{1}{3}}{z + 2} + \frac{4}{z - 3}$$

and for  $n \geq 0$ , we have

$$\begin{aligned} x(nT) &= \delta(nT) - \frac{2}{3}u(nT - T) - \frac{1}{3}u(nT - T)(-2)^{n-1} + 4u(nT - T)(3)^{n-1} \\ &= \delta(nT) + u(nT - T)\left[-\frac{2}{3} - \frac{1}{3}(-2)^{n-1} + 4(3)^{n-1}\right] \quad \blacksquare \end{aligned}$$

SA.4 (a) Find the  $z$  transform of the following discrete-time signal

$$x(nT) = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } 0 \leq n \leq 5 \\ 1 + (n - 5)T & \text{for } n > 5 \end{cases}$$

(b) The  $z$  transform of a discrete-time signal  $x(nT)$  is given by

$$X(z) = \frac{z(3z^2 - 2z + 1)}{(z^2 + 1)(z - 1)}$$

Using the initial-value theorem (Theorem 3.8), show that  $x(nT) = 0$  for  $n < 0$ . Then find  $x(nT)$  for  $n \geq 0$  using the general inversion formula

$$x(nT) = \frac{1}{2\pi j} \oint_{\Gamma} X(z)z^{n-1} dz.$$

### Solution

(a) The signal can be expressed as

$$x(nT) = u(nT) + r(nT - 5T)$$

Hence

$$\begin{aligned} \mathcal{Z}x(nT) &= \mathcal{Z}u(nT) + \mathcal{Z}r(nT - 5T) \\ &= \mathcal{Z}u(nT) + z^{-5}\mathcal{Z}r(nT) \\ &= \frac{z}{z - 1} - \frac{Tz^{-4}}{(z - 1)^2} \end{aligned}$$

- (b) The first nonzero value of  $x(nT)$  occurs at  $KT = (N - M)T$  where  $N$  is the denominator degree and  $M$  in the numerator degree in  $X(z)$ . Since  $N = M = 3$ , we have  $K = 0$ , i.e., the signal starts at  $KT = 0$ . Hence for  $n < 0$ , we have

$$x(nT) = 0 \quad \blacksquare$$

For  $n \geq 0$ , we can write

$$\begin{aligned} x(nT) &= \sum_{i=1}^M \Re_{eS} \left[ X(z) z^{n-1} \right] \\ &= \sum_{i=1}^M \Re_{eS} \frac{z(3z^2 - 2z + 1)z^{n-1}}{(z^2 + 1)(z - 1)} \\ &= \sum_{i=1}^M \Re_{eS} \frac{(3z^2 - 2z + 1)z^n}{(z - j)(z + j)(z - 1)} \\ &= \frac{R_1}{z - j} + \frac{R_1^*}{z + j} + \frac{R_2}{z - 1} \end{aligned} \quad (\text{B})$$

where  $M$  is the number of poles and  $R_1^*$  is the complex conjugate of  $R_1$  since the pole at  $z = j = e^{j\pi/2}$  is the complex conjugate of the pole at  $z = -j = e^{-j\pi/2}$ . Since the poles are simple, we have

$$\begin{aligned} R_1 &= \lim_{z \rightarrow j} \left[ (z - j) \frac{(3z^2 - 2z + 1)z^n}{(z - j)(z + j)(z - 1)} \right] \\ &= \lim_{z \rightarrow j} \frac{(3z^2 - 2z + 1)z^n}{(z + j)(z - 1)} = \frac{(-3 - 2j + 1)j^n}{2j(j - 1)} \\ &= \frac{-2 - 2j}{-2 - 2j} = j^n = e^{jn\pi/2} \end{aligned}$$

and

$$R_2 = \lim_{z \rightarrow 1} \frac{(3z^2 - 2z + 1)z^n}{(z^2 + 1)} = \frac{2}{2} = 1$$

Therefore, from Eq. (SA.2), we can write

$$\begin{aligned} x(nT) &= u(nT)e^{jn\pi/2} + u(nT)e^{-jn\pi/2} + u(nT) \\ &= u(nT)(2 \cos n\pi/2 + 1) \quad \blacksquare \end{aligned}$$

SA.5 An initially relaxed discrete-time system can be represented by the equation

$$y(nT) = \mathcal{R}x(nT) = 2.5x(nT) + |e^{0.1(nT+2T)}|x(nT - T) + x(nT - 2T)$$

By using *appropriate tests*, check the system for

- linearity,
- time invariance, and
- causality.

### Solution

- (a) **Linearity**

$$\begin{aligned} \mathcal{R}[\alpha x_1(nT) + \beta x_2(nT)] &= 2.5[\alpha x_1(nT) + \beta x_2(nT)] + |e^{0.1(nT+2T)}|[\alpha x_1(nT - T) \\ &\quad + \beta x_2(nT - T)] + [\alpha x_1(nT - 2T) + \beta x_2(nT - 2T)] \\ &= \alpha[2.5x_1(nT) + |e^{0.1(nT+2T)}|x_1(nT - T) + x_1(nT - 2T)] \\ &\quad + \beta[2.5x_2(nT) + |e^{0.1(nT+2T)}|x_2(nT - T) + x_2(nT - 2T)] \\ &= \alpha \mathcal{R}x_1(nT) + \beta \mathcal{R}x_2(nT) \end{aligned}$$

Therefore, the system is linear.  $\blacksquare$



(b) **Time invariance**

The response to a delayed excitation is

$$\mathcal{R}x(nT - kT) = 2.5x(nT - kT) + |e^{0.1(nT+2T)}|x(nT - kT - T) + x(nT - kT - 2T)$$

The delayed response is

$$y(nT - kT) = 2.5x(nT - kT) + |e^{0.1(nT-kT+2T)}|x(nT - kT - T) + x(nT - kT - 2T)$$

For any  $k \neq 0$ , we have

$$|e^{0.1nT+2T}| \neq |e^{0.1(nT-kT+2T)}|$$

Thus

$$y(nT - kT) \neq \mathcal{R}x(nT - kT)$$

and, therefore, the system is time dependent. ■

(c) Let  $x_1(nT)$  and  $x_2(nT)$  be arbitrary discrete-time signals such that

$$\begin{aligned} x_1(nT) &= x_2(nT) & \text{for } n \leq k \\ x_1(nT) &\neq x_2(nT) & \text{for } n > k \end{aligned}$$

We have

$$\mathcal{R}x_1(nT) = 2.5x_1(nT) + |e^{0.1(nT+2T)}|x_1(nT - T) + x_1(nT - 2T) \quad (\text{C})$$

and

$$\mathcal{R}x_2(nT) = 2.5x_2(nT) + |e^{0.1(nT+2T)}|x_2(nT - T) + x_2(nT - 2T) \quad (\text{D})$$

Since

$$x_1(nT) = x_2(nT) \quad \text{for } n \leq k$$

then

$$x_1(nT - T) = x_2(nT - T) \quad \text{for } n \leq k$$

and

$$x_1(nT - 2T) = x_2(nT - 2T) \quad \text{for } n \leq k$$

Hence the right-hand side in Eq. (SA.3) is equal to the right-hand side in Eq. (SA.4) for  $n \leq k$  and thus

$$\mathcal{R}x_1(nT) = \mathcal{R}x_2(nT) \quad \text{for } n \leq k$$

Therefore, the filter is causal. ■

SA.6 (a) Derive a state-space representation for the filter shown in Fig. SA.6.

(b) Using the state-space representation obtained in part (a), compute the impulse response of the filter at  $nT = 5T$ .

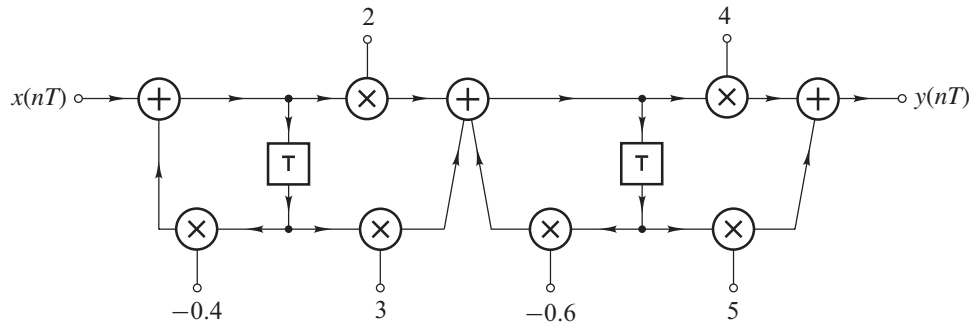


Figure SA.6

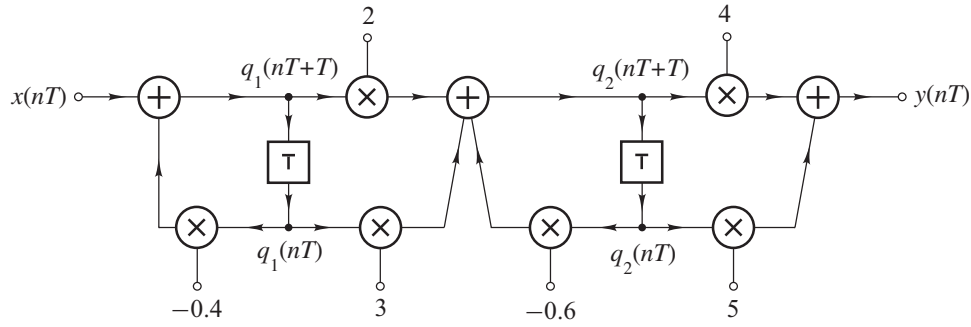


Figure SA.7

**Solution**

(a) State variables can be assigned as shown in Fig. SA.7. Hence we can write

$$q_1(nT + T) = x(nT) - 0.4q_1(nT) \quad (\text{E})$$

$$q_2(nT + T) = 2q_1(nT + T) + 3q_1(nT) - 0.6q_2(nT) \quad (\text{F})$$

On eliminating  $q_1(nT + T)$  in Eq. (SA.6) using Eq. (SA.5), we get

$$\begin{aligned} q_2(nT + T) &= 2x(nT) - 0.8q_1(nT) + 3q_1(nT) - 0.6q_2(nT) \\ &= 2.2q_1(nT) - 0.6q_2(nT) + 2x(nT) \end{aligned} \quad (\text{G})$$

From Eqs. (SA.5) and (SA.7)

$$\begin{bmatrix} q_1(nT + T) \\ q_2(nT + T) \end{bmatrix} = \begin{bmatrix} -0.4 & 0 \\ 2.2 & -0.6 \end{bmatrix} \begin{bmatrix} q_1(nT) \\ q_2(nT) \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} x(nT) \quad (\text{H})$$

The output is given by

$$y(nT) = 4q_2(nT + T) + 5q_2(nT) \quad (\text{I})$$

and from Eqs. (SA.6) and (SA.9), we have

$$\begin{aligned} y(nT) &= 4 \times 2.2q_1(nT) - 4 \times 0.6q_2(nT) + 4 \times 2x(nT) + 5q_2(nT) \\ &= 8.8q_1(nT) + 2.6q_2(nT) + 8x(nT) \\ &= [8.8 \quad 2.6] \begin{bmatrix} q_1(nT) \\ q_2(nT) \end{bmatrix} + 8x(nT) \end{aligned} \quad (\text{J})$$

Therefore, Eqs. (SA.8) and (SA.10) can be written as

$$\begin{aligned} \mathbf{q}(nT + T) &= \mathbf{A}\mathbf{q}(nT) + \mathbf{b}x(nT) \\ y(nT) &= \mathbf{c}^T \mathbf{q}(nT) = dx(nT) \end{aligned}$$

where

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} -0.4 & 0 \\ 2.2 & -0.6 \end{bmatrix} & \mathbf{b} &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \blacksquare \\ \mathbf{c}^T &= [8.8 \quad 2.6] & d &= 8 \quad \blacksquare \end{aligned}$$

(b) The impulse response is given by

$$h(nT) = \begin{cases} a_1 & \text{for } n = 0 \\ \mathbf{c}^T \mathbf{A}^{n-1} \mathbf{b} & \text{otherwise} \end{cases}$$

For  $n = 5$

$$h(5T) = \mathbf{c}^T \mathbf{A}^4 \mathbf{b} = [8.8 \quad 2.6] \begin{bmatrix} -0.4 & 0 \\ 2.2 & -0.6 \end{bmatrix}^4 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Since

$$\begin{aligned} \begin{bmatrix} -0.4 & 0 \\ 2.2 & -0.6 \end{bmatrix}^2 &= \begin{bmatrix} -0.4 & 0 \\ 2.2 & -0.6 \end{bmatrix} \begin{bmatrix} -0.4 & 0 \\ 2.2 & -0.6 \end{bmatrix} = \begin{bmatrix} 0.16 & 0.0 \\ -2.20 & 0.36 \end{bmatrix} \\ \begin{bmatrix} -0.4 & 0 \\ 2.2 & -0.6 \end{bmatrix}^4 &= \begin{bmatrix} 0.16 & 0.0 \\ -2.20 & 0.36 \end{bmatrix} \begin{bmatrix} 0.16 & 0.0 \\ -2.20 & 0.36 \end{bmatrix} = \begin{bmatrix} 0.0256 & 0.0 \\ -1.144 & 0.1296 \end{bmatrix} \end{aligned}$$

we get

$$h(5T) = [8.8 \quad 2.6] \begin{bmatrix} 0.0256 & 0.0 \\ -1.144 & 0.1296 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = [8.8 \quad 2.6] \begin{bmatrix} 0.0256 \\ -0.8848 \end{bmatrix} = -2.0752 \quad \blacksquare$$

SA.7 Fig. SA.8 shows a recursive digital filter.

- Find its transfer function.
- By using the Jury-Marden stability criterion, determine whether the filter is stable or unstable.

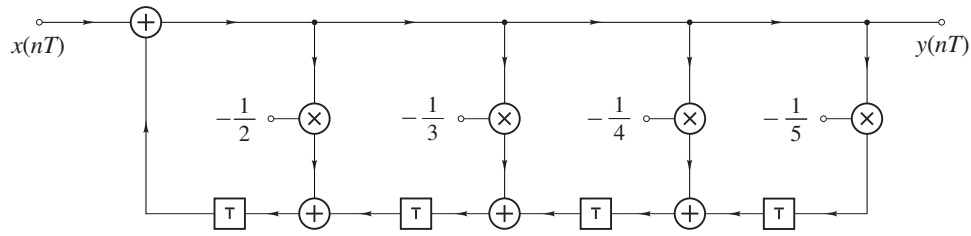


Figure SA.8

**Solution**

- From Fig. SA.8, we get

$$y(nT) = x(nT) - \frac{1}{2}y(nT - T) - \frac{1}{3}y(nT - 2T) - \frac{1}{4}y(nT - 3T) - \frac{1}{5}y(nT - 4T)$$

Hence the  $z$  transform gives

$$Y(z) = X(z) - \frac{1}{2}z^{-1}Y(z) - \frac{1}{3}z^{-2}Y(z) - \frac{1}{4}z^{-3}Y(z) - \frac{1}{5}z^{-4}Y(z)$$

or

$$Y(z) + \frac{1}{2}z^{-1}Y(z) + \frac{1}{3}z^{-2}Y(z) + \frac{1}{4}z^{-3}Y(z) + \frac{1}{5}z^{-4}Y(z) = X(z)$$

and so

$$\frac{Y(z)}{X(z)} = \frac{1}{1 + \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2} + \frac{1}{4}z^{-3} + \frac{1}{5}z^{-4}}$$

In effect,

$$H(z) = \frac{N(z)}{D(z)} = \frac{z^4}{z^4 + \frac{1}{2}z^3 + \frac{1}{3}z^2 + \frac{1}{4}z + \frac{1}{5}}$$

where

$$D(z) = z^4 + \frac{1}{2}z^3 + \frac{1}{3}z^2 + \frac{1}{4}z + \frac{1}{5}$$

We note that

$$\begin{aligned} D(1) &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = 2.283 > 0 && \text{(Ka)} \\ (-1)^4 D(-1) &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = 0.783 > 0 && \text{(Kb)} \end{aligned}$$

(b) The Jury-Marden array can be constructed as shown in Table SA.4.

**Table SA.4** The Jury-Marden array

	$b_0$	$b_1$	$b_2$	$b_3$	$b_4$
1	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
2	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1
3	$\frac{24}{25}$	$\frac{9}{20}$	$\frac{4}{15}$	$\frac{3}{20}$	
4	$\frac{3}{20}$	$\frac{4}{15}$	$\frac{9}{20}$	$\frac{24}{25}$	
5	0.8991	0.392	0.1885		

From Eqs. (SA.11a) and (SA.11b) and Table SA.4, we have

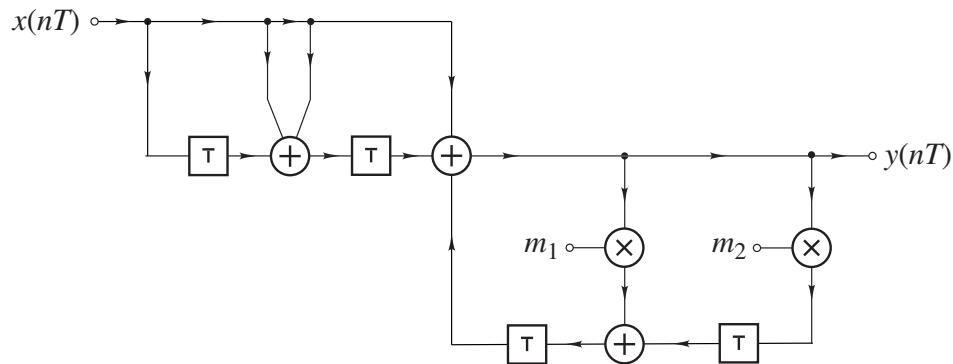
$$\begin{aligned}
 D(1) &> 0 & (-1)^4 D(-1) &> 0 \\
 b_0 &= 1 > \frac{1}{5} = |b_4| \\
 |c_0| &= \frac{24}{25} > \frac{3}{20} = |c_3| \\
 |d_0| &= 0.8991 > 0.1885 = |d_2|
 \end{aligned}$$

Therefore, conditions (i) to (iii) of the Jury-Marden stability criterion (see p. 220) are satisfied and the filter is stable. ■

SA.8 The filter of Fig. SA.9 is subjected to an input

$$x(nT) = \begin{cases} 1 & \text{for } n = 0 \text{ and } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the time-domain response in *closed* form if  $m_1 = -\frac{3}{4}$  and  $m_2 = -\frac{1}{8}$ . The filter is linear and time-invariant.



**Figure SA.9**

### Solution

From Fig. SA.9

$$Y(z) = X(z) + 2z^{-1}X(z) + z^{-2}X(z) + m_1z^{-1}Y(z) + m_2z^{-2}Y(z)$$

Hence

$$Y(z)(1 - m_1z^{-1} - m_2z^{-2}) = (1 + 2z^{-1} + z^{-2})X(z)$$

or

$$Y(z) = \frac{(1 + 2z^{-1} + z^{-2})}{(1 - m_1z^{-1} - m_2z^{-2})}X(z)$$

Therefore,

$$\frac{Y(z)}{X(z)} = H(z) = \frac{z^2 + 2z + 1}{z^2 - m_1z - m_2} = \frac{z^2 + 2z + 1}{z^2 + \frac{3}{4}z + \frac{1}{8}} = \frac{z^2 + 2z + 1}{(z + \frac{1}{2})(z + \frac{1}{4})}$$

The time-domain response can be obtained in a number of ways, as detailed below.

**Method 1:**

The input is the sum of two impulse functions, i.e.,

$$x(nT) = \delta(nT) + \delta(nT - T)$$

Hence

$$\mathcal{R}x(nT) = \mathcal{R}[\delta(nT) + \delta(nT - T)]$$

Since the filter is linear, we have

$$\mathcal{R}x(nT) = \mathcal{R}\delta(nT) + \mathcal{R}\delta(nT - T)$$

and if  $h(nT)$  is the impulse response, i.e.,

$$h(nT) = \mathcal{R}\delta(nT)$$

then by virtue of the fact that the filter is time-invariant, we get

$$y(nT) = h(nT) + h(nT - T)$$

In effect, all we have to do is find the impulse response. Expanding  $H(z)/z$  into partial fractions gives

$$\frac{H(z)}{z} = \frac{z^2 + 2z + 1}{z(z + \frac{1}{2})(z + \frac{1}{4})} = \frac{R_1}{z} + \frac{R_2}{z + \frac{1}{2}} + \frac{R_2}{z + \frac{1}{4}}$$

where

$$\begin{aligned} R_1 &= \left. \frac{z^2 + 2z + 1}{(z + \frac{1}{2})(z + \frac{1}{4})} \right|_{z=0} = \frac{1}{\frac{1}{2} \times \frac{1}{4}} = 8 \\ R_2 &= \left. \frac{z^2 + 2z + 1}{z(z + \frac{1}{4})} \right|_{z=-\frac{1}{2}} = \frac{\frac{1}{4} - \frac{2}{2} + 1}{(-\frac{1}{2})(-\frac{1}{2} + \frac{1}{4})} = \frac{\frac{1}{4}}{(-\frac{1}{2})(-\frac{1}{4})} = 2 \\ R_3 &= \left. \frac{z^2 + 2z + 1}{z(z + \frac{1}{2})} \right|_{z=-\frac{1}{4}} = \frac{\frac{1}{16} - \frac{2}{4} + 1}{(-\frac{1}{4})(-\frac{1}{4} + \frac{1}{2})} = \frac{\frac{1}{16} + \frac{8}{16}}{-\frac{1}{16}} = -9 \end{aligned}$$

i.e.,

$$H(z) = R_1 + \frac{R_2z}{z + \frac{1}{2}} + \frac{R_3z}{z + \frac{1}{4}}$$

and

$$\begin{aligned} h(nT) &= R_1\delta(nT) + [R_2(-\frac{1}{2})^n + R_3(-\frac{1}{4})^n]u(nT) \\ &= 8\delta(nT) + [2(-\frac{1}{2})^n - 9(-\frac{1}{4})^n]u(nT) \end{aligned}$$

Now

$$y(nT) = h(nT) + h(nT - T)$$

Hence

$$\begin{aligned} y(nT) &= 8[\delta(nT) + \delta(nT - T)] + [2(-\frac{1}{2})^n - 9(-\frac{1}{4})^n]u(nT) \\ &\quad + [2(-\frac{1}{2})^{n-1} - 9(-\frac{1}{4})^{n-1}]u(nT - T) \quad \blacksquare \end{aligned}$$

**Method 2**

Since

$$x(nT) = \delta(nT) + \delta(nT - T)$$

the  $z$  transform gives

$$X(z) = 1 + z^{-1} = \frac{z+1}{z}$$

Hence

$$Y(z) = H(z)X(z) = \frac{(z^2 + 2z + 1)}{(z + \frac{1}{2})(z + \frac{1}{4})} \times \frac{(z+1)}{z}$$

Expanding  $H(z)X(z)$  into partial fractions, gives

$$H(z)X(z) = R_1 + \frac{R_2}{z} + \frac{R_3}{z + \frac{1}{2}} + \frac{R_4}{z + \frac{1}{4}}$$

where

$$\begin{aligned} R_1 &= \lim_{z \rightarrow \infty} H(z)X(z) = 1 \\ R_2 &= \left. \frac{(z^2 + 2z + 1)(z + 1)}{(z + \frac{1}{2})(z + \frac{1}{4})} \right|_{z=0} = \frac{1}{\frac{1}{2} \times \frac{1}{4}} = 8 \\ R_3 &= \left. \frac{(z^2 + 2z + 1)(z + 1)}{(z + \frac{1}{4})z} \right|_{z=-\frac{1}{2}} = \frac{(\frac{1}{4} - 1 + 1)(-\frac{1}{2} + 1)}{(-\frac{1}{2} + \frac{1}{4})(-\frac{1}{2})} = \frac{(\frac{1}{4})(\frac{1}{2})}{(-\frac{1}{4})(-\frac{1}{2})} = 1 \\ R_4 &= \left. \frac{(z^2 + 2z + 1)(z + 1)}{(z + \frac{1}{2})z} \right|_{z=-\frac{1}{4}} = \frac{(\frac{1}{16} - \frac{2}{4} + 1)(-\frac{1}{4} + 1)}{(-\frac{1}{4} + \frac{1}{2})(-\frac{1}{4})} = \frac{(\frac{1}{16} + \frac{8}{16})(\frac{3}{4})}{(\frac{1}{4})(-\frac{1}{4})} \\ &= \frac{\frac{9}{16} \times \frac{3}{4}}{-\frac{1}{16}} = -\frac{27}{4} \end{aligned}$$

Therefore,

$$\begin{aligned} y(nT) &= R_1\delta(nT) + R_2\delta(nT - T) + R_3u(nT - T)\left(-\frac{1}{2}\right)^{n-1} + R_4u(nT - T)\left(-\frac{1}{4}\right)^{n-1} \\ &= \delta(nT) + 8\delta(nT - T) + u(nT - T)\left(-\frac{1}{2}\right)^{n-1} - \frac{27}{4}u(nT - T)\left(-\frac{1}{4}\right)^{n-1} \quad \blacksquare \end{aligned}$$

**Method 3**

The inverse of  $Y(z)$  is obtained from first principles as

$$y(nT) = \sum_{\text{res}} Y_0(z)$$

where

$$Y_0(z) = Y(z)z^{n-1} = H(z)X(z)z^{n-1} = \frac{(z^2 + 2z + 1)(z + 1)}{z(z + \frac{1}{2})(z + \frac{1}{4})} z^{n-1}$$

However, watch out for pitfalls at the origin. In this case, we have a second-order pole at the origin if  $n = 0$ , a first-order pole at the origin if  $n = 1$ , and no poles at the origin if  $n \geq 2$ . Hence, we have to find  $y(0)$  and  $y(T)$  individually and then proceed to  $y(nT)$  for  $n \geq 2$ . This would make this method quite long.

**Method 4**

We can express  $Y(z)$  into partial fractions as

$$Y(z) = \frac{(z^2 + 2z + 1)(z + 1)}{z(z + \frac{1}{2})(z + \frac{1}{4})} = R_1 + \frac{R_2}{z} + \frac{R_3z}{z + \frac{1}{2}} + \frac{R_4z}{z + \frac{1}{4}}$$

where

$$\begin{aligned}
 R_2 &= \lim_{z=0} zY(z) = \frac{z(z^2 + 2z + 1)(z + 1)}{z(z + \frac{1}{2})(z + \frac{1}{4})} \Big|_{z=0} = \frac{1 \times 1}{\frac{1}{2} \times \frac{1}{4}} = 8 \\
 R_3 &= \lim_{z=-\frac{1}{2}} \frac{z + \frac{1}{2}}{z} \times Y(z) = \frac{(z^2 + 2z + 1)(z + 1)}{z^2(z + \frac{1}{4})} \Big|_{z=-\frac{1}{2}} \\
 &= \frac{\left[(-\frac{1}{2})^2 + 2(-\frac{1}{2}) + 1\right](-\frac{1}{2} + 1)}{(-\frac{1}{2})^2(-\frac{1}{2} + \frac{1}{4})} = \frac{(\frac{1}{4} - 1 + 1)(\frac{1}{2})}{(\frac{1}{4})(-\frac{1}{4})} = \frac{\frac{1}{8}}{-\frac{1}{16}} = -2 \\
 R_4 &= \lim_{z=-\frac{1}{4}} \frac{z + \frac{1}{4}}{z} \times Y(z) = \frac{(z^2 + 2z + 1)(z + 1)}{z^2(z + \frac{1}{2})} \Big|_{z=-\frac{1}{4}} \\
 &= \frac{\left[(-\frac{1}{4})^2 + 2(-\frac{1}{4}) + 1\right](-\frac{1}{4} + 1)}{(-\frac{1}{4})^2(-\frac{1}{4} + \frac{1}{2})} = \frac{(\frac{1}{16} - \frac{2}{4} + 1)(\frac{3}{4})}{\frac{1}{16} \times \frac{1}{4}} = \frac{\frac{9}{16} \times \frac{3}{4}}{\frac{1}{64}} = 27
 \end{aligned}$$

Constant  $A$  can be obtained by noting that

$$\lim_{z \rightarrow \infty} Y(z) = R_1 + R_3 + R_4 = \frac{z^3}{z^3} = 1$$

Thus

$$R_1 = 1 - R_3 - R_4 = 1 - (-2) - 27 = 3 - 27 = -24$$

Hence

$$Y(z) = -24 + \frac{8}{z} - \frac{2z}{z + \frac{1}{2}} + \frac{27z}{z + \frac{1}{4}}$$

Therefore,

$$y(nT) = -24\delta(nT) + 8\delta(nT - T) + [27(-\frac{1}{4})^n - 2(-\frac{1}{2})^n] u(nT) \quad \blacksquare$$

### Method 5

One could expand  $Y(z)/z$  into partial fractions as

$$\frac{Y(z)}{z} = \frac{R_1}{z} + \frac{R_2}{z^2} + \frac{R_3}{z + \frac{1}{2}} + \frac{R_4}{z + \frac{1}{4}}$$

However, this is essentially the same as method 4.

SA.9 A discrete-time system has a transfer function

$$H(z) = \frac{z^2 + 2}{z^2 - (2r \cos \theta)z + r^2}$$

- Find the unit-step response in closed form.
- Using MATLAB, D-Filter, or similar software, plot the unit-step response for  $r = 0.3$  and  $\theta = \pi/4$ .
- Repeat part (b) for  $r = 0.6$  and  $\theta = \pi/4$ .
- Repeat part (b) for  $r = 0.9$  and  $\theta = \pi/4$ .
- Compare the unit-step responses in parts (b) to (d).

### Solution

- The  $z$  transform of the output of the system is given by

$$Y(z) = H(z)X(z) \tag{L}$$

where  $H(z)$  is the transfer function and  $X(z)$  is the  $z$  transform of the input. Since the input is a unit step, we have

$$X(z) = \mathcal{Z}u(nT) = \frac{z}{z-1} \quad (\text{M})$$

Thus Eqs. (SA.12) and (SA.13) give

$$Y(z) = \frac{z^2 + 2}{z^2 - (2r \cos \theta)z + r^2} \cdot \frac{z}{z-1} = \frac{z^3 + 2z}{(z-1)[z^2 - (2r \cos \theta)z + r^2]}$$

The general inversion formula gives

$$y(nT) = \frac{1}{2\pi j} \oint_{\Gamma} Y(z)z^{n-1}dz = \sum_{i=1}^M \mathfrak{Res}_{z=p_i} Y_0(z) \quad (\text{N})$$

where

$$\begin{aligned} Y_0(z) &= Y(z)z^{n-1} = \frac{(z^3 + 2z)z^{n-1}}{(z-1)[z^2 - (2r \cos \theta)z + r^2]} \\ &= \frac{(z^3 + 2z)z^{n-1}}{(z-1)[z^2 - r(e^{j\theta} + e^{-j\theta})z + r^2]} \\ &= \frac{(z^2 + 2)z^n}{(z-1)(z - re^{j\theta})(z - re^{-j\theta})} \end{aligned}$$

and  $M = 3$ . The residues of  $Y_0(z)$  can be obtained as follows:

$$\begin{aligned} R_1 &= \lim_{z=1} (z-1)Y_0(z) = \frac{(z^2 + 2)z^n}{z^2 - (2r \cos \theta)z + r^2} \Big|_{z=1} \\ &= \frac{3}{1 - (2r \cos \theta) + r^2} \end{aligned} \quad (\text{O})$$

$$\begin{aligned} R_2 &= \lim_{z=re^{j\theta}} (z - re^{j\theta})Y_0(z) = \frac{(z^2 + 2)z^n}{(z-1)(z - re^{-j\theta})} \Big|_{z=re^{j\theta}} \\ &= \frac{(r^2 e^{j2\theta} + 2)r^n e^{jn\theta}}{(re^{j\theta} - 1)(re^{j\theta} - re^{-j\theta})} = \frac{(r^2 e^{j2\theta} + 2)r^{n-1} e^{jn\theta}}{(re^{j\theta} - 1)2j \sin \theta} \\ &= \frac{(r^2 \cos 2\theta + 2 + jr^2 \sin 2\theta)r^{n-1} e^{jn\theta}}{(r \cos \theta - 1 + jr \sin \theta)2j \sin \theta} \\ &= \frac{(r^2 \cos 2\theta + 2 + jr^2 \sin 2\theta)r^{n-1} e^{jn\theta}}{[(r \cos \theta - 1) + jr \sin \theta]2e^{j\pi/2} \sin \theta} \\ &= Me^{j\phi} r^{n-1} e^{jn\theta} = Mr^{n-1} e^{j(n\theta + \phi)} \end{aligned} \quad (\text{P})$$

where

$$\begin{aligned} M &= \left| \frac{r^2 \cos 2\theta + 2 + jr^2 \sin 2\theta}{[j(r \cos \theta - 1) - r \sin \theta]2 \sin \theta} \right| \\ &= \sqrt{\frac{(r^2 \cos 2\theta + 2)^2 + (r^2 \sin 2\theta)^2}{4[(r \cos \theta - 1)^2 + (r \sin \theta)^2] \sin^2 \theta}} \end{aligned}$$

and

$$\begin{aligned} \phi &= \tan^{-1} \frac{r^2 \sin 2\theta}{r^2 \cos 2\theta + 2} - \tan^{-1} \frac{(r \cos \theta - 1)}{-r \sin \theta} \\ &= \tan^{-1} \frac{r^2 \sin 2\theta}{r^2 \cos 2\theta + 2} - \tan^{-1} \frac{r \sin \theta}{(r \cos \theta - 1)} - \frac{\pi}{2} \end{aligned}$$

Similarly,

$$R_3 = R_2^* \quad (\text{Q})$$



since complex conjugate poles give complex conjugate residues. We note that there is no additional pole at the origin when  $n = 0$  and hence for  $n \geq 0$  Eq. (SA.14) gives

$$y(nT) = R_1 + R_2 + R_2^* \quad (R)$$

Since the numerator degree in  $Y(z)$  does not exceed the denominator degree, we have

$$y(nT) = 0 \quad \text{for } n < 0$$

Therefore, for any  $n$ , Eqs. (SA.15)–(SA.18) give

$$\begin{aligned} y(nT) &= u(nT) \left[ R_1 + Mr^{n-1}e^{j(n\theta+\phi)} + Mr^{n-1}e^{-j(n\theta+\phi)} \right] \\ &= u(nT) \left[ R_1 + 2Mr^{n-1} \cos(n\theta + \phi) \right] \quad \blacksquare \end{aligned}$$

(b-d) The step responses for the three cases are illustrated in Fig. SA.10a-c.  $\blacksquare$

(e) On the basis of the step responses obtained, the system in part (b) is what they call *overdamped*, the one in part (c) is *critically damped*, and the one in part (d) is referred to as *underdamped*.  $\blacksquare$

SA.10 A second-order digital filter has zeros  $z_1 = e^{j\pi/3}$  and  $z_2 = e^{-j\pi/3}$  and poles  $p_1 = 0.5e^{j\pi/4}$  and  $p_2 = 0.5e^{-j\pi/4}$  and its multiplier constant is 2.

- (a) Obtain the transfer function of the filter.
- (b) Obtain an expression for the gain.
- (c) Assuming that the sampling frequency is  $2\pi$ , calculate the gain at  $\omega = 0, \pi/4, \pi/3$ , and  $\pi$ .

**Solution**

(a) Since we have the zeros, poles, and multiplier constant of the filter, the transfer function can be readily constructed as

$$\begin{aligned} H(z) &= \frac{2(z - e^{j\pi/3})(z - e^{-j\pi/3})}{(z - \frac{1}{2}e^{j\pi/4})(z - \frac{1}{2}e^{-j\pi/4})} = \frac{2[z^2 - (e^{j\pi/3} + e^{-j\pi/3})z + 1]}{z^2 - \frac{1}{2}(e^{j\pi/4} + e^{-j\pi/4})z + \frac{1}{4}} \\ &= \frac{2[z^2 - 2(\cos \pi/3)z + 1]}{z^2 - (\cos \pi/4)z + \frac{1}{4}} = \frac{2[z^2 - z + 1]}{z^2 - \frac{\sqrt{2}}{2}z + \frac{1}{4}} \quad \blacksquare \end{aligned}$$

(b) The gain is given by

$$\begin{aligned} M(\omega) &= |H(e^{j\omega T})| = \left| \frac{2[e^{2j\omega T} - e^{j\omega T} + 1]}{e^{j2\omega T} - \frac{\sqrt{2}}{2}e^{j\omega T} + \frac{1}{4}} \right| \\ &= 2 \left| \frac{\cos 2\omega T + j \sin 2\omega T - \cos \omega T - j \sin \omega T + 1}{\cos 2\omega T + j \sin 2\omega T - \frac{\sqrt{2}}{2}(\cos \omega T + j \sin \omega T) + \frac{1}{4}} \right| \\ &= 2 \sqrt{\frac{(\cos 2\omega T - \cos \omega T + 1)^2 + (\sin 2\omega T - \sin \omega T)^2}{(\cos 2\omega T - \frac{\sqrt{2}}{2} \cos \omega T + \frac{1}{4})^2 + (\sin 2\omega T - \frac{\sqrt{2}}{2} \sin \omega T)^2}} \quad \blacksquare \end{aligned}$$

(c) Since  $\omega_s = 2\pi$ , we have  $2\pi f_s = 2\pi/T = 2\pi$ . Hence  $T = 1$  s. For the frequencies given, the numerical values in Table SA.5 can be readily calculated.

**Table SA.5**

$\omega$	$\cos \omega T$	$\sin \omega T$	$\cos 2\omega T$	$\sin 2\omega T$
0	1	0	1	0
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	0	1
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\pi$	-1	0	1	0

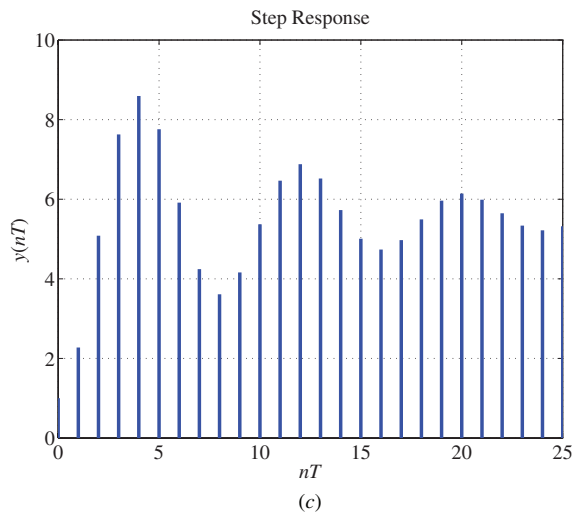
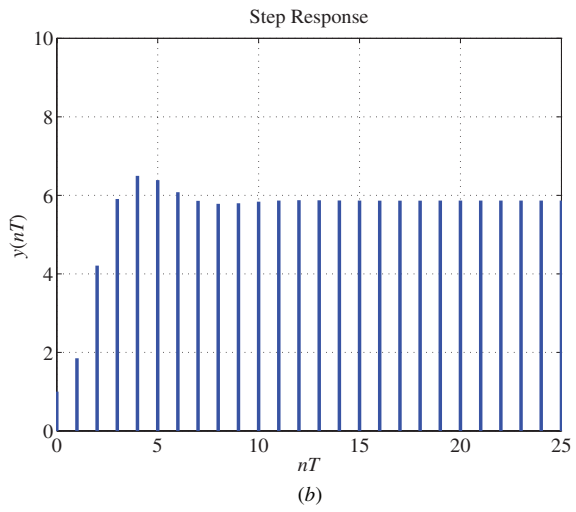
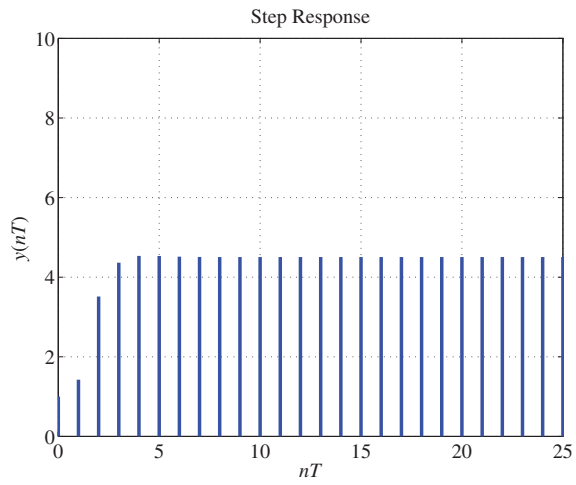


Figure SA.10

Thus

$$M(0) = 2\sqrt{\frac{(1-1+1)^2 + (0-0)^2}{(1-\frac{\sqrt{2}}{2}+\frac{1}{4})^2 + (0-0)^2}} = \frac{2}{\frac{5-2\sqrt{2}}{4}} = 3.6840 \quad \blacksquare$$

$$\begin{aligned} M\left(\frac{\pi}{4}\right) &= 2\sqrt{\frac{(0-\frac{\sqrt{2}}{2}+1)^2 + (1-\frac{\sqrt{2}}{2})^2}{(0-\frac{\sqrt{2}}{2}\times\frac{\sqrt{2}}{2}+\frac{1}{4})^2 + (1-\frac{\sqrt{2}}{2}\times\frac{\sqrt{2}}{2})^2}} \\ &= 2\sqrt{\frac{(2-\sqrt{2})^2 + (2-\sqrt{2})^2}{(\frac{1}{2}-1)^2 + 1}} = 2\sqrt{\frac{2(2-\sqrt{2})^2}{(-\frac{1}{2})^2 + 1}} = 1.489 \quad \blacksquare \end{aligned}$$

$$\begin{aligned} M\left(\frac{\pi}{3}\right) &= 2\sqrt{\frac{(-\frac{1}{2}-\frac{1}{2}+1)^2 + (\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2})^2}{(-\frac{1}{2}-\frac{\sqrt{2}}{2}\times\frac{1}{2}+\frac{1}{4})^2 + (\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2})^2}} \\ &= 2\sqrt{\frac{0+0}{\left(\frac{1-2-\sqrt{2}}{4}\right)^2 + \dots}} = 0 \quad \blacksquare \end{aligned}$$

$$\begin{aligned} M(\pi) &= 2\sqrt{\frac{(1+1+1)^2 + (0-0)}{(1+\frac{\sqrt{2}}{2}+\frac{1}{4})^2 + (0-0)}} = 2\sqrt{\frac{3^2 \times 16}{(4+2\sqrt{2}+1)^2}} \\ &= \frac{2 \times 3 \times 4}{4+2\sqrt{2}+1} = 3.0658 \quad \blacksquare \end{aligned}$$

SA.11 A digital filter characterized by the transfer function

$$H(z) = \frac{2(z^2 - z + 1)}{z^2 - \frac{\sqrt{2}}{2}z + \frac{1}{4}}$$

and a practical D/A converter are connected in cascade as shown in Fig. SA.11a. The output waveform of the D/A converter is of the form illustrated in Fig. S11b where  $\tau = 0.01T$  s and  $T$  is the sampling period. The sampling frequency is  $2\pi$  rad/s.

- Obtain an expression for the gain of just the digital filter.
- Obtain an expression for the overall gain of the digital filter in cascade with the D/A converter.
- Calculate the gain of just the digital filter at  $\omega = 0, \pi/4, \pi/3$ , and  $\pi$ .
- Calculate the overall gain of the digital filter in cascade with the D/A converter at  $\omega = 0, \pi/4, \pi/3$ , and  $\pi$ .
- Sketch (i) the gain of just the digital filter and (ii) the overall gain of the digital filter in cascade with the D/A converter, and explain the effect of the D/A converter on the amplitude response.

**Solution**

- The gain is given by

$$\begin{aligned} M(\omega) &= |H(e^{j\omega T})| = \left| \frac{2[e^{2j\omega T} - e^{j\omega T} + 1]}{e^{j2\omega T} - \frac{\sqrt{2}}{2}e^{j\omega T} + \frac{1}{4}} \right| \\ &= 2 \left| \frac{\cos 2\omega T + j \sin 2\omega T - \cos \omega T - j \sin \omega T + 1}{\cos 2\omega T + j \sin 2\omega T - \frac{\sqrt{2}}{2}(\cos \omega T + j \sin \omega T) + \frac{1}{4}} \right| \\ &= 2\sqrt{\frac{(\cos 2\omega T - \cos \omega T + 1)^2 + (\sin 2\omega T - \sin \omega T)^2}{(\cos 2\omega T - \frac{\sqrt{2}}{2}\cos \omega T + \frac{1}{4})^2 + (\sin 2\omega T - \frac{\sqrt{2}}{2}\sin \omega T)^2}} \quad \blacksquare \end{aligned}$$

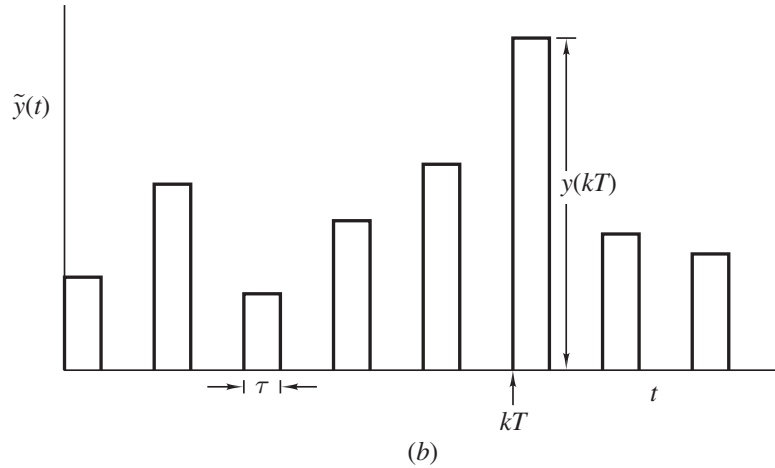
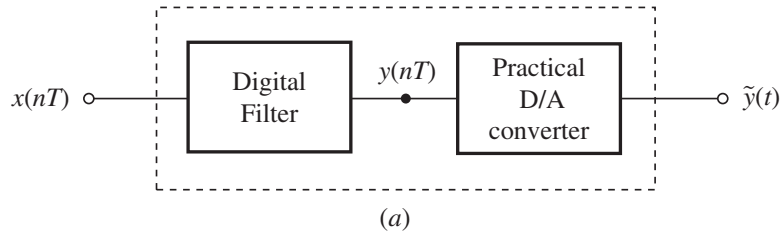


Figure SA.11

(b) The practical D/A converter has a gain

$$|H_p(j\omega)| = \tau \left| \frac{\sin(\omega\tau/2)}{\omega\tau/2} \right| \quad (\text{S})$$

where  $\tau = 0.01T$  and  $T = 2\pi/\omega_s = 2\pi/2\pi = 1$  s (see Eq. (6.60) in textbook). Hence the overall gain of the digital filter in cascade with the D/A converter is given by

$$M_T(\omega) = 2 \sqrt{\frac{(\cos 2\omega T - \cos \omega T + 1)^2 + (\sin 2\omega T - \sin \omega T)^2}{(\cos 2\omega T - \frac{\sqrt{2}}{2} \cos \omega T + \frac{1}{4})^2 + (\sin 2\omega T - \frac{\sqrt{2}}{2} \sin \omega T)^2}} \cdot \tau \left| \frac{\sin \omega\tau/2}{\omega\tau/2} \right|$$

(c) Since  $\omega_s = 2\pi$ , we have  $2\pi f_s = 2\pi/T = 2\pi$ . Hence  $T = 1$  s. For the frequencies given, the numerical values in Table SA.6 can be readily calculated.

Table SA.6

$\omega$	$\cos \omega T$	$\sin \omega T$	$\cos 2\omega T$	$\sin 2\omega T$
0	1	0	1	0
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	0	1
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\pi$	-1	0	1	0

Thus

$$\begin{aligned}
 M(0) &= 2\sqrt{\frac{(1-1+1)^2 + (0-0)^2}{(1-\frac{\sqrt{2}}{2} + \frac{1}{4})^2 + (0-0)^2}} = \frac{2}{\frac{5-2\sqrt{2}}{4}} = 3.6840 \quad \blacksquare \\
 M\left(\frac{\pi}{4}\right) &= 2\sqrt{\frac{(0-\frac{\sqrt{2}}{2} + 1)^2 + (1-\frac{\sqrt{2}}{2})^2}{(0-\frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} + \frac{1}{4})^2 + (1-\frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2})^2}} \\
 &= 2\sqrt{\frac{(2-\sqrt{2})^2 + (2-\sqrt{2})^2}{(\frac{1}{2}-1)^2 + 1}} = 2\sqrt{\frac{2(2-\sqrt{2})^2}{(-\frac{1}{2})^2 + 1}} = 1.4890 \quad \blacksquare \\
 M\left(\frac{\pi}{3}\right) &= 2\sqrt{\frac{(-\frac{1}{2}-\frac{1}{2} + 1)^2 + (\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2})^2}{(-\frac{1}{2}-\frac{\sqrt{2}}{2} \times \frac{1}{2} + \frac{1}{4})^2 + (\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2})^2}} \\
 &= 2\sqrt{\frac{0+0}{\left(\frac{1-2-\sqrt{2}}{4}\right)^2 + \dots}} = 0 \quad \blacksquare \\
 M(\pi) &= 2\sqrt{\frac{(1+1+1)^2 + (0-0)}{(1+\frac{\sqrt{2}}{2} + \frac{1}{4})^2 + (0-0)}} = 2\sqrt{\frac{3^2 \times 16}{(4+2\sqrt{2}+1)^2}} \\
 &= \frac{2 \times 3 \times 4}{4+2\sqrt{2}+1} = 3.0657 \quad \blacksquare
 \end{aligned}$$

(d) Since  $T = 1\text{s}$ , Eq. (SA.19) gives

$$\begin{aligned}
 |H_P(j\omega)| &= 0.01T \left| \frac{\sin(0.01T\omega/2)}{0.01T\omega/2} \right| = 0.01 \left| \frac{\sin(0.01\omega/2)}{0.01\omega/2} \right| \\
 &= p0.01
 \end{aligned}$$

Hence the overall gain of the digital filter in cascade with the D/A converter is obtained from the above numerical values as follows:

$$\begin{aligned}
 M_T(0) &= 3.6840 \times 0.01 = 3.6840 \times 10^{-2} \quad \blacksquare \\
 M_T\left(\frac{\pi}{4}\right) &= 1.4890 \times 0.01 = 1.4890 \times 10^{-2} \quad \blacksquare \\
 M_T\left(\frac{\pi}{3}\right) &= 0 \times 0.0100 = 0.0 \quad \blacksquare \\
 M_T(\pi) &= 3.0557 \times 0.01 = 3.0557 \times 10^{-2} \quad \blacksquare
 \end{aligned}$$

SA.12 Realize the transfer function

$$H(z) = \left( \frac{z}{z-0.5+j0.3} + \frac{z}{z-0.5-j0.3} \right) \cdot \left( \frac{3z}{z+0.4} + \frac{5z}{z+0.5} \right)$$

using two second-order canonic filter sections in cascade.

### Solution

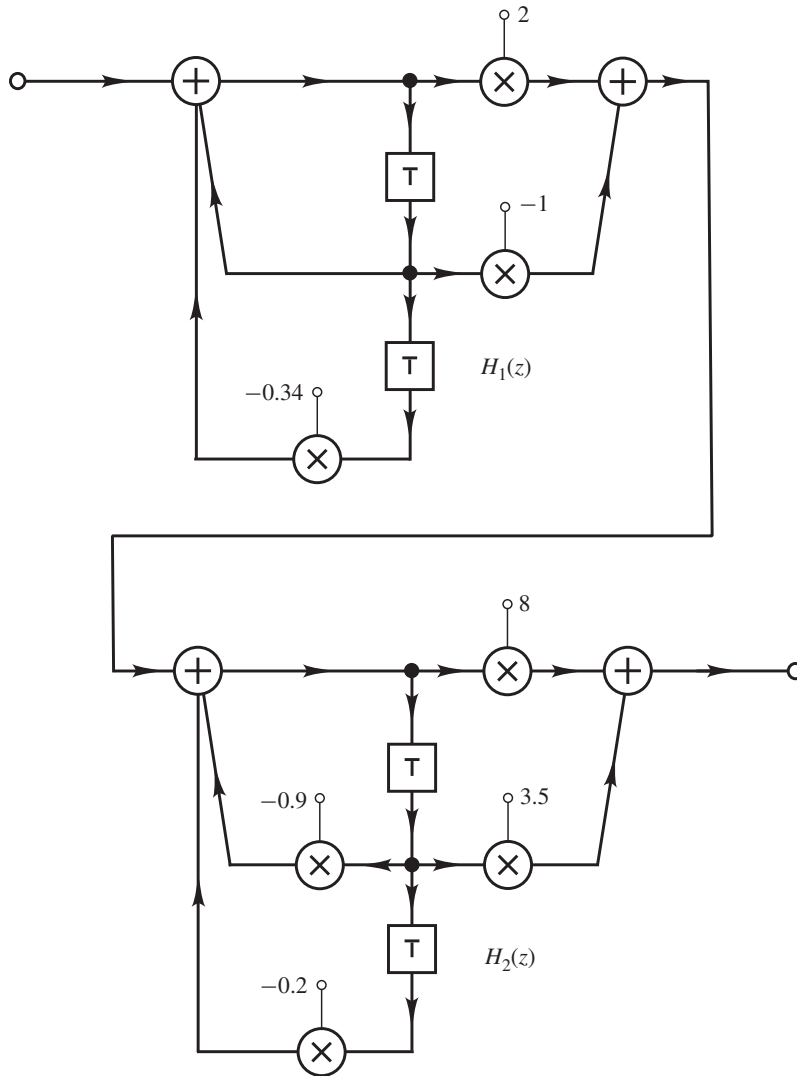
The transfer function can be expressed as

$$\begin{aligned}
 H(z) &= \frac{2z^2 - 0.5z - 0.5z}{z^2 - z + 0.5^2 + 0.3^2} \cdot \frac{3z^2 + 1.5z + 5z^2 + 2z}{z^2 + 0.9z + 0.2} \\
 &= \frac{2z^2 - z}{z^2 - z + 0.34} \cdot \frac{8z^2 + 3.5z}{z^2 + 0.9z + 0.2} \\
 &= \frac{2 - z^{-1}}{1 - z^{-1} + 0.34z^{-2}} \cdot \frac{8 + 3.5z^{-1}}{1 + 0.9z^{-1} + 0.2z^{-2}} \\
 &= H_1(z) \cdot H_2(z)
 \end{aligned}$$

where

$$H_1(z) = \frac{2 - z^{-1}}{1 - z^{-1} + 0.34z^{-2}} \quad \text{and} \quad H_2(z) = \frac{8 + 3.5z^{-1}}{1 + 0.9z^{-1} + 0.2z^{-2}}$$

Now if we realize  $H_1(z)$  and  $H_2(z)$  in terms of direct canonic sections, the cascade realization shown in Fig. SA.12 can be readily obtained. ■



**Figure SA.12**

SA.13 An analog elliptic lowpass filter with a cutoff frequency of 1 rad/s has a transfer function of the form

$$H(s) = \frac{0.075(s^2 + 2.6)}{(s + 0.38)(s^2 + 0.31s + 0.51)}$$

(a) By applying the lowpass-to-highpass transformation

$$s = \frac{1}{\bar{s}}$$

get a continuous-time highpass transfer function.

(b) Construct the zero-pole plot of the continuous-time highpass transfer function.

(c) Using the zeros and poles obtained in part (b), get a corresponding discrete-time highpass transfer function using the matched- $z$ -transformation method. The sampling frequency is  $\omega_s = 20$  rad/s.

- (d) How does the matched- $z$ -transformation method compare with the invariant-impulse-response method?

**Solution**

The transfer function has zeros at

$$z = z_1, z_1^*$$

where

$$z_1 = 0 + j1.6125$$

and poles at

$$z = p_0, p_1, p_1^*$$

where

$$p_0 = -0.3800 + j0.0000$$

$$p_1 = -0.1550 + j0.6971$$

- (a) The highpass transfer function is obtained as

$$\begin{aligned} H_{HP}(\bar{s}) &= H(s) \Big|_{\bar{s} \rightarrow \frac{1}{\bar{s}}} = \frac{0.075(\bar{s}^2 + 2.6)}{(s + 0.38)(s^2 + 0.31s + 0.51)} \Big|_{s=1/\bar{s}} \\ &= \frac{0.075(\frac{1}{\bar{s}^2} + 2.6)}{(\frac{1}{\bar{s}} + 0.38)(\frac{1}{\bar{s}^2} + 0.31\frac{1}{\bar{s}} + 0.51)} \\ &= \frac{0.075\bar{s}(1 + 2.6\bar{s}^2)}{(1 + 0.38\bar{s})(0.51\bar{s}^2 + 0.31\bar{s} + 1)} \\ &= \frac{0.075 \times 2.6}{0.38 \times 0.51} \times \frac{\bar{s}(\bar{s}^2 + \frac{1}{2.6})}{(\bar{s} + \frac{1}{0.38})(\bar{s}^2 + \frac{0.31}{0.51}\bar{s} + \frac{1}{0.51})} \\ &= 1.0063 \times \frac{\bar{s}(\bar{s}^2 + 0.3846)}{(\bar{s} + 2.6316)(\bar{s}^2 + 0.6079\bar{s} + 1.9609)} \quad \blacksquare \end{aligned}$$

- (b) Therefore, the highpass filter has zeros at

$$\bar{s} = \bar{s}_0, \bar{s}_1, \bar{s}_2$$

where

$$\bar{s}_0 = 0, \quad \bar{s}_1, \bar{s}_2 = 0 \pm j0.6202$$

and poles at

$$\bar{s} = \bar{p}_0, \bar{p}_1, \bar{p}_2$$

where

$$\bar{p}_0 = -2.6316, \quad \bar{p}_1, \bar{p}_2 = -0.3040 \pm j1.3669$$

- (c) The transfer function of the digital filter is given by

$$H_D(z) = (z + 1)^L \frac{H_0 \prod_{i=1}^M (z - e^{\bar{s}_i T})}{\prod_{i=1}^N (z - e^{\bar{p}_i T})}$$

where  $L = 0$  for a highpass filter. Hence

$$H_D(z) = \frac{\prod_{i=1}^3 (z - \tilde{z}_i)}{3 \prod_{i=1}^3 (z - \tilde{p}_i)} \quad \blacksquare$$

where

$$\begin{aligned}\tilde{z}_0 &= e^{0T} = 1, & \tilde{z}_1, \tilde{z}_2 &= e^{\pm j0.6202T} \\ \tilde{p}_0 &= e^{-2.6316T}, & \tilde{p}_1, \tilde{p}_2 &= e^{(-0.3040 \pm j1.3669)T}\end{aligned}$$

with

$$T = \frac{2\pi}{\omega_s} = \frac{2\pi}{20} = \frac{\pi}{10}$$

- (d) The method works with all types of filters, i.e., LP, HP, BP, and BS, and is easy to apply. However, it tends to increase the passband ripple. ■

SA.14 A lowpass analog filter has a transfer function

$$H_A(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

- (a) Assuming a sampling frequency of  $10\pi$ , design a digital filter using the bilinear transformation method.  
 (b) Find the 1-dB and 30-dB frequencies of the analog filter.  
 (c) Find the 1-dB and 30-dB frequencies of the digital filter.  
 (d) What should be the 3-dB frequency of the analog filter to get a 3-dB frequency in the digital filter at 1 rad/s?

### Solution

- (a) The sampling period is given by

$$T = \frac{1}{f_s} = \frac{2\pi}{\omega_s} = \frac{2\pi}{10\pi} = \frac{1}{5}$$

The discrete-time transfer function is given by

$$\begin{aligned}H_D(z) &= H_A(s) \Big|_{s=\frac{2}{T} \frac{z-1}{z+1}} \\ &= \frac{1}{s^2 + \sqrt{2}s + 1} \Big|_{s=\frac{10(z-1)}{z+1}} \\ &= \frac{1}{\left[\frac{10(z-1)}{z+1}\right]^2 + \sqrt{2} \left[\frac{10(z-1)}{z+1}\right] + 1} \\ &= \frac{z^2 + 2z + 1}{100(z^2 - 2z + 1) + 10\sqrt{2}(z^2 - 1) + (z^2 + 2z + 1)} \\ &= \frac{z^2 + 2z + 1}{b_2 z^2 + b_1 z + b_0} \quad \blacksquare\end{aligned}$$

where

$$\begin{aligned}b_0 &= 100 - 10\sqrt{2} + 1 = 86.86 \\ b_1 &= -200 + 2 = -198.00 \\ b_2 &= 100 + 10\sqrt{2} + 1 = 115.14\end{aligned}$$

Alternatively, by factorizing constant  $b_2$  in the denominator, the transfer function can be expressed as

$$H_D(z) = H_0 \frac{z^2 + 2z + 1}{z^2 + b'_1 z + b'_0} \quad \blacksquare$$

where

$$H_0 = 8.685 \times 10^{-3}, \quad b'_0 = 0.7544, \quad b'_1 = -1.720$$



(b) The frequency response of the analog filter

$$H_A(j\omega) = \frac{1}{-\omega^2 + j\sqrt{2}\omega + 1}$$

Hence the loss is given by

$$\begin{aligned} A(\omega) &= 20 \log \sqrt{(1 - \omega^2)^2 + 2\omega^2} \\ &= 10 \log(1 - 2\omega^2 + \omega^4 + 2\omega^2) \end{aligned}$$

Thus

$$1 + \omega^4 = 10^{0.1 \times A(\omega)}$$

By letting  $A(\omega) = 1$  dB, the 1-dB frequency is obtained as

$$\omega_1 = (10^{0.1} - 1)^{1/4} = 0.7133 \text{ rad/s} \quad \blacksquare$$

By letting  $A(\omega) = 30$  dB, the 30-dB frequency is obtained as

$$\omega_2 = (10^{0.1 \times 30} - 1)^{1/4} = 5.622 \text{ rad/s} \quad \blacksquare$$

(c) A frequency  $\omega_i$  in the analog filter transforms into a frequency  $\Omega_i$  given by

$$\Omega_i = \frac{2}{T} \tan^{-1} \frac{\omega_i T}{2} \quad (\text{T})$$

Hence the 1- and 30-dB frequencies in the digital filter are obtained as

$$\Omega_1 = 10 \times \tan^{-1} \frac{0.7133}{10} = 0.7121 \text{ rad/s} \quad \blacksquare$$

and

$$\Omega_2 = 10 \times \tan^{-1} \frac{5.622}{10} = 5.122 \text{ rad/s} \quad \blacksquare$$

respectively.

(d) Now if the 3-dB frequency in the digital filter is required to be 1 rad/s, then according to Eq. (SA.20) the 3-dB frequency in the analog filter should be

$$\omega_3 = \frac{2}{T} \tan \frac{\Omega_3 T}{2} = 10 \times \tan \frac{1}{10} = 1.003 \text{ rad/s} \quad \blacksquare$$