SYMMETRIC EXTENSION FOR QUINCUNX FILTER BANKS

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ABSTRACT

Symmetric extension is a commonly used technique for constructing nonexpansive transforms for one-dimensional signals of finite length. In this paper, we show how to extend this technique to the two-dimensional case with perfect reconstruction quincunx filter banks composed of quadrantally-symmetric linear-phase filters. We derive the constraints on the group delays and symmetry types of the analysis filters, in particular those with non-integer vector group delays.

1. INTRODUCTION

Quincunx filter banks are nonseparable two-dimensional (2-D) two-channel filter banks that are used to compute subband transforms for many image processing applications. Fig. 1 shows the block diagram of a quincunx filter bank, where H_0, H_1 and G_0, G_1 are the analysis and synthesis filters, respectively. The subband sequences y_0 and y_1 are obtained by downsampling the analysis filter outputs in a nonseparable pattern such that the sampling density of each filter output is reduced by a factor of two. The combined sampling rate of y_0 and y_1 is then the same as that of the input sequence x. Although the filter bank is defined to operate on sequences of infinite extent, in practice, we frequently deal with sequences of finite extent. Therefore, we often require some means for adapting the filter bank to such sequences. This leads to the well known boundary filtering problem that can arise any time a finite-extent sequence is filtered. Furthermore, in many applications, it is desirable to employ a transform that is nonexpansive (i.e., maps a sequence of N samples to a new sequence of no more than N samples). Consequently, we seek a solution to the boundary problem that yields nonexpansive transforms.

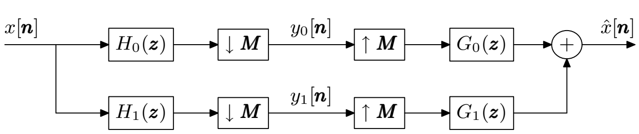


Fig. 1. 2-D two-channel filter bank.

In the 1-D case, symmetric extension [1, 2] is a commonly used technique to build nonexpansive transforms for finite-length sequences. It extends the input sequence symmetrically and periodically, maintaining continuity at the splice points between periods. In this paper, we show how this technique can be extended to (2-D) quincunx filter banks, provided that the analysis and synthesis filters are chosen appropriately.

The remainder of this paper is structured as follows. Section 2 briefly comments on some of the notational conventions used herein. Section 3 discusses symmetry in the 2-D case, and some results related to symmetry, periodicity, and filter banks. Then, these results are used in Section 4 to derive our new symmetric extension algorithms. Finally, Section 5 summarizes our work.

This work was supported, in part, by the Natural Sciences and Engineering Research Council of Canada.

2. NOTATION AND TERMINOLOGY

In this paper, matrices and vectors are denoted by upper and lower case boldface letters, respectively. The set of integers and the set of ordered pairs of integers are denoted as $\mathbb{Z}$ and $\mathbb{Z}^2$, respectively. For a set S and a scalar k, the notation kS denotes the set $\{ks\}_{s \in S}$. The difference of two sets A and B is denoted as $A \setminus B$. An element of a sequence x defined on $\mathbb{Z}^2$ is denoted either as $x[\boldsymbol{n}]$ or $x[n_0, n_1]$ (whichever is more convenient), where $\boldsymbol{n} = [n_0 \quad n_1]^T$ and $n_0, n_1 \in \mathbb{Z}$. The convolution of two sequences x and y is denoted as $x * y$.

For the most part, the multidimensional multirate systems notation employed in this paper follows that used in [3]. A sequence x defined on $\mathbb{Z}^2$ is periodic with a periodicity matrix $\boldsymbol{P}$ if $x[\boldsymbol{n}] = x[\boldsymbol{n} + \boldsymbol{Pk}]$ for all $\boldsymbol{n}, \boldsymbol{k} \in \mathbb{Z}^2$. The lattice generated by sampling matrix $\boldsymbol{M}$ is denoted as $\mathrm{LAT}(\boldsymbol{M})$ (i.e., $\mathrm{LAT}(\boldsymbol{M}) = \{\boldsymbol{Mn}\}_{\boldsymbol{n} \in \mathbb{Z}^2}$). The quincunx lattice can be associated with the generating matrix $\boldsymbol{M} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ and the two representative coset vectors $\boldsymbol{k}_0 = [0 \ \ 0]^T$ and $\boldsymbol{k}_1 = [1 \ \ 0]^T$. Throughout this work, the quantity $\boldsymbol{M}$ should be understood to be this particular generating matrix of the quincunx lattice (unless explicitly noted otherwise).

3. SYMMETRIC EXTENSION PRELIMINARIES

With our proposed symmetric extension scheme, we use a structure for the forward transform like that shown in Fig. 2. The input sequence $\tilde{x}$ is first extended to an infinite-extent periodic symmetric sequence x. The periodicity and symmetry properties may propagate across the nonseparable downsampler by carefully constraining the choice of the analysis filters H_0 and H_1. Thus, the independent samples of the subbands y_0 and y_1 are each located in a finite region, and then we can extract these samples from y_0 and y_1. As we will later show, using this structure, it is possible to obtain a nonexpansive transform.

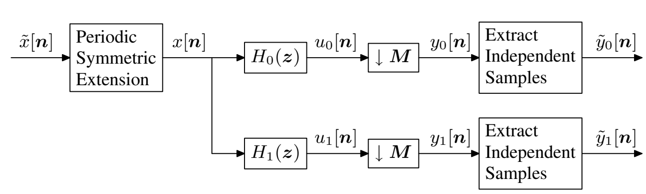


Fig. 2. Analysis side of the filter bank with symmetric extension.

The notion of symmetry is of fundamental importance in this paper. For a 1-D sequence x, if there is $c \in \frac{1}{2}\mathbb{Z}$ and $S \in \{-1, 1\}$ such that $x[n] = Sx[2c - n]$ for all $n \in \mathbb{Z}$, x is said to be either symmetric about c if $S = 1$ or antisymmetric about c if $S = -1$. The sequence x is said to have whole-sample symmetry/antisymmetry if $c \in \mathbb{Z}$, and half-sample symmetry/antisymmetry if $c \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. We can see that only a very limited number of symmetry types is possible in the 1-D case. In the 2-D case, however, there are considerably more possibilities. We begin by introducing several types of symmetries as given by the definitions below.

Definition 1 (Centrosymmetry)**.** A sequence x defined on $\mathbb{Z}^2$ is said to be **centrosymmetric** about $\boldsymbol{c}$ (i.e., has **linear phase** with **group delay** $\boldsymbol{c}$) if, for some $\boldsymbol{c} \in \frac{1}{2}\mathbb{Z}^2$ and $S \in \{-1, 1\}$,

$$x[\boldsymbol{n}] = Sx[2\boldsymbol{c} - \boldsymbol{n}] \quad \text{for all } \boldsymbol{n} \in \mathbb{Z}^2. \tag{1}$$

In the 2-D case, sequences can have higher order symmetries such as four-fold symmetry. Below, we define two types of four-fold symmetry relevant to this work.

Definition 2 (Quadrantal centrosymmetry)**.** A sequence x defined on $\mathbb{Z}^2$ is said to be **quadrantally centrosymmetric** about $\boldsymbol{c}$ (quadrantally symmetric linear phase with group delay $\boldsymbol{c}$) if for some $S, T \in \{-1, 1\}$ and $\boldsymbol{c} = \begin{bmatrix} c_0 & c_1 \end{bmatrix}^T \in \frac{1}{2}\mathbb{Z}^2$,

$$\begin{aligned} x[n_0, n_1] &= STx[2c_0 - n_0, 2c_1 - n_1] \\ &= Sx[2c_0 - n_0, n_1] \\ &= Tx[n_0, 2c_1 - n_1] \end{aligned} \tag{2}$$

for all $n_0, n_1 \in \mathbb{Z}$. In terms of S and T, four types of quadrantal centrosymmetry are possible [4]:

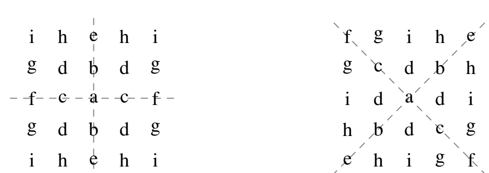
Type	S	T
even-even	1	1
odd-odd	-1	-1
even-odd	1	-1
odd-even	-1	1

Definition 3 (Rotated quadrantal centrosymmetry)**.** A sequence x defined on $\mathbb{Z}^2$ is said to be **rotated quadrantally centrosymmetric** about $\boldsymbol{c}$ if, for some $S, T \in \{-1, 1\}$ and $\boldsymbol{c} = \begin{bmatrix} c_0 & c_1 \end{bmatrix}^T \in \frac{1}{2}\mathbb{Z}^2$ satisfying $c_0 + c_1 \in \mathbb{Z}$,

$$\begin{aligned} x[n_0, n_1] &= STx[2c_0 - n_0, 2c_1 - n_1] \\ &= Sx[c_0 - c_1 + n_1, c_1 - c_0 + n_0] \\ &= Tx[c_0 + c_1 - n_1, c_0 + c_1 - n_0] \end{aligned} \tag{3}$$

for all $n_0, n_1 \in \mathbb{Z}$.

Examples of quadrantally-centrosymmetric and rotated-quadrantally-centrosymmetric sequences are shown in Figs. 3(a) and (b), respectively. One can see from the diagrams that quadrantal centrosymmetry and rotated quadrantal centrosymmetry are each a type of four-fold symmetry (i.e., only approximately $\frac{1}{4}$ of the samples are independent).



(a) (b)

Fig. 3. Types of four-fold symmetry. (a) Quadrantal centrosymmetry and (b) rotated quadrantal centrosymmetry.

We now introduce a scheme for mapping a finite-extent (2-D) sequence defined on a rectangular region to an infinite-extent sequence that is both quadrantally centrosymmetric and periodic. This process is called symmetric extension.

In the 1-D case, there are two ways to extend a finite-length sequence, such that the extended sequence has whole-sample or half-sample symmetry. (Herein, we do not consider antisymmetric extension as this extension scheme produces signals with large discontinuities, something that is undesirable in most applications.) Since the symmetric extension of a 2-D sequence can be viewed as 1-D extension operations applied independently along each dimension of the sequence, there are four types of symmetric extension for a 2-D sequence as defined below. In this paper, the type-2 and type-3 symmetric extension are of most interest.

Definition 4 (Symmetric extension of sequence)**.** Let $\tilde{x}$ be a (2-D) sequence defined on the rectangular region $\{0, 1, \ldots, L_0 - 1\} \times \{0, 1, \ldots, L_1 - 1\}$. Then, the symmetric extension x of $\tilde{x}$ is defined as

$$x[n_0, n_1] = \begin{cases} \tilde{x}[f_w[n_0, L_0], f_w[n_1, L_1]] & \text{type 1} \\ \tilde{x}[f_h[n_0, L_0], f_w[n_1, L_1]] & \text{type 2} \\ \tilde{x}[f_w[n_0, L_0], f_h[n_1, L_1]] & \text{type 3} \\ \tilde{x}[f_h[n_0, L_0], f_h[n_1, L_1]] & \text{type 4,} \end{cases} \tag{4}$$

where the functions f_w and f_h are given by

$$f_w[n, L] = \min\{\operatorname{mod}(n, 2L-2), 2L-2-\operatorname{mod}(n, 2L-2)\},$$
$$\text{and } f_h[n, L] = \min\{\operatorname{mod}(n, 2L), 2L-1-\operatorname{mod}(n, 2L)\}.$$

The rows (i.e., 1-D horizontal slices) of the 2-D sequence are whole- or half-sample symmetric and $(2L_0 - 2)$- or $2L_0$-periodic in the horizontal direction depending on whether f_w or f_h is used in (4). Similarly, the columns (i.e., 1-D vertical slices) of the 2-D sequence are also symmetric and periodic in the vertical direction. This leads to the symmetry and periodicity properties of a 2-D symmetrically extended sequence as summarized by the below lemma.

Lemma 1 (Properties of symmetrically extended sequences)**.** *Let $\tilde{x}$ be a sequence defined on the rectangular region $\{0, 1, \ldots, L_0 - 1\} \times \{0, 1, \ldots, L_1 - 1\}$, and x be the symmetric extension of $\tilde{x}$ as defined by (4). Let $\boldsymbol{M}$ denote the quincunx generating matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Then, x is $\boldsymbol{P}$-periodic with $\boldsymbol{M}^{-1}\boldsymbol{P}$ being an integer matrix and even-even quadrantally centrosymmetric about $\boldsymbol{c}_x$, where $\boldsymbol{c}_x$ and $\boldsymbol{P}$ are given by the following table:*

Type	$\boldsymbol{c}_x$	$\boldsymbol{P}$
1	$\begin{bmatrix} 0 & 0 \end{bmatrix}^T$	$\begin{bmatrix} 2L_0-2 & 0 \\ 0 & 2L_1-2 \end{bmatrix}$
2	$\begin{bmatrix} -\frac{1}{2} & 0 \end{bmatrix}^T$	$\begin{bmatrix} 2L_0 & 0 \\ 0 & 2L_1-2 \end{bmatrix}$
3	$\begin{bmatrix} 0 & -\frac{1}{2} \end{bmatrix}^T$	$\begin{bmatrix} 2L_0-2 & 0 \\ 0 & 2L_1 \end{bmatrix}$
4	$\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}^T$	$\begin{bmatrix} 2L_0 & 0 \\ 0 & 2L_1 \end{bmatrix}$

Proof. We only prove the properties of the type-2 symmetric extension. First, we show that x is $\boldsymbol{P}$-periodic with $\boldsymbol{P} = \boldsymbol{M}\begin{bmatrix} L_0 & L_1-1 \\ L_0 & -L_1+1 \end{bmatrix}$. Since $\operatorname{mod}(u + kv, v) = \operatorname{mod}(u, v)$ for $k \in \mathbb{Z}$, we have $f_h[n_0 + 2L_0k_0, L_0] = f_h[n_0, L_0]$ and $f_w[n_1 + (2L_1 - 2)k_1, L_1] = f_w[n_1, L_1]$, for k_0, $k_1 \in \mathbb{Z}$. This implies that $x[\boldsymbol{n} + \boldsymbol{P}\boldsymbol{k}] = x[\boldsymbol{n}]$ for $\boldsymbol{k} = \begin{bmatrix} k_0 & k_1 \end{bmatrix}^T$ with $\boldsymbol{P} = \begin{bmatrix} 2L_0 & 0 \\ 0 & 2L_1-2 \end{bmatrix}$. Therefore, x is $\boldsymbol{P}$-periodic, and $\boldsymbol{M}^{-1}\boldsymbol{P} = \begin{bmatrix} L_0 & L_1-1 \\ L_0 & -L_1+1 \end{bmatrix}$ is an integer matrix.

Now, we show that x is quadrantally centrosymmetric about $\begin{bmatrix} -\frac{1}{2} & 0 \end{bmatrix}^T$. For $u, v \in \mathbb{Z}$, if $v \nmid u$, $\operatorname{mod}(-u, v) = v - \operatorname{mod}(u, v)$, otherwise, $\operatorname{mod}(-u, v) = 0$. It follows that $f_h[-n_0, L_0] = f_h[n_0 - 1, L_0]$

and $f_w[-n_1, L_1] = f_w[n_1, L_1]$. Therefore,

$$\begin{aligned} x[-1-n_0, -n_1] &= \tilde{x}[f_h[-1-n_0, L_0], f_w[-n_1, L_1]] \\ &= \tilde{x}[f_h[n_0, L_0], f_w[n_1, L_1]] \\ &= x[n_0, n_1]. \end{aligned}$$

Similarly, we have $x[-1-n_0, n_1] = x[n_0, n_1]$ and $x[n_0, -n_1] = x[n_0, n_1]$. Thus, from Definition 2, x is quadrantally centrosymmetric about $\boldsymbol{c}_x = \begin{bmatrix} -\frac{1}{2} & 0 \end{bmatrix}^T$. Since S and T in (2) are both 1, x has the even-even symmetry. (Due to $\boldsymbol{P}$-periodicity, x is also quadrantally centrosymmetric about $\boldsymbol{Pk} + \boldsymbol{c}_x$ for $\boldsymbol{k} \in \frac{1}{2}\mathbb{Z}^2$.) $\square$

As shown in Fig. 2, the symmetrically extended sequence x is fed into the analysis side of the filter bank, which consists of filters followed by downsamplers. In order for the subband transform to be nonexpansive, the preservation of the four-fold symmetry and periodicity under convolution and downsampling is important. In what follows, we consider the effects of these operations on symmetry and periodicity.

Lemma 2 (Preservation of symmetry under convolution)**.** *Let x and h be sequences defined on $\mathbb{Z}^2$, and define $y = x * h$. If x and h are quadrantally centrosymmetric about $\boldsymbol{c}_x$ and $\boldsymbol{c}_h$, respectively, then y is quadrantally centrosymmetric about $\boldsymbol{c}_y = \boldsymbol{c}_x + \boldsymbol{c}_h$. Moreover, if x has the even-even symmetry, then y has the same type of symmetry as h [5].*

Lemma 3 (Preservation of periodicity under convolution)**.** *Let x and h be sequences defined on $\mathbb{Z}^2$, with x being $\boldsymbol{P}$-periodic. Then, $y = x * h$ is $\boldsymbol{P}$-periodic [5].*

Lemma 4 (Downsampling of periodic sequence)**.** *Let $\boldsymbol{M}$ be an arbitrary sampling (i.e., nonsingular integer) matrix. Let x be $\boldsymbol{P}$-periodic such that $\boldsymbol{M}^{-1}\boldsymbol{P}$ is an integer matrix. Then, $(\downarrow \boldsymbol{M})x$ is $(\boldsymbol{M}^{-1}\boldsymbol{P})$-periodic [5].*

Lemma 5 (Downsampling of quadrantally centrosymmetric sequence)**.** *Let x be a quadrantally centrosymmetric sequence with symmetry center $\boldsymbol{c}_x \in \mathbb{Z}^2$, and $\boldsymbol{M}$ be the quincunx generating matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Define $y = (\downarrow \boldsymbol{M})x$. Then, y is rotated quadrantally centrosymmetric about $\boldsymbol{M}^{-1}\boldsymbol{c}_x$ [5].*

The above results show that the symmetry and periodicity properties may be preserved under convolution and downsampling. These results will be used in later derivations of our new symmetric extension algorithms.

4. SYMMETRIC EXTENSION ALGORITHM

We will now derive a scheme based on symmetric extension that allows for the construction of nonexpansive transforms based on quincunx filter banks like the one shown in Fig. 2. For nonexpansive transforms, the number of independent samples in each of the subbands y_0 and y_1 is approximately half of that in the extended input sequence x, which suggests that the subband sequences also have four-fold symmetry and periodicity. Therefore, from Lemmas 2 and 5, the analysis filters H_0 and H_1 should have quadrantal centrosymmetry and their group delays should be chosen such that the symmetry centers of u_0 and u_1 are both on the integer lattice. Below, we present in detail the constraints on the quincunx filter banks that are compatible with each type of symmetric extension defined in (4).

For type-1 symmetric extension, where the extended sequence has symmetry center $\boldsymbol{0}$, it has been shown that all perfect reconstruction quincunx filter banks having quadrantally centrosymmetric filters with group delays on the integer lattice lead to nonexpansive transforms [5]. For type-2 symmetric extension, the extended sequence x is half-sample symmetric in the horizontal direction and whole-sample symmetric in the vertical direction. In order for u_0 and u_1 to be quadrantally centrosymmetric with symmetry centers on the integer lattice, the analysis filters must have the same kind of symmetry. Based on this observation, we have the theorem below.

Theorem 1 (Type-2 symmetric extension algorithm)**.** *Consider the filter bank shown in Fig. 2, where $\tilde{x}$ is defined on the rectangular region $\{0, 1, \ldots, L_0 - 1\} \times \{0, 1, \ldots, L_1 - 1\}$ and x is the type-2 symmetric extension of $\tilde{x}$ as given by (4). Let $\boldsymbol{M}$ denote the quincunx generating matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Suppose that the analysis filters H_0 and H_1 satisfy the following conditions: 1) H_0 has even-even quadrantal centrosymmetry with group delay $\boldsymbol{d}_0 = [d_{0,0}\ d_{0,1}]^T$, $d_{0,0} \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ and $d_{0,1} \in \mathbb{Z}$; 2) H_1 has odd-even quadrantal centrosymmetry with group delay $\boldsymbol{d}_1 = \begin{bmatrix} d_{1,0} & d_{1,1} \end{bmatrix}^T$, $d_{1,0} \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ and $d_{1,1} \in \mathbb{Z}$; 3) $\boldsymbol{d}_0 - \boldsymbol{d}_1 \in \mathrm{LAT}(\boldsymbol{M})$. In this case, the subband output y_0 can be completely characterized by N_0 samples with indices $\boldsymbol{n} = \begin{bmatrix} n_0 & n_1 \end{bmatrix}^T$ given by*

$$\begin{aligned} &\left\lceil \frac{d_{0,0} + d_{0,1} - \frac{1}{2}}{2} \right\rceil \leq n_0 \leq \left\lfloor \frac{d_{0,0} + d_{0,1} + L_0 + L_1 - \frac{3}{2}}{2} \right\rfloor, \\ &\qquad \textit{and } \max\{d_{0,0} - n_0 - \tfrac{1}{2}, n_0 - d_{0,1} - L_1 + 1\} \leq n_1 \\ &\qquad\qquad \leq \min\{d_{0,0} + L_0 - n_0 - \tfrac{1}{2}, n_0 - d_{0,1}\}; \end{aligned} \tag{5}$$

y_1 can be completely characterized by N_1 samples given by

$$\begin{aligned} &\left\lceil \frac{d_{1,0} + d_{1,1} + \frac{1}{2}}{2} \right\rceil \leq n_0 \leq \left\lfloor \frac{d_{1,0} + d_{1,1} + L_0 + L_1 - \frac{5}{2}}{2} \right\rfloor, \\ &\qquad \textit{and } \max\{d_{1,0} - n_0 + \tfrac{1}{2}, n_0 - d_{1,1} - L_1 + 1\} \leq n_1 \\ &\qquad\qquad \leq \min\{d_{1,0} + L_0 - n_0 - \tfrac{3}{2}, n_0 - d_{1,1}\}; \end{aligned} \tag{6}$$

and $N_0 + N_1 = L_0 L_1$ (i.e., the transform is nonexpansive).

Proof. In what follows, we refer to the sequences in Fig. 2. From Lemma 1, we know that x is $\boldsymbol{P}$-periodic with $\boldsymbol{P} = \begin{bmatrix} 2L_0 & 0 \\ 0 & 2L_1 - 2 \end{bmatrix}$ and quadrantally centrosymmetric about $\begin{bmatrix} -\frac{1}{2} & 0 \end{bmatrix}^T$. Consider the first channel, where H_0 is quadrantally centrosymmetric with group delay $\boldsymbol{d}_0 = \begin{bmatrix} d_{0,0} & d_{0,1} \end{bmatrix}^T$ satisfying $d_{0,0} \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ and $d_{0,1} \in \mathbb{Z}$. Then, the analysis filter output u_0 is $\boldsymbol{P}$-periodic from Lemma 3 and quadrantally centrosymmetric about $\boldsymbol{c}_{u_0} = \begin{bmatrix} d_{0,0} - \frac{1}{2} & d_{0,1} \end{bmatrix}^T$ from Lemma 2. Since $\boldsymbol{M}^{-1}\boldsymbol{P} = \begin{bmatrix} L_0 & L_1 - 1 \\ L_0 & -L_1 + 1 \end{bmatrix}$ is an integer matrix and $\boldsymbol{c}_{u_0} \in \mathbb{Z}^2$, y_0 is $\boldsymbol{M}^{-1}\boldsymbol{P}$-periodic from Lemma 4 and rotated quadrantally centrosymmetric about $\boldsymbol{M}^{-1}\boldsymbol{c}_{u_0}$ from Lemma 5. Therefore, y_0 can be completely characterized by samples with indices $\boldsymbol{n} = \begin{bmatrix} n_0 & n_1 \end{bmatrix}^T$ given by

$$\begin{aligned} \boldsymbol{Mn} \in &\{d_{0,0} - \tfrac{1}{2}, d_{0,0} + \tfrac{1}{2}, \ldots, d_{0,0} + L_0 - \tfrac{1}{2}\} \times \\ &\{d_{0,1}, d_{0,1} + 1, \ldots, d_{0,1} + L_1 - 1\}. \end{aligned} \tag{7}$$

Solving (7), we obtain the conditions for n_0 and n_1 as shown in (5). The number N_0 of characteristic samples of y_0 is given by

$$N_0 = \begin{cases} \frac{1}{2}(L_0 + 1)L_1 & \text{for } (L_0 + 1)L_1 \text{ even} \\ \frac{1}{2}((L_0 + 1)L_1 + 1) & \text{for } (L_0 + 1)L_1 \text{ odd}, \boldsymbol{c}_{u_0} \in \Gamma \\ \frac{1}{2}((L_0 + 1)L_1 - 1) & \text{for } (L_0 + 1)L_1 \text{ odd}, \boldsymbol{c}_{u_0} \notin \Gamma, \end{cases} \tag{8}$$

where $\Gamma = \mathrm{LAT}(\boldsymbol{M})$. The above equation can be equivalently written as

$$N_0 = \left\lfloor \tfrac{1}{2}((L_0+1)L_1+d_{0,0}+d_{0,1}+\tfrac{1}{2}) \right\rfloor - \left\lceil \tfrac{1}{2}(d_{0,0}+d_{0,1}-\tfrac{1}{2}) \right\rceil .$$

Now we consider the second channel. Similar to the case of the first channel, we know that u_1 is $\boldsymbol{P}$-periodic and quadrantally centrosymmetric about $\boldsymbol{c}_{u_1} = \begin{bmatrix} d_{1,0}-\frac{1}{2} & d_{1,1} \end{bmatrix}^T$. Since H_1 has odd-even symmetry, u_1 also has odd-even symmetry. Note that each 1-D horizontal slice of u_1 has whole-sample antisymmetry. Then, y_1 can be completely characterized by samples with indices $\boldsymbol{n} = \begin{bmatrix} n_0 & n_1 \end{bmatrix}^T$ given by

$$\begin{aligned} \boldsymbol{Mn} \in &\{d_{1,0}+\tfrac{1}{2}, d_{1,0}+\tfrac{3}{2}, \ldots, d_{1,0}+L_0-\tfrac{3}{2}\} \times \\ &\{d_{1,1}, d_{1,1}+1, \ldots, d_{1,1}+L_1-1\}. \end{aligned} \tag{9}$$

Solving (9), we obtain the conditions for n_0 and n_1 as shown in (6). The number of characteristic samples of y_1 is given by

$$N_1 = \begin{cases} \frac{1}{2}(L_0-1)L_1 & \text{for } (L_0-1)L_1 \text{ even} \\ \frac{1}{2}((L_0-1)L_1-1) & \text{for } (L_0-1)L_1 \text{ odd}, \boldsymbol{c}_{u_1} \in \Gamma \\ \frac{1}{2}((L_0-1)L_1+1) & \text{for } (L_0-1)L_1 \text{ odd}, \boldsymbol{c}_{u_1} \notin \Gamma, \end{cases} \tag{10}$$

where $\Gamma = \mathrm{LAT}(\boldsymbol{M})$. The above equation is equivalent to

$$N_1 = \left\lfloor \tfrac{1}{2}((L_0-1)L_1+d_{1,0}+d_{1,1}+\tfrac{3}{2}) \right\rfloor - \left\lceil \tfrac{1}{2}(d_{1,0}+d_{1,1}+\tfrac{1}{2}) \right\rceil .$$

Since $\boldsymbol{d}_0 - \boldsymbol{d}_1 \in \mathrm{LAT}(\boldsymbol{M})$, $\boldsymbol{c}_{u_0}$ and $\boldsymbol{c}_{u_1}$ are in the same coset of the quincunx lattice. Therefore, from (8) and (10), we have $N_0 + N_1 = L_0 L_1$. □

Next we examine the existence of perfect reconstruction (PR) filter banks that satisfy the three conditions in Theorem 1. Recall that the PR condition for a quincunx filter bank is given by

$$H_0(\boldsymbol{z})G_0(\boldsymbol{z}) + H_1(\boldsymbol{z})G_1(\boldsymbol{z}) = 2\boldsymbol{z}^{-\boldsymbol{l}} \quad \text{and} \tag{11a}$$

$$H_0(-\boldsymbol{z})G_0(\boldsymbol{z}) + H_1(-\boldsymbol{z})G_1(\boldsymbol{z}) = 0, \tag{11b}$$

where $\boldsymbol{l} = [l_0 \;\; l_1]^T$ is an integer vector, $\boldsymbol{z}^{-\boldsymbol{l}} = z_0^{-l_0} z_1^{-l_1}$, and $H_0(\boldsymbol{z})$, $H_1(\boldsymbol{z})$ and $G_0(\boldsymbol{z})$, $G_1(\boldsymbol{z})$ are the analysis and synthesis filter transfer functions, respectively. If we let $G_0(\boldsymbol{z}) = H_1(-\boldsymbol{z})$ and $G_1(\boldsymbol{z}) = -H_0(-\boldsymbol{z})$, then (11b) is satisfied. If we further define $P(\boldsymbol{z}) = H_0(\boldsymbol{z})G_0(\boldsymbol{z}) = H_0(\boldsymbol{z})H_1(-\boldsymbol{z})$, then (11a) becomes $P(\boldsymbol{z}) - P(-\boldsymbol{z}) = 2\boldsymbol{z}^{-\boldsymbol{l}}$. It can be verified that if H_0 and H_1 satisfy the three conditions in Theorem 1, then according to Lemma 2, $P(\boldsymbol{z})$ has even-even quadrantal centrosymmetry with group delay $\boldsymbol{d}_p = \boldsymbol{d}_0 + \boldsymbol{d}_1 \in \mathbb{Z}^2 \setminus \mathrm{LAT}(\boldsymbol{M})$. Therefore, $P(\boldsymbol{z}) - P(-\boldsymbol{z})$ becomes a monomial if the coefficients of P are zero on all non-lattice points except at the point $\boldsymbol{d}_p$. In this case, the filter bank satisfies the PR condition. A simple example of this type of PR filter bank is given by $H_0(z_0, z_1) = \frac{1}{2}(1+z_0)$, $H_1(z_0, z_1) = 1 - z_0$, $G_0(z_0, z_1) = 1 + z_0$, and $G_1(z_0, z_1) = \frac{1}{2}(-1+z_0)$.

For type-3 symmetric extension as defined in (4), the extended sequence x is whole-sample symmetric in the horizontal direction and half-sample symmetric in the vertical direction. We show the filter banks compatible with type-3 symmetric extension below.

Theorem 2 (Type-3 symmetric extension algorithm)**.** *Consider the filter bank shown in Fig. 2, where $\tilde{x}$ is defined on the rectangular region $\{0, 1, \ldots, L_0-1\} \times \{0, 1, \ldots, L_1-1\}$ and x is the type-3 symmetric extension of $\tilde{x}$ as given by (4). Let $\boldsymbol{M}$ denote the quincunx generating matrix $\left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$. Suppose that the analysis filters H_0 and H_1 satisfy the following conditions: 1) H_0 has even-even quadrantal centrosymmetry with group delay $\boldsymbol{d}_0 = \begin{bmatrix} d_{0,0} & d_{0,1} \end{bmatrix}^T$, $d_{0,0} \in \mathbb{Z}$ and $d_{0,1} \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$; 2) H_1 has even-odd quadrantal centrosymmetry with group delay $\boldsymbol{d}_1 = \begin{bmatrix} d_{1,0} & d_{1,1} \end{bmatrix}^T$, $d_{1,0} \in \mathbb{Z}$ and $d_{1,1} \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$; 3) $\boldsymbol{d}_0 - \boldsymbol{d}_1 \in \mathrm{LAT}(\boldsymbol{M})$. In this case, the subband output y_0 can be completely characterized by N_0 samples with indices $\boldsymbol{n} = \begin{bmatrix} n_0 & n_1 \end{bmatrix}^T$ given by*

$$\begin{aligned} &\left\lceil \frac{d_{0,0}+d_{0,1}-\frac{1}{2}}{2} \right\rceil \le n_0 \le \left\lfloor \frac{d_{0,0}+d_{0,1}+L_0+L_1-\frac{3}{2}}{2} \right\rfloor, \\ &\quad \textit{and } \max\{d_{0,0}-n_0, n_0-d_{0,1}-L_1+\tfrac{1}{2}\} \le n_1 \\ &\quad \le \min\{d_{0,0}+L_0-n_0-1, n_0-d_{0,1}+\tfrac{1}{2}\}; \end{aligned} \tag{12}$$

y_1 can be completely characterized by N_1 samples given by

$$\begin{aligned} &\left\lceil \frac{d_{1,0}+d_{1,1}+\frac{1}{2}}{2} \right\rceil \le n_0 \le \left\lfloor \frac{d_{1,0}+d_{1,1}+L_0+L_1-\frac{5}{2}}{2} \right\rfloor, \\ &\quad \textit{and } \max\{d_{1,0}-n_0, n_0-d_{1,1}-L_1+\tfrac{3}{2}\} \le n_1 \\ &\quad \le \min\{d_{1,0}+L_0-n_0-1, n_0-d_{1,1}-\tfrac{1}{2}\}; \end{aligned} \tag{13}$$

and $N_0 + N_1 = L_0 L_1$ (i.e., the transform is nonexpansive).

The proof of this theorem is similar to that of Theorem 1. It can also be shown that there exist PR filter banks satisfying the three conditions in the above theorem.

For type-4 symmetric extension defined in (4), there are no compatible PR quincunx filter banks that lead to nonexpansive transforms. The total number of independent samples in y_0 and y_1 combined is always more than the number of samples in the original sequence $\tilde{x}$.

5. CONCLUSIONS

In this paper, we have shown four ways to extend a 2-D finite-extent input sequence of a quincunx filter bank to an infinite-extent periodic symmetric sequence, and discussed how the periodicity and symmetry properties of the extended sequence can be preserved by the convolution and downsampling operations of the filter bank. Then, based on two types of symmetric extension, we have proposed new algorithms which can be used to construct nonexpansive transforms associated with quincunx filter banks. These schemes are potentially useful in any application that processes finite-extent sequences using such filter banks.

6. REFERENCES

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