





# Continuous-Time Signals and Systems

Edition 2.0



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Published by the University of Victoria, Victoria, British Columbia, Canada

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#### Library and Archives Canada Cataloguing in Publication

Adams, Michael D., 1969–, author

Continuous-time signals and systems / Michael D.

Adams.

Includes index.

ISBN 978-1-55058-657-2 (pbk.)

ISBN 978-1-55058-658-9 (PDF)

1. Signal theory (Telecommunication)—Textbooks.

2. System analysis—Textbooks. 3. MATLAB—Textbooks.

I. Title.

TK5102.5.A33 2013

621.382'23

C2013-904334-9

To my students, past, present, and future



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# Preface

This book is primarily intended to be used as a text for undergraduate students in engineering (and related) disciplines. The book provides a basic introduction to continuous-time signals and systems. Since many engineering curricula use MATLAB as a teaching tool, the book also includes a detailed tutorial on MATLAB as an appendix. This book evolved from a detailed set of lecture notes that the author prepared in order to teach two different undergraduate courses on signals and systems at the University of Victoria. The first version of these lectures notes was developed while the author was teaching ELEC 260 in the Summer 2003 term. These lecture notes were then augmented in order to accommodate the teaching of ELEC 255. Over time, the lecture notes underwent many changes, eventually evolving into the book that you are now reading.

More recently, the author has started to develop some material on discrete-time signals and systems. With a little luck, the author will complete this draft material, and include it in a future version of this book.

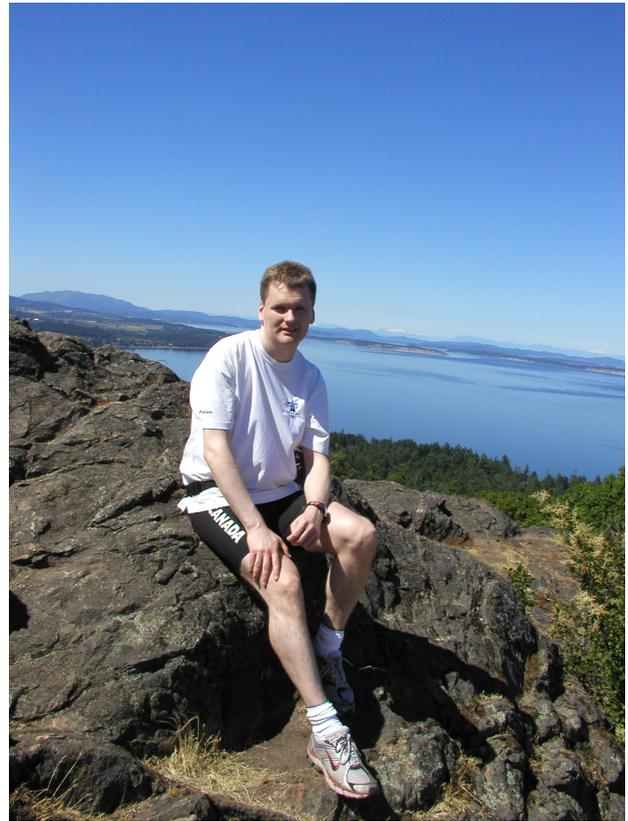
## Acknowledgments

I would like to thank my colleague, Dr. Wu-Sheng Lu, for many interesting technical discussions that helped to clarify some of the finer points of the mathematics behind signals and systems. Also, I would like to thank my past students for their feedback regarding earlier revisions of this manuscript. They have helped me to eliminate numerous errors in this manuscript that would have otherwise gone undetected. In particular, I would like to thank Ioana Sevcenco for her careful proofreading of much of an earlier version of the manuscript.

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Victoria, BC  
2020-02-29



## About the Author



Michael Adams received the B.A.Sc. degree in computer engineering from the University of Waterloo, Waterloo, ON, Canada in 1993, the M.A.Sc. degree in electrical engineering from the University of Victoria, Victoria, BC, Canada in 1998, and the Ph.D. degree in electrical engineering from the University of British Columbia, Vancouver, BC, Canada in 2002. From 1993 to 1995, Michael was a member of technical staff at Bell-Northern Research in Ottawa, ON, Canada, where he developed real-time software for fiber-optic telecommunication systems. Since 2003, Michael has been on the faculty of the Department of Electrical and Computer Engineering at the University of Victoria, Victoria, BC, Canada, first as an Assistant Professor and currently as an Associate Professor.

Michael is the recipient of a Natural Sciences and Engineering Research Council (of Canada) Postgraduate Scholarship. He has served as a voting member of the Canadian Delegation to ISO/IEC JTC 1/SC 29 (i.e., Coding of Audio, Picture, Multimedia and Hypermedia Information), and been an active participant in the JPEG-2000 standardization effort, serving as co-editor of the JPEG-2000 Part-5 standard and principal author of one of the first JPEG-2000 implementations (i.e., JasPer). His research interests include software, signal processing, image/video/audio processing and coding, multiresolution signal processing (e.g., filter banks and wavelets), geometry processing, and data compression.



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# Chapter 1

## Introduction

### 1.1 Signals and Systems

Mathematics has a very broad scope, encompassing many areas such as: linear algebra, calculus, probability and statistics, geometry, differential equations, and numerical methods. For engineers, however, an area of mathematics of particular importance is the one that pertains to signals and systems (which is, loosely speaking, the branch of mathematics known as functional analysis). It is this area of mathematics that is the focus of this book. Before we can treat this topic in any meaningful way, however, we must first explain precisely what signals and systems are. This is what we do next.

### 1.2 Signals

A **signal** is a function of one or more variables that conveys information about some (usually physical) phenomenon. Some examples of signals include:

- a human voice
- a voltage in an electronic circuit
- the temperature of a room controlled by a thermostat system
- the position, velocity, and acceleration of an aircraft
- the acceleration measured by an accelerometer in a cell phone
- the force measured by a force sensor in a robotic system
- the electromagnetic waves used to transmit information in wireless computer networks
- a digitized photograph
- a digitized music recording
- the evolution of a stock market index over time

#### 1.2.1 Classification of Signals

Signals can be classified based on the number of independent variables with which they are associated. A signal that is a function of only one variable is said to be **one dimensional**. Similarly, a signal that is a function of two or more variables is said to be **multi-dimensional**. Human speech is an example of a one-dimensional signal. In this case, we have a signal associated with fluctuations in air pressure as a function of time. An example of a two-dimensional signal is a monochromatic image. In this case, we have a signal that corresponds to a measure of light intensity as a function of horizontal and vertical position.

A signal can also be classified on the basis of whether it is a function of continuous or discrete variables. A signal that is a function of continuous variables (e.g., a real variable) is said to be **continuous time**. Similarly, a signal that is a function of discrete variables (e.g., an integer variable) is said to be **discrete time**. Although the independent variable need not represent time, for matters of convenience, much of the terminology is chosen as if this were so.

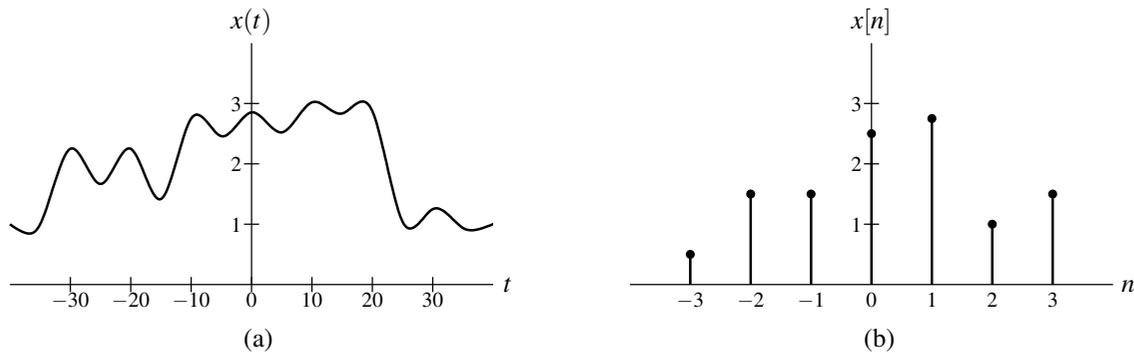


Figure 1.1: Graphical representations of (a) continuous-time and (b) discrete-time signals.

For example, a digital image (which consists of a rectangular array of pixels) would be referred to as a discrete-time signal, even though the independent variables (i.e., horizontal and vertical position) do not actually correspond to time.

If a signal is a function of discrete variables (i.e., discrete-time) and the value of the function itself is also discrete, the signal is said to be **digital**. Similarly, if a signal is a function of continuous variables, and the value of the function itself is also continuous, the signal is said to be **analog**.

Many phenomena in our physical world can be described in terms of continuous-time signals. Some examples of continuous-time signals include: voltage or current waveforms in an electronic circuit; electrocardiograms, speech, and music recordings; position, velocity, and acceleration of a moving body; forces and torques in a mechanical system; and flow rates of liquids or gases in a chemical process. Any signals processed by digital computers (or other digital devices) are discrete-time in nature. Some examples of discrete-time signals include digital video, digital photographs, and digital audio data.

A discrete-time signal may be inherently discrete or correspond to a sampled version of a continuous-time signal. An example of the former would be a signal corresponding to the Dow Jones Industrial Average stock market index (which is only defined on daily intervals), while an example of the latter would be the sampled version of a (continuous-time) speech signal.

## 1.2.2 Notation and Graphical Representation of Signals

In the case of discrete-time signals, we sometimes refer to the signal as a **sequence**. The  $n$ th element of a sequence  $x$  is denoted as either  $x(n)$  or  $x_n$ . Figure 1.1 shows how continuous-time and discrete-time signals are represented graphically.

## 1.2.3 Examples of Signals

A number of examples of signals have been suggested previously. Here, we provide some graphical representations of signals for illustrative purposes. Figure 1.2 depicts a digitized speech signal. Figure 1.3 shows an example of a monochromatic image. In this case, the signal represents light intensity as a function of two variables (i.e., horizontal and vertical position).

## 1.3 Systems

A **system** is an entity that processes one or more input signals in order to produce one or more output signals, as shown in Figure 1.4. Such an entity is represented mathematically by a system of one or more equations.

In a communication system, the input might represent the message to be sent, and the output might represent the received message. In a robotics system, the input might represent the desired position of the end effector (e.g., gripper), while the output could represent the actual position.

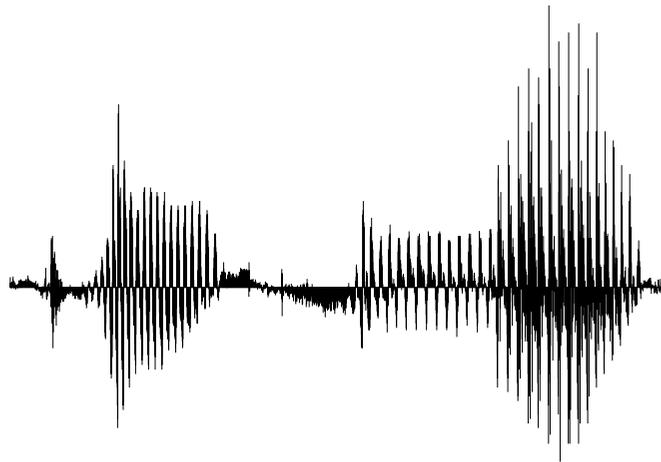


Figure 1.2: Segment of digitized human speech.



Figure 1.3: A monochromatic image.

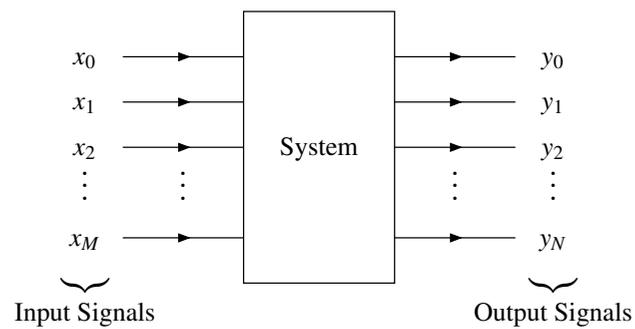


Figure 1.4: System with one or more inputs and one or more outputs.

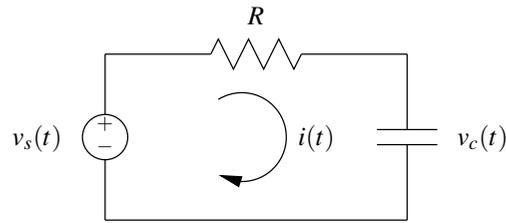


Figure 1.5: A simple RC network.

### 1.3.1 Classification of Systems

A system can be classified based on the number of inputs and outputs it has. A system with only one input is described as **single input**, while a system with multiple inputs is described as **multi-input**. Similarly, a system with only one output is said to be **single output**, while a system with multiple outputs is said to be **multi-output**. Two commonly occurring types of systems are single-input single-output (SISO) and multi-input multi-output (MIMO).

A system can also be classified based on the types of signals with which it interacts. A system that deals with continuous-time signals is called a **continuous-time system**. Similarly, a system that deals with discrete-time signals is said to be a **discrete-time system**. A system that handles both continuous- and discrete-time signals, is sometimes referred to as a **hybrid system** (or sampled-data system). Similarly, systems that deal with digital signals are referred to as **digital**, while systems that handle analog signals are referred to as **analog**. If a system interacts with one-dimensional signals, the system is referred to as **one-dimensional**. Likewise, if a system handles multi-dimensional signals, the system is said to be **multi-dimensional**.

### 1.3.2 Examples of Systems

Systems can manipulate signals in many different ways and serve many useful purposes. Sometimes systems serve to extract information from their input signals. For example, in the case of speech signals, systems can be used in order to perform speaker identification or voice recognition. A system might analyze electrocardiogram signals in order to detect heart abnormalities. Amplification and noise reduction are other functionalities that systems could offer.

One very basic system is the resistor-capacitor (RC) network shown in Figure 1.5. Here, the input would be the source voltage  $v_s$ , and the output would be the capacitor voltage  $v_c$ .

Consider the signal-processing systems shown in Figure 1.6. The system in Figure 1.6(a) uses a discrete-time system (such as a digital computer) to process a continuous-time signal. The system in Figure 1.6(b) uses a continuous-time system (such as an analog computer) to process a discrete-time signal. The first of these types of systems is ubiquitous in the world today.

Consider the communication system shown in Figure 1.7. This system takes a message at one location and reproduces this message at another location. In this case, the system input is the message to be sent, and the output is the estimate of the original message. Usually, we want the message reproduced at the receiver to be as close as possible to the original message sent by the transmitter.

A system of the general form shown in Figure 1.8 frequently appears in control applications. Often, in such applications, we would like an output to track some reference input as closely as possible. Consider, for example, a robotics application. The reference input might represent the desired position of the end effector, while the output represents the actual position.

## 1.4 Why Study Signals and Systems?

As can be seen from the earlier examples, there are many practical applications in which we need to develop systems that manipulate signals. In order to do this, we need a formal mathematical framework for the study of such systems. Such a framework can be used to guide the design of new systems as well as to analyze the behavior of already existing systems. Over time, the complexity of systems designed by engineers has continued to grow. Today, most systems

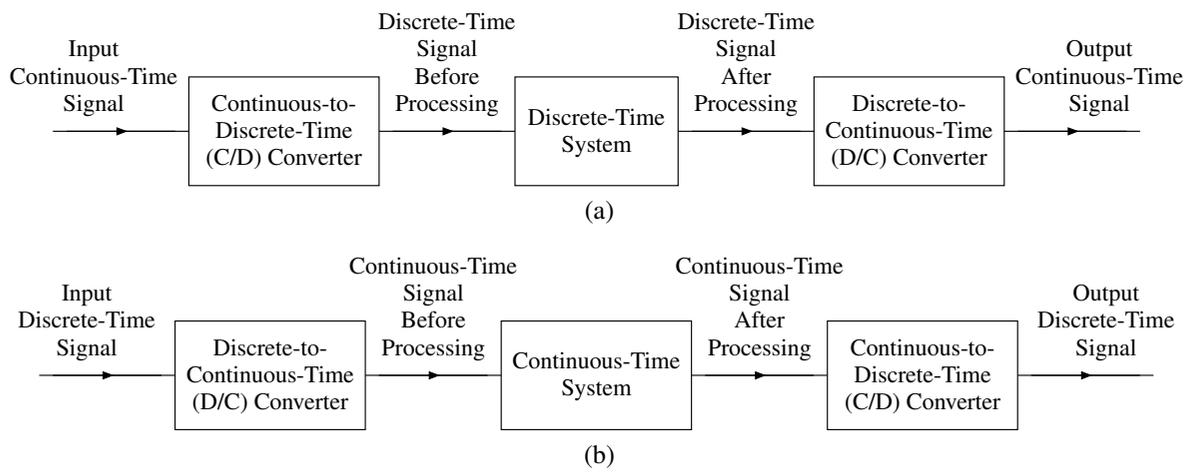


Figure 1.6: Signal processing systems. (a) Processing a continuous-time signal with a discrete-time system. (b) Processing a discrete-time signal with a continuous-time system.

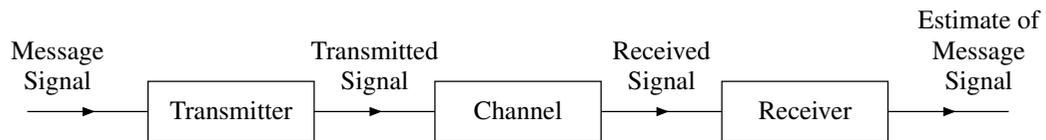


Figure 1.7: Communication system.

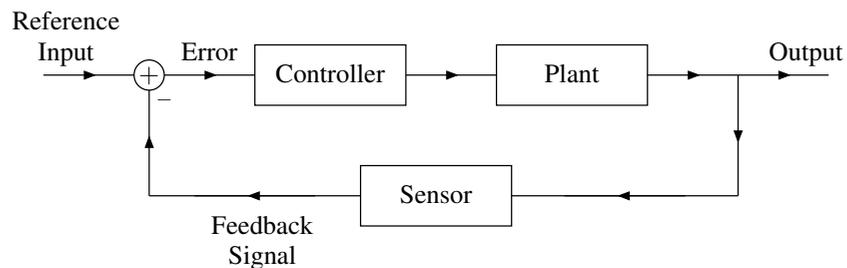


Figure 1.8: Feedback control system.

of practical interest are highly complex. For this reason, a formal mathematical framework to guide the design of systems is absolutely critical. Without such a framework, there is little hope that any system that we design would operate as desired, meeting all of the required specifications.

## 1.5 Overview of This Book

This book presents the mathematical tools used for studying both signals and systems. In its current incarnation, this book focuses primarily on the study of signals and systems in the continuous-time case. Although most systems considered herein are SISO, the mathematics extends in a very straightforward manner to systems that have multiple inputs and/or multiple outputs. Only the one-dimensional case is considered herein, however, as the multi-dimensional case is considerably more complicated and beyond the scope of this book.

The remainder of this book is organized as follows. Chapter 2 presents some mathematical preliminaries that are essential for both the continuous-time and discrete-time cases. Then, Chapters 3, 4, 5, 6, and 7 cover material that relates primarily to the continuous-time case.

The material on continuous-time signals and systems consists of the following. Chapter 3 examines signals and systems in more depth than covered in earlier chapters. A particular class of systems known as (continuous-time) linear time-invariant (LTI) systems is introduced. Chapter 4 then studies LTI systems in depth. Finally, Chapters 5, 6, and 7 introduce continuous-time Fourier series, the continuous-time Fourier transform, and the Laplace transform, respectively, which are important mathematical tools for studying continuous-time signals and systems.

This book also includes several appendices, which contain supplemental material related to the topics covered in the main chapters of the book. Appendix A provides a review of complex analysis. Appendix B introduces partial fraction expansions. Appendix C presents time-domain methods for solving for differential equations. Appendix D offers a list of some useful mathematical formulas. Appendix E presents a detailed introduction to MATLAB. Finally, Appendix F provides some additional exercises.

## Chapter 2

# Preliminaries

### 2.1 Mappings

A **mapping** is a relationship involving two sets that associates each element in one set, called the **domain**, with an element from the other set, called the **codomain**. For example, the sine function (i.e.,  $\sin$ ) is an example of a mapping. For each real number  $t$ , the sine function maps  $t$  to the sine of  $t$  (i.e.,  $\sin t$ ). In the case of the sine function both the domain and codomain are the real numbers. As another example consider the function `trunc` that rounds a real number  $t$  to an integer by discarding the fractional part of  $t$  (i.e., `trunc` rounds towards zero). The `trunc` function is a mapping where the domain and codomain are the real numbers and integers, respectively.

Although very many types of mappings exist, the types of most relevance to our study of signals and systems are: functions, sequences, system operators, and transforms. In the sections that follow, we will introduce (or further elaborate upon) each of these types of mappings.

### 2.2 Functions

A **function** is a mapping where the domain is a set that is continuous in nature, such as the real numbers, complex numbers, or a subset of either of these. In practice, the codomain is typically either the real numbers or complex numbers. Functions are also commonly referred to as **continuous-time (CT) signals**. Herein, we focus mostly on functions of a single independent variable (i.e., one-dimensional functions). The trigonometric sine (i.e.,  $\sin$ ) function is an example of a function. It associates each real number  $t$  with the real number  $\sin t$ . In this case, the domain and codomain are the real numbers.

For a function  $f$ , the expression “ $f(t)$ ” denotes the value of the function  $f$  evaluated at  $t$ . Note that “ $f(t)$ ” does not denote the function  $f$  itself. This distinction is an extremely important one to make—a point that we shall revisit shortly. Normally, the parameters to a function are placed in (round, square, or curly) brackets. For standard mathematical functions with only a single parameter (e.g.,  $\sin$  and  $\cos$ ), brackets generally tend to be omitted when not needed for grouping. That is, we typically write “ $\sin t$ ” instead of “ $\sin(t)$ ”. In the case of a non-standard function like  $f$ , we would always write “ $f(t)$ ”, not “ $ft$ ”.

Many notational schemes can be used to specify a function mathematically. In engineering, however, it is arguably most common to specify a function in terms of a defining equation. For example, we can specify a function  $f$  that maps the real number  $t$  to  $t^2$  (i.e., a squaring function) using the equation  $f(t) = t^2$ . Here,  $f$  is the function, and  $t$  is a dummy variable that is used only for the purposes of writing the equation  $f(t) = t^2$  that defines the function  $f$ . Since  $t$  is only a dummy variable, the equations  $f(t) = t^2$  and  $f(x) = x^2$  define exactly the same function. Note that “ $f$ ” and “ $f(t) = t^2$ ” are, strictly speaking, different things. That is, “ $f$ ” is the function itself whereas “ $f(t) = t^2$ ” is not the function  $f$  but rather an equation that defines the function  $f$ . So, strictly speaking, to define a function  $f$ , we should use wording like: “let  $f$  be the function defined by the equation  $f(t) = \dots$ ”. Often, in practice, we abuse notation somewhat and simply write “the function  $f(t) = \dots$ ” or simply “ $f(t) = \dots$ ”. Fortunately, this type notational abuse does not usually lead to problems in most cases.

Since notational conventions play a crucial role, it is worthwhile to take some time in order to clearly introduce such conventions. In what follows, we present several examples to illustrate various aspects of the notation associated with functions.

**Example 2.1.** Let  $f$  and  $g$  each denote a real-valued function of a real variable. Let  $t$  denote an arbitrary real number. The expression  $f + g$  denotes a *function*, namely, the function formed by adding the functions  $f$  and  $g$ . The expression  $f(t) + g(t)$  denotes a *number*, namely, the sum of: 1) the value of the function  $f$  evaluated at  $t$ ; and 2) the value of the function  $g$  evaluated at  $t$ . The expression  $(f + g)(t)$  denotes the result obtained by: 1) first computing a new function  $h$  that is the sum of the functions  $f$  and  $g$ ; and 2) then evaluating  $h$  at  $t$ . Note that the meanings of the expressions  $(f + g)(t)$  and  $f(t) + g(t)$  are subtly different. In the first case, the addition operation is being applied to two functions, while in the second case, the addition operation is being applied to two numbers. Although the meanings of these two expressions are subtly different, they are always equal. In other words, it is always true that

$$(f + g)(t) = f(t) + g(t) \quad \text{for all } t.$$

This is due to the fact that preceding equation is precisely how the addition of functions is defined. In other words, we add functions by adding their values at corresponding points (i.e., addition is defined in a pointwise manner). ■

**Example 2.2.** For two functions  $x_1$  and  $x_2$ , the expression  $x_1 + x_2$  denotes the function that is the sum of the functions  $x_1$  and  $x_2$ . The expression  $(x_1 + x_2)(t)$  denotes the function  $x_1 + x_2$  evaluated at  $t$ . Since the addition of functions can be defined pointwise (i.e., we can add two functions by adding their values at corresponding pairs of points), the following relationship always holds:

$$(x_1 + x_2)(t) = x_1(t) + x_2(t) \quad \text{for all } t.$$

Similarly, since subtraction, multiplication, and division can also be defined pointwise, the following relationships also hold:

$$\begin{aligned} (x_1 - x_2)(t) &= x_1(t) - x_2(t) \quad \text{for all } t, \\ (x_1 x_2)(t) &= x_1(t) x_2(t) \quad \text{for all } t, \quad \text{and} \\ (x_1/x_2)(t) &= x_1(t)/x_2(t) \quad \text{for all } t. \end{aligned}$$

It is important to note, however, that not all mathematical operations involving functions can be defined in a pointwise manner. That is, some operations fundamentally require that their operands be functions. The convolution operation (for functions), which will be considered later, is one such example. If some operator, which we denote for illustrative purposes as “ $\diamond$ ”, is defined in such a way that it can only be applied to functions, then the expression  $(x_1 \diamond x_2)(t)$  is mathematically valid, but the expression  $x_1(t) \diamond x_2(t)$  is not. The latter expression is not valid since the  $\diamond$  operator requires two functions as operands, but the provided operands  $x_1(t)$  and  $x_2(t)$  are numbers (namely, the values of the functions  $x_1$  and  $x_2$  each evaluated at  $t$ ). Due to issues like this, one must be careful in the use of mathematical notation related to functions. Otherwise, it is easy to fall into the trap of writing expressions that are ambiguous, contradictory, or nonsensical. ■

## 2.3 Sequences

A **sequence** is a mapping where the domain is a set that is discrete in nature, such as the integers, or a subset thereof. In practice, the codomain is typically either the real numbers or complex numbers. Sequences are also commonly referred to as **discrete-time (DT) signals**. Herein, we focus mostly on sequences with a single independent variable (i.e., one-dimensional sequences). The sequence  $1^2, 2^2, 3^2, 4^2, \dots$  of perfect squares is an example of a sequence. It associates each (strictly) positive integer  $n$  with the integer  $n^2$ . In this case, the domain and codomain are the positive integers.

Similar comments as above also apply to expressions involving sequences. For a sequence  $f$ , the expression “ $f(n)$ ” denotes the value of the sequence evaluated at  $n$ , whereas the expression “ $f$ ” denotes the sequence itself. It is often critically important to make a clear distinction between a sequence and its value. A sequence is typically

specified by a defining equation. For example, a sequence  $f$  that maps the integer  $n$  to  $n^2$  (i.e., a sequence of squares) can be specified using the equation  $f(n) = n^2$ .

Since notational conventions play a crucial role, it is worthwhile to take some time in order to clearly introduce such conventions. In what follows, we present several examples to illustrate various aspects of the notation associated with sequences.

**Example 2.3.** Let  $f$  and  $g$  each denote a real-valued sequence with an integer index. Let  $n$  denote an arbitrary integer. The expression  $fg$  denotes a *sequence*, namely, the sequence formed by multiplying the sequences  $f$  and  $g$ . The expression  $f(n)g(n)$  denotes a *number*, namely, the product of: 1) the value of the sequence  $f$  evaluated at  $n$ ; and 2) the value of the sequence  $g$  evaluated at  $n$ . The expression  $(fg)(n)$  denotes the result obtained by: 1) first computing a new sequence  $h$  that is the product of the sequences  $f$  and  $g$ ; and 2) then evaluating  $h$  at  $n$ . Note that the meanings of the expressions  $(fg)(n)$  and  $f(n)g(n)$  are subtly different. In the first case, the multiplication operation is being applied to two sequences, while in the second case, the multiplication operation is being applied to two numbers. Although the meanings of these two expressions are subtly different, they are always equal. In other words, it is always true that

$$(fg)(n) = f(n)g(n) \quad \text{for all } n.$$

This is due to the fact that preceding equation is precisely how the multiplication of sequences is defined. In other words, we multiply sequences by multiplying their values at corresponding points (i.e., multiplication is defined in a pointwise manner). ■

**Example 2.4.** For two sequences  $x_1$  and  $x_2$ , we have two ways in which we can express the equality of these sequences. First, we can simply write that  $x_1 = x_2$ . Second, we can write that  $x_1(n) = x_2(n)$  for all  $n$ . The first approach is probably preferable since it is less verbose. ■

**Example 2.5.** For two sequences  $x_1$  and  $x_2$ , the expression  $x_1 + x_2$  denotes the sequence that is the sum of the sequences  $x_1$  and  $x_2$ . The expression  $(x_1 + x_2)(n)$  denotes the sequence  $x_1 + x_2$  evaluated at  $n$ . Since the addition of sequences can be defined pointwise (i.e., we can add two sequences by adding their values at corresponding pairs of points), the following relationship always holds:

$$(x_1 + x_2)(n) = x_1(n) + x_2(n) \quad \text{for all } n.$$

Similarly, since subtraction, multiplication, and division can also be defined pointwise, the following relationships also hold:

$$\begin{aligned} (x_1 - x_2)(n) &= x_1(n) - x_2(n) \quad \text{for all } n, \\ (x_1 x_2)(n) &= x_1(n)x_2(n) \quad \text{for all } n, \quad \text{and} \\ (x_1/x_2)(n) &= x_1(n)/x_2(n) \quad \text{for all } n. \end{aligned}$$

It is important to note, however, that not all mathematical operations involving sequences can be defined in a pointwise manner. That is, some operations fundamentally require that their operands be sequences. The convolution operation (for sequences), which will be considered later, is one such example. If some operator, which we denote for illustrative purposes as “ $\diamond$ ”, is defined in such a way that it can only be applied to sequences, then the expression  $(x_1 \diamond x_2)(n)$  is mathematically valid, but the expression  $x_1(n) \diamond x_2(n)$  is not. The latter expression is not valid since the  $\diamond$  operator requires two sequences as operands, but the provided operands  $x_1(n)$  and  $x_2(n)$  are numbers (namely, the values of the sequences  $x_1$  and  $x_2$  each evaluated at  $n$ ). Due to issues like this, one must be careful in the use of mathematical notation related to sequences. Otherwise, it is easy to fall into the trap of writing expressions that are ambiguous, contradictory, or nonsensical. ■

## 2.4 Remarks on Abuse of Notation

Unfortunately, for a function  $f$ , it is common practice for engineers to abuse mathematical notation and write “ $f(t)$ ” to refer to the function  $f$  itself. A similar issue also exists for the case of sequences. The abuse of mathematical notation,

however, can often lead to trouble. In some contexts, it is critically important to make a clear distinction between a function (or sequence) and its value, and failing to do so can lead to many problems, such as writing mathematical expressions that are ambiguous, contradictory, or nonsensical. For this reason, it is strongly recommended that one try to maintain a clear distinction between a function (or sequence) and its value.

With the above said, notational abuse in trivial cases is not likely to cause problems. For example, if we simply write “the function  $f(t)$ ” instead of more correctly writing the “the function  $f$ ”, this is unlikely to cause confusion. Where notational abuse is much more likely to become problematic is when the expressions that are being referred to as functions contain mathematical operations or more than one variable, such as expressions like:  $f(t-1)$ ,  $f(7t)$ ,  $f(at-b)$ ,  $tf(t)$ , and  $f(t-\tau)$ . In cases like these, abuse of notation makes the intended meaning much less clear, opening the possibility of misinterpretation. For example, in the case of “ $f(t-\tau)$ ”, more than one reasonable interpretation exists if one allows notation to be abused. In particular, “ $f(t-\tau)$ ” might mean:

1. a number that is equal to the function  $f$  evaluated at  $t-\tau$ ;
2. an anonymous (i.e., unnamed) function that is equal to the function  $g$ , where  $g(x) = f(x-\tau)$  (i.e.,  $t$  and  $\tau$  are interpreted as a variable and constant, respectively);
3. an anonymous function that is equal to the function  $g$ , where  $g(x) = f(t-x)$  (i.e.,  $t$  and  $\tau$  are interpreted as a constant and variable, respectively);
4. an anonymous function that is equal to the constant function  $g$ , where  $g(x) = f(t-\tau)$  (i.e.,  $t$  and  $\tau$  are both interpreted as constants).

In circumstances like this one, notational problems can be easily avoided by simply specifying the desired function in terms of an equation. In other words, we can give a name to the function being specified and then define the function in terms of an equation using the given name. For example, instead of saying “the function  $f(t-\tau)$ ”, we can say “the function  $g(t) = f(t-\tau)$ ”. This latter notation makes clear that  $\tau$  is a constant, for example. As another example, instead of saying “the function  $f(t-1)$ ”, we can say “the function  $g(t) = f(t-1)$ ”.

Due to problems like those above, great care must be exercised when using anonymous functions in order to avoid ambiguous notation. Since ambiguous notation is a frequent source of problems, the author would suggest that anonymous functions are best avoided in most circumstances.

## 2.5 Dot Notation for Functions and Sequences

Sometimes a situation may arise where one would like to distinguish a function from the value of a function, but without resorting to giving the function a name or other more verbose notational approaches. A similar comment also applies for the case of sequences. In this regard, the dot notation for functions and sequences is quite useful. If we wish to indicate that an expression corresponds to a function (as opposed to the value of a function), we can denote this using the interpunct symbol (i.e., “ $\cdot$ ”). In each place where the variable for the function would normally appear, we simply replace it with an interpunct symbol (i.e., “ $\cdot$ ”). For example,  $\sqrt{\cdot}$  denotes the square root function, whereas  $\sqrt{t}$  denotes the value of the square root function evaluated at  $t$ . Some additional examples of the dot notation for functions can be found in Table 2.1(a). A similar convention can also be applied to sequences. Some examples of the dot notation for sequences can be found in Table 2.1(b). Since some readers may find this dot notation to be somewhat strange, this book minimizes the use of this notation. It is, however, used in a few limited number of places in order to achieve clarity without the need for being overly verbose. Although the dot notation may appear strange at first, it is a very commonly used notation by mathematicians. Sadly, it is not used as much by engineers, in spite of its great utility.

## 2.6 System Operators

A system operator is a mapping used to represent a system. In what follows, we will focus exclusively on the case of single-input single-output systems, since this case is our primary focus herein. A (single-input single-output) **system operator** maps a function or sequence representing the input of a system to a function or sequence representing the output of the system. For example, the system  $\mathcal{H}$  that maps a function to a function and is given by

$$\mathcal{H}x(t) = 2x(t)$$

Table 2.1: Examples of dot notation for functions and sequences. Examples for (a) functions and (b) sequences.

(a)

Named Function $f$	Corresponding Unnamed Function
$f(t) = g(t)$	$g(\cdot)$
$f(t) = t^2$	$(\cdot)^2$
$f(t) = \sqrt[3]{t}$	$\sqrt[3]{\cdot}$
$f(t) = e^t$	$e^{(\cdot)}$
$f(t) =  t $	$ \cdot $
$f(t) = t^2 + 3t + 1$	$(\cdot)^2 + 3 \cdot + 1$
$f(t) = g(at - b)$	$g(a \cdot - b)$
$f(t) = g(t - 1)$	$g(\cdot - 1)$
$f(t) = g(3t)$	$g(3 \cdot)$

(b)

Named Sequence $f$	Corresponding Unnamed Sequence
$f(n) = g(n)$	$g(\cdot)$
$f(n) = n^2$	$(\cdot)^2$
$f(n) = \sin\left(\frac{2\pi}{3}n\right)$	$\sin\left[\frac{2\pi}{3}(\cdot)\right]$
$f(n) = g(3n)$	$g(3 \cdot)$
$f(n) = g(n - 1)$	$g(\cdot - 1)$
$f(n) = ng(n)$	$(\cdot)g(\cdot)$
$f(n) = 2n^2 + n + 5$	$2(\cdot)^2 + (\cdot) + 5$

multiplies its input function  $x$  by a factor of 2 in order to produce its output function. The system  $\mathcal{H}$  that maps a sequence to a sequence and is given by

$$\mathcal{H}x(n) = x(n) + 1$$

adds one to its input sequence  $x$  in order to produce its output sequence.

For a system operator  $\mathcal{H}$  and function  $x$ , the expression  $\mathcal{H}(x)$  denotes the output produced by the system  $\mathcal{H}$  when the input is the function  $x$ . Since only a single symbol  $x$  follows the operator  $\mathcal{H}$ , there is only one way to group the operations in this expression. Therefore, the parentheses can be omitted without any risk of changing the meaning of the expression. In other words, we can equivalently write  $\mathcal{H}(x)$  as  $\mathcal{H}x$ . Since  $\mathcal{H}x$  is a function, we can evaluate this function at  $t$ , which corresponds to the expression  $(\mathcal{H}x)(t)$ . We can omit the first set of parentheses in this expression without changing its meaning. In other words, the expressions  $(\mathcal{H}x)(t)$  and  $\mathcal{H}x(t)$  have identical meanings. This is due to the fact that there is only one possible way to group the operations in  $\mathcal{H}x(t)$  that is mathematically valid. For example, the grouping  $\mathcal{H}[x(t)]$  is not mathematically valid since  $\mathcal{H}$  must be provided a function as an operand, but the provided operand  $x(t)$  is a number (namely, the value of the function  $x$  evaluated at  $t$ ).

Again, since notational conventions play a crucial role, it is worthwhile to take some time in order to clearly introduce such conventions. In what follows, we present several examples to illustrate various aspects of the notation associated with system operators.

**Example 2.6.** For a system operator  $\mathcal{H}$ , a function  $x$ , and a real constant  $t_0$ , the expression  $\mathcal{H}x(t - t_0)$  denotes the result obtained by taking the function  $y$  produced as the output of the system  $\mathcal{H}$  when the input is the function  $x$  and then evaluating  $y$  at  $t - t_0$ . ■

**Example 2.7.** For a system operator  $\mathcal{H}$ , function  $x'$ , and real number  $t$ , the expression  $\mathcal{H}x'(t)$  denotes result of taking the function  $y$  produced as the output of the system  $\mathcal{H}$  when the input is the function  $x'$  and then evaluating  $y$  at  $t$ . ■

**Example 2.8.** For a system operator  $\mathcal{H}$ , function  $x$ , and a complex constant  $a$ , the expression  $\mathcal{H}(ax)$  denotes the output from the system  $\mathcal{H}$  when the input is the function  $ax$  (i.e.,  $a$  times  $x$ ). ■

Table 2.2: Examples of transforms

Name	Domain	Codomain
CT Fourier Series	$T$ -periodic functions (with domain $\mathbb{R}$ )	sequences (with domain $\mathbb{Z}$ )
CT Fourier Transform	functions (with domain $\mathbb{R}$ )	functions (with domain $\mathbb{R}$ )
Laplace Transform	functions (with domain $\mathbb{R}$ )	functions (with domain $\mathbb{C}$ )
DT Fourier Series	$N$ -periodic sequences (with domain $\mathbb{Z}$ )	$N$ -periodic sequences (with domain $\mathbb{Z}$ )
DT Fourier Transform	sequences (with domain $\mathbb{Z}$ )	$2\pi$ -periodic functions (with domain $\mathbb{R}$ )
Z Transform	sequences (with domain $\mathbb{Z}$ )	functions (with domain $\mathbb{C}$ )

**Example 2.9.** For a system operator  $\mathcal{H}$  and the functions  $x_1$  and  $x_2$ , the expression  $\mathcal{H}(x_1 + x_2)$  denotes the output produced by the system  $\mathcal{H}$  when the input is the function  $x_1 + x_2$ . Note that, in this case, we cannot omit the parentheses without changing the meaning of the expression. That is,  $\mathcal{H}x_1 + x_2$  means  $(\mathcal{H}x_1) + x_2$ , which denotes the sum of the function  $x_2$  and the output produced by the system  $\mathcal{H}$  when the input is the function  $x_1$ . ■

**Example 2.10.** For a system operator  $\mathcal{H}$  and the functions  $x_1$  and  $x_2$ , and the complex constants  $a_1$  and  $a_2$ , the expression  $\mathcal{H}(a_1x_1 + a_2x_2)$  denotes the output produced by the system  $\mathcal{H}$  when the input is the function  $a_1x_1 + a_2x_2$ . ■

**Example 2.11.** Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  denote the operators representing two systems and let  $x$  denote a function. Consider the expression  $\mathcal{H}_2\mathcal{H}_1x$ . The implied grouping of operations in this expression is  $\mathcal{H}_2(\mathcal{H}_1x)$ . So, this expression denotes the output produced by the system  $\mathcal{H}_2$  when its input is the function  $y$ , where  $y$  is the output produced by the system  $\mathcal{H}_1$  when its input is the function  $x$ . ■

## 2.7 Transforms

Later, we will be introduced to several types of mappings known as transforms. Transforms have a mathematical structure similar to system operators. That is, transforms map functions/sequences to functions/sequences. Due to this similar structure, many of the earlier comments about system operators also apply to the case of transforms. Some examples of transforms of interest in the study of signals and systems are listed in Table 2.2. For example, the Fourier transform (introduced later) is denoted as  $\mathcal{F}$  and the result of applying the Fourier transform operator to the function/sequence  $x$  is denoted as  $\mathcal{F}x$ .

## 2.8 Basic Signal Properties

Signals can possess a number of interesting properties. In what follows, we introduce a few very basic properties that are frequently useful.

### 2.8.1 Symmetry of Functions and Sequences

A function  $x$  is said to be **even** if it satisfies

$$x(t) = x(-t) \quad \text{for all } t \text{ (where } t \text{ is a real number).} \quad (2.1)$$

Similarly, a sequence  $x$  is said to be **even** if it satisfies

$$x(n) = x(-n) \quad \text{for all } n \text{ (where } n \text{ is an integer).} \quad (2.2)$$

Geometrically, an even function or sequence is symmetric about the origin. Examples of an even function and sequence are given in Figures 2.1 and 2.2. Some other examples of even functions include the cosine, absolute value, and square functions. Some other examples of even sequences include the unit-impulse and rectangular sequences (to be introduced later).

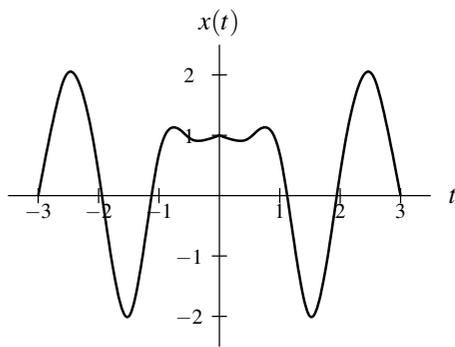


Figure 2.1: Example of an even function.

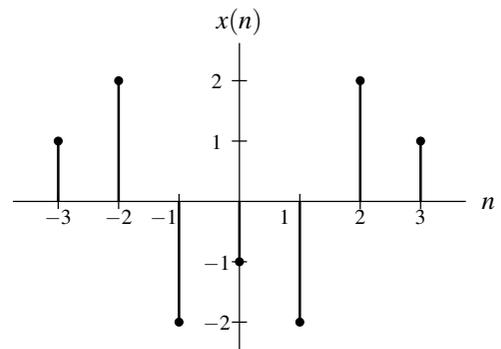


Figure 2.2: Example of an even sequence.

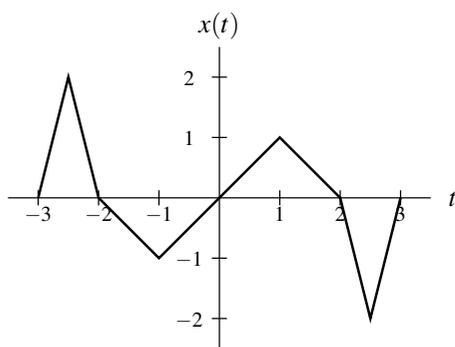


Figure 2.3: Example of an odd function.

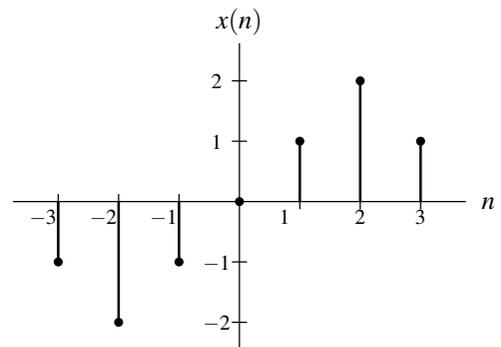


Figure 2.4: Example of an odd sequence.

A function  $x$  is said to be **odd** if it satisfies

$$x(t) = -x(-t) \quad \text{for all } t \text{ (where } t \text{ is a real number).} \quad (2.3)$$

Similarly, a sequence  $x$  is said to be **odd** if it satisfies

$$x(n) = -x(-n) \quad \text{for all } n \text{ (where } n \text{ is an integer).} \quad (2.4)$$

Geometrically, an odd function or sequence is antisymmetric about the origin. Examples of an odd function and odd sequence are shown in Figures 2.3 and 2.4. One can easily show that an odd function or sequence  $x$  must be such that  $x(0) = 0$ , assuming that the domain of  $x$  includes 0 (i.e.,  $x(0)$  is defined). Some other examples of odd functions include the sine, signum, and cube (i.e.,  $x(t) = t^3$ ) functions.

A function  $x$  is said to be **conjugate symmetric** if it satisfies

$$x(t) = x^*(-t) \quad \text{for all } t \text{ (where } t \text{ is a real number).} \quad (2.5)$$

Similarly, a sequence  $x$  is said to be **conjugate symmetric** if it satisfies

$$x(n) = x^*(-n) \quad \text{for all } n \text{ (where } n \text{ is an integer).} \quad (2.6)$$

An example of a conjugate symmetric function is a complex sinusoid  $x(t) = \cos \omega t + j \sin \omega t$ , where  $\omega$  is a real constant. The real part of a conjugate symmetric function or sequence is even. The imaginary part of a conjugate symmetric function or sequence is odd.

### 2.8.2 Periodicity of Functions and Sequences

A function  $x$  is said to be **periodic** with **period**  $T$  (or simply  **$T$ -periodic**) if, for some strictly positive real constant  $T$ ,

$$x(t) = x(t+T) \quad \text{for all } t \text{ (where } t \text{ is a real number).} \quad (2.7)$$

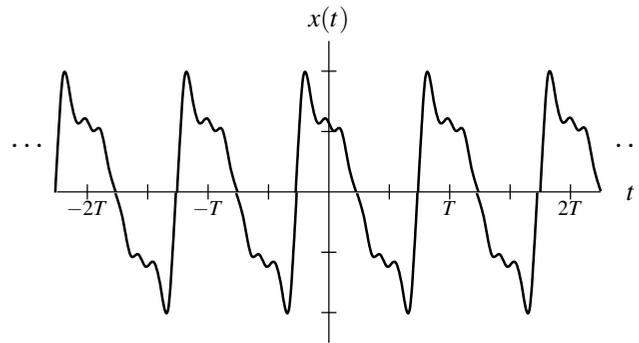


Figure 2.5: Example of a periodic function (with a fundamental period of  $T$ ).

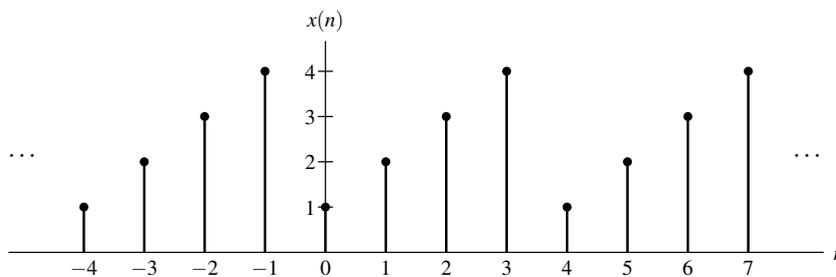


Figure 2.6: Example of a periodic sequence (with a fundamental period of 4).

In other words, the graph of a  $T$ -periodic function repeats in value every  $T$  units along the horizontal axis. A  $T$ -periodic function  $x$  is said to have the **frequency**  $\frac{1}{T}$  and **angular frequency**  $\frac{2\pi}{T}$ . Similarly, a sequence  $x$  is said to be **periodic** with **period**  $T$  (or simply  **$T$ -periodic**) if, for some strictly positive integer  $N$ ,

$$x(n) = x(n + N) \quad \text{for all } n \text{ (where } n \text{ is an integer).} \quad (2.8)$$

An  $N$ -periodic sequence  $x$  is said to have **frequency**  $\frac{1}{N}$  and **angular frequency**  $\frac{2\pi}{N}$ . A function or sequence that is not periodic is said to be **aperiodic**. Examples of a periodic function and sequence are shown in Figures 2.5 and 2.6. Some other examples of periodic functions include the cosine and sine functions.

The period of a periodic function or sequence is not uniquely determined. In particular, a function or sequence that is periodic with period  $T$  is also periodic with period  $kT$ , for every strictly positive integer  $k$ . In most cases, we are interested in the smallest (positive) value of  $T$  or  $N$  for which (2.7) or (2.8) is satisfied, respectively. We refer to this value as the **fundamental period**. Moreover, the frequency corresponding to the fundamental period is called the **fundamental frequency**.

Consider, for example, the periodic function  $x$  shown in Figure 2.5. The function  $x$  is periodic with periods  $T$ ,  $2T$ ,  $3T$ , and so on. The fundamental period of  $x$ , however, is uniquely determined and has the value  $T$ .

## 2.9 Exercises

### 2.9.1 Exercises Without Answer Key

**2.1** Let each of  $\mathcal{G}$  and  $\mathcal{H}$  denote a system operator that maps a function to a function; let  $x$  and  $y$  denote functions; and assume that all other variables denote numbers. Fully parenthesize each of the expressions below in order to show the implied grouping of all operations.

- (a)  $\mathcal{H}x(t) = t^2 + 1$ ;
- (b)  $\mathcal{G}\mathcal{H}y(t)$ ;
- (c)  $\mathcal{H}x + y$ ; and
- (d)  $x\mathcal{H}\mathcal{G}y$ .

**2.2** Let  $\mathcal{H}$  denote a system operator that maps a function to a function; let  $x$  and  $y$  denote functions; and let all other variables denote numbers. Using strictly-correct mathematical notation, write an expression for each quantity specified below. Only use brackets for grouping when strictly required. Use  $\mathcal{D}$  to denote the derivative operator.

- (a) the output of the system  $\mathcal{H}$  when its input is  $y$ ;
- (b) the output of the system  $\mathcal{H}$  evaluated at  $2t - 1$  when the input to the system is  $x$ ;
- (c) the output of the system  $\mathcal{H}$  evaluated at  $t$  when the input to the system is  $ax$ ;
- (d) the output of the system  $\mathcal{H}$  evaluated at  $5t$  when the input to the system is  $x + y$ ;
- (e) the derivative of the output of the system  $\mathcal{H}$  when its input is  $ax$ ;
- (e) the output of the system  $\mathcal{H}$  when its input is the derivative of  $ax$ ;
- (f) the sum of: 1) the output of the system  $\mathcal{H}$  when its input is  $x$ ; and 2) the output of the system  $\mathcal{H}$  when its input is  $y$ ;
- (g) the output of the system  $\mathcal{H}$  when its input is  $x + y$ ; and
- (h) the derivative of  $x$  evaluated at  $5t - 3$ .

**2.3** Let  $\mathcal{D}$  denote the derivative operator; let  $\mathcal{H}_1$ ,  $\mathcal{H}_2$ , and  $\mathcal{H}_3$  denote the system operators  $\mathcal{H}_1x(t) = tx(t)$ ,  $\mathcal{H}_2x(t) = x(t - 1)$ , and  $\mathcal{H}_3x(t) = x(3t)$ ; and let  $x$  denote the function  $x(t) = t - 1$ . Evaluate each of the expressions given below.

- (a)  $\mathcal{H}_2\mathcal{H}_1x(t)$ ;
- (b)  $\mathcal{H}_1\mathcal{H}_2x(t)$ ;
- (c)  $\mathcal{H}_1x(t - 1)$ ;
- (d)  $\mathcal{D}\mathcal{H}_3x(t)$ ;
- (e)  $\mathcal{H}_3\mathcal{D}x(t)$ ; and
- (f)  $\mathcal{D}\{3x\}(t)$ .

### 2.9.2 Exercises With Answer Key

Currently, there are no exercises available with an answer key.



## Chapter 3

# Continuous-Time Signals and Systems

### 3.1 Transformations of the Independent Variable

An important concept in the study of signals and systems is the transformation of a signal. Here, we introduce several elementary signal transformations. Each of these transformations involves a simple modification of the independent variable.

#### 3.1.1 Time Shifting (Translation)

The first type of signal transformation that we shall consider is known as time shifting. **Time shifting** (also known as **translation**) maps a function  $x$  to the function  $y$  given by

$$y(t) = x(t - b), \quad (3.1)$$

where  $b$  is a real constant. In other words, the function  $y$  is formed by replacing  $t$  by  $t - b$  in the expression for  $x(t)$ . Geometrically, the transformation (3.1) shifts the function  $x$  (to the left or right) along the time axis to yield  $y$ . If  $b > 0$ ,  $y$  is shifted to the right relative to  $x$  (i.e., delayed in time). If  $b < 0$ ,  $y$  is shifted to the left relative to  $x$  (i.e., advanced in time).

The effects of time shifting are illustrated in Figure 3.1. By applying a time-shifting transformation to the function  $x$  shown in Figure 3.1(a), each of the functions in Figures 3.1(b) and (c) can be obtained.

#### 3.1.2 Time Reversal (Reflection)

The next type of signal transformation that we consider is called time reversal. **Time reversal** (also known as **reflection**) maps a function  $x$  to the function  $y$  given by

$$y(t) = x(-t). \quad (3.2)$$

In other words, the function  $y$  is formed by replacing  $t$  with  $-t$  in the expression for  $x(t)$ . Geometrically, the transformation (3.2) reflects the function  $x$  about the origin to yield  $y$ .

To illustrate the effects of time reversal, an example is provided in Figure 3.2. Applying a time-reversal transformation to the function  $x$  in Figure 3.2(a) yields the function in Figure 3.2(b).

#### 3.1.3 Time Compression/Expansion (Dilation)

The next transformation to be considered is called time compression/expansion. **Time compression/expansion** (also known as **dilation**) maps a function  $x$  to the function  $y$  given by

$$y(t) = x(at), \quad (3.3)$$

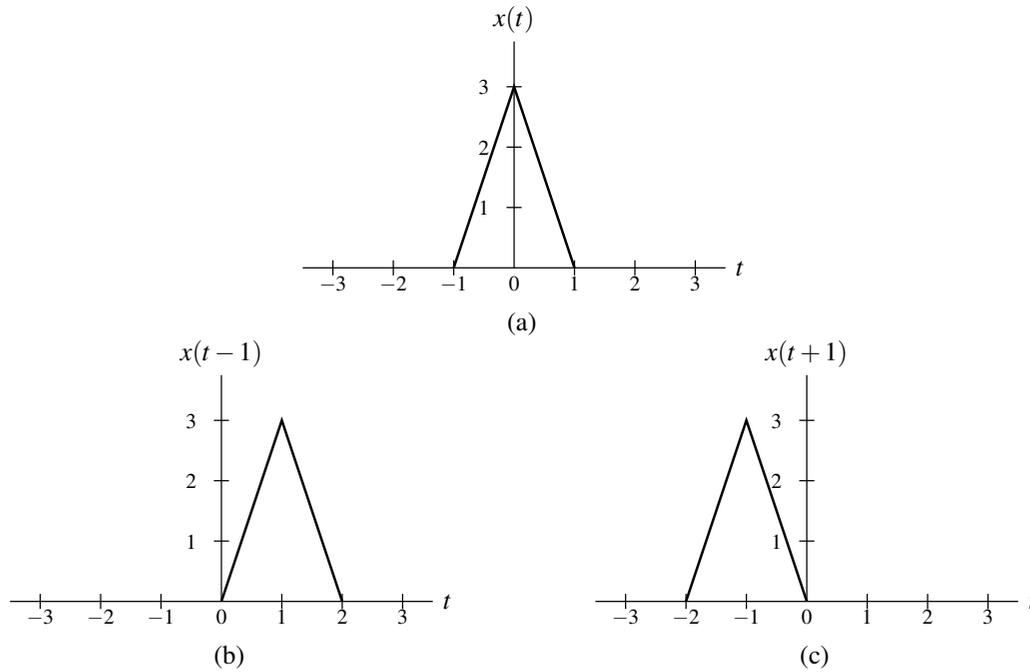


Figure 3.1: Example of time shifting. (a) The function  $x$ ; and the result of applying a time-shifting transformation to  $x$  with a shift of (b) 1 and (c)  $-1$ .

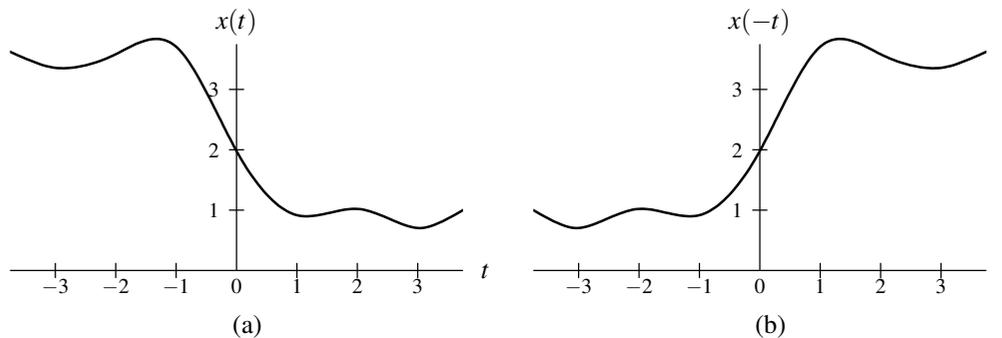


Figure 3.2: Example of time reversal. (a) The function  $x$ ; and (b) the result of applying a time-reversal transformation to  $x$ .

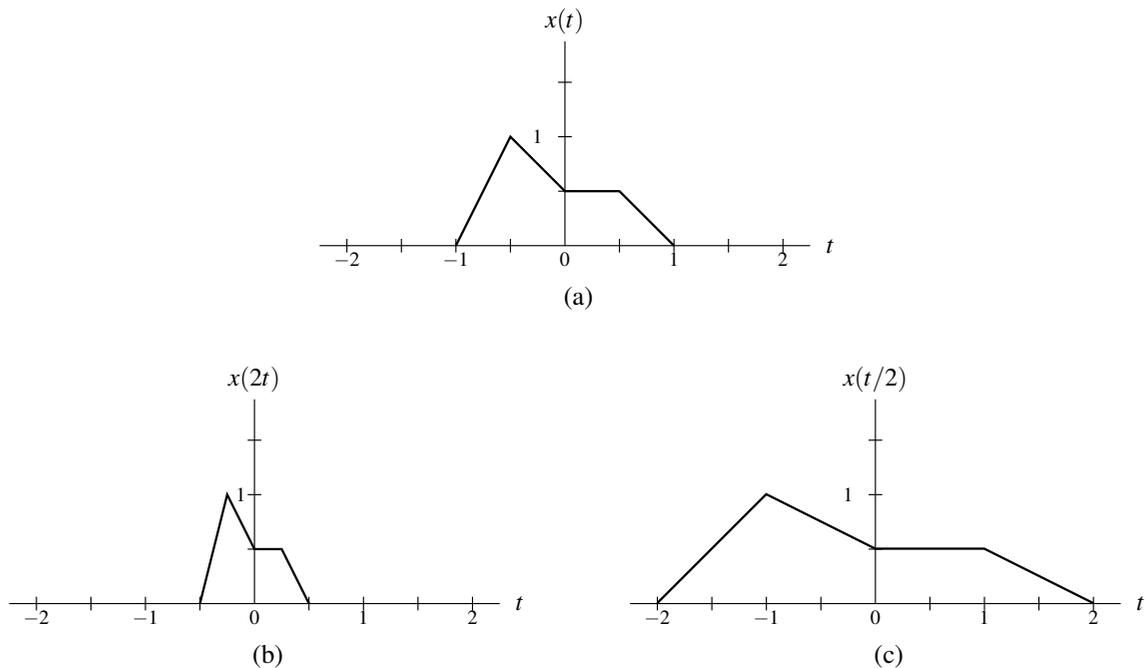


Figure 3.3: Example of time compression/expansion. (a) The function  $x$ ; and the result of applying a time compression/expansion transformation to  $x$  with a scaling factor of (b) 2 and (c)  $\frac{1}{2}$ .

where  $a$  is a strictly positive real constant. In other words, the function  $y$  is formed by replacing  $t$  by  $at$  in the expression for  $x(t)$ . The constant  $a$  is referred to as the scaling factor. The transformation in (3.3) is associated with a compression/expansion along the time axis. If  $a > 1$ ,  $y$  is compressed along the horizontal axis by a factor of  $a$ , relative to  $x$ . If  $a < 1$ ,  $y$  is expanded (i.e., stretched) along the horizontal axis by a factor of  $\frac{1}{a}$ , relative to  $x$ .

To illustrate the effects of time compression/expansion, an example is provided in Figure 3.3. By applying a time compression/expansion transformation to the function  $x$  in Figure 3.3(a), each of the functions shown in Figures 3.3(b) and (c) can be obtained.

### 3.1.4 Time Scaling (Dilation/Reflection)

Another type of signal transformation is called time scaling. **Time scaling** maps a function  $x$  to the function  $y$  given by

$$y(t) = x(at), \quad (3.4)$$

where  $a$  is a nonzero real constant. In other words, the function  $y$  is formed by replacing  $t$  with  $at$  in the expression for  $x(t)$ . The quantity  $a$  is referred to as the scaling factor. Geometrically, the transformation (3.4) is associated with a compression/expansion along the time axis and/or reflection about the origin. If  $|a| < 1$ , the function is expanded (i.e., stretched) along the time axis. If  $|a| > 1$ , the function is instead compressed. If  $|a| = 1$ , the function is neither expanded nor compressed. Lastly, if  $a < 0$ , the function is reflected about the origin. Observe that time scaling includes both time compression/expansion and time reversal as special cases.

To illustrate the behavior of time scaling, we provide an example in Figure 3.4. Each of the functions shown in Figures 3.4(b), (c), and (d) can be obtained by applying a time-scaling transformation to the function  $x$  given in Figure 3.4(a).

### 3.1.5 Combining Time Shifting and Time Scaling

In the preceding sections, we introduced the time shifting and time scaling transformations. Moreover, we observed that time scaling includes time compression/expansion and time reversal as special cases. Some independent-variable

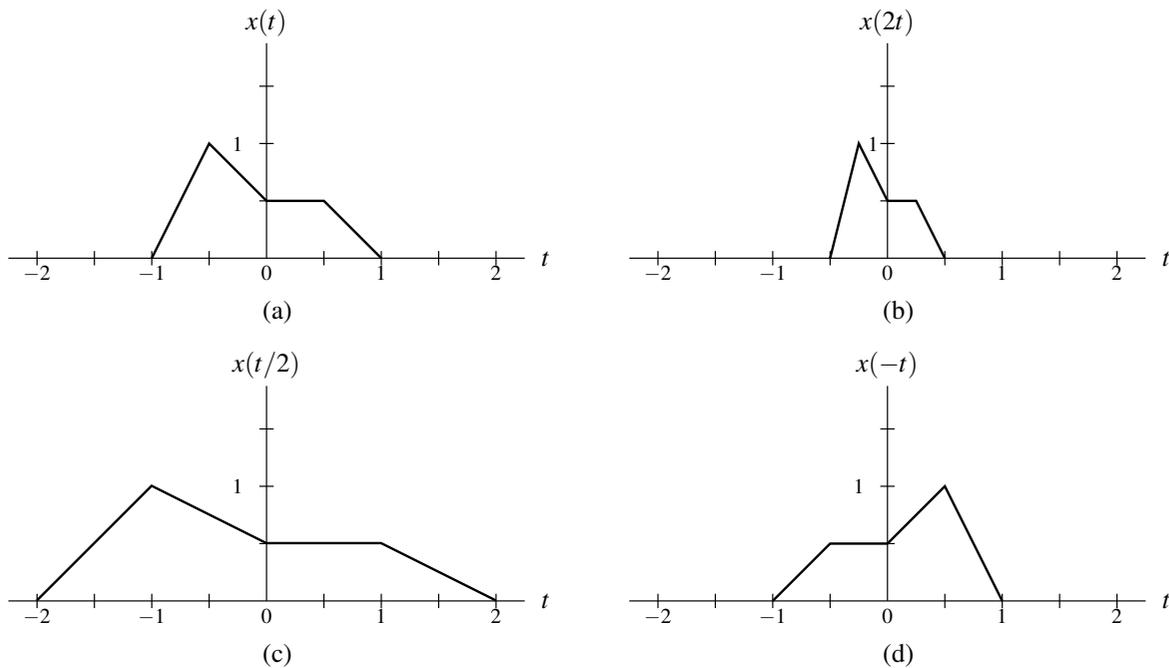


Figure 3.4: Example of time scaling. (a) The function  $x$ ; and the result of applying a time-scaling transformation to  $x$  with a scaling factor of (b) 2, (c)  $\frac{1}{2}$ , and (d)  $-1$ .

transformations commute, while other do not. The issue of commutativity is important, for example, when trying to simplify or manipulate expressions involving combined transformations. Time-scaling, time-reversal, and time-compression/expansion operations commute. Time-shifting (with a nonzero shift) and each of time-scaling, time-reversal, and time-compression/expansion operations do not commute.

Now, we introduce a more general transformation that combines the effects of time shifting and time scaling. This new transformation maps a function  $x$  to the function  $y$  given by

$$y(t) = x(at - b), \quad (3.5)$$

where  $a$  and  $b$  are real constants and  $a \neq 0$ . In other words, the function  $y$  is formed by replacing  $t$  with  $at - b$  in the expression for  $x(t)$ . One can show that the transformation (3.5) is equivalent to first time shifting  $x$  by  $b$ , and then time scaling the resulting function by  $a$ . Geometrically, this transformation preserves the shape of  $x$  except for a possible expansion/compression along the time axis and/or a reflection about the origin. If  $|a| < 1$ , the function is stretched along the time axis. If  $|a| > 1$ , the function is instead compressed. If  $a < 0$ , the function is reflected about the origin.

The above transformation has two distinct but equivalent interpretations. That is, it is equivalent to each of the following:

1. first, time shifting  $x$  by  $b$ , and then time scaling the result by  $a$ .
2. first, time scaling  $x$  by  $a$ , and then time shifting the result by  $\frac{b}{a}$ .

Observe that the shifting amount differs in these two interpretations (i.e.,  $b$  versus  $\frac{b}{a}$ ). This is due to the fact that time shifting and time scaling do not commute. The proof that the above two interpretations are valid is left as an exercise for the reader in Exercise 3.3.

**Example 3.1.** To illustrate the two equivalent interpretations of this combined transformation, we consider a simple example. Consider the function  $x$  shown in Figure 3.5(a). Let us now determine the transformed function  $y(t) = x(at - b)$ , where  $a = 2$  and  $b = 1$ .

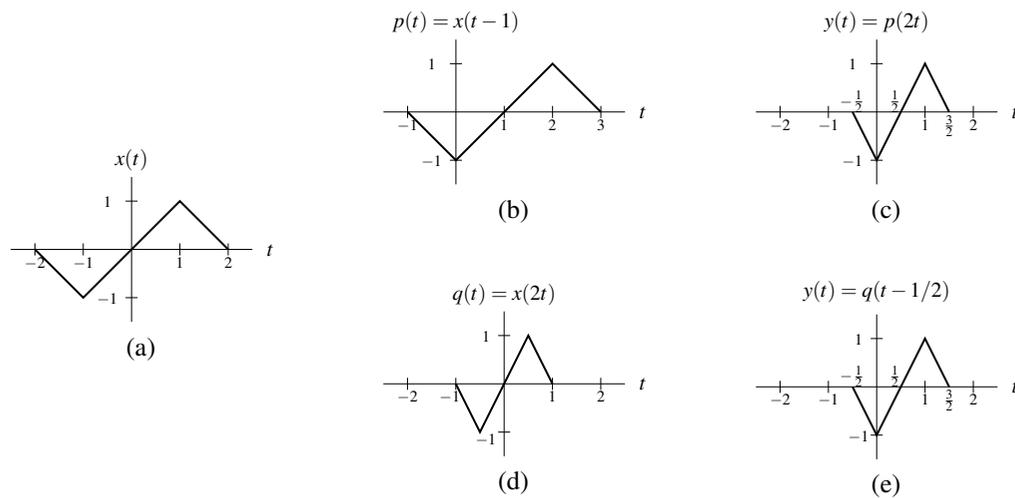


Figure 3.5: Two different interpretations of a combined time-shifting and time-scaling transformation. (a) Original function. Results obtained by shifting followed by scaling: (b) intermediate result and (c) final result. Results obtained by scaling followed by shifting: (d) intermediate result and (e) final result.

*Solution.* First, we consider the shift-then-scale method. In this case, we first shift the function  $x$  by  $b$  (i.e., 1). This yields the function in Figure 3.5(b). Then, we scale this new function by  $a$  (i.e., 2) in order to obtain  $y$  as shown in Figure 3.5(c). Second, we consider the scale-then-shift method. In this case, we first scale the function  $x$  by  $a$  (i.e., 2). This yields the function in Figure 3.5(d). Then, we shift this new function by  $\frac{b}{a}$  (i.e.,  $\frac{1}{2}$ ) in order to obtain  $y$  as shown in Figure 3.5(e). ■

### 3.1.6 Two Perspectives on Independent-Variable Transformations

A transformation of the independent variable can be viewed in terms of:

1. the effect that the transformation has on the *function*; or
2. the effect that the transformation has on the *horizontal axis*.

This distinction is important because such a transformation has *opposite* effects on the function and horizontal axis. For example, the (time-shifting) transformation that replaces  $t$  by  $t - b$  (where  $b$  is a real number) in the expression for  $x(t)$  can be viewed as a transformation that

1. shifts the function  $x$  *right* by  $b$  units; or
2. shifts the horizontal axis *left* by  $b$  units.

In our treatment of independent-variable transformations, we are only interested in the effect that a transformation has on the *function*. If one is not careful to consider that we are interested in the function perspective (as opposed to the axis perspective), many aspects of independent-variable transformations will not make sense.

## 3.2 Transformations of the Dependent Variable

In the preceding sections, we examined several transformations of the independent variable. Now, we consider some transformations of the dependent variable.

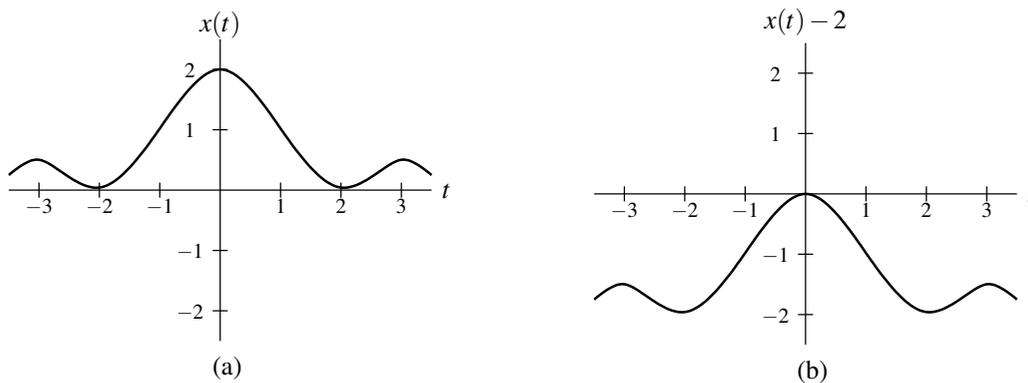


Figure 3.6: Example of amplitude shifting. (a) The function  $x$ ; and the result obtained by applying an amplitude-shifting transformation to  $x$  with a shifting value of  $-2$ .

### 3.2.1 Amplitude Shifting

The first transformation that we consider is referred to as amplitude shifting. **Amplitude shifting** maps a function  $x$  to the function  $y$  given by

$$y(t) = x(t) + b,$$

where  $b$  is a scalar constant. Geometrically, the function  $y$  is displaced vertically relative to  $x$ . If  $b > 0$ ,  $y$  is shifted upwards by  $|b|$  relative to  $x$ . If  $b < 0$ ,  $y$  is shifted downwards by  $|b|$  relative to  $x$ .

The effects of amplitude shifting are illustrated by Figure 3.6. The function shown in Figure 3.6(b) can be obtained by applying an amplitude-shifting transformation to the function  $x$  given in Figure 3.6(a).

### 3.2.2 Amplitude Scaling

The next transformation that we consider is referred to as amplitude scaling. **Amplitude scaling** maps a function  $x$  to the function  $y$  given by

$$y(t) = ax(t),$$

where  $a$  is a scalar constant. Geometrically, the function  $y$  is expanded/compressed in amplitude and/or reflected about the time axis, relative to  $x$ .

To illustrate the effects of amplitude scaling, an example is given in Figure 3.7. Each of the functions shown in Figures 3.7(b), (c), and (d) can be obtained by applying an amplitude-scaling transformation to the function  $x$  given in Figure 3.7(a).

### 3.2.3 Combining Amplitude Shifting and Scaling

In the previous sections, we considered the amplitude-shifting and amplitude-scaling transformations. We can define a new transformation that combines the effects of amplitude shifting and amplitude scaling. This transformation maps the function  $x$  to the function  $y$  given by

$$y(t) = ax(t) + b, \tag{3.6}$$

where  $a$  and  $b$  are scalar constants. One can show that this transformation is equivalent to first scaling  $x$  by  $a$  and then shifting the resulting function by  $b$ . Moreover, since (3.6) can be rewritten as  $y(t) = a[x(t) + \frac{b}{a}]$ , this transformation is also equivalent to first amplitude shifting  $x$  by  $\frac{b}{a}$  and then amplitude scaling the resulting function by  $a$ .

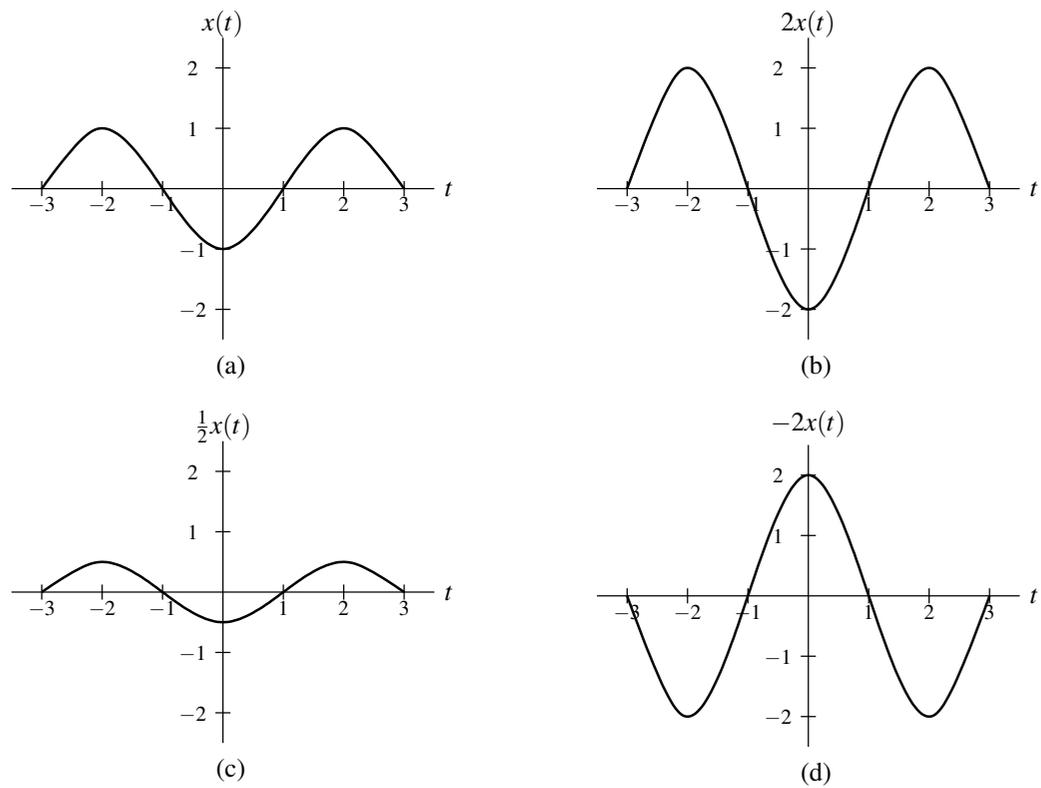


Figure 3.7: Example of amplitude scaling. (a) The function  $x$ ; and the result of applying an amplitude-scaling transformation to  $x$  with a scaling factor of (b) 2, (c)  $\frac{1}{2}$ , and (d)  $-2$ .

### 3.3 Properties of Functions

Functions can possess a number of interesting properties. In what follows, we consider the properties of symmetry and periodicity (introduced earlier) in more detail. Also, we present several other function properties. The properties considered are frequently useful in the analysis of signals and systems.

#### 3.3.1 Remarks on Symmetry

At this point, we make some additional comments about even and odd functions (introduced earlier). Since functions are often summed or multiplied, one might wonder what happens to the even/odd symmetry properties of functions under these operations. In what follows, we introduce a few results in this regard.

Sums involving even and odd functions have the following properties:

- The sum of two even functions is even.
- The sum of two odd functions is odd.
- The sum of an even function and odd function is neither even nor odd, provided that the functions are not identically zero.

Products involving even and odd functions have the following properties:

- The product of two even functions is even.
- The product of two odd functions is even.
- The product of an even function and an odd function is odd.

(The proofs of the above properties involving sums and products of even and odd functions is left as an exercise for the reader in Exercise 3.8.)

As it turns out, any arbitrary function can be expressed as the sum of an even and odd function, as elaborated upon by the theorem below.

**Theorem 3.1** (Decomposition of function into even and odd parts). *Any arbitrary function  $x$  can be uniquely represented as the sum of the form*

$$x(t) = x_e(t) + x_o(t), \quad (3.7)$$

where  $x_e$  and  $x_o$  are even and odd, respectively, and given by

$$x_e(t) = \frac{1}{2} [x(t) + x(-t)] \quad \text{and} \quad (3.8)$$

$$x_o(t) = \frac{1}{2} [x(t) - x(-t)]. \quad (3.9)$$

As a matter of terminology,  $x_e$  is called the **even part** of  $x$  and is denoted  $\text{Even}\{x\}$ , and  $x_o$  is called the **odd part** of  $x$  and is denoted  $\text{Odd}\{x\}$ .

*Proof.* From (3.8) and (3.9), we can easily confirm that  $x_e + x_o = x$  as follows:

$$\begin{aligned} x_e(t) + x_o(t) &= \frac{1}{2}[x(t) + x(-t)] + \frac{1}{2}[x(t) - x(-t)] \\ &= \frac{1}{2}x(t) + \frac{1}{2}x(-t) + \frac{1}{2}x(t) - \frac{1}{2}x(-t) \\ &= x(t). \end{aligned}$$

Furthermore, we can easily verify that  $x_e$  is even and  $x_o$  is odd. From the definition of  $x_e$  in (3.8), we have

$$\begin{aligned} x_e(-t) &= \frac{1}{2}[x(-t) + x(-[-t])] \\ &= \frac{1}{2}[x(t) + x(-t)] \\ &= x_e(t). \end{aligned}$$

Thus,  $x_e$  is even. From the definition of  $x_o$  in (3.9), we have

$$\begin{aligned} x_o(-t) &= \frac{1}{2}[x(-t) - x(-[-t])] \\ &= \frac{1}{2}[-x(t) + x(-t)] \\ &= -x_o(t). \end{aligned}$$

Thus,  $x_o$  is odd.

Lastly, we show that the decomposition of  $x$  into the sum of an even function and odd function is unique. Suppose that  $x$  can be written as the sum of an even function and odd function in two ways as

$$x(t) = f_e(t) + f_o(t) \quad \text{and} \quad (3.10a)$$

$$x(t) = g_e(t) + g_o(t), \quad (3.10b)$$

where  $f_e$  and  $g_e$  are even and  $f_o$  and  $g_o$  are odd. Equating these two expressions for  $x$ , we have

$$f_e(t) + f_o(t) = g_e(t) + g_o(t).$$

Rearranging this equation, we have

$$f_e(t) - g_e(t) = g_o(t) - f_o(t).$$

Now, we consider the preceding equation more carefully. Since the sum of even functions is even and the sum of odd functions is odd, we have that the left- and right-hand sides of the preceding equation correspond to even and odd functions, respectively. Thus, we have that the even function  $f_e(t) - g_e(t)$  is equal to the odd function  $g_o(t) - f_o(t)$ . The only function, however, that is both even and odd is the zero function. (A proof of this fact is left as an exercise for the reader in Exercise 3.14.) Therefore, we have that

$$f_e(t) - g_e(t) = g_o(t) - f_o(t) = 0.$$

In other words, we have that

$$f_e(t) = g_e(t) \quad \text{and} \quad f_o(t) = g_o(t).$$

This implies that the two decompositions of  $x$  given by (3.10a) and (3.10b) must be the same decomposition (i.e., they cannot be distinct). Thus, the decomposition of  $x$  into the sum of an even function and odd function is unique. ■

### 3.3.2 Remarks on Periodicity

Since we often add functions, it is helpful to know if the sum of periodic functions is also periodic. In this regard, the theorem below is useful.

**Theorem 3.2** (Sum of periodic functions). *Suppose that  $x_1$  and  $x_2$  are periodic functions with fundamental periods  $T_1$  and  $T_2$ , respectively. Then, the sum  $y = x_1 + x_2$  is a periodic function if and only if the ratio  $T_1/T_2$  is a rational number (i.e., the quotient of two integers). Suppose that  $T_1/T_2 = q/r$  where  $q$  and  $r$  are integers and coprime (i.e., have no common factors), then the fundamental period of  $y$  is  $rT_1$  (or equivalently,  $qT_2$ , since  $rT_1 = qT_2$ ). (Note that  $rT_1$  is simply the least common multiple of  $T_1$  and  $T_2$ .)*

In passing, we note that the above result can be extended to the more general case of the sum of  $N$  periodic functions. The sum of  $N$  periodic functions  $x_1, x_2, \dots, x_N$  with periods  $T_1, T_2, \dots, T_N$ , respectively, is periodic if and only if the ratios of the periods are rational numbers (i.e.,  $T_1/T_k$  is rational for  $k = 2, 3, \dots, N$ ). If the sum is periodic, then the fundamental period is simply the least common multiple of  $\{T_1, T_2, \dots, T_N\}$ . (Note that the **least common multiple** of the set of positive real numbers  $\{T_1, T_2, \dots, T_N\}$  is the smallest positive real number  $T$  that is an integer multiple of each  $T_k$  for  $k = 1, 2, \dots, N$ .)

**Example 3.2.** Let  $x_1(t) = \sin \pi t$  and  $x_2(t) = \sin t$ . Determine whether the function  $y = x_1 + x_2$  is periodic.

*Solution.* Denote the fundamental periods of  $x_1$  and  $x_2$  as  $T_1$  and  $T_2$ , respectively. We then have

$$T_1 = \frac{2\pi}{\pi} = 2 \quad \text{and} \quad T_2 = \frac{2\pi}{1} = 2\pi.$$

Here, we used the fact that the fundamental period of  $\sin \alpha t$  is  $\frac{2\pi}{|\alpha|}$ . Thus, we have

$$\frac{T_1}{T_2} = \frac{2}{2\pi} = \frac{1}{\pi}.$$

Since  $\pi$  is an irrational number,  $\frac{T_1}{T_2}$  is not rational. Therefore,  $y$  is not periodic. ■

**Example 3.3.** Let  $x_1(t) = \cos(2\pi t + \frac{\pi}{4})$  and  $x_2(t) = \sin(7\pi t)$ . Determine if the function  $y = x_1 + x_2$  is periodic, and if it is, find its fundamental period.

*Solution.* Let  $T_1$  and  $T_2$  denote the fundamental periods of  $x_1$  and  $x_2$ , respectively. Thus, we have

$$T_1 = \frac{2\pi}{2\pi} = 1 \quad \text{and} \quad T_2 = \frac{2\pi}{7\pi} = \frac{2}{7}.$$

Taking the ratio of  $T_1$  to  $T_2$ , we have

$$\frac{T_1}{T_2} = \frac{7}{2}.$$

Since  $\frac{T_1}{T_2}$  is a rational number,  $y$  is periodic. Let  $T$  denote the fundamental period of  $y$ . Since 7 and 2 are coprime,

$$T = 2T_1 = 7T_2 = 2. \quad \blacksquare$$

**Example 3.4.** Let  $x_1(t) = \cos 6\pi t$  and  $x_2(t) = \sin 30\pi t$ . Determine if the function  $y = x_1 + x_2$  is periodic, and if it is, find its fundamental period.

*Solution.* Let  $T_1$  and  $T_2$  denote the fundamental periods of  $x_1$  and  $x_2$ , respectively. We have

$$T_1 = \frac{2\pi}{6\pi} = \frac{1}{3} \quad \text{and} \quad T_2 = \frac{2\pi}{30\pi} = \frac{1}{15}.$$

Thus, we have

$$\frac{T_1}{T_2} = \left(\frac{1}{3}\right) / \left(\frac{1}{15}\right) = \frac{15}{3} = \frac{5}{1}.$$

Since  $\frac{T_1}{T_2}$  is a rational number,  $y$  is periodic. Let  $T$  denote the fundamental period of  $y$ . Since 5 and 1 are coprime, we have

$$T = 1T_1 = 5T_2 = \frac{1}{3}. \quad \blacksquare$$

### 3.3.3 Support of Functions

We can classify functions based on the interval over which their function value is nonzero. This is sometimes referred to as the support of a function. In what follows, we introduce some terminology related to the support of functions.

A function  $x$  is said to be **left sided** if, for some (finite) real constant  $t_0$ , the following condition holds:

$$x(t) = 0 \quad \text{for all } t > t_0.$$

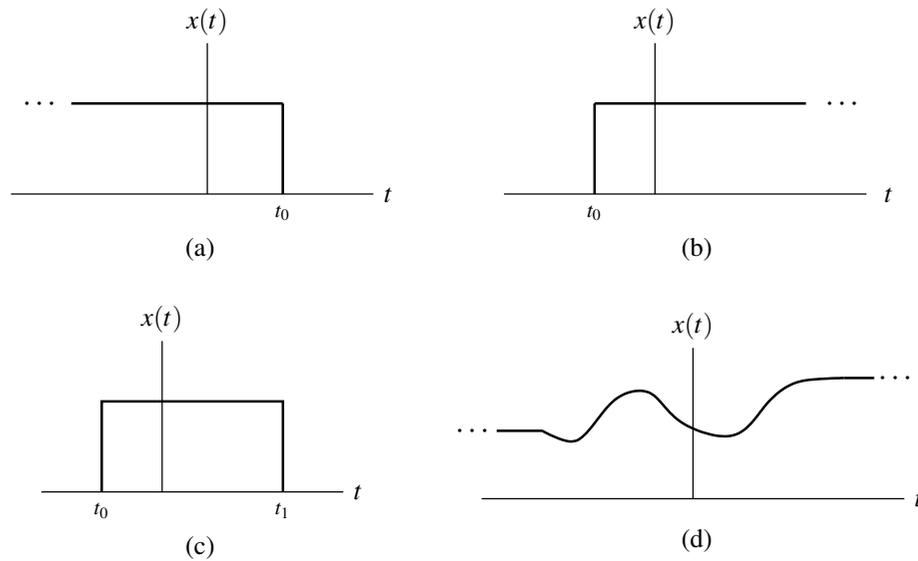


Figure 3.8: Examples of (a) left-sided, (b) right-sided, (c) finite-duration, and (d) two-sided functions.

In other words, the value of the function is always zero to the right of some point. A function  $x$  is said to be **right sided** if, for some (finite) real constant  $t_0$ , the following condition holds:

$$x(t) = 0 \quad \text{for all } t < t_0.$$

In other words, the value of the function is always zero to the left of some point. A function that is both left sided and right sided is said to be **time limited** or **finite duration**. A function that is neither left sided nor right sided is said to be **two sided**. Examples of left-sided, right-sided, finite-duration, and two-sided functions are shown in Figure 3.8.

A function  $x$  is said to be **causal** if

$$x(t) = 0 \quad \text{for all } t < 0.$$

A causal function is a special case of a right-sided function. Similarly, a function  $x$  is said to be **anticausal** if

$$x(t) = 0 \quad \text{for all } t > 0.$$

An anticausal function is a special case of a left-sided function. Note that the qualifiers “causal” and “anticausal”, when applied to functions, have nothing to do with cause and effect. In this sense, this choice of terminology is arguably not the best.

### 3.3.4 Bounded Functions

A function  $x$  is said to be **bounded** if there exists some (finite) nonnegative real constant  $A$  such that

$$|x(t)| \leq A \quad \text{for all } t$$

(i.e.,  $x(t)$  is finite for all  $t$ ). For example, the sine and cosine functions are bounded, since

$$|\sin t| \leq 1 \quad \text{for all } t \quad \text{and} \quad |\cos t| \leq 1 \quad \text{for all } t.$$

In contrast, the tangent function and any nonconstant polynomial function  $p$  (e.g.,  $p(t) = t^2$ ) are unbounded, since

$$\lim_{t \rightarrow \pi/2} |\tan t| = \infty \quad \text{and} \quad \lim_{|t| \rightarrow \infty} |p(t)| = \infty.$$

### 3.3.5 Signal Energy and Power

The **energy**  $E$  contained in the function  $x$  is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt.$$

As a matter of terminology, a function  $x$  with finite energy is said to be an **energy signal**. The **average power**  $P$  contained in the function  $x$  is given by

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt.$$

As a matter of terminology, a function  $x$  with (nonzero) finite average power is said to be a **power signal**.

### 3.3.6 Examples

**Example 3.5.** Let  $x$  be a function with the following properties:

$$\begin{aligned} v(t) = x(t-3) \text{ is causal; and} \\ x \text{ is odd.} \end{aligned}$$

Determine for what values of  $t$  the quantity  $x(t)$  must be zero.

*Solution.* Since  $v$  is causal, we know that  $v(t) = 0$  for all  $t < 0$ . Since  $v(t) = x(t-3)$ , this implies that

$$x(t) = 0 \text{ for } t < -3. \quad (3.11)$$

Since  $x$  is odd, we know that

$$x(t) = -x(-t) \text{ for all } t. \quad (3.12)$$

From (3.11) and (3.12), we have  $-x(-t) = 0$  for  $t < -3$  which implies that

$$x(t) = 0 \text{ for } t > 3. \quad (3.13)$$

Substituting  $t = 0$  into (3.12) yields  $x(0) = -x(0)$  which implies that

$$x(0) = 0. \quad (3.14)$$

Combining (3.11), (3.13), and (3.14), we conclude that  $x(t)$  must be zero for

$$t < -3, \quad t > 3, \quad \text{or} \quad t = 0. \quad \blacksquare$$

**Example 3.6.** Consider a function  $x$  with the following properties:

- $v_1(t) = x(t-3)$  is even;
- $x(t) = t+5$  for  $-5 \leq t \leq -3$ ;
- $v_2(t) = x(t-1)$  is anticausal; and
- $v_3(t) = x(t-5)$  is causal.

Find  $x(t)$  for all  $t$ .

*Solution.* Since  $v_2(t) = x(t-1)$  is anticausal, we know that  $v_2(t) = 0$  for  $t > 0$ . This implies that

$$x(t) = 0 \text{ for } t > -1. \quad (3.15)$$

Since  $v_3(t) = x(t-5)$  is causal, we know that  $v_3(t) = 0$  for  $t < 0$ . This implies that

$$x(t) = 0 \text{ for } t < -5. \quad (3.16)$$

We are given that

$$x(t) = t + 5 \text{ for } -5 \leq t \leq -3. \quad (3.17)$$

Combining (3.15), (3.16), and (3.17), we have

$$x(t) = \begin{cases} 0 & t < -5 \\ t + 5 & -5 \leq t \leq -3 \\ 0 & t > -1. \end{cases} \quad (3.18)$$

So, we only need to determine  $x(t)$  for  $-3 \leq t \leq -1$ . Since  $v_1(t) = x(t-3)$  is even, we know that  $v_1(t) = v_1(-t)$  for all  $t$ , which is equivalent to

$$x(t-3) = x(-t-3) \text{ for all } t.$$

This implies that

$$x(t) = x(-t-6) \text{ for all } t. \quad (3.19)$$

Combining (3.17) and (3.19), we have

$$\begin{aligned} x(t) &= x(-t-6) \\ &= (-t-6) + 5 \text{ for } -5 \leq -t-6 \leq -3 \\ &= -t-1 \text{ for } 1 \leq -t \leq 3 \\ &= -t-1 \text{ for } -3 \leq t \leq -1. \end{aligned}$$

Combining this with our earlier result (3.18), we have

$$x(t) = \begin{cases} 0 & t < -5 \\ t + 5 & -5 \leq t < -3 \\ -t - 1 & -3 \leq t < -1 \\ 0 & t \geq -1. \end{cases} \quad \blacksquare$$

## 3.4 Elementary Functions

A number of elementary signals are particularly useful in the study of signals and systems. In what follows, we introduce some of the more beneficial ones for our purposes.

### 3.4.1 Real Sinusoidal Functions

One important class of functions is the real sinusoids. A **real sinusoidal function**  $x$  has the general form

$$x(t) = A \cos(\omega t + \theta),$$

where  $A$ ,  $\omega$ , and  $\theta$  are real constants. Such a function is periodic with fundamental period  $T = \frac{2\pi}{|\omega|}$ , and has a plot resembling that shown in Figure 3.9.

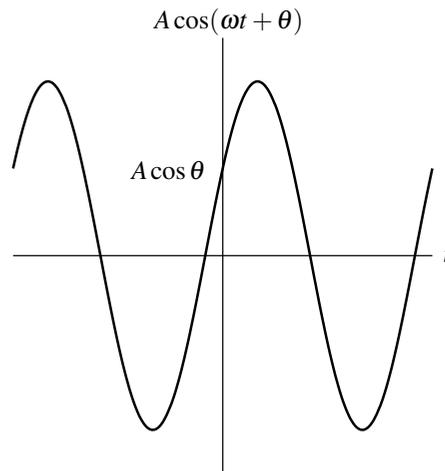
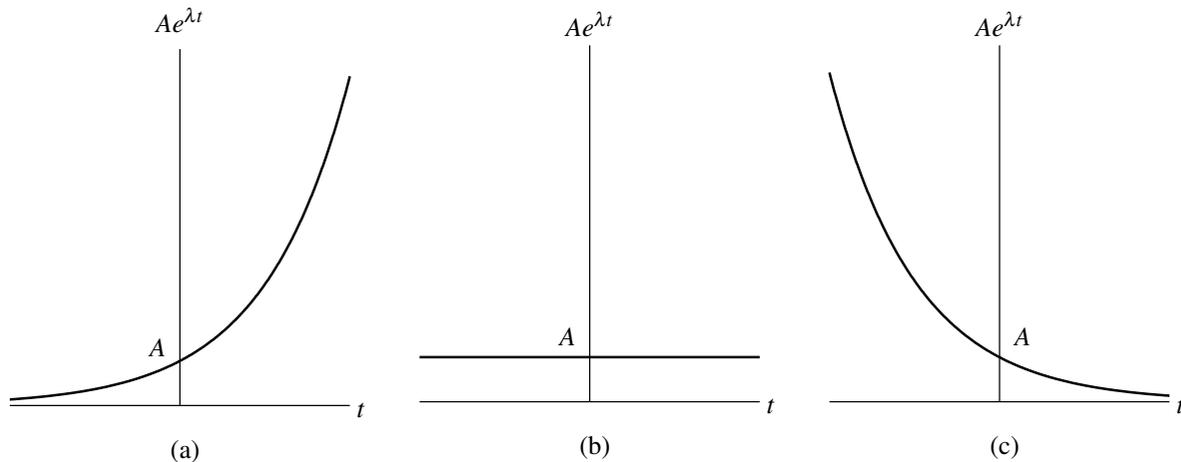


Figure 3.9: Real sinusoidal function.

Figure 3.10: Real exponential function for (a)  $\lambda > 0$ , (b)  $\lambda = 0$ , and (c)  $\lambda < 0$ .

### 3.4.2 Complex Exponential Functions

Another important class of functions is the complex exponentials. A **complex exponential function**  $x$  has the general form

$$x(t) = Ae^{\lambda t}, \quad (3.20)$$

where  $A$  and  $\lambda$  are complex constants. Complex exponentials are of fundamental importance to systems theory, and also provide a convenient means for representing a number of other classes of functions. A complex exponential can exhibit one of a number of distinctive modes of behavior, depending on the values of its parameters  $A$  and  $\lambda$ . In what follows, we examine some special cases of complex exponentials, in addition to the general case.

#### Real Exponential Functions

The first special case of the complex exponentials to be considered is the real exponentials. In the case of a **real exponential function**, we restrict  $A$  and  $\lambda$  in (3.20) to be real. A real exponential can exhibit one of three distinct modes of behavior, depending on the value of  $\lambda$ , as illustrated in Figure 3.10. If  $\lambda > 0$ ,  $x(t)$  increases exponentially as  $t$  increases (i.e., a growing exponential). If  $\lambda < 0$ ,  $x(t)$  decreases exponentially as  $t$  increases (i.e., a decaying or damped exponential). If  $\lambda = 0$ ,  $x(t)$  simply equals the constant  $A$  for all  $t$ .

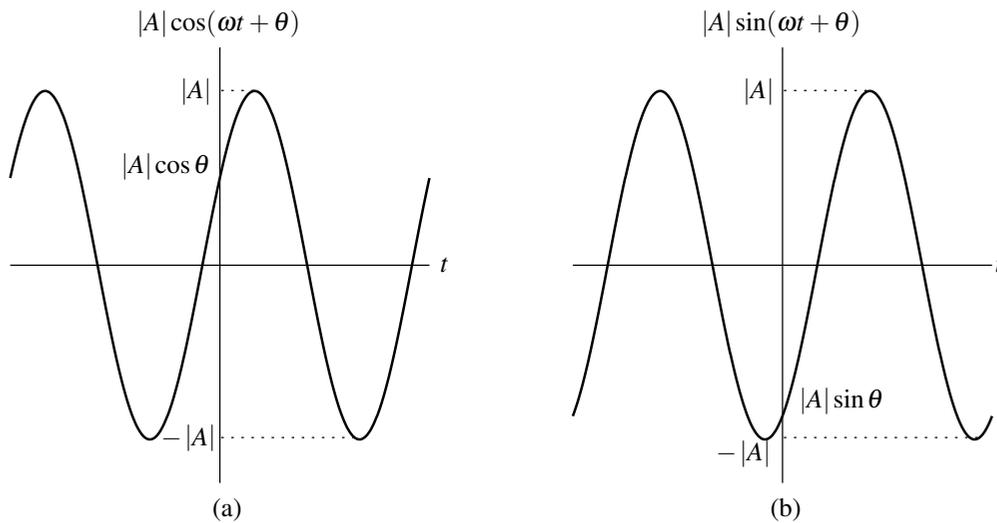


Figure 3.11: Complex sinusoidal function. (a) Real and (b) imaginary parts.

### Complex Sinusoidal Functions

The second special case of the complex exponentials that we shall consider is the complex sinusoids. In the case of a **complex sinusoidal function**, the parameters in (3.20) are such that  $A$  is complex and  $\lambda$  is purely imaginary (i.e.,  $\text{Re}(\lambda) = 0$ ). For convenience, let us re-express  $A$  in polar form and  $\lambda$  in Cartesian form as follows:

$$A = |A| e^{j\theta} \quad \text{and} \quad \lambda = j\omega,$$

where  $\theta$  and  $\omega$  are real constants. Using Euler's relation (A.3), we can rewrite (3.20) as

$$\begin{aligned} x(t) &= A e^{\lambda t} \\ &= |A| e^{j\theta} e^{j\omega t} \\ &= |A| e^{j(\omega t + \theta)} \\ &= |A| \cos(\omega t + \theta) + j |A| \sin(\omega t + \theta). \end{aligned}$$

From the above equation, we can see that each of the real and imaginary parts of  $x$  is periodic with fundamental period  $\frac{2\pi}{|\omega|}$ . Furthermore, from this, we can deduce that  $x$  is also periodic with fundamental period  $\frac{2\pi}{|\omega|}$ . To illustrate the form of a complex sinusoid, we plot its real and imaginary parts in Figure 3.11. The real and imaginary parts are the same except for a phase difference.

### General Complex Exponential Functions

Lastly, we consider general complex exponential functions. That is, we consider the general case of (3.20) where  $A$  and  $\lambda$  are both complex. For convenience, let us re-express  $A$  in polar form and  $\lambda$  in Cartesian form as

$$A = |A| e^{j\theta} \quad \text{and} \quad \lambda = \sigma + j\omega,$$

where  $\theta$ ,  $\sigma$ , and  $\omega$  are real constants. Substituting these expressions for  $A$  and  $\lambda$  into (3.20), we obtain

$$\begin{aligned} x(t) &= A e^{\lambda t} \\ &= |A| e^{j\theta} e^{(\sigma + j\omega)t} \\ &= |A| e^{\sigma t} e^{j(\omega t + \theta)} \\ &= |A| e^{\sigma t} \cos(\omega t + \theta) + j |A| e^{\sigma t} \sin(\omega t + \theta). \end{aligned}$$

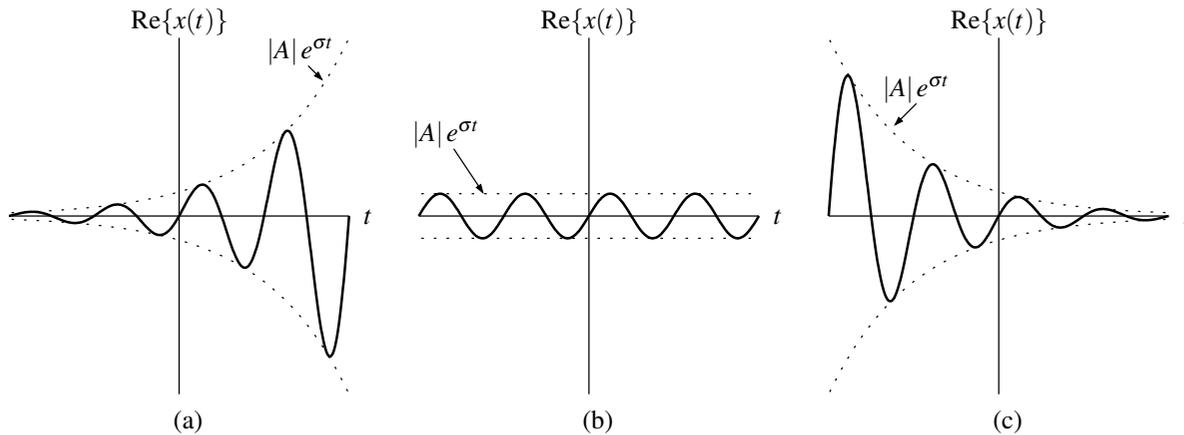


Figure 3.12: Real part of a general complex exponential function for (a)  $\sigma > 0$ , (b)  $\sigma = 0$ , and (c)  $\sigma < 0$ .

We can see that  $\text{Re}[x(t)]$  and  $\text{Im}[x(t)]$  have a similar form. Each is the product of a real exponential and real sinusoid. One of three distinct modes of behavior is exhibited by  $x$ , depending on the value of  $\sigma$ . If  $\sigma = 0$ ,  $\text{Re}(x)$  and  $\text{Im}(x)$  are real sinusoids. If  $\sigma > 0$ ,  $\text{Re}(x)$  and  $\text{Im}(x)$  are each the product of a real sinusoid and a growing real exponential. If  $\sigma < 0$ ,  $\text{Re}(x)$  and  $\text{Im}(x)$  are each the product of a real sinusoid and a decaying real exponential. These three cases are illustrated for  $\text{Re}(x)$  in Figure 3.12.

### 3.4.3 Relationship Between Complex Exponential and Real Sinusoidal Functions

A real sinusoid can be expressed as the sum of two complex sinusoids using the identity

$$A \cos(\omega t + \theta) = \frac{A}{2} \left( e^{j(\omega t + \theta)} + e^{-j(\omega t + \theta)} \right) \quad \text{and} \quad (3.21)$$

$$A \sin(\omega t + \theta) = \frac{A}{2j} \left( e^{j(\omega t + \theta)} - e^{-j(\omega t + \theta)} \right). \quad (3.22)$$

This result follows from Euler's relation and is simply a restatement of (A.7).

### 3.4.4 Unit-Step Function

Another elementary function often used in systems theory is the unit-step function. The **unit-step function** (also known as the **Heaviside function**) is denoted as  $u$  and defined as

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

A plot of this function is given in Figure 3.13.

Clearly,  $u$  is discontinuous at the origin. At this point of discontinuity, we have chosen to define  $u$  such that its value is 1 (i.e.,  $u(0) = 1$ ). As it turns out, however, this choice is somewhat arbitrary. That is, for most practical purposes, due to technical reasons beyond the scope of this book, what is important is that  $u(0)$  is finite, not its specific value. For this reason, some authors choose to leave the value of  $u(0)$  as an unspecified (but finite) constant, while others choose to assign a specific value to  $u(0)$ . In cases where a specific value is chosen for  $u(0)$ , the values most commonly used are 0,  $\frac{1}{2}$ , and 1. Obviously, in the case of this book, the author has chosen to use  $u(0) = 1$ .

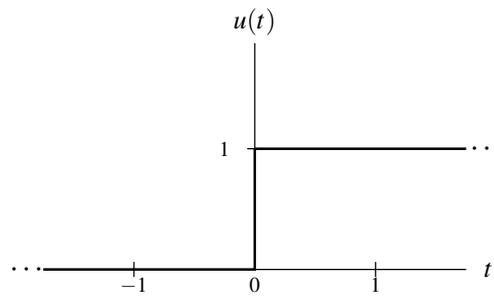


Figure 3.13: Unit-step function.

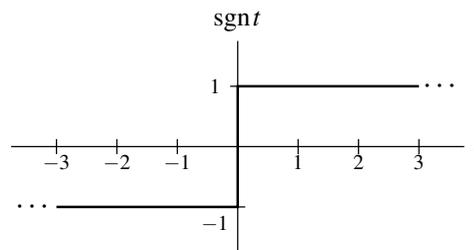


Figure 3.14: Signum function.

### 3.4.5 Signum Function

Another function closely related to the unit-step function is the so called signum function. The **signum function**, denoted  $\text{sgn}$ , is defined as

$$\text{sgn } t = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0. \end{cases} \quad (3.23)$$

A plot of this function is given in Figure 3.14. From (3.23), one can see that the signum function simply computes the sign of a number.

### 3.4.6 Rectangular Function

Another useful function is the rectangular function. The **rectangular function** (also known as the **unit-rectangular pulse function**) is denoted as  $\text{rect}$  and is defined as

$$\text{rect } t = \begin{cases} 1 & -\frac{1}{2} \leq t < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

A plot of this function is shown in Figure 3.15.

**Example 3.7** (Extracting part of a function with a rectangular pulse). Use the rectangular function to extract one period of the function  $x$  shown in Figure 3.16(a).

*Solution.* Let us choose to extract the period of  $x(t)$  for  $-\frac{T}{2} < t \leq \frac{T}{2}$ . In order to extract this period, we want to multiply  $x$  by a function that is one over this interval and zero elsewhere. Such a function is simply  $v(t) = \text{rect}(t/T)$  as shown in Figure 3.16(b). Multiplying  $v$  and  $x$  results in the function shown in Figure 3.16(c). ■

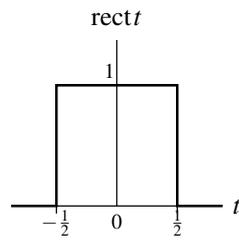
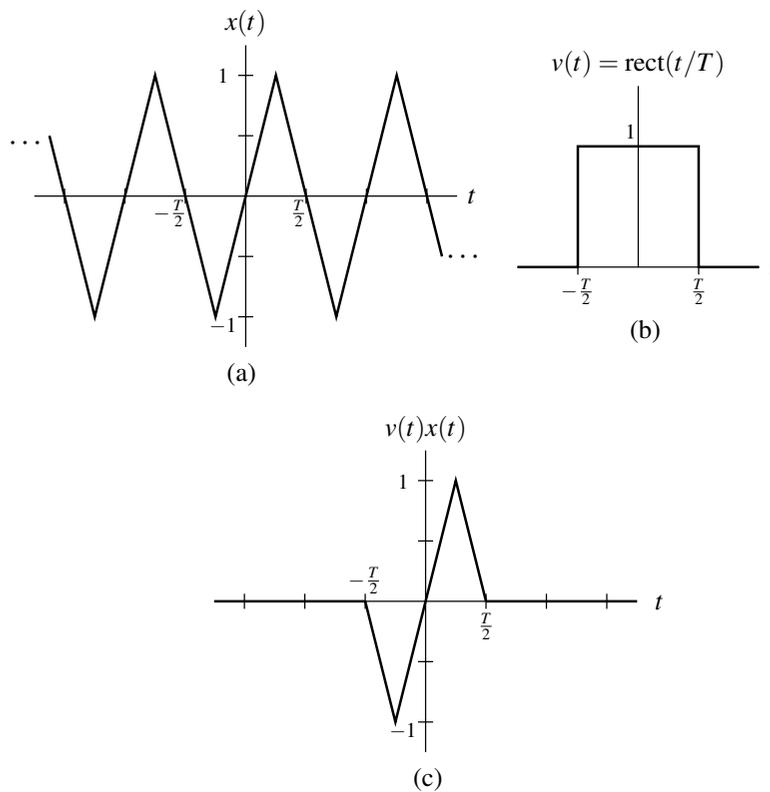


Figure 3.15: Rectangular function.

Figure 3.16: Using the rectangular function to extract one period of a periodic function  $x$ . (a) The function  $x$ . (b) A time-scaled rectangular function  $v$ . (c) The product of  $x$  and  $v$ .

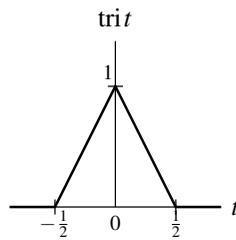


Figure 3.17: Triangular function.

### 3.4.7 Triangular Function

Another useful elementary function is the **triangular function** (also known as the **unit triangular pulse function**), which is denoted as  $\text{tri}$  and defined as

$$\text{tri } t = \begin{cases} 1 - 2|t| & |t| \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

A plot of this function is given in Figure 3.17.

### 3.4.8 Cardinal Sine Function

In the study of signals and systems, a function of the form  $x(t) = \frac{\sin t}{t}$  frequently appears. Therefore, as a matter of convenience, this particular function is given a special name, the cardinal sine function. More formally, the **cardinal sine function** (also known as the **sinc function**) is denoted  $\text{sinc}$  and defined as

$$\text{sinc } t = \frac{\sin t}{t}. \quad (3.24)$$

By using l'Hopital's rule, one can confirm that  $\text{sinc } t$  is well defined for  $t = 0$ . That is,  $\text{sinc } 0 = 1$ . In passing, we note that the name “sinc” is simply a contraction of the function's full Latin name “sinus cardinalis” (cardinal sine).

In passing, we note that an alternative normalization of the sinc function is sometimes used in certain contexts. In particular, the following normalization is sometimes used:

$$x(t) = \frac{\sin \pi t}{\pi t}. \quad (3.25)$$

In order to avoid any possible confusion, we will refer to the function  $x$  defined by (3.25) as the **normalized sinc function**.

### 3.4.9 Unit-Impulse Function

In systems theory, one elementary function of fundamental importance is the unit-impulse function. Instead of defining this function explicitly, it is defined in terms of its properties. In particular, the **unit-impulse function** (also known as **Dirac delta function** or **delta function**) is denoted as  $\delta$  and defined as the function with the following two properties:

$$\delta(t) = 0 \quad \text{for } t \neq 0 \quad \text{and} \quad (3.26a)$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1. \quad (3.26b)$$

From these properties, we can see that the function is zero everywhere, except at  $t = 0$  where it is undefined. Indeed, this is an unusual function. Although it is zero everywhere except at a single point, it has a nonzero integral. Technically, the unit-impulse function is not a function in the ordinary sense. Rather, it is what is known as a **generalized function**.

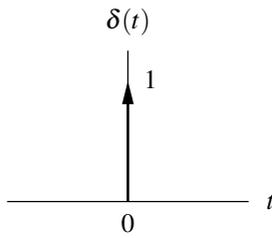


Figure 3.18: Unit impulse function.

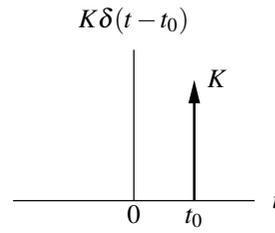


Figure 3.19: Scaled and time-shifted unit impulse function.

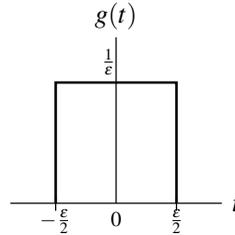


Figure 3.20: Unit-impulse function as limit of rectangular pulse.

Graphically, we represent the unit-impulse function as shown in Figure 3.18. Since the function assumes an infinite value at the origin, we cannot plot the true value of the function. Instead, we use a vertical arrow to represent this infinite value. To show the strength of the impulse, its weight is also indicated. In Figure 3.19, we plot a scaled and time-shifted version of the unit-impulse function.

We can view the unit-impulse function  $\delta$  as a limiting case involving a rectangular pulse. More specifically, let us define the following rectangular pulse function

$$g(t) = \begin{cases} 1/\varepsilon & |t| < \varepsilon/2 \\ 0 & \text{otherwise.} \end{cases}$$

This function is plotted in Figure 3.20. Clearly, the area under the curve is unity for any choice of  $\varepsilon$ . The function  $\delta$  is obtained by taking the following limit:

$$\delta(t) = \lim_{\varepsilon \rightarrow 0} g(t).$$

Thus,  $\delta$  can be viewed as a limiting case of a rectangular pulse where the pulse width becomes infinitesimally small and the pulse height becomes infinitely large in such a way that the integral of the resulting function remains unity.

Informally, one can also think of the unit-impulse function  $\delta$  as the derivative of the unit-step function  $u$ . Strictly speaking, however, the derivative of  $u$  does not exist in the ordinary sense, since  $u$  is discontinuous at 0. To be more precise,  $\delta$  is what is called the **generalized derivative** of  $u$ . The generalized derivative is essentially an extension of the notion of (ordinary) derivative, which can be well defined even for functions with discontinuities.

The unit-impulse function has two important properties that follow from its operational definition in (3.26). These properties are given by the theorems below.

**Theorem 3.3** (Equivalence property). *For any function  $x$  that is continuous at the point  $t_0$ ,*

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0) \quad \text{for all } t. \quad (3.27)$$

*This result is known as the **equivalence property**. This property is illustrated graphically in Figure 3.21.*

*Proof.* The proof follows immediately from the fact that the unit-impulse function is only nonzero at a single point. ■

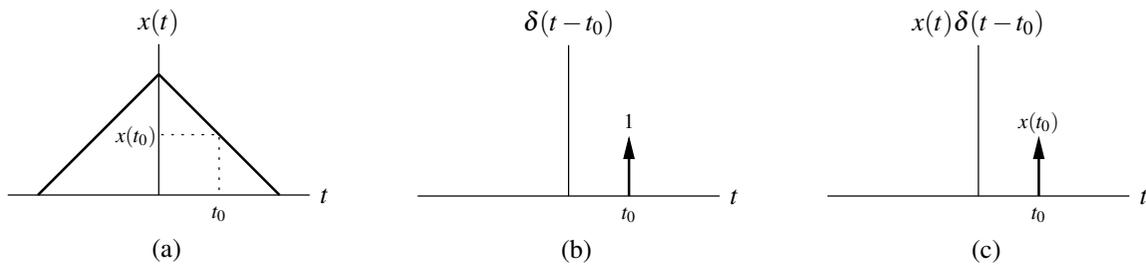


Figure 3.21: Graphical interpretation of equivalence property. (a) A function  $x$ ; (b) a time-shifted unit-impulse function; and (c) the product of these two functions.

**Theorem 3.4** (Sifting property). *For any function  $x$  that is continuous at the point  $t_0$ ,*

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0). \quad (3.28)$$

*This result is known as the **sifting property**.*

*Proof.* From (3.26b), we can write

$$\int_{-\infty}^{\infty} \delta(t-t_0)dt = 1.$$

Multiplying both sides of the preceding equation by  $x(t_0)$  yields

$$\int_{-\infty}^{\infty} x(t_0)\delta(t-t_0)dt = x(t_0).$$

Then, by using the equivalence property in (3.27), we can write

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0). \quad \blacksquare$$

As we shall see later, the equivalence and sifting properties are extremely helpful. Lastly, note that it follows from the definition of  $\delta$  that integrating this function over any interval not containing the origin will result in the value of zero.

**Example 3.8** (Sifting property example). Evaluate the integral

$$\int_{-\infty}^{\infty} [\sin t]\delta(t-\pi/4)dt.$$

*Solution.* Using the sifting property of the unit impulse function, we have

$$\begin{aligned} \int_{-\infty}^{\infty} [\sin t]\delta(t-\pi/4)dt &= \sin \pi/4 \\ &= \frac{1}{\sqrt{2}}. \quad \blacksquare \end{aligned}$$

**Example 3.9** (Sifting property example). Evaluate the integral

$$\int_{-\infty}^{\infty} [\sin 2\pi t]\delta(4t-1)dt.$$

*Solution.* First, we observe that the integral to be evaluated does not quite have the same form as (3.28). So, we need to perform a change of variable. Let  $\tau = 4t$  so that  $t = \tau/4$  and  $dt = d\tau/4$ . Performing the change of variable, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} [\sin 2\pi t] \delta(4t - 1) dt &= \int_{-\infty}^{\infty} \frac{1}{4} [\sin 2\pi \tau/4] \delta(\tau - 1) d\tau \\ &= \int_{-\infty}^{\infty} \left[ \frac{1}{4} \sin \pi \tau/2 \right] \delta(\tau - 1) d\tau.\end{aligned}$$

Now the integral has the desired form, and we can use the sifting property of the unit-impulse function to write

$$\begin{aligned}\int_{-\infty}^{\infty} [\sin 2\pi t] \delta(4t - 1) dt &= \left[ \frac{1}{4} \sin \pi \tau/2 \right] \Big|_{\tau=1} \\ &= \frac{1}{4} \sin \pi/2 \\ &= \frac{1}{4}.\end{aligned}$$

■

**Example 3.10.** Evaluate the integral  $\int_{-\infty}^t (\tau^2 + 1) \delta(\tau - 2) d\tau$ .

*Solution.* Using the equivalence property of the delta function given by (3.27), we can write

$$\begin{aligned}\int_{-\infty}^t (\tau^2 + 1) \delta(\tau - 2) d\tau &= \int_{-\infty}^t (2^2 + 1) \delta(\tau - 2) d\tau \\ &= 5 \int_{-\infty}^t \delta(\tau - 2) d\tau.\end{aligned}$$

Using the defining properties of the delta function given by (3.26), we have that

$$\begin{aligned}\int_{-\infty}^t \delta(\tau - 2) d\tau &= \begin{cases} 1 & t \geq 2 \\ 0 & t < 2 \end{cases} \\ &= u(t - 2).\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}\int_{-\infty}^t (\tau^2 + 1) \delta(\tau - 2) d\tau &= \begin{cases} 5 & t \geq 2 \\ 0 & t < 2 \end{cases} \\ &= 5u(t - 2).\end{aligned}$$

■

### 3.5 Representation of Arbitrary Functions Using Elementary Functions

In the earlier sections, we introduced a number of elementary functions. Often in signal analysis, it is convenient to represent arbitrary functions in terms of elementary functions. Here, we consider how the unit-step function can be exploited in order to obtain alternative representations of functions.

**Example 3.11** (Rectangular function). Show that the rect function can be expressed in terms of  $u$  as

$$\text{rect}t = u\left(t + \frac{1}{2}\right) - u\left(t - \frac{1}{2}\right).$$

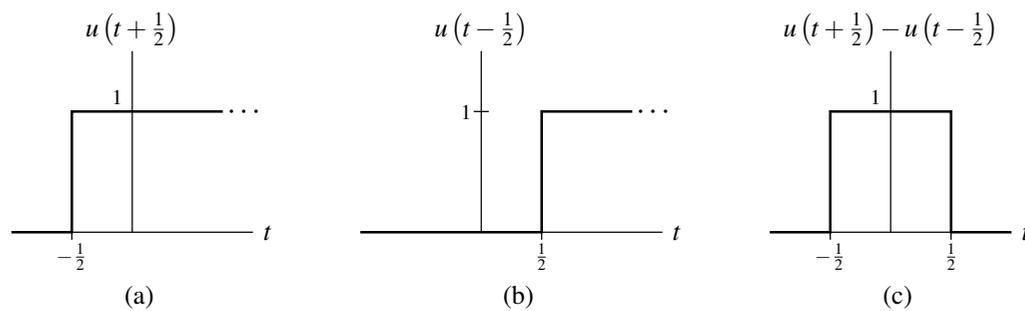


Figure 3.22: Representing the rectangular function using unit-step functions. (a) A shifted unit-step function, (b) another shifted unit-step function, and (c) their difference (which is the rectangular function).

*Solution.* Using the definition of  $u$  and time-shift transformations, we have

$$u\left(t + \frac{1}{2}\right) = \begin{cases} 1 & t \geq -\frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad u\left(t - \frac{1}{2}\right) = \begin{cases} 1 & t \geq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}.$$

Thus, we have

$$\begin{aligned} u\left(t + \frac{1}{2}\right) - u\left(t - \frac{1}{2}\right) &= \begin{cases} 0 & t < -\frac{1}{2} \\ 1 & -\frac{1}{2} \leq t < \frac{1}{2} \\ 0 & t \geq \frac{1}{2} \end{cases} \\ &= \begin{cases} 1 & -\frac{1}{2} \leq t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= \text{rect } t. \end{aligned}$$

Graphically, we have the scenario depicted in Figure 3.22.

In passing, we note that the above result can be generalized. For a rectangular pulse  $x$  of height 1 with a rising edge at  $a$  and falling edge at  $b$ , one can show that

$$\begin{aligned} x(t) &= u(t - a) - u(t - b) \\ &= \begin{cases} 1 & a \leq t < b \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

**Example 3.12** (Piecewise-linear function). Consider the piecewise-linear function  $x$  given by

$$x(t) = \begin{cases} t & 0 \leq t < 1 \\ 1 & 1 \leq t < 2 \\ 3 - t & 2 \leq t < 3 \\ 0 & \text{otherwise} \end{cases}.$$

Find a single expression for  $x(t)$  (involving unit-step functions) that is valid for all  $t$ .

*Solution.* A plot of  $x$  is shown in Figure 3.23(a). We consider each segment of the piecewise-linear function separately. The first segment (i.e., for  $0 \leq t < 1$ ) can be expressed as

$$v_1(t) = t[u(t) - u(t - 1)].$$

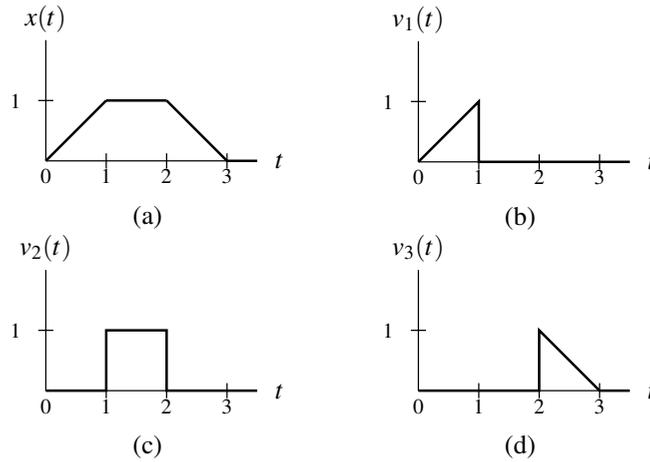


Figure 3.23: Representing a piecewise-linear function using unit-step functions. (a) The function  $x$ . (b), (c), and (d) Three functions whose sum is  $x$ .

This function is plotted in Figure 3.23(b). The second segment (i.e., for  $1 \leq t < 2$ ) can be expressed as

$$v_2(t) = u(t-1) - u(t-2).$$

This function is plotted in Figure 3.23(c). The third segment (i.e., for  $2 \leq t < 3$ ) can be expressed as

$$v_3(t) = (3-t)[u(t-2) - u(t-3)].$$

This function is plotted in Figure 3.23(d). Now, we observe that  $x = v_1 + v_2 + v_3$ . That is, we have

$$\begin{aligned} x(t) &= v_1(t) + v_2(t) + v_3(t) \\ &= t[u(t) - u(t-1)] + [u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)] \\ &= tu(t) + (1-t)u(t-1) + (3-t-1)u(t-2) + (t-3)u(t-3) \\ &= tu(t) + (1-t)u(t-1) + (2-t)u(t-2) + (t-3)u(t-3). \end{aligned}$$

Thus, we have found a single expression for  $x(t)$  that is valid for all  $t$ . ■

**Example 3.13** (Piecewise-polynomial function). Consider the piecewise-polynomial function  $x$  given by

$$x(t) = \begin{cases} 1 & 0 \leq t < 1 \\ (t-2)^2 & 1 \leq t < 3 \\ 4-t & 3 \leq t < 4 \\ 0 & \text{otherwise.} \end{cases}$$

Find a single expression for  $x(t)$  (involving unit-step functions) that is valid for all  $t$ .

*Solution.* A plot of  $x$  is shown in Figure 3.24(a). We consider each segment of the piecewise-polynomial function separately. The first segment (i.e., for  $0 \leq t < 1$ ) can be written as

$$v_1(t) = u(t) - u(t-1).$$

This function is plotted in Figure 3.24(b). The second segment (i.e., for  $1 \leq t < 3$ ) can be written as

$$v_2(t) = (t-2)^2[u(t-1) - u(t-3)] = (t^2 - 4t + 4)[u(t-1) - u(t-3)].$$

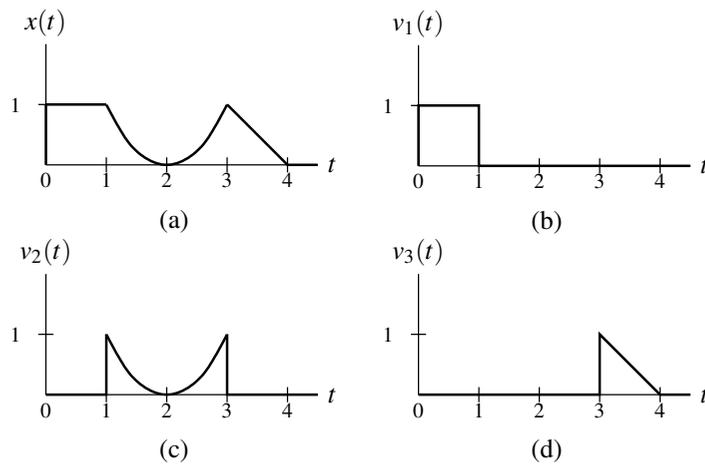


Figure 3.24: Representing a piecewise-polynomial function using unit-step functions. (a) The function  $x$ ; and (b), (c), and (d) three functions whose sum is  $x$ .

This function is plotted in Figure 3.24(c). The third segment (i.e., for  $3 \leq t < 4$ ) can be written as

$$v_3(t) = (4-t)[u(t-3) - u(t-4)].$$

This function is plotted in Figure 3.24(d). Now, we observe that  $x = v_1 + v_2 + v_3$ . So, we have

$$\begin{aligned} x(t) &= v_1(t) + v_2(t) + v_3(t) \\ &= [u(t) - u(t-1)] + (t^2 - 4t + 4)[u(t-1) - u(t-3)] + (4-t)[u(t-3) - u(t-4)] \\ &= u(t) + (t^2 - 4t + 4 - 1)u(t-1) + (4-t - [t^2 - 4t + 4])u(t-3) - (4-t)u(t-4) \\ &= u(t) + (t^2 - 4t + 3)u(t-1) + (-t^2 + 3t)u(t-3) + (t-4)u(t-4). \end{aligned}$$

Thus, we have found a single expression for  $x(t)$  that is valid for all  $t$ . ■

**Example 3.14** (Periodic function). Consider the periodic function  $x$  shown in Figure 3.25(a). Find a single expression for  $x(t)$  (involving unit-step functions) that is valid for all  $t$ .

*Solution.* We begin by finding an expression for a single period of  $x$ . Let us denote this expression as  $v$ . We can then write:

$$v(t) = u\left(t + \frac{1}{2}\right) - u\left(t - \frac{1}{2}\right).$$

This function is plotted in Figure 3.25(b). In order to obtain the periodic function  $x$ , we must repeat  $v$  every 2 units (since the period of  $x$  is 2). This can be accomplished by adding an infinite number of shifted copies of  $v$  as given by

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} v(t-2k) \\ &= \sum_{k=-\infty}^{\infty} \left[ u\left(t + \frac{1}{2} - 2k\right) - u\left(t - \frac{1}{2} - 2k\right) \right]. \end{aligned}$$

Thus, we have found a single expression for  $x$  that is valid for all  $t$ . ■

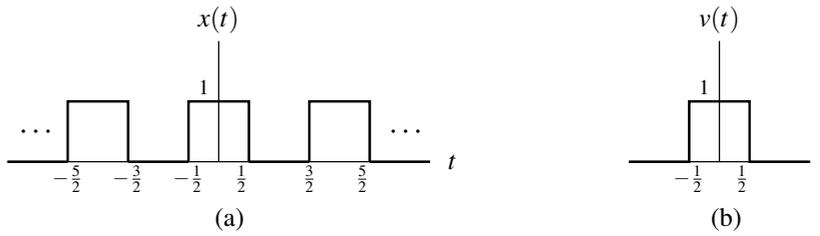


Figure 3.25: Representing a periodic function using unit-step functions. (a) The periodic function  $x$ ; and (b) a function  $v$  that contains a single period of  $x$ .

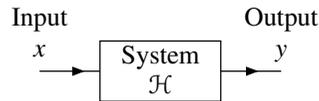


Figure 3.26: Block diagram of system.

## 3.6 Continuous-Time Systems

Suppose that we have a system with input  $x$  and output  $y$ . Such a system can be described mathematically by the equation

$$y = \mathcal{H}x, \quad (3.29)$$

where  $\mathcal{H}$  is an operator (i.e., transformation) representing the system. The operator  $\mathcal{H}$  simply maps the input function  $x$  to the output function  $y$ . Alternatively, we sometimes express the relationship (3.29) using the notation

$$x \xrightarrow{\mathcal{H}} y.$$

Furthermore, if clear from the context, the operator  $\mathcal{H}$  is often omitted, yielding the abbreviated notation

$$x \rightarrow y.$$

Note that the symbols “ $\rightarrow$ ” and “ $=$ ” have very different meanings. For example, the notation  $x \rightarrow y$  does not in any way imply that  $x = y$ . The symbol “ $\rightarrow$ ” should be read as “produces” (not as “equals”). That is, “ $x \rightarrow y$ ” should be read as “the input  $x$  produces the output  $y$ ”.

### 3.6.1 Block Diagram Representation

Suppose that we have a system defined by the operator  $\mathcal{H}$  and having the input  $x$  and output  $y$ . Often, we represent such a system using a block diagram as shown in Figure 3.26.

### 3.6.2 Interconnection of Systems

Systems may be interconnected in a number of ways. Two basic types of connections are as shown in Figure 3.27. The first type of connection, as shown in Figure 3.27(a), is known as a **series** or **cascade** connection. In this case, the overall system is defined by

$$y = \mathcal{H}_2 \mathcal{H}_1 x. \quad (3.30)$$

The second type of connection, as shown in Figure 3.27(b), is known as a **parallel** connection. In this case, the overall system is defined by

$$y = \mathcal{H}_1 x + \mathcal{H}_2 x. \quad (3.31)$$

The system equations in (3.30) and (3.31) cannot be simplified further unless the definitions of the operators  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are known.

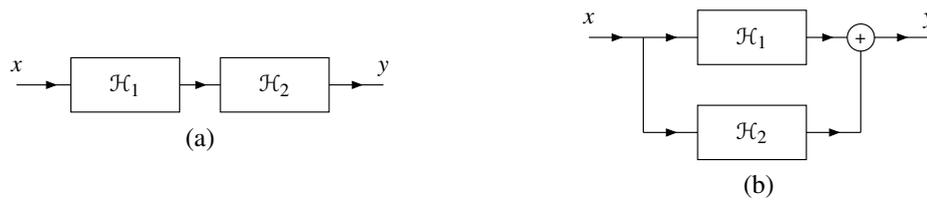


Figure 3.27: Interconnection of systems. (a) Series interconnection of the systems  $\mathcal{H}_1$  and  $\mathcal{H}_2$ . (b) Parallel interconnection of the systems  $\mathcal{H}_1$  and  $\mathcal{H}_2$ .

## 3.7 Properties of Systems

In what follows, we will define a number of important properties that a system may possess. These properties are useful in classifying systems, as well as characterizing their behavior.

### 3.7.1 Memory

A system  $\mathcal{H}$  is said to be **memoryless** if, for every real constant  $t_0$ ,  $\mathcal{H}x(t)$  does not depend on  $x(t)$  for some  $t \neq t_0$ . In other words, a memoryless system is such that the value of its output at any given point in time can depend on the value of its input at only the *same* point in time. A system that is not memoryless is said to have **memory**. Although simple, a memoryless system is not very flexible, since its current output value cannot rely on past or future values of the input.

**Example 3.15** (Ideal amplifier). Determine whether the system  $\mathcal{H}$  is memoryless, where

$$\mathcal{H}x(t) = Ax(t)$$

and  $A$  is a nonzero real constant.

*Solution.* Consider the calculation of  $\mathcal{H}x(t)$  at any arbitrary point  $t = t_0$ . We have

$$\mathcal{H}x(t_0) = Ax(t_0).$$

Thus,  $\mathcal{H}x(t_0)$  depends on  $x(t)$  only for  $t = t_0$ . Therefore, the system is memoryless. ■

**Example 3.16** (Ideal integrator). Determine whether the system  $\mathcal{H}$  is memoryless, where

$$\mathcal{H}x(t) = \int_{-\infty}^t x(\tau) d\tau.$$

*Solution.* Consider the calculation of  $\mathcal{H}x(t)$  at any arbitrary point  $t = t_0$ . We have

$$\mathcal{H}x(t_0) = \int_{-\infty}^{t_0} x(\tau) d\tau.$$

Thus,  $\mathcal{H}x(t_0)$  depends on  $x(t)$  for  $-\infty < t \leq t_0$ . So,  $\mathcal{H}x(t_0)$  is dependent on  $x(t)$  for some  $t \neq t_0$  (e.g.,  $t_0 - 1$ ). Therefore, the system has memory (i.e., is not memoryless). ■

**Example 3.17.** Determine whether the system  $\mathcal{H}$  is memoryless, where

$$\mathcal{H}x(t) = e^{x(t)}.$$

*Solution.* Consider the calculation of  $\mathcal{H}x(t)$  at any arbitrary point  $t = t_0$ . We have

$$\mathcal{H}x(t_0) = e^{x(t_0)}.$$

Thus,  $\mathcal{H}x(t_0)$  depends on  $x(t)$  only for  $t = t_0$ . Therefore, the system is memoryless. ■

**Example 3.18.** Determine whether the system  $\mathcal{H}$  is memoryless, where

$$\mathcal{H}x(t) = \text{Odd}(x)(t) = \frac{1}{2}[x(t) - x(-t)].$$

*Solution.* Consider the calculation of  $\mathcal{H}x(t)$  at any arbitrary point  $t = t_0$ . We have

$$\mathcal{H}x(t_0) = \frac{1}{2}[x(t_0) - x(-t_0)].$$

Thus, for any  $x$  and any real  $t_0$ , we have that  $\mathcal{H}x(t_0)$  depends on  $x(t)$  for  $t = t_0$  and  $t = -t_0$ . Since  $\mathcal{H}x(t_0)$  depends on  $x(t)$  for some  $t \neq t_0$ , the system has memory (i.e., the system is not memoryless). ■

### 3.7.2 Causality

A system  $\mathcal{H}$  is said to be **causal** if, for every real constant  $t_0$ ,  $\mathcal{H}x(t)$  does not depend on  $x(t)$  for some  $t > t_0$ . In other words, a causal system is such that the value of its output at any given point in time can depend on the value of its input at only the *same or earlier* points in time (i.e., *not later* points in time). A memoryless system is always causal, although the converse is not necessarily true.

If the independent variable represents time, a system must be causal in order to be physically realizable. Noncausal systems can sometimes be useful in practice, however, as the independent variable need not always represent time.

**Example 3.19** (Ideal integrator). Determine whether the system  $\mathcal{H}$  is causal, where

$$\mathcal{H}x(t) = \int_{-\infty}^t x(\tau) d\tau.$$

*Solution.* Consider the calculation of  $\mathcal{H}x(t_0)$  for arbitrary  $t_0$ . We have

$$\mathcal{H}x(t_0) = \int_{-\infty}^{t_0} x(\tau) d\tau.$$

Thus, we can see that  $\mathcal{H}x(t_0)$  depends only on  $x(t)$  for  $-\infty < t \leq t_0$ . Since all of the values in this interval are less than or equal to  $t_0$ , the system is causal. ■

**Example 3.20.** Determine whether the system  $\mathcal{H}$  is causal, where

$$\mathcal{H}x(t) = \int_{t-1}^{t+1} x(\tau) d\tau.$$

*Solution.* Consider the calculation of  $\mathcal{H}x(t_0)$  for arbitrary  $t_0$ . We have

$$\mathcal{H}x(t_0) = \int_{t_0-1}^{t_0+1} x(\tau) d\tau.$$

Thus, we can see that  $\mathcal{H}x(t_0)$  only depends on  $x(t)$  for  $t_0 - 1 \leq t \leq t_0 + 1$ . Since some of the values in this interval are greater than  $t_0$  (e.g.,  $t_0 + 1$ ), the system is not causal. ■

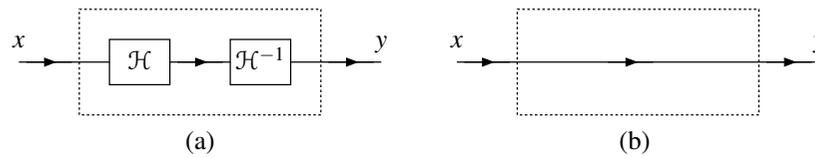


Figure 3.28: Systems that are equivalent (assuming  $\mathcal{H}^{-1}$  exists). (a) First and (b) second system.

**Example 3.21.** Determine whether the system  $\mathcal{H}$  is causal, where

$$\mathcal{H}x(t) = (t + 1)e^{x(t-1)}.$$

*Solution.* Consider the calculation of  $\mathcal{H}x(t_0)$  for arbitrary  $t_0$ . We have

$$\mathcal{H}x(t_0) = (t_0 + 1)e^{x(t_0-1)}.$$

Thus, we can see that  $\mathcal{H}x(t_0)$  depends only on  $x(t)$  for  $t = t_0 - 1$ . Since  $t_0 - 1 \leq t_0$ , the system is causal. ■

**Example 3.22.** Determine whether the system  $\mathcal{H}$  is causal, where

$$\mathcal{H}x(t) = \text{Odd}(x)(t) = \frac{1}{2}[x(t) - x(-t)].$$

*Solution.* For any  $x$  and any real constant  $t_0$ , we have that  $\mathcal{H}x(t_0)$  depends only on  $x(t)$  for  $t = t_0$  and  $t = -t_0$ . Suppose that  $t_0 = -1$ . In this case, we have that  $\mathcal{H}x(t_0)$  (i.e.,  $\mathcal{H}x(-1)$ ) depends on  $x(t)$  for  $t = 1$  but  $t = 1 > t_0$ . Therefore, the system is not causal. ■

### 3.7.3 Invertibility

The **inverse** of a system  $\mathcal{H}$  is a system  $\mathcal{G}$  such that, for every function  $x$ ,

$$\mathcal{G}\mathcal{H}x = x$$

(i.e., the system formed by the cascade interconnection of  $\mathcal{H}$  followed by  $\mathcal{G}$  is a system whose input and output are equal). In other words, the effect of  $\mathcal{H}$  is cancelled by  $\mathcal{G}$ . As a matter of notation, the inverse of  $\mathcal{H}$  is denoted  $\mathcal{H}^{-1}$ . The relationship between a system and its inverse is illustrated in Figure 3.28. The two systems in this figure must be equivalent, due to the relationship between  $\mathcal{H}$  and  $\mathcal{H}^{-1}$  (i.e.,  $\mathcal{H}^{-1}$  cancels  $\mathcal{H}$ ).

A system  $\mathcal{H}$  is said to be **invertible** if it has a corresponding inverse system (i.e., its inverse exists). An invertible system must be such that its input  $x$  can always be uniquely determined from its output  $\mathcal{H}x$ . From this definition, it follows that an invertible system will always produce distinct outputs from any two distinct inputs.

To show that a system is invertible, we simply find the inverse system. To show that a system is not invertible, it is sufficient to find two distinct inputs to that system that result in identical outputs. In practical terms, invertible systems are nice in the sense that their effects can be undone.

**Example 3.23.** Determine whether the system  $\mathcal{H}$  is invertible, where

$$\mathcal{H}x(t) = x(t - t_0)$$

and  $t_0$  is a real constant.

*Solution.* Let  $y = \mathcal{H}x$ . By substituting  $t + t_0$  for  $t$  in  $y(t) = x(t - t_0)$ , we obtain

$$\begin{aligned} y(t + t_0) &= x(t + t_0 - t_0) \\ &= x(t). \end{aligned}$$

Thus, we have shown that

$$x(t) = y(t + t_0).$$

This, however, is simply the equation of the inverse system  $\mathcal{H}^{-1}$ . In particular, we have that

$$x(t) = \mathcal{H}^{-1}y(t)$$

where

$$\mathcal{H}^{-1}y(t) = y(t + t_0).$$

Thus, we have found  $\mathcal{H}^{-1}$ . Therefore, the system  $\mathcal{H}$  is invertible. ■

**Example 3.24.** Determine whether the system  $\mathcal{H}$  is invertible, where

$$\mathcal{H}x(t) = \sin[x(t)].$$

*Solution.* Consider an input of the form  $x(t) = 2\pi k$  where  $k$  is an arbitrary integer. The response  $\mathcal{H}x$  to such an input is given by

$$\begin{aligned} \mathcal{H}x(t) &= \sin[x(t)] \\ &= \sin 2\pi k \\ &= 0. \end{aligned}$$

Thus, we have found an infinite number of distinct inputs (i.e.,  $x(t) = 2\pi k$  for  $k = 0, \pm 1, \pm 2, \dots$ ) that all result in the same output. Therefore, the system is not invertible. ■

**Example 3.25.** Determine whether the system  $\mathcal{H}$  is invertible, where

$$\mathcal{H}x(t) = \text{Odd}(x)(t) = \frac{1}{2}[x(t) - x(-t)].$$

*Solution.* Consider the response  $\mathcal{H}x$  of the system to an input  $x$  of the form

$$x(t) = \alpha,$$

where  $\alpha$  is a real constant. We have that

$$\begin{aligned} \mathcal{H}x(t) &= \frac{1}{2}[x(t) - x(-t)] \\ &= \frac{1}{2}(\alpha - \alpha) \\ &= 0. \end{aligned}$$

Therefore, any constant input yields the same zero output. This, however, implies that distinct inputs can yield identical outputs. Therefore, the system is not invertible. ■

**Example 3.26.** Determine whether the system  $\mathcal{H}$  is invertible, where

$$\mathcal{H}x(t) = 3x(3t + 3).$$

*Solution.* Let  $y = \mathcal{H}x$ . From the definition of  $\mathcal{H}$ , we can write

$$\begin{aligned} y(t) &= 3x(3t + 3) \\ \Rightarrow y\left(\frac{1}{3}t - 1\right) &= 3x(t) \\ \Rightarrow x(t) &= \frac{1}{3}y\left(\frac{1}{3}t - 1\right). \end{aligned}$$

In other words,  $\mathcal{H}^{-1}$  is given by  $\mathcal{H}^{-1}y(t) = \frac{1}{3}y\left(\frac{1}{3}t - 1\right)$ . Since we have just found  $\mathcal{H}^{-1}$ ,  $\mathcal{H}^{-1}$  exists. Therefore, the system  $\mathcal{H}$  is invertible. ■

### 3.7.4 Bounded-Input Bounded-Output (BIBO) Stability

Although stability can be defined in numerous ways, in systems theory, we are often most interested in bounded-input bounded-output (BIBO) stability.

A system  $\mathcal{H}$  is **BIBO stable** if, for every bounded function  $x$ ,  $\mathcal{H}x$  is also bounded (i.e.,  $|x(t)| < \infty$  for all  $t$  implies that  $|\mathcal{H}x(t)| < \infty$  for all  $t$ ). In other words, a BIBO stable system is such that it guarantees to always produce a bounded output as long as its input is bounded.

To prove that a system is BIBO stable, we must show that every bounded input leads to a bounded output. To show that a system is not BIBO stable, we simply need to find one counterexample (i.e., a single bounded input that leads to an unbounded output).

In practical terms, a BIBO stable system is well behaved in the sense that, as long as the system input remains finite for all time, the output will also remain finite for all time. Usually, a system that is not BIBO stable will have serious safety issues. For example, a portable music player with a battery input of 3.7 volts and headset output of  $\infty$  volts would result in one vaporized human (and likely one big lawsuit as well).

**Example 3.27** (Ideal integrator). Determine whether the system  $\mathcal{H}$  is BIBO stable, where

$$\mathcal{H}x(t) = \int_{-\infty}^t x(\tau) d\tau.$$

*Solution.* Suppose that we choose the input  $x = u$  (where  $u$  denotes the unit-step function). Clearly,  $u$  is bounded (i.e.,  $|u(t)| \leq 1$  for all  $t$ ). Calculating the response  $\mathcal{H}x$  to this input, we have

$$\begin{aligned} \mathcal{H}x(t) &= \int_{-\infty}^t u(\tau) d\tau \\ &= \int_0^t d\tau \\ &= [\tau]_0^t \\ &= t. \end{aligned}$$

From this result, however, we can see that as  $t \rightarrow \infty$ ,  $\mathcal{H}x(t) \rightarrow \infty$ . Thus, the output  $\mathcal{H}x$  is unbounded for the bounded input  $x$ . Therefore, the system is not BIBO stable. ■

**Example 3.28** (Squarer). Determine whether the system  $\mathcal{H}$  is BIBO stable, where

$$\mathcal{H}x(t) = x^2(t).$$

*Solution.* Suppose that the input  $x$  is bounded such that (for all  $t$ )

$$|x(t)| \leq A,$$

where  $A$  is a finite real constant. Squaring both sides of the inequality, we obtain

$$|x(t)|^2 \leq A^2.$$

Interchanging the order of the squaring and magnitude operations on the left-hand side of the inequality, we have

$$|x^2(t)| \leq A^2.$$

Using the fact that  $\mathcal{H}x(t) = x^2(t)$ , we can write

$$|\mathcal{H}x(t)| \leq A^2.$$

Since  $A$  is finite,  $A^2$  is also finite. Thus, we have that  $\mathcal{H}x$  is bounded (i.e.,  $|\mathcal{H}x(t)| \leq A^2 < \infty$  for all  $t$ ). Therefore, the system is BIBO stable. ■

**Example 3.29.** Determine whether the system  $\mathcal{H}$  is BIBO stable, where

$$\mathcal{H}x(t) = \text{Odd}(x)(t) = \frac{1}{2}[x(t) - x(-t)].$$

*Solution.* Suppose that  $x$  is bounded. Then,  $x(-t)$  is also bounded. Since the difference of two bounded functions is bounded,  $x(t) - x(-t)$  is bounded. Multiplication of a bounded function by a finite constant yields a bounded result. So, the function  $\frac{1}{2}[x(t) - x(-t)]$  is bounded. Thus,  $\mathcal{H}x(t)$  is bounded. Since a bounded input must yield a bounded output, the system is BIBO stable. ■

**Example 3.30.** Determine whether the system  $\mathcal{H}$  is BIBO stable, where

$$\mathcal{H}x(t) = 3x(3t + 3).$$

*Solution.* Suppose that  $x$  is bounded such that  $|x(t)| \leq A < \infty$  for all  $t$ . Then,  $|x(3t + 3)| \leq A$  for all  $t$ . Furthermore, we have that  $|3x(3t + 3)| = 3|x(3t + 3)|$ . So,  $|3x(3t + 3)| \leq 3A$  for all  $t$ , where  $3A$  is finite. Therefore,  $|\mathcal{H}x(t)| \leq 3A < \infty$  (i.e.,  $\mathcal{H}x$  is bounded). Consequently, the system is BIBO stable. ■

### 3.7.5 Time Invariance

A system  $\mathcal{H}$  is said to be **time invariant (TI)** (or **shift invariant (SI)**) if, for every function  $x$  and every real number  $t_0$ , the following condition holds:

$$\mathcal{H}(x(t - t_0)) = \mathcal{H}x'(t) \text{ for all } t \text{ where } x'(t) = x(t - t_0)$$

(i.e.,  $\mathcal{H}$  commutes with time shifts). In other words, a system is time invariant, if a time shift (i.e., advance or delay) in the input function results in an identical time shift in the output function. A system that is not time invariant is said to be **time varying** (or **shift varying**). In effect, time invariance means that the two systems shown in Figure 3.29 are equivalent, where  $\mathcal{S}_{t_0}$  denotes an operator that applies a time shift of  $t_0$  to a function (i.e.,  $\mathcal{S}_{t_0}x(t) = x(t - t_0)$ ).

In simple terms, a time invariant system is a system whose behavior does not change with respect to time. Practically speaking, compared to time-varying systems, time-invariant systems are much easier to design and analyze, since their behavior does not change with respect to time.

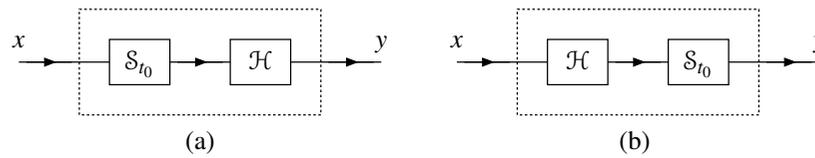


Figure 3.29: Systems that are equivalent if  $\mathcal{H}$  is time invariant (i.e.,  $\mathcal{H}$  commutes with  $\mathcal{S}_{t_0}$ ). (a) A system that first time shifts by  $t_0$  and then applies  $\mathcal{H}$  (i.e.,  $y = \mathcal{H}\mathcal{S}_{t_0}x$ ); and (b) a system that first applies  $\mathcal{H}$  and then time shifts by  $t_0$  (i.e.,  $y = \mathcal{S}_{t_0}\mathcal{H}x$ ).

**Example 3.31.** Determine whether the system  $\mathcal{H}$  is time invariant, where

$$\mathcal{H}x(t) = tx(t).$$

*Solution.* Let  $x'(t) = x(t - t_0)$ , where  $t_0$  is an arbitrary real constant. From the definition of  $\mathcal{H}$ , we have

$$\begin{aligned}\mathcal{H}x(t - t_0) &= (t - t_0)x_1(t - t_0) \quad \text{and} \\ \mathcal{H}x'(t) &= tx'(t) \\ &= tx(t - t_0).\end{aligned}$$

Since  $\mathcal{H}x(t - t_0) = \mathcal{H}x'(t)$  does not hold for all  $x$  and  $t_0$ , the system is not time invariant (i.e., the system is time varying). ■

**Example 3.32.** Determine whether the system  $\mathcal{H}$  is time invariant, where

$$\mathcal{H}x(t) = \sin[x(t)].$$

*Solution.* Let  $x'(t) = x(t - t_0)$ , where  $t_0$  is an arbitrary real constant. From the definition of  $\mathcal{H}$ , we can easily deduce that

$$\begin{aligned}\mathcal{H}x(t - t_0) &= \sin[x(t - t_0)] \quad \text{and} \\ \mathcal{H}x'(t) &= \sin x'(t) \\ &= \sin[x(t - t_0)].\end{aligned}$$

Since  $\mathcal{H}x(t - t_0) = \mathcal{H}x'(t)$  for all  $x$  and  $t_0$ , the system is time invariant. ■

**Example 3.33.** Determine whether the system  $\mathcal{H}$  is time invariant, where

$$\mathcal{H}x(t) = \text{Odd}(x)(t) = \frac{1}{2}[x(t) - x(-t)].$$

*Solution.* Let  $x'(t) = x(t - t_0)$ , where  $t_0$  is an arbitrary real constant. From the definition of  $\mathcal{H}$ , we have

$$\begin{aligned}\mathcal{H}x(t - t_0) &= \frac{1}{2}[x(t - t_0) - x(-(t - t_0))] \\ &= \frac{1}{2}[x(t - t_0) - x(-t + t_0)] \quad \text{and} \\ \mathcal{H}x'(t) &= \frac{1}{2}[x'(t) - x'(-t)] \\ &= \frac{1}{2}[x(t - t_0) - x(-t - t_0)].\end{aligned}$$

Since  $\mathcal{H}x(t - t_0) = \mathcal{H}x'(t)$  does not hold for all  $x$  and  $t_0$ , the system is not time invariant. ■

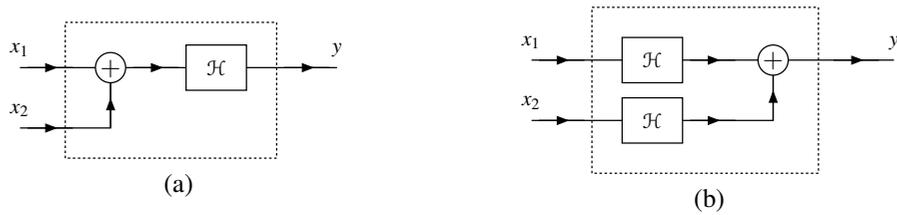


Figure 3.30: Systems that are equivalent if  $\mathcal{H}$  is additive (i.e.,  $\mathcal{H}$  commutes with addition). (a) A system that first performs addition and then applies  $\mathcal{H}$  (i.e.,  $y = \mathcal{H}(x_1 + x_2)$ ); and (b) a system that first applies  $\mathcal{H}$  and then performs addition (i.e.,  $y = \mathcal{H}x_1 + \mathcal{H}x_2$ ).

**Example 3.34.** Determine whether the system  $\mathcal{H}$  is time invariant, where

$$\mathcal{H}x(t) = 3x(3t + 3).$$

*Solution.* Let  $x'(t) = x(t - t_0)$ , where  $t_0$  is an arbitrary real constant. From the definition of  $\mathcal{H}$ , we have

$$\begin{aligned} \mathcal{H}x(t - t_0) &= 3x(3(t - t_0) + 3) \\ &= 3x(3t - 3t_0 + 3) \quad \text{and} \\ \mathcal{H}x'(t) &= 3x'(3t + 3) \\ &= 3x(3t + 3 - t_0). \end{aligned}$$

Since  $\mathcal{H}x(t - t_0) = \mathcal{H}x'(t)$  does not hold for all  $x$  and  $t_0$ , the system is not time invariant. ■

### 3.7.6 Linearity

Two of the most and frequently-occurring mathematical operations are addition and scalar multiplication. For this reason, it is often extremely helpful to know if these operations commute with the operation performed by a given system. The system properties to be introduced next relate to this particular issue.

A system  $\mathcal{H}$  is said to be **additive** if, for all functions  $x_1$  and  $x_2$ , the following condition holds:

$$\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$$

(i.e.,  $\mathcal{H}$  commutes with addition). Essentially, a system  $\mathcal{H}$  being additive means that the two systems shown in Figure 3.30 are equivalent.

A system  $\mathcal{H}$  is said to be **homogeneous** if, for every function  $x$  and every complex constant  $a$ , the following condition holds:

$$\mathcal{H}(ax) = a\mathcal{H}x$$

(i.e.,  $\mathcal{H}$  commutes with scalar multiplication). Essentially, a system  $\mathcal{H}$  being homogeneous means that the two systems shown in Figure 3.31 are equivalent.

The additivity and homogeneity properties can be combined into a single property known as superposition. In particular, a system  $\mathcal{H}$  is said to have the **superposition** property, if for all functions  $x_1$  and  $x_2$  and all complex constants  $a_1$  and  $a_2$ , the following condition holds:

$$\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$$

(i.e.,  $\mathcal{H}$  commutes with linear combinations). A system that is both additive and homogeneous (or equivalently satisfies superposition) is said to be **linear**. Essentially, a system  $\mathcal{H}$  being linear means that the two systems shown in Figure 3.32 are equivalent. To show that a system is linear, we can show that it possesses both the additivity and homogeneity properties, or we can simply show that the superposition property holds. Practically speaking, linear systems are much easier to design and analyze than nonlinear systems.

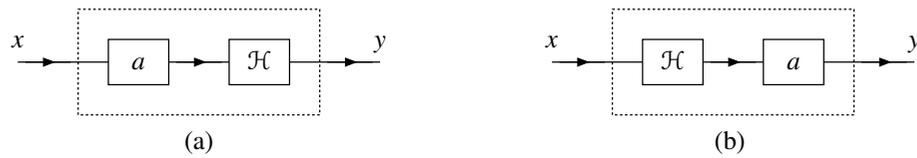


Figure 3.31: Systems that are equivalent if  $\mathcal{H}$  is homogeneous (i.e.,  $\mathcal{H}$  commutes with scalar multiplication). (a) A system that first performs scalar multiplication and then applies  $\mathcal{H}$  (i.e.,  $y = \mathcal{H}(ax)$ ); and (b) a system that first applies  $\mathcal{H}$  and then performs scalar multiplication (i.e.,  $y = a\mathcal{H}(x)$ ).

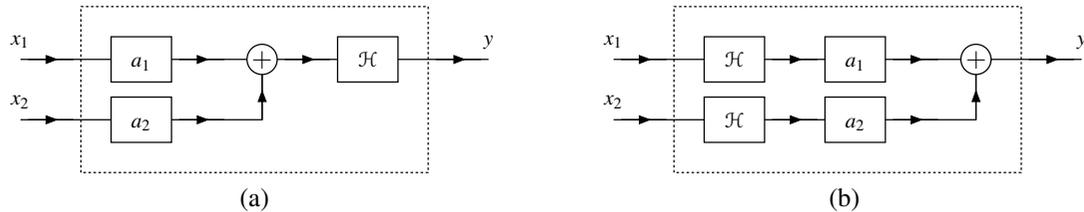


Figure 3.32: Systems that are equivalent if  $\mathcal{H}$  is linear (i.e.,  $\mathcal{H}$  commutes with linear combinations). (a) A system that first computes a linear combination and then applies  $\mathcal{H}$  (i.e.,  $y = \mathcal{H}(a_1x_1 + a_2x_2)$ ); and (b) a system that first applies  $\mathcal{H}$  and then computes a linear combination (i.e.,  $y = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$ ).

**Example 3.35.** Determine whether the system  $\mathcal{H}$  is linear, where

$$\mathcal{H}x(t) = tx(t).$$

*Solution.* Let  $x'(t) = a_1x_1(t) + a_2x_2(t)$ , where  $x_1$  and  $x_2$  are arbitrary functions and  $a_1$  and  $a_2$  are arbitrary complex constants. From the definition of  $\mathcal{H}$ , we can write

$$\begin{aligned} a_1\mathcal{H}x_1(t) + a_2\mathcal{H}x_2(t) &= a_1tx_1(t) + a_2tx_2(t) \quad \text{and} \\ \mathcal{H}x'(t) &= tx'(t) \\ &= t[a_1x_1(t) + a_2x_2(t)] \\ &= a_1tx_1(t) + a_2tx_2(t). \end{aligned}$$

Since  $\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$  for all  $x_1, x_2, a_1,$  and  $a_2$ , the superposition property holds and the system is linear. ■

**Example 3.36.** Determine whether the system  $\mathcal{H}$  is linear, where

$$\mathcal{H}x(t) = |x(t)|.$$

*Solution.* Let  $x'(t) = a_1x_1(t) + a_2x_2(t)$ , where  $x_1$  and  $x_2$  are arbitrary functions and  $a_1$  and  $a_2$  are arbitrary complex constants. From the definition of  $\mathcal{H}$ , we have

$$\begin{aligned} a_1\mathcal{H}x_1(t) + a_2\mathcal{H}x_2(t) &= a_1|x_1(t)| + a_2|x_2(t)| \quad \text{and} \\ \mathcal{H}x'(t) &= |x'(t)| \\ &= |a_1x_1(t) + a_2x_2(t)|. \end{aligned}$$

At this point, we recall the triangle inequality (i.e., for  $a, b \in \mathbb{C}$ ,  $|a + b| \leq |a| + |b|$ ). Thus,  $\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$  cannot hold for all  $x_1, x_2, a_1,$  and  $a_2$  due, in part, to the triangle inequality. For example, this condition fails to hold for

$$a_1 = -1, \quad x_1(t) = 1, \quad a_2 = 0, \quad \text{and} \quad x_2(t) = 0,$$

in which case

$$a_1\mathcal{H}x_1(t) + a_2\mathcal{H}x_2(t) = -1 \quad \text{and} \quad \mathcal{H}x'(t) = 1.$$

Therefore, the superposition property does not hold and the system is not linear. ■

**Example 3.37.** Determine whether the system  $\mathcal{H}$  is linear, where

$$\mathcal{H}x(t) = \text{Odd}(x)(t) = \frac{1}{2}[x(t) - x(-t)].$$

*Solution.* Let  $x'(t) = a_1x_1(t) + a_2x_2(t)$ , where  $x_1$  and  $x_2$  are arbitrary functions and  $a_1$  and  $a_2$  are arbitrary complex constants. From the definition of  $\mathcal{H}$ , we have

$$\begin{aligned} a_1\mathcal{H}x_1(t) + a_2\mathcal{H}x_2(t) &= \frac{1}{2}a_1[x_1(t) - x_1(-t)] + \frac{1}{2}a_2[x_2(t) - x_2(-t)] \quad \text{and} \\ \mathcal{H}x'(t) &= \frac{1}{2}[x'(t) - x'(-t)] \\ &= \frac{1}{2}[a_1x_1(t) + a_2x_2(t) - [a_1x_1(-t) + a_2x_2(-t)]] \\ &= \frac{1}{2}[a_1x_1(t) - a_1x_1(-t) + a_2x_2(t) - a_2x_2(-t)] \\ &= \frac{1}{2}a_1[x_1(t) - x_1(-t)] + \frac{1}{2}a_2[x_2(t) - x_2(-t)]. \end{aligned}$$

Since  $\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$  for all  $x_1, x_2, a_1$ , and  $a_2$ , the system is linear. ■

**Example 3.38.** Determine whether the system  $\mathcal{H}$  is linear, where

$$\mathcal{H}x(t) = 3x(3t + 3).$$

*Solution.* Let  $x'(t) = a_1x_1(t) + a_2x_2(t)$ , where  $x_1$  and  $x_2$  are arbitrary functions and  $a_1$  and  $a_2$  are arbitrary complex constants. From the definition of  $\mathcal{H}$ , we have

$$\begin{aligned} a_1\mathcal{H}x_1(t) + a_2\mathcal{H}x_2(t) &= 3a_1x_1(3t + 3) + 3a_2x_2(3t + 3) \quad \text{and} \\ \mathcal{H}x'(t) &= 3x'(3t + 3) \\ &= 3[a_1x_1(3t + 3) + a_2x_2(3t + 3)] \\ &= 3a_1x_1(3t + 3) + 3a_2x_2(3t + 3). \end{aligned}$$

Since  $\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$  for all  $x_1, x_2, a_1$ , and  $a_2$ , the system is linear. ■

**Example 3.39** (Ideal integrator). A system  $\mathcal{H}$  is defined by the equation

$$\mathcal{H}x(t) = \int_{-\infty}^t x(\tau)d\tau.$$

Determine whether this system is additive and/or homogeneous. Determine whether this system is linear.

*Solution.* First, we consider the additivity property. From the definition of  $\mathcal{H}$ , we have

$$\begin{aligned} \mathcal{H}x_1(t) + \mathcal{H}x_2(t) &= \int_{-\infty}^t x_1(\tau)d\tau + \int_{-\infty}^t x_2(\tau)d\tau \quad \text{and} \\ \mathcal{H}(x_1 + x_2)(t) &= \int_{-\infty}^t (x_1 + x_2)(\tau)d\tau \\ &= \int_{-\infty}^t [x_1(\tau) + x_2(\tau)]d\tau \\ &= \int_{-\infty}^t x_1(\tau)d\tau + \int_{-\infty}^t x_2(\tau)d\tau. \end{aligned}$$

Since  $\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$  for all  $x_1$  and  $x_2$ , the system is additive.

Second, we consider the homogeneity property. Let  $a$  denote an arbitrary complex constant. From the definition of  $\mathcal{H}$ , we can write

$$\begin{aligned} a\mathcal{H}x(t) &= a \int_{-\infty}^t x(\tau) d\tau \quad \text{and} \\ \mathcal{H}(ax)(t) &= \int_{-\infty}^t (ax)(\tau) d\tau \\ &= \int_{-\infty}^t ax(\tau) d\tau \\ &= a \int_{-\infty}^t x(\tau) d\tau. \end{aligned}$$

Since  $\mathcal{H}(ax) = a\mathcal{H}x$  for all  $x$  and  $a$ , the system is homogeneous.

Lastly, we consider the linearity property. The system is linear since it has both the additivity and homogeneity properties. ■

**Example 3.40.** A system  $\mathcal{H}$  is given by

$$\mathcal{H}x(t) = \operatorname{Re}[x(t)].$$

Determine whether this system is additive and/or homogeneous. Determine whether this system is linear.

*Solution.* First, we check if the additivity property is satisfied. From the definition of  $\mathcal{H}$ , we have

$$\begin{aligned} \mathcal{H}x_1(t) + \mathcal{H}x_2(t) &= \operatorname{Re}[x_1(t)] + \operatorname{Re}[x_2(t)] \quad \text{and} \\ \mathcal{H}(x_1 + x_2)(t) &= \operatorname{Re}[(x_1 + x_2)(t)] \\ &= \operatorname{Re}[x_1(t) + x_2(t)] \\ &= \operatorname{Re}[x_1(t)] + \operatorname{Re}[x_2(t)]. \end{aligned}$$

Since  $\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$  for all  $x_1$  and  $x_2$ , the system is additive.

Second, we check if the homogeneity property is satisfied. Let  $a$  denote an arbitrary complex constant. From the definition of  $\mathcal{H}$ , we have

$$\begin{aligned} a\mathcal{H}x(t) &= a\operatorname{Re}[x(t)] \quad \text{and} \\ \mathcal{H}(ax)(t) &= \operatorname{Re}[(ax)(t)] \\ &= \operatorname{Re}[ax(t)]. \end{aligned}$$

In order for  $\mathcal{H}$  to be homogeneous,  $a\mathcal{H}x(t) = \mathcal{H}(ax)(t)$  must hold for all  $x$  and all complex  $a$ . Suppose that  $a = j$  and  $x$  is not identically zero (i.e.,  $x$  is not the function  $x(t) = 0$ ). In this case, we have

$$\begin{aligned} a\mathcal{H}x(t) &= j\operatorname{Re}[x(t)] \quad \text{and} \\ \mathcal{H}(ax)(t) &= \operatorname{Re}[(jx)(t)] \\ &= \operatorname{Re}[jx(t)] \\ &= \operatorname{Re}[j(\operatorname{Re}[x(t)] + j\operatorname{Im}[x(t)])] \\ &= \operatorname{Re}[-\operatorname{Im}[x(t)] + j\operatorname{Re}[x(t)]] \\ &= -\operatorname{Im}[x(t)]. \end{aligned}$$

Thus, the quantities  $\mathcal{H}(ax)$  and  $a\mathcal{H}x$  are clearly not equal. Therefore, the system is not homogeneous.

Lastly, we consider the linearity property. Since the system does not possess both the additivity and homogeneity properties, it is not linear. ■

### 3.7.7 Eigenfunctions

An **eigenfunction** of a system  $\mathcal{H}$  is a function  $x$  that satisfies

$$\mathcal{H}x = \lambda x,$$

for some complex constant  $\lambda$ , which is called an **eigenvalue**. Essentially, a system behaves as an ideal amplifier (i.e., performs amplitude scaling) when presented with one of its eigenfunctions as input. The significance of the eigenfunction property cannot be overstated. No matter how complicated a system might be, it exhibits extremely simple behavior for its eigenfunctions. We can often exploit this simplicity to reduce the complexity of solving many types of problems involving systems. In fact, as we will see later, eigenfunctions essentially form the basis for many of the mathematical tools that we use for studying systems.

**Example 3.41.** Consider the system  $\mathcal{H}$  characterized by the equation

$$\mathcal{H}x(t) = \mathcal{D}^2x(t),$$

where  $\mathcal{D}$  denotes the derivative operator. For each function  $x$  given below, determine if  $x$  is an eigenfunction of  $\mathcal{H}$ , and if it is, find the corresponding eigenvalue.

(a)  $x(t) = \cos 2t$ ; and

(b)  $x(t) = t^3$ .

*Solution.* (a) We have

$$\begin{aligned} \mathcal{H}x(t) &= \mathcal{D}^2\{\cos 2t\}(t) \\ &= \mathcal{D}\{-2\sin 2t\}(t) \\ &= -4\cos 2t \\ &= -4x(t). \end{aligned}$$

Therefore,  $x$  is an eigenfunction of  $\mathcal{H}$  with the eigenvalue  $-4$ .

(b) We have

$$\begin{aligned} \mathcal{H}x(t) &= \mathcal{D}^2\{t^3\}(t) \\ &= \mathcal{D}\{3t^2\}(t) \\ &= 6t \\ &= \frac{6t}{\cos 2t}x(t). \end{aligned}$$

Therefore,  $x$  is not an eigenfunction of  $\mathcal{H}$ . ■

## 3.8 Exercises

### 3.8.1 Exercises Without Answer Key

3.1 Identify the independent- and dependent-variable transformations that must be applied to the function  $x$  in order to obtain each function  $y$  given below. Choose the transformations such that time shifting precedes time scaling and amplitude scaling precedes amplitude shifting. Be sure to clearly indicate the order in which the transformations are to be applied.

(a)  $y(t) = x(2t - 1)$ ;

(b)  $y(t) = x(\frac{1}{2}t + 1)$ ;

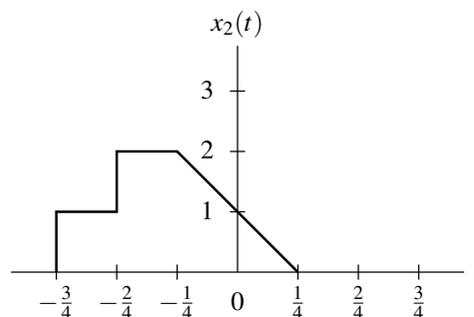
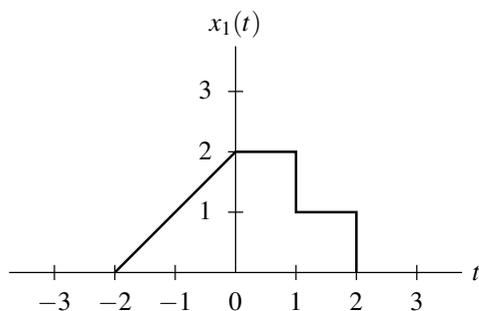
(c)  $y(t) = 2x(-\frac{1}{2}t + 1) + 3$ ;

(d)  $y(t) = -\frac{1}{2}x(-t + 1) - 1$ ; and

(e)  $y(t) = -3x(2[t - 1]) - 1$ ;

(f)  $y(t) = x(7[t + 3])$ .

3.2 Given the functions  $x_1$  and  $x_2$  shown in the figures below, express  $x_2$  in terms of  $x_1$ .



3.3 Suppose that we have two functions  $x$  and  $y$  related as

$$y(t) = x(at - b),$$

where  $a$  and  $b$  are real constants and  $a \neq 0$ .

(a) Show that  $y$  can be formed by first time shifting  $x$  by  $b$  and then time scaling the result by  $a$ .

(b) Show that  $y$  can also be formed by first time scaling  $x$  by  $a$  and then time shifting the result by  $\frac{b}{a}$ .

3.4 Given the function  $x$  shown in the figure below, plot and label each of the following functions:

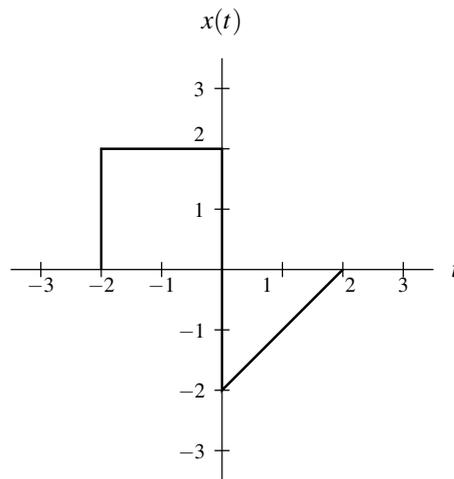
(a)  $x(t - 1)$ ;

(b)  $x(2t)$ ;

(c)  $x(-t)$ ;

(d)  $x(2t + 1)$ ; and

(e)  $\frac{1}{4}x(-\frac{1}{2}t + 1) - \frac{1}{2}$ .



**3.5** Determine if each function  $x$  given below is periodic, and if it is, find its fundamental period.

- (a)  $x(t) = \cos 2\pi t + \sin 5t$ ;
- (b)  $x(t) = [\cos(4t - \frac{\pi}{3})]^2$ ;
- (c)  $x(t) = e^{j2\pi t} + e^{j3\pi t}$ ;
- (d)  $x(t) = 1 + \cos 2t + e^{j5t}$ ;
- (e)  $x(t) = \cos(14t - 1) + \cos(77t - 3)$ ;
- (f)  $x(t) = \cos et + \sin 42t$ ; and
- (g)  $x(t) = |\sin \pi t|$ .

**3.6** If the function  $x$  is  $T$ -periodic, show that the function  $y$  is also  $T$ -periodic in the case that:

- (a)  $y(t) = cx(t)$ , where  $c$  is a complex constant;
- (b)  $y(t) = x(t) + c$ , where  $c$  is a complex constant; and
- (c)  $y(t) = x(t - c)$ , where  $c$  is a real constant.

**3.7** Determine whether each function  $x$  given below is even, odd, or neither even nor odd.

- (a)  $x(t) = t^3$ ;
- (b)  $x(t) = t^3 |t|$ ;
- (c)  $x(t) = |t^3|$ ;
- (d)  $x(t) = (\cos 2\pi t)(\sin 2\pi t)$ ;
- (e)  $x(t) = e^{j2\pi t}$ ; and
- (f)  $x(t) = \frac{1}{2}[e^t + e^{-t}]$ .

**3.8** Prove each of the following assertions:

- (a) The sum of two even functions is even.
- (b) The sum of two odd functions is odd.
- (c) The sum of an even function and an odd function, where neither function is identically zero, is neither even nor odd.
- (d) The product of two even functions is even.
- (e) The product of two odd functions is even.
- (f) The product of an even function and an odd function is odd.

**3.9** Show that, if  $x$  is an odd function, then

$$\int_{-A}^A x(t) dt = 0,$$

where  $A$  is a positive real constant.

**3.10** Show that, for any function  $x$ ,

$$\int_{-\infty}^{\infty} x^2(t) dt = \int_{-\infty}^{\infty} x_e^2(t) dt + \int_{-\infty}^{\infty} x_o^2(t) dt,$$

where  $x_e$  and  $x_o$  denote the even and odd parts of  $x$ , respectively.

**3.11** Let  $x$  denote a function with derivative  $y$ .

- (a) Show that if  $x$  is even then  $y$  is odd.  
 (b) Show that if  $x$  is odd then  $y$  is even.

**3.12** For an arbitrary function  $x$  with even and odd parts  $x_e$  and  $x_o$ , respectively, show that:

- (a)  $\int_{-\infty}^{\infty} x_e(t)x_o(t) dt = 0$ ; and  
 (b)  $\int_{-\infty}^{\infty} x(t) dt = \int_{-\infty}^{\infty} x_e(t) dt$ .

**3.13** Show that:

- (a) if a function  $x$  is  $T$ -periodic and even, then  $\int_0^T x(t) dt = 2 \int_0^{T/2} x(t) dt$ .  
 (b) if a function  $x$  is  $T$ -periodic and odd, then  $\int_0^T x(t) dt = 0$ .

**3.14** Show that the only function that is both even and odd is the zero function (i.e., the function  $x$  satisfying  $x(t) = 0$  for all  $t$ ).

**3.15** Suppose that the function  $h$  is causal and has the even part  $h_e$  given by

$$h_e(t) = t[u(t) - u(t-1)] + u(t-1) \quad \text{for } t \geq 0.$$

Find  $h$ .

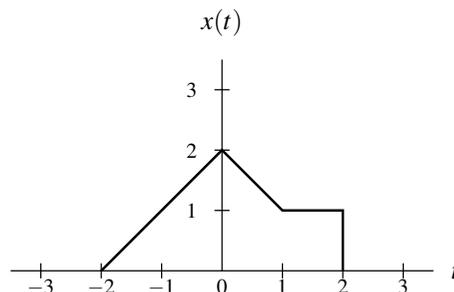
**3.16** Let  $Ex$  denote the energy of the function  $x$ . Show that:

- (a)  $E\{ax\} = a^2Ex$ , where  $a$  is a real constant;  
 (b)  $Ex' = Ex$ , where  $x'(t) = x(t-t_0)$  and  $t_0$  is a real constant; and  
 (c)  $Ex' = \frac{1}{|a|}Ex$ , where  $x'(t) = x(at)$  and  $a$  is a nonzero real constant.

**3.17** Fully simplify each of the expressions below.

- (a)  $\int_{-\infty}^{\infty} \sin\left(2t + \frac{\pi}{4}\right) \delta(t) dt$ ;  
 (b)  $\int_{-\infty}^t [\cos \tau] \delta(\tau + \pi) d\tau$ ;  
 (c)  $\int_{-\infty}^{\infty} x(t) \delta(at - b) dt$ , where  $a$  and  $b$  are real constants and  $a \neq 0$ ;  
 (d)  $\int_0^2 e^{j2t} \delta(t-1) dt$ ;  
 (e)  $\int_{-\infty}^t \delta(\tau) d\tau$ ; and  
 (f)  $\int_0^{\infty} (\tau^2 \cos \tau) \delta(\tau + 42) d\tau$ .

**3.18** Suppose that we have the function  $x$  shown in the figure below. Use unit-step functions to find a single expression for  $x(t)$  that is valid for all  $t$ .



**3.19** For each function  $x$  given below, find a single expression for  $x$  (i.e., an expression that does not involve multiple cases). Group similar unit-step function terms together in the expression for  $x$ .

$$(a) x(t) = \begin{cases} -t-3 & -3 \leq t < -2 \\ -1 & -2 \leq t < -1 \\ t^3 & -1 \leq t < 1 \\ 1 & 1 \leq t < 2 \\ -t+3 & 2 \leq t < 3 \\ 0 & \text{otherwise;} \end{cases}$$

$$(b) x(t) = \begin{cases} -1 & t < -1 \\ t & -1 \leq t < 1 \\ 1 & t \geq 1; \end{cases}$$

$$(c) x(t) = \begin{cases} 4t+4 & -1 \leq t < -\frac{1}{2} \\ 4t^2 & -\frac{1}{2} \leq t < \frac{1}{2} \\ -4t+4 & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise;} \end{cases}$$

**3.20** Determine whether each system  $\mathcal{H}$  given below is memoryless.

- (a)  $\mathcal{H}x(t) = \int_{-\infty}^{2t} x(\tau) d\tau$ ;
- (b)  $\mathcal{H}x(t) = \text{Even}(x)(t)$ ;
- (c)  $\mathcal{H}x(t) = x(t-1) + 1$ ;
- (d)  $\mathcal{H}x(t) = \int_t^{\infty} x(\tau) d\tau$ ;
- (e)  $\mathcal{H}x(t) = \int_{-\infty}^t x(\tau) \delta(\tau) d\tau$ ;
- (f)  $\mathcal{H}x(t) = tx(t)$ ; and
- (g)  $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) d\tau$ .

**3.21** Determine whether each system  $\mathcal{H}$  given below is causal.

- (a)  $\mathcal{H}x(t) = \int_{-\infty}^{2t} x(\tau) d\tau$ ;
- (b)  $\mathcal{H}x(t) = \text{Even}(x)(t)$ ;
- (c)  $\mathcal{H}x(t) = x(t-1) + 1$ ;
- (d)  $\mathcal{H}x(t) = \int_t^{\infty} x(\tau) d\tau$ ;
- (e)  $\mathcal{H}x(t) = \int_{-\infty}^t x(\tau) \delta(\tau) d\tau$ ; and
- (f)  $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau) u(t-\tau) d\tau$ .

**3.22** For each system  $\mathcal{H}$  given below, determine if  $\mathcal{H}$  is invertible, and if it is, specify its inverse.

- (a)  $\mathcal{H}x(t) = x(at-b)$  where  $a$  and  $b$  are real constants and  $a \neq 0$ ;
- (b)  $\mathcal{H}x(t) = e^{x(t)}$ , where  $x$  is a real function;
- (c)  $\mathcal{H}x(t) = \text{Even}(x)(t) - \text{Odd}(x)(t)$ ;
- (d)  $\mathcal{H}x(t) = \mathcal{D}x(t)$ , where  $\mathcal{D}$  denotes the derivative operator; and
- (e)  $\mathcal{H}x(t) = x^2(t)$ .

**3.23** Determine whether each system  $\mathcal{H}$  given below is BIBO stable.

- (a)  $\mathcal{H}x(t) = \int_t^{t+1} x(\tau) d\tau$  [Hint: For any function  $f$ ,  $|\int_a^b f(x) dx| \leq \int_a^b |f(x)| dx$ .];
- (b)  $\mathcal{H}x(t) = \frac{1}{2}x^2(t) + x(t)$ ;
- (c)  $\mathcal{H}x(t) = 1/x(t)$ ;
- (d)  $\mathcal{H}x(t) = e^{-|t|}x(t)$ ; and
- (e)  $\mathcal{H}x(t) = \left(\frac{1}{t-1}\right)x(t)$ .

**3.24** Determine whether each system  $\mathcal{H}$  given below is time invariant.

- (a)  $\mathcal{H}x(t) = \mathcal{D}x(t)$ ; where  $\mathcal{D}$  denotes the derivative operator;
- (b)  $\mathcal{H}x(t) = \text{Even}(x)(t)$ ;
- (c)  $\mathcal{H}x(t) = \int_t^{t+1} x(\tau - \alpha) d\tau$ , where  $\alpha$  is a constant;
- (d)  $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$ , where  $h$  is an arbitrary (but fixed) function;
- (e)  $\mathcal{H}x(t) = x(-t)$ ; and
- (f)  $\mathcal{H}x(t) = \int_{-\infty}^{2t} x(\tau)d\tau$ .

**3.25** Determine whether each system  $\mathcal{H}$  given below is linear.

- (a)  $\mathcal{H}x(t) = \int_{t-1}^{t+1} x(\tau)d\tau$ ;
- (b)  $\mathcal{H}x(t) = e^{x(t)}$ ;
- (c)  $\mathcal{H}x(t) = \text{Even}(x)(t)$ ;
- (d)  $\mathcal{H}x(t) = x^2(t)$ ; and
- (e)  $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$ , where  $h$  is an arbitrary (but fixed) function.

**3.26** Determine whether each system  $\mathcal{H}$  given below is additive and/or homogeneous.

- (a)  $\mathcal{H}x(t) = x^*(t)$ ; and
- (b)  $\mathcal{H}x(t) = \text{Im}[x(t)]$ .

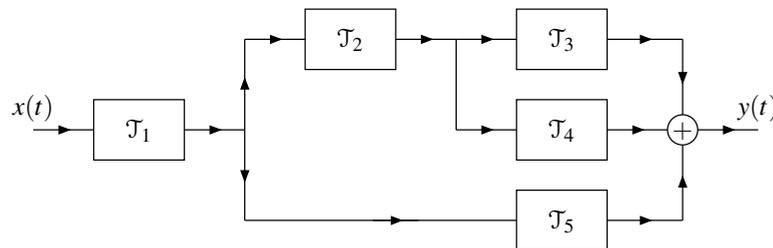
**3.27** Show that if a system  $\mathcal{H}$  is either additive or homogeneous, it has the property that, for a function  $x$  that is identically zero (i.e.,  $x(t) = 0$  for all  $t$ ),  $\mathcal{H}x$  is identically zero (i.e.,  $\mathcal{H}x(t) = 0$  for all  $t$ ).

**3.28** Given the function

$$x(t) = u(t+2) + u(t+1) + u(t) - 2u(t-1) - u(t-2),$$

find and sketch  $y(t) = x(-4t - 1)$ .

**3.29** For the system shown in the figure below, express the output  $y$  in terms of the input  $x$  and the transformations  $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_5$ .



**3.30** Let  $z$  denote a function that is identically zero (i.e.,  $z(t) = 0$  for all  $t$ ). Show that a linear system  $\mathcal{H}$  is invertible if and only if  $\mathcal{H}x = z$  implies that  $x = z$  (i.e., the only input  $x$  that produces an output  $y = z$  is  $x = z$ ).

**3.31** For each system  $\mathcal{H}$  and the functions  $\{x_k\}$  given below, determine if each of the  $x_k$  is an eigenfunction of  $\mathcal{H}$ , and if it is, also state the corresponding eigenvalue.

- (a)  $\mathcal{H}x(t) = x^2(t)$ ,  $x_1(t) = a$ ,  $x_2(t) = e^{-at}$ , and  $x_3(t) = \cos t$ , where  $a$  is a complex constant;
- (b)  $\mathcal{H}x(t) = \mathcal{D}x(t)$ ,  $x_1(t) = e^{at}$ ,  $x_2(t) = e^{at^2}$ , and  $x_3(t) = 42$ , where  $\mathcal{D}$  denotes the derivative operator and  $a$  is a real constant;
- (c)  $\mathcal{H}x(t) = \int_t^t x(\tau)d\tau$ ,  $x_1(t) = e^{at}$ ,  $x_2(t) = t$ , and  $x_3(t) = \sin t$ , where  $a$  is a nonzero complex constant; and
- (d)  $\mathcal{H}x(t) = |x(t)|$ ,  $x_1(t) = a$ ,  $x_2(t) = t$ ,  $x_3(t) = t^2$ , where  $a$  is a strictly positive real constant.

### 3.8.2 Exercises With Answer Key

3.32 For each function  $x$  given below, determine whether  $x$  is periodic, and if it is, find its fundamental period  $T$ .

(a)  $x(t) = 3 \cos(\sqrt{2}t) + 7 \cos 2t$ ;

(b)  $x(t) = (3 \cos 2t)^3$ ; and

(c)  $x(t) = 7 \cos(35t + 3) + 5 \sin(15t - 2)$ .

**Short Answer.** (a) not periodic; (b)  $\pi$ -periodic; (c)  $(2\pi/5)$ -periodic

3.33 Simplify each of the following expressions:

(a)  $\frac{(\omega^2+1)\delta(\omega-1)}{\omega^2+9}$ ;

(b)  $\frac{(\sin k\omega)\delta(\omega)}{\omega}$ ;

(c)  $\int_{-\infty}^{\infty} e^{t-1} \cos\left[\frac{\pi}{2}(t-5)\right] \delta(t-3) dt$ ;

(d)  $\int_{-\infty}^{\infty} \delta(2t-3) \sin \pi t dt$ ; and

(e)  $\int_t^{\infty} (\tau^2 + 1) \delta(\tau - 2) d\tau$ .

**Short Answer.** (a)  $\frac{1}{5} \delta(\omega - 1)$ ; (b)  $k \delta(\omega)$ ; (c)  $-e^2$ ; (d)  $-\frac{1}{2}$ ; (e)  $5u(2-t)$ .

3.34 Determine whether each system  $\mathcal{H}$  given below is memoryless.

(a)  $\mathcal{H}x(t) = u[x(t)]$ ;

(b)  $\mathcal{H}x(t) = x[u(t)]$ ;

(c)  $\mathcal{H}x(t) = 42$ ;

(d)  $\mathcal{H}x(t) = x(t^2)$ ;

(e)  $\mathcal{H}x(t) = \int_{-\infty}^{\infty} 2x(\tau) \delta(\tau - t) d\tau$ ;

(f)  $\mathcal{H}x(t) = [x(t+1)]^{-1}$ ; and

(g)  $\mathcal{H}x(t) = x(-t)$ .

**Short Answer.** (a) memoryless; (b) has memory; (c) memoryless; (d) has memory; (e) memoryless; (f) has memory; (g) has memory

3.35 Determine whether each system  $\mathcal{H}$  given below is causal.

(a)  $\mathcal{H}x(t) = x(at)$ , where  $a$  is a nonzero real constant;

(b)  $\mathcal{H}x(t) = tu(t)x(t)$ ;

(c)  $\mathcal{H}x(t) = x(t-a)$ , where  $a$  is a strictly negative real constant;

(d)  $\mathcal{H}x(t) = [x(t+1)]^{-1}$ ; and

(e)  $\mathcal{H}x(t) = x(-t)$ .

**Short Answer.** (a) causal if and only if  $a = 1$ ; (b) causal; (c) not causal; (d) not causal; (e) not causal

3.36 Determine whether each system  $\mathcal{H}$  given below is invertible.

(a)  $\mathcal{H}x(t) = \cos x(t)$ ;

(b)  $\mathcal{H}x(t) = x * x(t)$ , where  $f * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau) d\tau$ ; and

(c)  $\mathcal{H}x(t) = \text{Even}x(t)$ .

**Short Answer.** (a) not invertible; (b) not invertible; (c) not invertible

**3.37** Determine whether each system  $\mathcal{H}$  given below is BIBO stable.

- (a)  $\mathcal{H}x(t) = u(t)x(t)$ ;
- (b)  $\mathcal{H}x(t) = \ln x(t)$ ;
- (c)  $\mathcal{H}x(t) = e^{x(t)}$ ;
- (d)  $\mathcal{H}x(t) = e^t x(t)$ ;
- (e)  $\mathcal{H}x(t) = \cos x(t)$ ; and
- (f)  $\mathcal{H}x(t) = x * x(t)$ , where  $f * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau)d\tau$ .

**Short Answer.** (a) stable; (b) not stable; (c) stable; (d) not stable; (e) stable; (f) not stable

**3.38** Determine whether each system  $\mathcal{H}$  given below is time invariant.

- (a)  $\mathcal{H}x(t) = \int_{-4}^4 x(\tau)d\tau$ ;
- (b)  $\mathcal{H}x(t) = \mathcal{D}\{x^2\}(t)$ , where  $\mathcal{D}$  denotes the derivative operator;
- (c)  $\mathcal{H}x(t) = tu(t)x(t)$ ;
- (d)  $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)x(t - \tau)d\tau$ ; and
- (e)  $\mathcal{H}x(t) = t^2\mathcal{D}^2x(t) + t\mathcal{D}x(t)$ , where  $\mathcal{D}$  denotes the derivative operator.

**Short Answer.** (a) not time invariant; (b) time invariant; (c) not time invariant; (d) not time invariant; (e) not time invariant

**3.39** Determine whether each system  $\mathcal{H}$  given below is linear.

- (a)  $\mathcal{H}x(t) = \frac{1}{3}[x(t) - 2]$ ;
- (b)  $\mathcal{H}x(t) = \mathcal{D}x(t)$ , where  $\mathcal{D}$  denotes the derivative operator;
- (c)  $\mathcal{H}x(t) = tu(t)x(t)$ ;
- (d)  $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)x(t - \tau)d\tau$ ; and
- (e)  $\mathcal{H}x(t) = t^2\mathcal{D}^2x(t) + t\mathcal{D}x(t)$ , where  $\mathcal{D}$  denotes the derivative operator.

**Short Answer.** (a) not linear; (b) linear; (c) linear; (d) not linear; (e) linear

**3.40** Determine whether each system  $\mathcal{H}$  given below is additive and/or homogeneous.

- (a)  $\mathcal{H}x(t) = \frac{x^2(t)}{\mathcal{D}x(t)}$ , where  $\mathcal{D}$  denotes the derivative operator.

**Short Answer.** (a) homogeneous but not additive

**3.41** For each system  $\mathcal{H}$  and the functions  $\{x_k\}$  given below, determine if each of the  $x_k$  is an eigenfunction of  $\mathcal{H}$ , and if it is, also state the corresponding eigenvalue.

- (a)  $\mathcal{H}x(t) = \mathcal{D}^2x(t)$ ,  $x_1(t) = \cos t$ ,  $x_2(t) = \sin t$ , and  $x_3(t) = 42$ , where  $\mathcal{D}$  denotes the derivative operator;
- (b)  $\mathcal{H}x(t) = \int_{-\infty}^t x(\tau)d\tau$ ,  $x_1(t) = e^{2t}$ , and  $x_2(t) = e^t u(-t)$ ; and
- (c)  $\mathcal{H}x(t) = t^2\mathcal{D}^2x(t) + t\mathcal{D}x(t)$  and  $x_1(t) = t^k$ , where  $k$  is an integer constant such that  $k \geq 2$ , and  $\mathcal{D}$  denotes the derivative operator.

**Short Answer.** (a)  $x_1$  is an eigenfunction with eigenvalue  $-1$ ,  $x_2$  is an eigenfunction with eigenvalue  $-1$ ,  $x_3$  is an eigenfunction with eigenvalue  $0$ ; (b)  $x_1$  is an eigenfunction with eigenvalue  $\frac{1}{2}$ ,  $x_2$  is not an eigenfunction; (c)  $x_1$  is an eigenfunction with eigenvalue  $k^2$ .



## Chapter 4

# Continuous-Time Linear Time-Invariant Systems

### 4.1 Introduction

In the previous chapter, we identified a number of properties that a system may possess. Two of these properties were linearity and time invariance. In this chapter, we focus our attention exclusively on systems with both of these properties. Such systems are referred to as **linear time-invariant (LTI)** systems.

### 4.2 Continuous-Time Convolution

In the context of LTI systems, a mathematical operation known as convolution turns out to be particularly important. The **convolution** of the functions  $x$  and  $h$ , denoted  $x * h$ , is defined as the function

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau. \quad (4.1)$$

Herein, the asterisk (or star) symbol (i.e., “\*”) will be used to denote convolution, not multiplication. It is important to make a distinction between convolution and multiplication, since these two operations are quite different and do not generally yield the same result.

Notationally,  $x * h$  denotes a function, namely the function that results from convolving  $x$  and  $h$ . In contrast,  $x * h(t)$  denotes the function  $x * h$  evaluated at  $t$ . Although we could equivalently write  $x * h(t)$  with an extra pair of brackets as  $(x * h)(t)$ , we usually omit this extra pair of brackets, since doing so does not introduce any ambiguity and leads to more compact notation. That is, there is only one sensible way to group operations in the expression  $x * h(t)$ . The grouping  $x * [h(t)]$  would not make sense since a convolution requires two functions as operands and  $h(t)$  is not a function, but rather the value of  $h$  evaluated at  $t$ . Thus, the only sensible way to interpret the expression  $x * h(t)$  is as  $(x * h)(t)$ .

Since the convolution operation is used extensively in system theory, we need some practical means for evaluating a convolution integral. Suppose that, for the given functions  $x$  and  $h$ , we wish to compute

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau.$$

Of course, we could naively attempt to compute  $x * h$  by evaluating  $x * h(t)$  as a separate integral for each possible value of  $t$ . This approach, however, is not feasible, as  $t$  can assume an infinite number of values, and therefore, an infinite number of integrals would need to be evaluated. Instead, we consider a slightly different approach. Let us redefine the integrand in terms of the intermediate function  $w_t(\tau)$  where

$$w_t(\tau) = x(\tau)h(t - \tau).$$

(Note that  $w_t(\tau)$  is implicitly a function of  $t$ .) This means that we need to compute

$$x * h(t) = \int_{-\infty}^{\infty} w_t(\tau) d\tau.$$

Now, we observe that, for most functions  $x$  and  $h$  of practical interest, the form of  $w_t(\tau)$  typically remains fixed over particular ranges of  $t$ . Thus, we can compute the convolution result  $x * h$  by first identifying each of the distinct expressions for  $w_t(\tau)$  and the range over which each expression is valid. Then, for each range, we evaluate an integral. In this way, we typically only need to compute a small number of integrals instead of the infinite number required with the naive approach suggested above.

The above discussion leads us to propose the following general approach for computing a convolution:

1. Plot  $x(\tau)$  and  $h(t - \tau)$  as a function of  $\tau$ .
2. Initially, consider an arbitrarily large negative value for  $t$ . This will result in  $h(t - \tau)$  being shifted very far to the left on the time axis.
3. Write the mathematical expression for  $w_t(\tau)$ .
4. Increase  $t$  gradually until the expression for  $w_t(\tau)$  changes form. Record the interval over which the expression for  $w_t(\tau)$  was valid.
5. Repeat steps 3 and 4 until  $t$  is an arbitrarily large positive value. This corresponds to  $h(t - \tau)$  being shifted very far to the right on the time axis.
6. For each of the intervals identified above, integrate  $w_t(\tau)$  in order to find an expression for  $x * h(t)$ . This will yield an expression for  $x * h(t)$  for each interval.
7. The results for the various intervals can be combined in order to obtain the convolution result  $x * h(t)$  for all  $t$ .

**Example 4.1.** Compute the convolution  $x * h$  where

$$x(t) = \begin{cases} -1 & -1 \leq t < 0 \\ 1 & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h(t) = e^{-t}u(t).$$

*Solution.* We begin by plotting the functions  $x$  and  $h$  as shown in Figures 4.1(a) and (b), respectively. Next, we proceed to determine the time-reversed and time-shifted version of  $h$ . We can accomplish this in two steps. First, we time-reverse  $h(\tau)$  to obtain  $h(-\tau)$  as shown in Figure 4.1(c). Second, we time-shift the resulting function by  $t$  to obtain  $h(t - \tau)$  as shown in Figure 4.1(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of  $t$ , we must multiply  $x(\tau)$  by  $h(t - \tau)$  and integrate the resulting product with respect to  $\tau$ . Due to the form of  $x$  and  $h$ , we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 4.1(e) to (h).

First, we consider the case of  $t < -1$ . From Figure 4.1(e), we can see that

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau = 0. \quad (4.2)$$

Second, we consider the case of  $-1 \leq t < 0$ . From Figure 4.1(f), we can see that

$$\begin{aligned} x * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau = \int_{-1}^t -e^{\tau-t} d\tau \\ &= -e^{-t} \int_{-1}^t e^{\tau} d\tau \\ &= -e^{-t} [e^{\tau}]_{-1}^t \\ &= -e^{-t} [e^t - e^{-1}] \\ &= e^{-t-1} - 1. \end{aligned} \quad (4.3)$$

Third, we consider the case of  $0 \leq t < 1$ . From Figure 4.1(g), we can see that

$$\begin{aligned}
 x * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-1}^0 -e^{\tau-t}d\tau + \int_0^t e^{\tau-t}d\tau \\
 &= -e^{-t} \int_{-1}^0 e^{\tau}d\tau + e^{-t} \int_0^t e^{\tau}d\tau \\
 &= -e^{-t}[e^{\tau}]_{-1}^0 + e^{-t}[e^{\tau}]_0^t \\
 &= -e^{-t}[1 - e^{-1}] + e^{-t}[e^t - 1] \\
 &= e^{-t}[e^{-1} - 1 + e^t - 1] \\
 &= 1 + (e^{-1} - 2)e^{-t}.
 \end{aligned} \tag{4.4}$$

Fourth, we consider the case of  $t \geq 1$ . From Figure 4.1(h), we can see that

$$\begin{aligned}
 x * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-1}^0 -e^{\tau-t}d\tau + \int_0^1 e^{\tau-t}d\tau \\
 &= -e^{-t} \int_{-1}^0 e^{\tau}d\tau + e^{-t} \int_0^1 e^{\tau}d\tau \\
 &= -e^{-t}[e^{\tau}]_{-1}^0 + e^{-t}[e^{\tau}]_0^1 \\
 &= e^{-t}[e^{-1} - 1 + e - 1] \\
 &= (e - 2 + e^{-1})e^{-t}.
 \end{aligned} \tag{4.5}$$

Combining the results of (4.2), (4.3), (4.4), and (4.5), we have that

$$x * h(t) = \begin{cases} 0 & t < -1 \\ e^{-t-1} - 1 & -1 \leq t < 0 \\ (e^{-1} - 2)e^{-t} + 1 & 0 \leq t < 1 \\ (e - 2 + e^{-1})e^{-t} & 1 \leq t. \end{cases}$$

The convolution result  $x * h$  is plotted in Figure 4.1(i). ■

**Example 4.2.** Compute the convolution  $x * h$ , where

$$x(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h(t) = \begin{cases} t & 0 \leq t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

*Solution.* We begin by plotting the functions  $x$  and  $h$  as shown in Figures 4.2(a) and (b), respectively. Next, we proceed to determine the time-reversed and time-shifted version of  $h(\tau)$ . We can accomplish this in two steps. First, we time-reverse  $h(\tau)$  to obtain  $h(-\tau)$  as shown in Figure 4.2(c). Second, we time-shift the resulting function by  $t$  to obtain  $h(t - \tau)$  as shown in Figure 4.2(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of  $t$ , we must multiply  $x(\tau)$  by  $h(t - \tau)$  and integrate the resulting product with respect to  $\tau$ . Due to the form of  $x$  and  $h$ , we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 4.2(e) to (h).

First, we consider the case of  $t < 0$ . From Figure 4.2(e), we can see that

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{4.6}$$

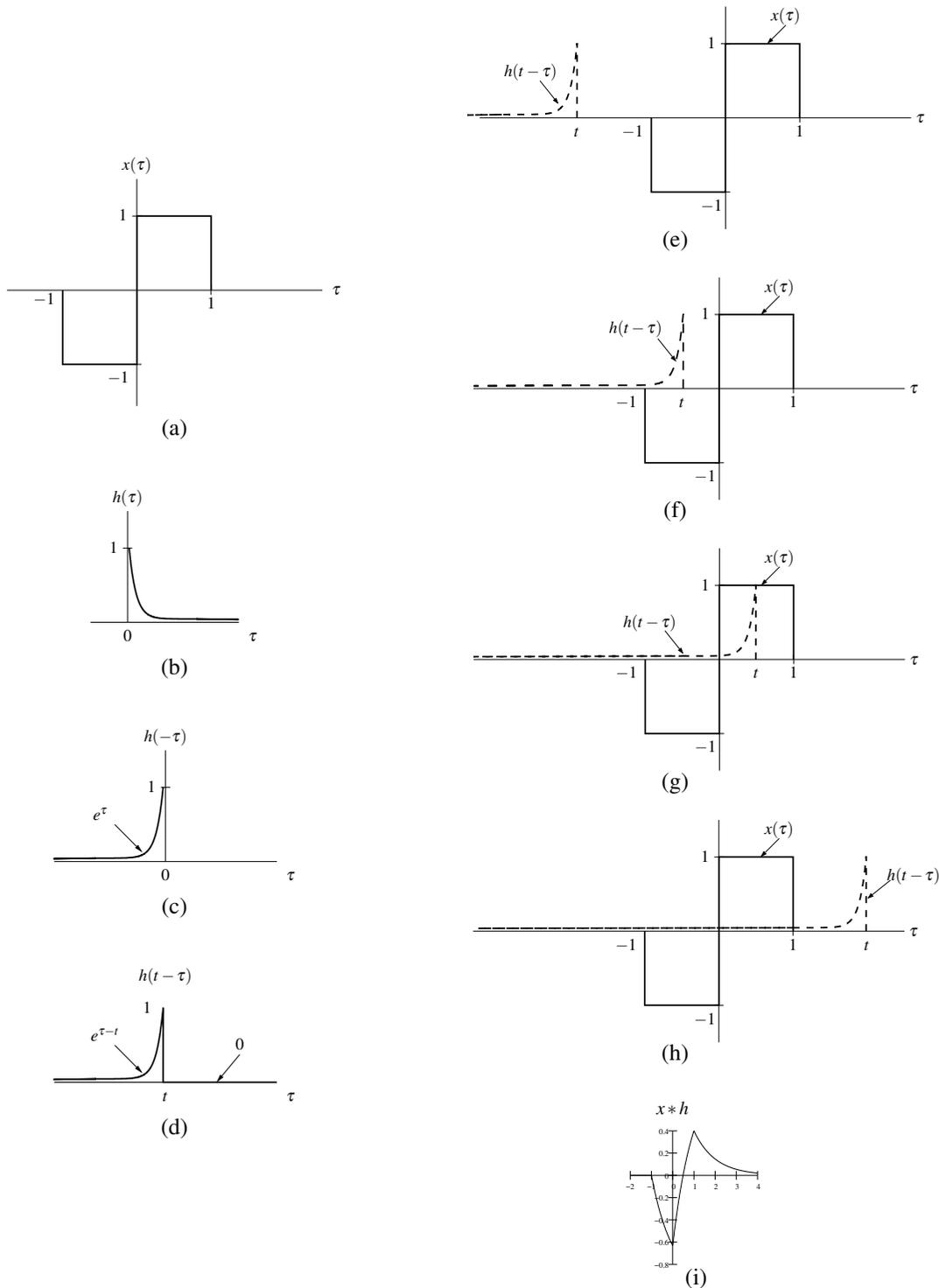


Figure 4.1: Evaluation of the convolution  $x * h$ . (a) The function  $x$ ; (b) the function  $h$ ; plots of (c)  $h(-\tau)$  and (d)  $h(t-\tau)$  versus  $\tau$ ; the functions associated with the product in the convolution integral for (e)  $t < -1$ , (f)  $-1 \leq t < 0$ , (g)  $0 \leq t < 1$ , and (h)  $t \geq 1$ ; and (i) the convolution result  $x * h$ .

Second, we consider the case of  $0 \leq t < 1$ . From Figure 4.2(f), we can see that

$$\begin{aligned} x * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_0^t (t - \tau)d\tau \\ &= [t\tau - \frac{1}{2}\tau^2]_0^t \\ &= t^2 - \frac{1}{2}t^2 \\ &= \frac{1}{2}t^2. \end{aligned} \tag{4.7}$$

Third, we consider the case of  $1 \leq t < 2$ . From Figure 4.2(g), we can see that

$$\begin{aligned} x * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{t-1}^1 (t - \tau)d\tau \\ &= [t\tau - \frac{1}{2}\tau^2]_{t-1}^1 \\ &= t - \frac{1}{2}(1)^2 - [t(t-1) - \frac{1}{2}(t-1)^2] \\ &= t - \frac{1}{2} - [t^2 - t - \frac{1}{2}(t^2 - 2t + 1)] \\ &= -\frac{1}{2}t^2 + t. \end{aligned} \tag{4.8}$$

Fourth, we consider the case of  $t \geq 2$ . From Figure 4.2(h), we can see that

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = 0. \tag{4.9}$$

Combining the results of (4.6), (4.7), (4.8), and (4.9), we have that

$$x * h(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2}t^2 & 0 \leq t < 1 \\ -\frac{1}{2}t^2 + t & 1 \leq t < 2 \\ 0 & t \geq 2. \end{cases}$$

The convolution result  $x * h$  is plotted in Figure 4.2(i). ■

**Example 4.3.** Compute the quantity  $x * h$ , where

$$x(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ -t + 2 & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases} \quad \text{and} \quad h(t) = u(t) - u(t - 1).$$

*Solution.* Due to the somewhat ugly nature of the expressions for  $x(t)$  and  $h(t)$ , this problem can be more easily solved if we use the graphical interpretation of convolution to guide us. We begin by plotting the functions  $x$  and  $h$ , as shown in Figures 4.3(a) and (b), respectively.

Next, we need to determine  $h(t - \tau)$ , the time-reversed and time-shifted version of  $h(\tau)$ . We can accomplish this in two steps. First, we time-reverse  $h(\tau)$  to obtain  $h(-\tau)$  as shown in Figure 4.3(c). Second, we time-shift the resulting signal by  $t$  to obtain  $h(t - \tau)$  as shown in Figure 4.3(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of  $t$ , we must multiply  $x(\tau)$  by  $h(t - \tau)$  and integrate the resulting product with respect to  $\tau$ . Due to the form of  $x$  and  $h$ , we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 4.3(e) to (i).

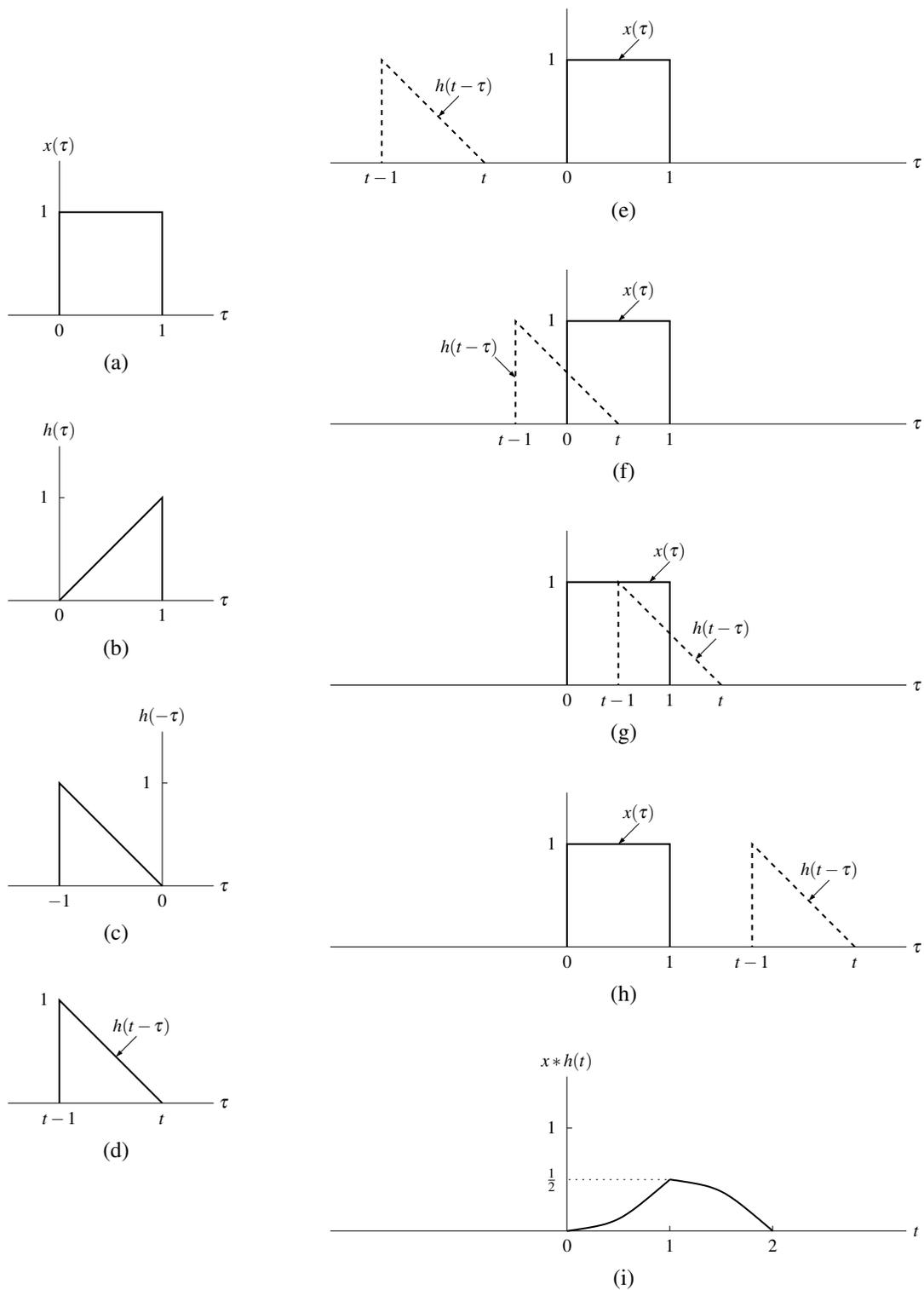


Figure 4.2: Evaluation of the convolution  $x * h$ . (a) The function  $x$ ; (b) the function  $h$ ; plots of (c)  $h(-\tau)$  and (d)  $h(t-\tau)$  versus  $\tau$ ; the functions associated with the product in the convolution integral for (e)  $t < 0$ , (f)  $0 \leq t < 1$ , (g)  $1 \leq t < 2$ , and (h)  $t \geq 2$ ; and (i) the convolution result  $x * h$ .

First, we consider the case of  $t < 0$ . From Figure 4.3(e), we can see that

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = 0. \quad (4.10)$$

Second, we consider the case of  $0 \leq t < 1$ . From Figure 4.3(f), we can see that

$$\begin{aligned} x * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_0^t \tau d\tau \\ &= \left[ \frac{1}{2} \tau^2 \right]_0^t \\ &= \frac{1}{2} t^2. \end{aligned} \quad (4.11)$$

Third, we consider the case of  $1 \leq t < 2$ . From Figure 4.3(g), we can see that

$$\begin{aligned} x * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{t-1}^1 \tau d\tau + \int_1^t (-\tau + 2)d\tau \\ &= \left[ \frac{1}{2} \tau^2 \right]_{t-1}^1 + \left[ -\frac{1}{2} \tau^2 + 2\tau \right]_1^t \\ &= \frac{1}{2} - \left[ \frac{1}{2} (t-1)^2 \right] - \frac{1}{2} t^2 + 2t - \left[ -\frac{1}{2} + 2 \right] \\ &= -t^2 + 3t - \frac{3}{2}. \end{aligned} \quad (4.12)$$

Fourth, we consider the case of  $2 \leq t < 3$ . From Figure 4.3(h), we can see that

$$\begin{aligned} x * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{t-1}^2 (-\tau + 2)d\tau \\ &= \left[ -\frac{1}{2} \tau^2 + 2\tau \right]_{t-1}^2 \\ &= 2 - \left[ -\frac{1}{2} t^2 + 3t - \frac{5}{2} \right] \\ &= \frac{1}{2} t^2 - 3t + \frac{9}{2}. \end{aligned} \quad (4.13)$$

Lastly, we consider the case of  $t \geq 3$ . From Figure 4.3(i), we can see that

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = 0. \quad (4.14)$$

Combining the results of (4.10), (4.11), (4.12), (4.13), and (4.14) together, we have that

$$x * h(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2} t^2 & 0 \leq t < 1 \\ -t^2 + 3t - \frac{3}{2} & 1 \leq t < 2 \\ \frac{1}{2} t^2 - 3t + \frac{9}{2} & 2 \leq t < 3 \\ 0 & t \geq 3. \end{cases}$$

The convolution result  $x * h$  is plotted in Figure 4.3(j). ■

**Example 4.4.** Compute the convolution  $x * h$ , where

$$x(t) = e^{-at} u(t), \quad h(t) = u(t),$$

and  $a$  is a strictly positive real constant.

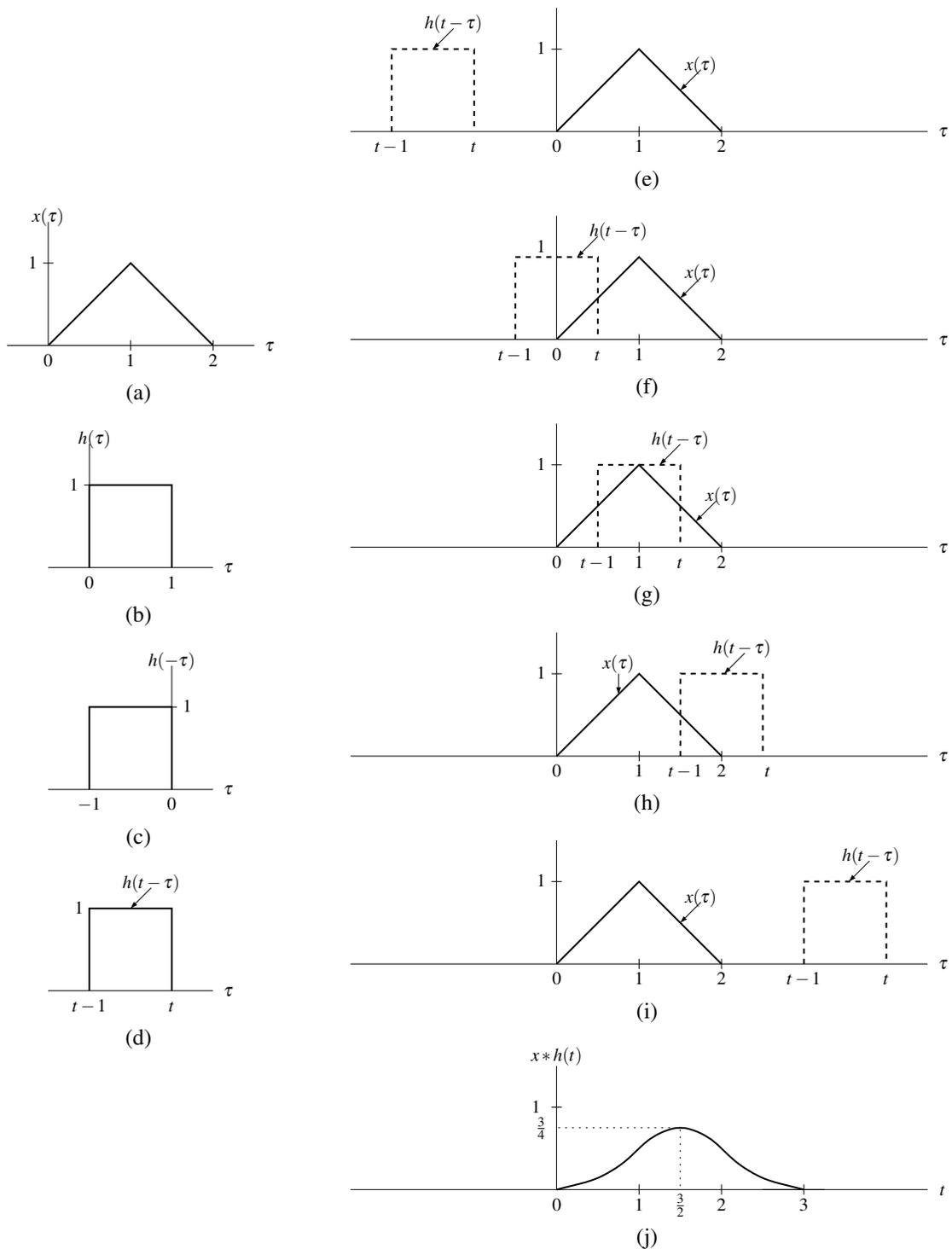


Figure 4.3: Evaluation of the convolution  $x*h$ . (a) The function  $x$ ; (b) the function  $h$ ; plots of (c)  $h(-\tau)$  and (d)  $h(t-\tau)$  versus  $\tau$ ; the functions associated with the product in the convolution integral for (e)  $t < 0$ , (f)  $0 \leq t < 1$ , (g)  $1 \leq t < 2$ , (h)  $2 \leq t < 3$ , and (i)  $t \geq 3$ ; and (j) the convolution result  $x*h$ .

*Solution.* Since  $x$  and  $h$  are relatively simple functions, we will solve this problem without the aid of graphs. Our objective in this example is twofold. First, we want to show that it is possible, if one is very careful, to perform simple convolutions without using graphs as aids. Second, we would like to show that this is actually somewhat tricky to do correctly, and probably it would have been better to draw graphs for guidance in this example in order to reduce the likelihood of errors.

From the definition of convolution, we have

$$\begin{aligned} x * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t - \tau)d\tau. \end{aligned} \quad (4.15)$$

Since

$$u(\tau)u(t - \tau) = \begin{cases} 1 & 0 \leq \tau \text{ and } \tau \leq t \\ 0 & \text{otherwise,} \end{cases}$$

the integrand can only be nonzero if  $0 \leq \tau$  and  $\tau \leq t$  (which necessarily requires that  $t \geq 0$ ). So, if  $t < 0$ , the integrand will be zero, and  $x * h(t) = 0$ . Now, let us consider the case of  $t > 0$ . From (4.15), we have

$$\begin{aligned} x * h(t) &= \int_0^t e^{-a\tau}d\tau \\ &= \left[-\frac{1}{a}e^{-a\tau}\right]_0^t \\ &= \frac{1}{a}(1 - e^{-at}). \end{aligned}$$

Thus, we have

$$\begin{aligned} x * h(t) &= \begin{cases} \frac{1}{a}(1 - e^{-at}) & t > 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \frac{1}{a}(1 - e^{-at})u(t). \end{aligned}$$

Note that, as the above solution illustrates, computing a convolution without graphs as aids can be somewhat tricky to do correctly, even when the functions being convolved are relatively simple like the ones in this example. If some steps in the above solution are unclear, it would be helpful to sketch graphs to assist in the convolution computation. For example, the use of graphs, like those shown in Figure 4.4, would likely make the above convolution much easier to compute correctly. ■

### 4.3 Properties of Convolution

Since convolution is frequently employed in the study of LTI systems, it is important for us to know some of its basic properties. In what follows, we examine some of these properties.

**Theorem 4.1** (Commutativity of convolution). *Convolution is commutative. That is, for any two functions  $x$  and  $h$ ,*

$$x * h = h * x. \quad (4.16)$$

*In other words, the result of a convolution is not affected by the order of its operands.*

*Proof.* We now provide a proof of the commutative property stated above. To begin, we expand the left-hand side of (4.16) as follows:

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau.$$

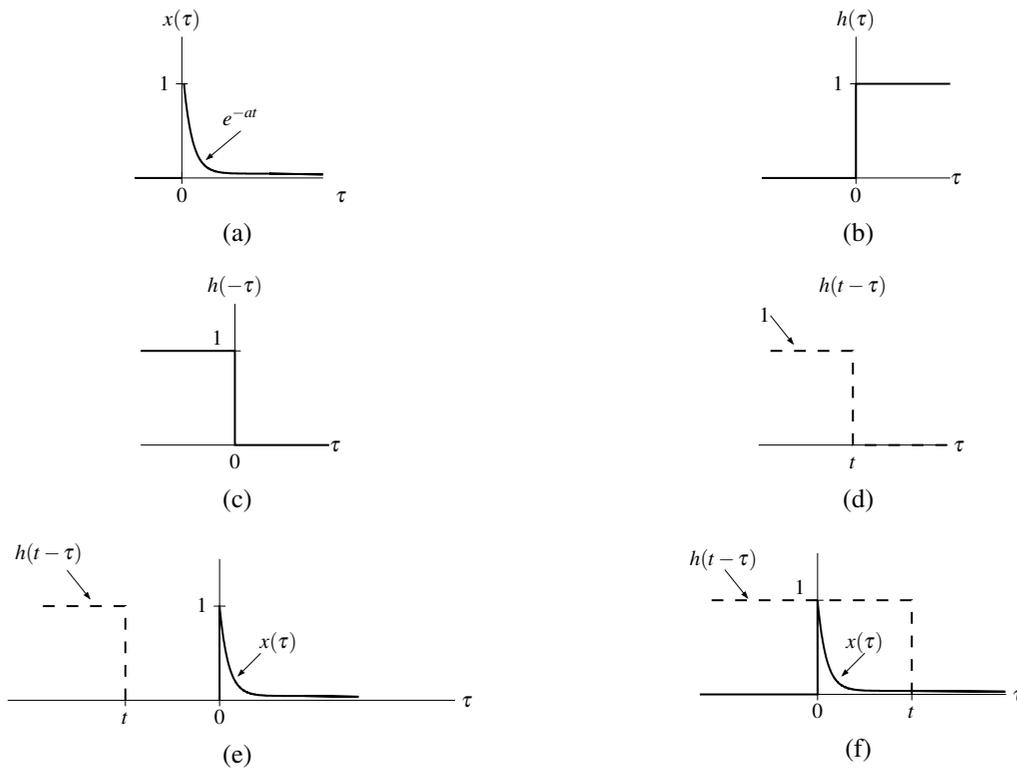


Figure 4.4: Evaluation of the convolution  $x * h$ . (a) The function  $x$ ; (b) the function  $h$ ; plots of (c)  $h(-\tau)$  and (d)  $h(t-\tau)$  versus  $\tau$ ; and the functions associated with the product in the convolution integral for (e)  $t < 0$  and (f)  $t > 0$ .

Next, we perform a change of variable. Let  $v = t - \tau$  which implies that  $\tau = t - v$  and  $d\tau = -dv$ . Using this change of variable, we can rewrite the previous equation as

$$\begin{aligned} x * h(t) &= \int_{t+\infty}^{t-\infty} x(t-v)h(v)(-dv) \\ &= \int_{\infty}^{-\infty} x(t-v)h(v)(-dv) \\ &= \int_{-\infty}^{\infty} x(t-v)h(v)dv \\ &= \int_{-\infty}^{\infty} h(v)x(t-v)dv \\ &= h * x(t). \end{aligned}$$

(Note that, above, we used the fact that, for any function  $f$ ,  $\int_a^b f(x)dx = -\int_b^a f(x)dx$ .) Thus, we have proven that convolution is commutative. ■

**Theorem 4.2** (Associativity of convolution). *Convolution is associative. That is, for any three functions  $x$ ,  $h_1$ , and  $h_2$ ,*

$$(x * h_1) * h_2 = x * (h_1 * h_2). \quad (4.17)$$

*In other words, the final result of multiple convolutions does not depend on how the convolution operations are grouped.*

*Proof.* To begin, we use the definition of the convolution operation to expand the left-hand side of (4.17) as follows:

$$\begin{aligned} ([x * h_1] * h_2)(t) &= \int_{-\infty}^{\infty} [x * h_1(v)]h_2(t-v)dv \\ &= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} x(\tau)h_1(v-\tau)d\tau \right) h_2(t-v)dv. \end{aligned}$$

Now, we change the order of integration to obtain

$$([x * h_1] * h_2)(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)h_1(v-\tau)h_2(t-v)dv d\tau.$$

Pulling the factor of  $x(\tau)$  out of the inner integral yields

$$([x * h_1] * h_2)(t) = \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} h_1(v-\tau)h_2(t-v)dv d\tau.$$

Next, we perform a change of variable. Let  $\lambda = v - \tau$  which implies that  $v = \lambda + \tau$  and  $d\lambda = dv$ . Using this change of variable, we can write

$$\begin{aligned} ([x * h_1] * h_2)(t) &= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} h_1(\lambda)h_2(t-\lambda-\tau)d\lambda d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) \left( \int_{-\infty}^{\infty} h_1(\lambda)h_2([t-\tau]-\lambda)d\lambda \right) d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) [h_1 * h_2(t-\tau)] d\tau \\ &= (x * [h_1 * h_2])(t). \end{aligned}$$

Thus, we have proven that convolution is associative. ■

**Theorem 4.3** (Distributivity of convolution). *Convolution is distributive. That is, for any three functions  $x$ ,  $h_1$ , and  $h_2$ ,*

$$x * (h_1 + h_2) = x * h_1 + x * h_2. \quad (4.18)$$

*In other words, convolution can be distributed across addition.*

*Proof.* The proof of this property is relatively simple. Expanding the left-hand side of (4.18), we have:

$$\begin{aligned} (x * [h_1 + h_2])(t) &= \int_{-\infty}^{\infty} x(\tau)[h_1(t - \tau) + h_2(t - \tau)]d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)h_1(t - \tau)d\tau + \int_{-\infty}^{\infty} x(\tau)h_2(t - \tau)d\tau \\ &= x * h_1(t) + x * h_2(t). \end{aligned}$$

Thus, we have shown that convolution is distributive. ■

The identity for an operation defined on elements of a set is often extremely helpful to know. Consider the operations of addition and multiplication as defined for real numbers. For any real number  $a$ ,  $a + 0 = a$ . Since adding zero to  $a$  has no effect (i.e., the result is  $a$ ), we call 0 the **additive identity**. For any real number  $a$ ,  $1 \cdot a = a$ . Since multiplying  $a$  by 1 has no effect (i.e., the result is  $a$ ), we call 1 the **multiplicative identity**. Imagine for a moment how difficult arithmetic would be if we did not know that  $a + 0 = a$  or  $1 \cdot a = a$ . For this reason, identity values are clearly of fundamental importance.

Earlier, we were introduced to a new operation known as convolution. So, in light of the above, it is natural to wonder if there is a convolutional identity. In fact, there is, as given by the theorem below.

**Theorem 4.4** (Convolutional identity). *For any function  $x$ ,*

$$x * \delta = x. \quad (4.19)$$

*In other words,  $\delta$  is the convolutional identity (i.e., convolving any function  $x$  with  $\delta$  simply yields  $x$ ).*

*Proof.* Suppose that we have an arbitrary function  $x$ . From the definition of convolution, we can write

$$x * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau.$$

Now, let us employ a change of variable. Let  $\lambda = -\tau$  so that  $\tau = -\lambda$  and  $d\tau = -d\lambda$ . Applying the change of variable, we obtain

$$\begin{aligned} x * \delta(t) &= \int_{\infty}^{-\infty} x(-\lambda)\delta(t + \lambda)(-1)d\lambda \\ &= \int_{-\infty}^{\infty} x(-\lambda)\delta(\lambda + t)d\lambda. \end{aligned} \quad (4.20)$$

From the equivalence property of  $\delta$ , we can rewrite the preceding equation as

$$\begin{aligned} x * \delta(t) &= \int_{-\infty}^{\infty} x(-[-t])\delta(\lambda + t)d\lambda \\ &= \int_{-\infty}^{\infty} x(t)\delta(\lambda + t)d\lambda. \end{aligned}$$

Factoring  $x(t)$  out of the integral, we obtain

$$x * \delta(t) = x(t) \int_{-\infty}^{\infty} \delta(\lambda + t)d\lambda.$$

Since  $\int_{-\infty}^{\infty} \delta(\lambda)d\lambda = 1$  implies that  $\int_{-\infty}^{\infty} \delta(\lambda + t)d\lambda = 1$ , we have

$$x * \delta(t) = x(t).$$

Thus,  $\delta$  is the convolutional identity (i.e.,  $x * \delta = x$ ). (Alternatively, we could have directly applied the sifting property to (4.20) to show the desired result.) ■

## 4.4 Periodic Convolution

The convolution of two periodic functions is usually not well defined. This motivates an alternative notion of convolution for periodic signals known as periodic convolution. The **periodic convolution** of the  $T$ -periodic functions  $x$  and  $h$ , denoted  $x \circledast h$ , is defined as

$$x \circledast h(t) = \int_T x(\tau)h(t - \tau)d\tau,$$

where  $\int_T$  denotes integration over an interval of length  $T$ . The periodic convolution and (linear) convolution of the  $T$ -periodic functions  $x$  and  $h$  are related as

$$x \circledast h(t) = x_0 * h(t) \quad \text{where} \quad x(t) = \sum_{k=-\infty}^{\infty} x_0(t - kT)$$

(i.e.,  $x_0(t)$  equals  $x(t)$  over a single period of  $x$  and is zero elsewhere).

## 4.5 Characterizing LTI Systems and Convolution

As a matter of terminology, the **impulse response**  $h$  of a system  $\mathcal{H}$  is defined as

$$h = \mathcal{H}\delta.$$

In other words, the impulse response of a system is the output that it produces when presented with  $\delta$  as an input. As it turns out, a LTI system has a very special relationship between its input, output, and impulse response, as given by the theorem below.

**Theorem 4.5** (LTI systems and convolution). *A LTI system  $\mathcal{H}$  with impulse response  $h$  is such that*

$$\mathcal{H}x = x * h.$$

*In other words, a LTI system computes a convolution. More specifically, it computes the convolution of its input with its impulse response.*

*Proof.* To begin, we assume that the system  $\mathcal{H}$  is LTI (i.e., the operator  $\mathcal{H}$  is both linear and time invariant). Using the fact that  $\delta$  is the convolutional identity, we have

$$\begin{aligned} y &= \mathcal{H}x \\ &= \mathcal{H}\{x * \delta\} \\ &= \mathcal{H}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(\cdot - \tau)d\tau\right\}. \end{aligned}$$

Since  $\mathcal{H}$  is a linear operator, we can pull the integral and  $x(\tau)$  (which is a constant with respect to the operation performed by  $\mathcal{H}$ ) outside  $\mathcal{H}$ , and simplify the result to obtain

$$\begin{aligned} y &= \int_{-\infty}^{\infty} \mathcal{H}\{x(\tau)\delta(\cdot - \tau)\}d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)\mathcal{H}\{\delta(\cdot - \tau)\}d\tau. \end{aligned} \tag{4.21}$$

Since the system is time invariant (by assumption), we know that

$$\mathcal{H}\{\delta(\cdot - \tau)\} = h(\cdot - \tau). \tag{4.22}$$

Thus, we can rewrite (4.21) as

$$\begin{aligned} y &= \int_{-\infty}^{\infty} x(\tau)h(\cdot - \tau)d\tau \\ &= x * h. \end{aligned}$$

Thus, the output  $y$  is simply the convolution of the input  $x$  and the impulse response  $h$ . ■

By Theorem 4.5 above, the behavior of a LTI system is completely characterized by its impulse response. That is, if the impulse response of a system is known, we can determine the response of the system to *any* input. Consequently, the impulse response provides a very powerful tool for the study of LTI systems.

**Example 4.5.** Consider a LTI system  $\mathcal{H}$  with impulse response

$$h(t) = u(t). \quad (4.23)$$

Show that  $\mathcal{H}$  is characterized by the equation

$$\mathcal{H}x(t) = \int_{-\infty}^t x(\tau) d\tau \quad (4.24)$$

(i.e.,  $\mathcal{H}$  corresponds to an ideal integrator).

*Solution.* Since the system is LTI, we have that

$$\mathcal{H}x(t) = x * h(t).$$

Substituting (4.23) into the preceding equation, and simplifying we obtain

$$\begin{aligned} \mathcal{H}x(t) &= x * h(t) \\ &= x * u(t) \\ &= \int_{-\infty}^{\infty} x(\tau) u(t - \tau) d\tau \\ &= \int_{-\infty}^t x(\tau) u(t - \tau) d\tau + \int_t^{\infty} x(\tau) u(t - \tau) d\tau \\ &= \int_{-\infty}^t x(\tau) d\tau. \end{aligned}$$

Therefore, the system with the impulse response  $h$  given by (4.23) is, in fact, the ideal integrator given by (4.24). ■

**Example 4.6.** Consider a LTI system with input  $x$ , output  $y$ , and impulse response  $h$ , where

$$h(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find and plot the response of the system to the particular input  $x$  given by

$$x(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

*Solution.* Plots of  $x$  and  $h$  are given in Figures 4.5(a) and (b), respectively. Since the system is LTI, we know that

$$y(t) = x * h(t).$$

Thus, in order to find the response of the system to the input  $x$ , we simply need to compute the convolution  $x * h$ .

We begin by plotting the functions  $x$  and  $h$  as shown in Figures 4.5(a) and (b), respectively. Next, we proceed to determine the time-reversed and time-shifted version of  $h(\tau)$ . We can accomplish this in two steps. First, we time-reverse  $h(\tau)$  to obtain  $h(-\tau)$  as shown in Figure 4.5(c). Second, we time-shift the resulting signal by  $t$  to obtain  $h(t - \tau)$  as shown in Figure 4.5(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of  $t$ , we must multiply  $x(\tau)$  by  $h(t - \tau)$  and integrate the resulting product with respect to  $\tau$ . Due to the form of  $x$

and  $h$ , we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 4.5(e) to (h).

First, we consider the case of  $t < 0$ . From Figure 4.5(e), we can see that

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = 0. \quad (4.25)$$

Second, we consider the case of  $0 \leq t < 1$ . From Figure 4.5(f), we can see that

$$\begin{aligned} x * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_0^t d\tau \\ &= [\tau]_0^t \\ &= t. \end{aligned} \quad (4.26)$$

Third, we consider the case of  $1 \leq t < 2$ . From Figure 4.5(g), we can see that

$$\begin{aligned} x * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{t-1}^1 d\tau \\ &= [\tau]_{t-1}^1 \\ &= 1 - (t - 1) \\ &= 2 - t. \end{aligned} \quad (4.27)$$

Fourth, we consider the case of  $t \geq 2$ . From Figure 4.5(h), we can see that

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = 0. \quad (4.28)$$

Combining the results of (4.25), (4.26), (4.27), and (4.28), we have that

$$x * h(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 2 - t & 1 \leq t < 2 \\ 0 & t \geq 2. \end{cases}$$

The convolution result  $x * h$  is plotted in Figure 4.5(i). The response of the system to the specified input is simply  $x * h$ . ■

## 4.6 Step Response of LTI Systems

The **step response**  $s$  of a system  $\mathcal{H}$  is defined as

$$s = \mathcal{H}u$$

(i.e., the step response of a system is the output it produces for a unit-step function input). In the case of a LTI system, it turns out that the step response is closely related to the impulse response, as given by the theorem below.

**Theorem 4.6.** *The step response  $s$  and impulse response  $h$  of a LTI system are related as*

$$h(t) = \frac{ds(t)}{dt} \quad \text{and} \quad s(t) = \int_{-\infty}^t h(\tau)d\tau.$$

*That is, the impulse response  $h$  is the derivative of the step response  $s$ .*

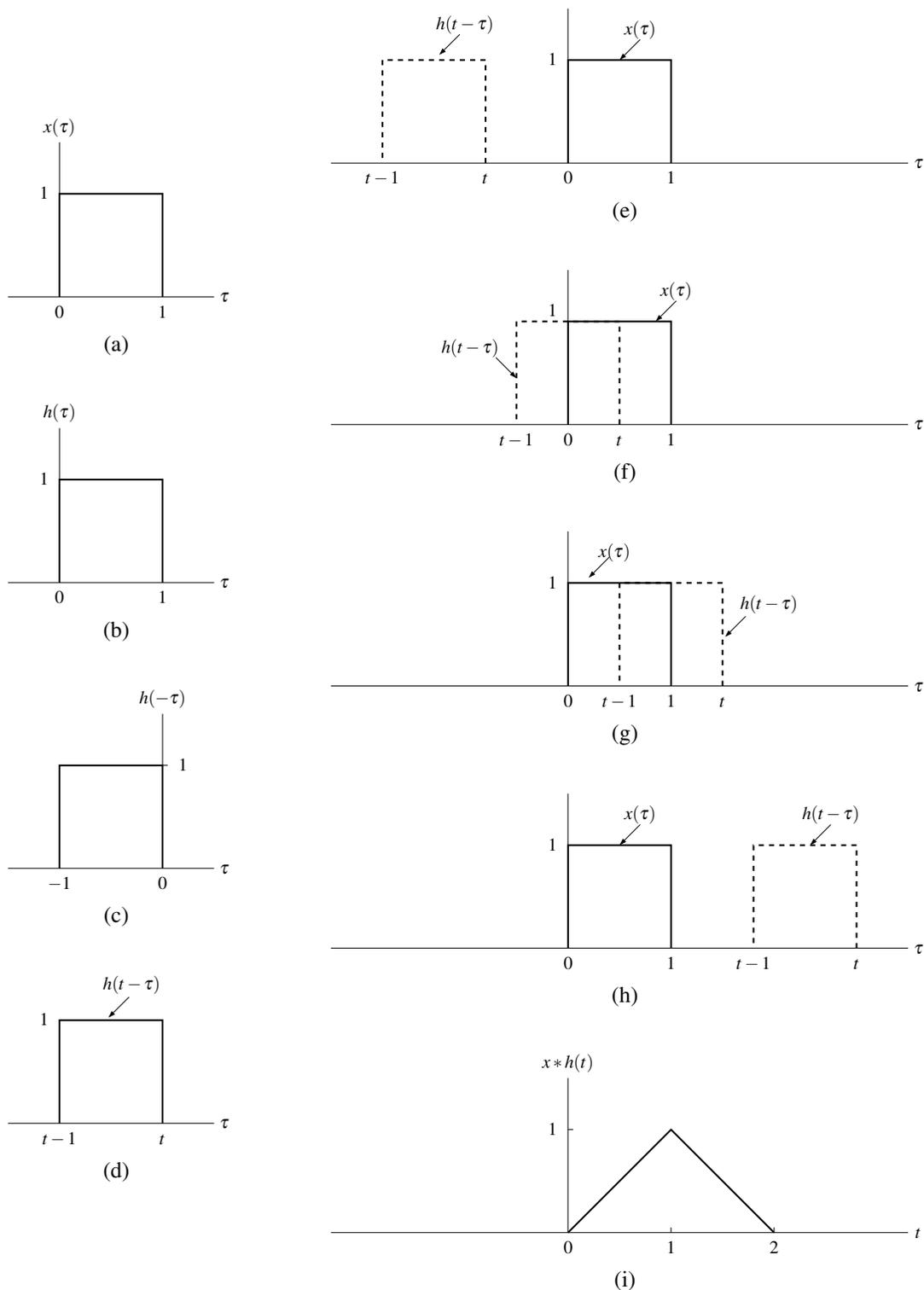


Figure 4.5: Evaluation of the convolution  $x * h$ . (a) The function  $x$ ; (b) the function  $h$ ; plots of (c)  $h(-\tau)$  and (d)  $h(t-\tau)$  versus  $\tau$ ; the functions associated with the product in the convolution integral for (e)  $t < 0$ , (f)  $0 \leq t < 1$ , (g)  $1 \leq t < 2$ , and (h)  $t \geq 2$ ; and (i) the convolution result  $x * h$ .

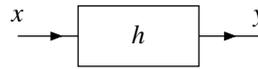


Figure 4.6: Block diagram representation of continuous-time LTI system with input  $x$ , output  $y$ , and impulse response  $h$ .

*Proof.* Using the fact that  $s = u * h$ , we can write

$$\begin{aligned} s(t) &= u * h(t) \\ &= h * u(t) \\ &= \int_{-\infty}^{\infty} h(\tau) u(t - \tau) d\tau \\ &= \int_{-\infty}^t h(\tau) d\tau. \end{aligned}$$

Thus,  $s$  can be obtained by integrating  $h$ . Taking the derivative of  $s$ , we obtain

$$\begin{aligned} \frac{ds(t)}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left[ \int_{-\infty}^{t + \Delta t} h(\tau) d\tau - \int_{-\infty}^t h(\tau) d\tau \right] \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_t^{t + \Delta t} h(\tau) d\tau \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} (h(t) \Delta t) \\ &= h(t). \end{aligned}$$

Thus,  $h$  is the derivative of  $s$ . ■

The step response is often of great practical interest, since it can be used to determine the impulse response of a LTI system. In particular, the impulse response can be determined from the step response via differentiation. From a practical point of view, the step response is more useful for characterizing a system based on experimental measurements. Obviously, we cannot directly measure the impulse response of a system because we cannot (in the real world) generate a unit-impulse function or an accurate approximation thereof. We can, however, produce a reasonably good approximation of the unit-step function in the real world. Thus, we can measure the step response and from it determine the impulse response.

## 4.7 Block Diagram Representation of Continuous-Time LTI Systems

Frequently, it is convenient to represent continuous-time LTI systems in block diagram form. Since a LTI system is completely characterized by its impulse response, we often label such a system with its impulse response in a block diagram. That is, we represent a LTI system with input  $x$ , output  $y$ , and impulse response  $h$ , as shown in Figure 4.6.

## 4.8 Interconnection of Continuous-Time LTI Systems

Suppose that we have a LTI system with input  $x$ , output  $y$ , and impulse response  $h$ . We know that  $x$  and  $y$  are related as  $y = x * h$ . In other words, the system can be viewed as performing a convolution operation. From the properties of convolution introduced earlier, we can derive a number of equivalences involving the impulse responses of series- and parallel-interconnected systems.

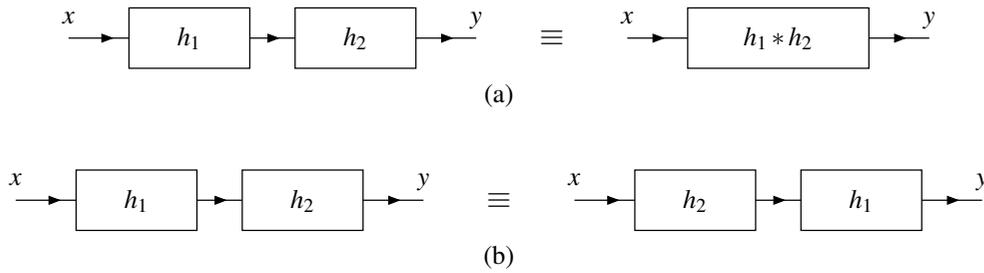


Figure 4.7: Equivalences for the series interconnection of continuous-time LTI systems. The (a) first and (b) second equivalences.

Consider two LTI systems with impulse responses  $h_1$  and  $h_2$  that are connected in a series configuration, as shown on the left-side of Figure 4.7(a). From the block diagram on the left side of Figure 4.7(a), we have

$$y = (x * h_1) * h_2.$$

Due to the associativity of convolution, however, this is equivalent to

$$y = x * (h_1 * h_2).$$

Thus, the series interconnection of two LTI systems behaves as a single LTI system with impulse response  $h_1 * h_2$ . In other words, we have the equivalence shown in Figure 4.7(a).

Consider two LTI systems with impulse responses  $h_1$  and  $h_2$  that are connected in a series configuration, as shown on the left-side of Figure 4.7(b). From the block diagram on the left side of Figure 4.7(b), we have

$$y = (x * h_1) * h_2.$$

Due to the associativity and commutativity of convolution, this is equivalent to

$$\begin{aligned} y &= x * (h_1 * h_2) \\ &= x * (h_2 * h_1) \\ &= (x * h_2) * h_1. \end{aligned}$$

Thus, interchanging the two LTI systems does not change the behavior of the overall system with input  $x$  and output  $y$ . In other words, we have the equivalence shown in Figure 4.7(b).

Consider two LTI systems with impulse responses  $h_1$  and  $h_2$  that are connected in a parallel configuration, as shown on the left-side of Figure 4.8. From the block diagram on the left side of Figure 4.8, we have

$$y = x * h_1 + x * h_2.$$

Due to convolution being distributive, however, this equation can be rewritten as

$$y = x * (h_1 + h_2).$$

Thus, the parallel interconnection of two LTI systems behaves as a single LTI system with impulse response  $h_1 + h_2$ . In other words, we have the equivalence shown in Figure 4.8.

**Example 4.7.** Consider the system with input  $x$ , output  $y$ , and impulse response  $h$  as shown in Figure 4.9. Each subsystem in the block diagram is LTI and labelled with its impulse response. Find  $h$ .

*Solution.* From the left half of the block diagram, we can write

$$\begin{aligned} v(t) &= x(t) + x * h_1(t) + x * h_2(t) \\ &= x * \delta(t) + x * h_1(t) + x * h_2(t) \\ &= (x * [\delta + h_1 + h_2])(t). \end{aligned}$$

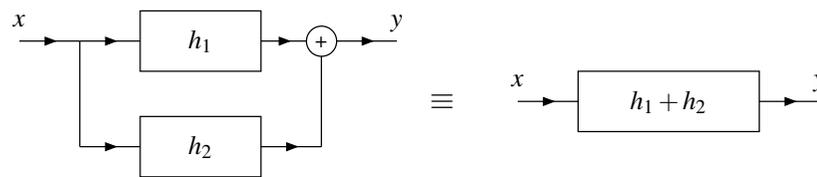


Figure 4.8: Equivalence for the parallel interconnection of continuous-time LTI systems.

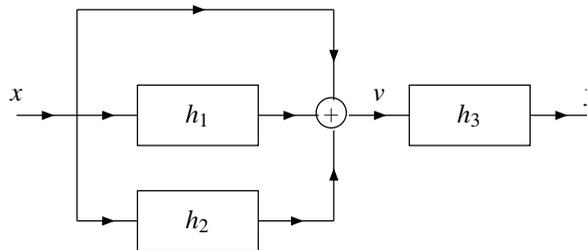


Figure 4.9: System interconnection example.

Similarly, from the right half of the block diagram, we can write

$$y(t) = v * h_3(t).$$

Substituting the expression for  $v$  into the preceding equation we obtain

$$\begin{aligned} y(t) &= v * h_3(t) \\ &= (x * [\delta + h_1 + h_2]) * h_3(t) \\ &= x * [h_3 + h_1 * h_3 + h_2 * h_3](t). \end{aligned}$$

Thus, the impulse response  $h$  of the overall system is

$$h(t) = h_3(t) + h_1 * h_3(t) + h_2 * h_3(t). \quad \blacksquare$$

## 4.9 Properties of Continuous-Time LTI Systems

In the previous chapter, we introduced a number of properties that might be possessed by a system (e.g., memory, causality, stability, and invertibility). Since a LTI system is completely characterized by its impulse response, one might wonder if there is a relationship between some of the properties introduced previously and the impulse response. In what follows, we explore some of these relationships.

### 4.9.1 Memory

The first system property to be considered is memory.

**Theorem 4.7** (Memorylessness of LTI system). *A LTI system with impulse response  $h$  is memoryless if and only if*

$$h(t) = 0 \text{ for all } t \neq 0.$$

*Proof.* Recall that a system is memoryless if its output  $y$  at any arbitrary time depends only on the value of its input  $x$  at that same time. Suppose now that we have a LTI system with input  $x$ , output  $y$ , and impulse response  $h$ . The output  $y$  at some arbitrary time  $t_0$  is given by

$$\begin{aligned} y(t_0) &= x * h(t_0) \\ &= h * x(t_0) \\ &= \int_{-\infty}^{\infty} h(\tau)x(t_0 - \tau)d\tau. \end{aligned}$$

Consider the integral in the above equation. In order for the system to be memoryless, the result of the integration must depend only on  $x(t)$  for  $t = t_0$ . This, however, is only possible if

$$h(t) = 0 \quad \text{for all } t \neq 0. \quad \blacksquare$$

From the preceding theorem, it follows that a memoryless LTI system must have an impulse response  $h$  of the form

$$h(t) = K\delta(t) \quad (4.29)$$

where  $K$  is a complex constant. As a consequence of this fact, we also have that all memoryless LTI systems must have an input-output relation of the form

$$\begin{aligned} y(t) &= x * (K\delta)(t) \\ &= K(x * \delta)(t) \\ &= Kx(t). \end{aligned}$$

In other words, a memoryless LTI system must be an ideal amplifier (i.e., a system that simply performs amplitude scaling).

**Example 4.8.** Consider the LTI system with the impulse response  $h$  given by

$$h(t) = e^{-at}u(t),$$

where  $a$  is a real constant. Determine whether this system has memory.

*Solution.* The system has memory since  $h(t) \neq 0$  for some  $t \neq 0$  (e.g.,  $h(1) = e^{-a} \neq 0$ ). ■

**Example 4.9.** Consider the LTI system with the impulse response  $h$  given by

$$h(t) = \delta(t).$$

Determine whether this system has memory.

*Solution.* Clearly,  $h$  is only nonzero at the origin. This follows immediately from the definition of the unit-impulse function  $\delta$ . Therefore, the system is memoryless (i.e., does not have memory). ■

## 4.9.2 Causality

The next system property to be considered is causality.

**Theorem 4.8** (Causality of LTI system). *A LTI system with impulse response  $h$  is causal if and only if*

$$h(t) = 0 \text{ for all } t < 0.$$

(i.e.,  $h$  is causal).

*Proof.* Recall that a system is causal if its output  $y$  at any arbitrary time  $t_0$  does not depend on its input  $x$  at a time later than  $t_0$ . Suppose that we have the LTI system with input  $x$ , output  $y$ , and impulse response  $h$ . The value of the output  $y$  at  $t_0$  is given by

$$\begin{aligned} y(t_0) &= x * h(t_0) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t_0 - \tau)d\tau \\ &= \int_{-\infty}^{t_0} x(\tau)h(t_0 - \tau)d\tau + \int_{t_0^+}^{\infty} x(\tau)h(t_0 - \tau)d\tau. \end{aligned} \quad (4.30)$$

In order for the expression for  $y(t_0)$  in (4.30) not to depend on  $x(t)$  for  $t > t_0$ , we must have that

$$h(t) = 0 \quad \text{for } t < 0 \quad (4.31)$$

(i.e.,  $h$  is causal). In this case, (4.30) simplifies to

$$y(t_0) = \int_{-\infty}^{t_0} x(\tau)h(t_0 - \tau)d\tau.$$

Clearly, the result of this integration does not depend on  $x(t)$  for  $t > t_0$  (since  $\tau$  varies from  $-\infty$  to  $t_0$ ). Therefore, a LTI system is causal if its impulse response  $h$  satisfies (4.31). ■

**Example 4.10.** Consider the LTI system with impulse response  $h$  given by

$$h(t) = e^{-at}u(t),$$

where  $a$  is a real constant. Determine whether this system is causal.

*Solution.* Clearly,  $h(t) = 0$  for  $t < 0$  (due to the  $u(t)$  factor in the expression for  $h(t)$ ). Therefore, the system is causal. ■

**Example 4.11.** Consider the LTI system with impulse response  $h$  given by

$$h(t) = \delta(t + t_0),$$

where  $t_0$  is a strictly positive real constant. Determine whether this system is causal.

*Solution.* From the definition of  $\delta$ , we can easily deduce that  $h(t) = 0$  except at  $t = -t_0$ . Since  $-t_0 < 0$ , the system is not causal. ■

### 4.9.3 Invertibility

The next system property to be considered is invertibility.

**Theorem 4.9** (Inverse of LTI system). *Let  $\mathcal{H}$  be a LTI system with impulse response  $h$ . If the inverse  $\mathcal{H}^{-1}$  of  $\mathcal{H}$  exists,  $\mathcal{H}^{-1}$  is LTI and has an impulse response  $h_{\text{inv}}$  that satisfies*

$$h * h_{\text{inv}} = \delta.$$

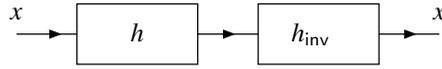


Figure 4.10: System in cascade with its inverse.

*Proof.* To begin, we need to show that the inverse of a LTI system, if it exists, must also be LTI. This part of the proof, however, is left as an exercise for the reader in Exercise 4.15. (The general approach to take for this problem is to show that: 1) the inverse of a linear system, if it exists, is linear; and 2) the inverse of a time-invariant system, if it exists, is time invariant.) We assume that this part of the proof has been demonstrated and proceed.

Suppose now that the inverse system  $\mathcal{H}^{-1}$  exists. We have that

$$\mathcal{H}x = x * h \quad \text{and} \quad \mathcal{H}^{-1}x = x * h_{\text{inv}}.$$

From the definition of an inverse system, we have that, for every function  $x$ ,

$$\mathcal{H}^{-1}\mathcal{H}x = x.$$

Expanding the left-hand side of the preceding equation, we obtain

$$\begin{aligned} \mathcal{H}^{-1}[x * h] &= x \\ \Leftrightarrow x * h * h_{\text{inv}} &= x. \end{aligned} \tag{4.32}$$

This relationship is expressed diagrammatically in Figure 4.10. Since the unit-impulse function is the convolutional identity, we can equivalently rewrite (4.32) as

$$x * h * h_{\text{inv}} = x * \delta.$$

This equation, however, must hold for arbitrary  $x$ . Thus, by comparing the left- and right-hand sides of this equation, we conclude

$$h * h_{\text{inv}} = \delta. \tag{4.33}$$

Therefore, if  $\mathcal{H}^{-1}$  exists, it must have an impulse response  $h_{\text{inv}}$  that satisfies (4.33). This completes the proof. ■

From the preceding theorem, we have the following result:

**Theorem 4.10** (Invertibility of LTI system). *A LTI system  $\mathcal{H}$  with impulse response  $h$  is invertible if and only if there exists a function  $h_{\text{inv}}$  satisfying*

$$h * h_{\text{inv}} = \delta.$$

*Proof.* The proof follows immediately from the result of Theorem 4.9 by simply observing that  $\mathcal{H}$  being invertible is equivalent to the existence of  $\mathcal{H}^{-1}$ . ■

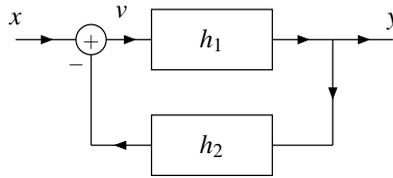
**Example 4.12.** Consider the LTI system  $\mathcal{H}$  with impulse response  $h$  given by

$$h(t) = A\delta(t - t_0),$$

where  $A$  and  $t_0$  are real constants and  $A \neq 0$ . Determine if  $\mathcal{H}$  is invertible, and if it is, find the impulse response  $h_{\text{inv}}$  of the system  $\mathcal{H}^{-1}$ .

*Solution.* If the system  $\mathcal{H}^{-1}$  exists, its impulse response  $h_{\text{inv}}$  is given by the solution to the equation

$$h * h_{\text{inv}} = \delta. \tag{4.34}$$

Figure 4.11: Feedback system with input  $x$  and output  $y$ .

So, let us attempt to solve this equation for  $h_{\text{inv}}$ . Substituting the given function  $h$  into (4.34) and using straightforward algebraic manipulation, we can write

$$\begin{aligned}
 h * h_{\text{inv}}(t) &= \delta(t) \\
 \Rightarrow \int_{-\infty}^{\infty} h(\tau) h_{\text{inv}}(t - \tau) d\tau &= \delta(t) \\
 \Rightarrow \int_{-\infty}^{\infty} A \delta(\tau - t_0) h_{\text{inv}}(t - \tau) d\tau &= \delta(t) \\
 \Rightarrow \int_{-\infty}^{\infty} \delta(\tau - t_0) h_{\text{inv}}(t - \tau) d\tau &= \frac{1}{A} \delta(t).
 \end{aligned}$$

Using the sifting property of the unit-impulse function, we can simplify the integral expression in the preceding equation to obtain

$$h_{\text{inv}}(t - t_0) = \frac{1}{A} \delta(t). \quad (4.35)$$

Substituting  $t + t_0$  for  $t$  in the preceding equation yields

$$\begin{aligned}
 h_{\text{inv}}([t + t_0] - t_0) &= \frac{1}{A} \delta(t + t_0) \quad \Leftrightarrow \\
 h_{\text{inv}}(t) &= \frac{1}{A} \delta(t + t_0).
 \end{aligned}$$

Since  $A \neq 0$ , the function  $h_{\text{inv}}$  is always well defined. Thus,  $\mathcal{H}^{-1}$  exists and consequently  $\mathcal{H}$  is invertible.  $\blacksquare$

**Example 4.13.** Consider the system with the input  $x$  and output  $y$  as shown in Figure 4.11. Each subsystem in the block diagram is LTI and labelled with its impulse response. Use the notion of an inverse system in order to express  $y$  in terms of  $x$ .

*Solution.* From Figure 4.11, we can write:

$$v = x - y * h_2 \quad \text{and} \quad (4.36)$$

$$y = v * h_1. \quad (4.37)$$

Substituting (4.36) into (4.37), and simplifying we obtain

$$\begin{aligned}
 y &= [x - y * h_2] * h_1 \\
 \Rightarrow y &= x * h_1 - y * h_2 * h_1 \\
 \Rightarrow y + y * h_2 * h_1 &= x * h_1 \\
 \Rightarrow y * \delta + y * h_2 * h_1 &= x * h_1 \\
 \Rightarrow y * [\delta + h_2 * h_1] &= x * h_1.
 \end{aligned} \quad (4.38)$$

For convenience, we now define the function  $g$  as

$$g = \delta + h_2 * h_1. \quad (4.39)$$

So, we can rewrite (4.38) as

$$y * g = x * h_1. \quad (4.40)$$

Thus, we have almost solved for  $y$  in terms of  $x$ . To complete the solution, we need to eliminate  $g$  from the left-hand side of the equation. To do this, we use the notion of an inverse system. Consider the inverse of the system with impulse response  $g$ . This inverse system has an impulse response  $g_{\text{inv}}$  given by

$$g * g_{\text{inv}} = \delta. \quad (4.41)$$

This relationship follows from the definition of an inverse system. Now, we use  $g_{\text{inv}}$  in order to simplify (4.40) as follows:

$$\begin{aligned} y * g &= x * h_1 \\ \Rightarrow y * g * g_{\text{inv}} &= x * h_1 * g_{\text{inv}} \\ \Rightarrow y * \delta &= x * h_1 * g_{\text{inv}} \\ \Rightarrow y &= x * h_1 * g_{\text{inv}}. \end{aligned}$$

Thus, we can express the output  $y$  in terms of the input  $x$  as

$$y = x * h_1 * g_{\text{inv}},$$

where  $g_{\text{inv}}$  is given by (4.41) and  $g$  is given by (4.39). ■

#### 4.9.4 BIBO Stability

The last system property to be considered is BIBO stability.

**Theorem 4.11** (BIBO Stability of LTI system). *A LTI system with impulse response  $h$  is BIBO stable if and only if*

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty \quad (4.42)$$

(i.e.,  $h$  is absolutely integrable).

*Proof.* Recall that a system is BIBO stable if any arbitrary bounded input produces a bounded output. Suppose that we have a LTI system with input  $x$ , output  $y$ , and impulse response  $h$ .

First, we consider the sufficiency of (4.42) for BIBO stability. Assume that  $|x(t)| \leq A < \infty$  for all  $t$  (i.e.,  $x$  is bounded). We can write

$$\begin{aligned} y(t) &= x * h(t) \\ &= h * x(t) \\ &= \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau. \end{aligned}$$

By taking the magnitude of both sides of the preceding equation, we obtain

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau \right|. \quad (4.43)$$

One can show, for any two functions  $f_1$  and  $f_2$ , that

$$\left| \int_{-\infty}^{\infty} f_1(t)f_2(t)dt \right| \leq \int_{-\infty}^{\infty} |f_1(t)f_2(t)|dt.$$

Using this inequality, we can rewrite (4.43) as

$$|y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)x(t-\tau)| d\tau = \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)| d\tau.$$

We know (by assumption) that  $|x(t)| \leq A$  for all  $t$ , so we can replace  $|x(t)|$  by its bound  $A$  in the above inequality to obtain

$$|y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)| d\tau \leq \int_{-\infty}^{\infty} A|h(\tau)| d\tau = A \int_{-\infty}^{\infty} |h(\tau)| d\tau. \quad (4.44)$$

Thus, we have

$$|y(t)| \leq A \int_{-\infty}^{\infty} |h(\tau)| d\tau. \quad (4.45)$$

Since  $A$  is finite, we can deduce from (4.45) that  $y$  is bounded if

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty \quad (4.46)$$

(i.e.,  $h$  is absolutely integrable). Thus, the absolute integrability of the impulse response  $h$  is a sufficient condition for BIBO stability.

Now, we consider the necessity of (4.42) for BIBO stability. Suppose that  $h$  is not absolutely integrable. That is, suppose that

$$\int_{-\infty}^{\infty} |h(t)| dt = \infty.$$

If such is the case, we can show that the system is not BIBO stable. To begin, consider the particular input  $x$  given by

$$x(t) = \text{sgn}(h(-t)).$$

(Note that the  $\text{sgn}$  function is as defined earlier by (3.23).) The function  $x$  can only assume the values  $-1$ ,  $0$ , and  $1$ . So,  $x$  is obviously bounded (i.e.,  $|x(t)| \leq 1$  for all  $t$ ). The output  $y$  is given by

$$\begin{aligned} y(t) &= x * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau) d\tau \\ &= \int_{-\infty}^{\infty} (\text{sgn } h(-\tau))h(t-\tau) d\tau. \end{aligned} \quad (4.47)$$

Now, let us consider the output's value  $y(t)$  at  $t = 0$ . From (4.47), we have

$$y(0) = \int_{-\infty}^{\infty} (\text{sgn } h(-\tau))h(-\tau) d\tau. \quad (4.48)$$

Then, we observe that

$$\alpha \text{sgn } \alpha = |\alpha| \quad \text{for any real } \alpha.$$

So  $(\text{sgn } h(-\tau))h(-\tau) = |h(-\tau)|$ , and we can simplify (4.48) to obtain

$$\begin{aligned} y(0) &= \int_{-\infty}^{\infty} |h(-\tau)| d\tau \\ &= \int_{-\infty}^{\infty} |h(\tau)| d\tau \\ &= \infty. \end{aligned}$$

Thus, we have shown that the bounded input  $x$  will result in an unbounded output  $y$  (where  $y(t)$  is unbounded for  $t = 0$ ). Thus, the absolute integrability of  $h$  is also necessary for BIBO stability. This completes the proof. ■

**Example 4.14.** Consider the LTI system with impulse response  $h$  given by

$$h(t) = e^{at}u(t),$$

where  $a$  is a real constant. Determine for what values of  $a$  the system is BIBO stable.

*Solution.* We need to determine for what values of  $a$  the impulse response  $h$  is absolutely integrable. Suppose that  $a \neq 0$ . We can write

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)| dt &= \int_{-\infty}^{\infty} |e^{at}u(t)| dt \\ &= \int_{-\infty}^0 0 dt + \int_0^{\infty} |e^{at}| dt \\ &= \int_0^{\infty} e^{at} dt. \end{aligned}$$

Assuming that  $a \neq 0$ , we obtain (after integration)

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)| dt &= \left[ \frac{1}{a} e^{at} \right]_0^{\infty} \\ &= \frac{1}{a} (e^{a\infty} - 1). \end{aligned}$$

We can see that the result of the above integration is finite if  $a < 0$  and infinite if  $a > 0$ . In particular, if  $a < 0$ , we have

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)| dt &= 0 - \frac{1}{a} \\ &= -\frac{1}{a}. \end{aligned}$$

Since we excluded the possibility that  $a = 0$  above, we must now consider the case of  $a = 0$ . In this case, we have

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)| dt &= \int_{-\infty}^{\infty} |u(t)| dt \\ &= \int_0^{\infty} 1 dt \\ &= [t]_0^{\infty} \\ &= \infty. \end{aligned}$$

Thus, we have shown that

$$\int_{-\infty}^{\infty} |h(t)| dt = \begin{cases} -\frac{1}{a} & a < 0 \\ \infty & a \geq 0. \end{cases}$$

In other words, the impulse response  $h$  is absolutely integrable if and only if  $a < 0$ . Consequently, the system is BIBO stable if and only if  $a < 0$ . ■

**Example 4.15.** Consider the LTI system with input  $x$  and output  $y$  defined by

$$y(t) = \int_{-\infty}^t x(\tau) d\tau$$

(i.e., an ideal integrator). Determine whether this system is BIBO stable.

*Solution.* First, we find the impulse response  $h$  of the system. We have

$$\begin{aligned} h(t) &= \int_{-\infty}^t \delta(\tau) d\tau \\ &= \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \\ &= u(t). \end{aligned}$$

Using this expression for  $h$ , we now check to see if  $h$  is absolutely integrable. We have

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)| dt &= \int_{-\infty}^{\infty} |u(t)| dt \\ &= \int_0^{\infty} 1 dt \\ &= \infty. \end{aligned}$$

Thus,  $h$  is not absolutely integrable. Therefore, the system is not BIBO stable. ■

## 4.10 Eigenfunctions of Continuous-Time LTI Systems

Earlier, in Section 3.7.7, we were introduced to notion of eigenfunctions of systems. Given that eigenfunctions have the potential to simplify the mathematics associated with systems, it is natural to wonder what eigenfunctions LTI systems might have. In this regard, the following theorem is enlightening.

**Theorem 4.12** (Eigenfunctions of LTI systems). *For an arbitrary LTI system  $\mathcal{H}$  with impulse response  $h$  and a function of the form  $x(t) = e^{st}$ , where  $s$  is an arbitrary complex constant (i.e.,  $x$  is an arbitrary complex exponential), the following holds:*

$$\mathcal{H}x(t) = H(s)e^{st},$$

where

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau} d\tau. \quad (4.49)$$

That is,  $x$  is an eigenfunction of  $\mathcal{H}$  with the corresponding eigenvalue  $H(s)$ .

*Proof.* We have

$$\begin{aligned} \mathcal{H}x(t) &= x * h(t) \\ &= h * x(t) \\ &= \int_{-\infty}^{\infty} h(\tau)x(t - \tau) d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)} d\tau \\ &= e^{st} \int_{-\infty}^{\infty} h(\tau)e^{-s\tau} d\tau \\ &= H(s)e^{st}. \end{aligned} \quad \blacksquare$$

As a matter of terminology, the function  $H$  that appears in the preceding theorem (i.e., Theorem 4.12) is referred to as the **system function** (or **transfer function**) of the system  $\mathcal{H}$ . The system function completely characterizes the behavior of a LTI system. Consequently, system functions are often useful when working with LTI systems. As it turns out, an integral of the form appearing in (4.49) is of great importance, as it defines what is called the Laplace transform. We will study the Laplace transform in great depth later in Chapter 7.

Let us now consider an application of eigenfunctions. Since convolution can often be quite painful to handle at the best of times, let us exploit eigenfunctions in order to devise a means to avoid having to deal with convolution directly in certain circumstances.

Suppose that we have a LTI system with input  $x$ , output  $y$ , and impulse response  $h$ . Suppose now that we can express some arbitrary input signal  $x$  as a sum of complex exponentials as follows:

$$x(t) = \sum_k a_k e^{s_k t}.$$

From the eigenvalue property, the response to the input  $a_k e^{s_k t}$  is  $a_k H(s_k) e^{s_k t}$ . By using this knowledge and the superposition property, we can write

$$\begin{aligned} y(t) &= \mathcal{H}x(t) \\ &= \mathcal{H} \left\{ \sum_k a_k e^{s_k t} \right\} (t) \\ &= \sum_k a_k \mathcal{H} \{ e^{s_k t} \} (t) \\ &= \sum_k a_k H(s_k) e^{s_k t}. \end{aligned}$$

Thus, if an input to a LTI system can be represented as a linear combination of complex exponentials, the output can also be represented as linear combination of the same complex exponentials.

**Example 4.16.** Consider the LTI system  $\mathcal{H}$  with the impulse response  $h$  given by

$$h(t) = \delta(t - 1). \quad (4.50)$$

(a) Find the system function  $H$  of the system  $\mathcal{H}$ . (b) Use the system function  $H$  to determine the response  $y$  of the system  $\mathcal{H}$  to the particular input  $x$  given by

$$x(t) = e^t \cos(\pi t).$$

*Solution.* (a) Substituting (4.50) into (4.49), we obtain

$$\begin{aligned} H(s) &= \int_{-\infty}^{\infty} h(t) e^{-st} dt \\ &= \int_{-\infty}^{\infty} \delta(t - 1) e^{-st} dt \\ &= e^{-s}. \end{aligned}$$

(b) To begin, we can rewrite  $x$  as

$$\begin{aligned} x(t) &= e^t \left[ \frac{1}{2} (e^{j\pi t} + e^{-j\pi t}) \right] \\ &= \frac{1}{2} e^{(1+j\pi)t} + \frac{1}{2} e^{(1-j\pi)t}. \end{aligned}$$

So, the input  $x$  is now expressed in the form

$$x(t) = \sum_{k=0}^1 a_k e^{s_k t},$$

where

$$a_0 = a_1 = \frac{1}{2}, \quad s_0 = 1 + j\pi, \quad \text{and} \quad s_1 = s_0^* = 1 - j\pi.$$

In part (a), we found the system function  $H$  to be  $H(s) = e^{-s}$ . So we can calculate  $y$  by using the system function as follows:

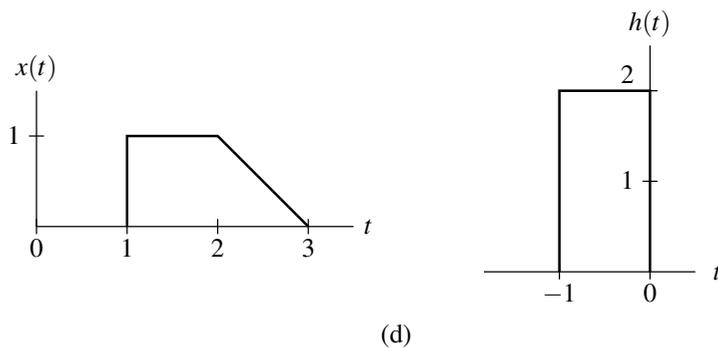
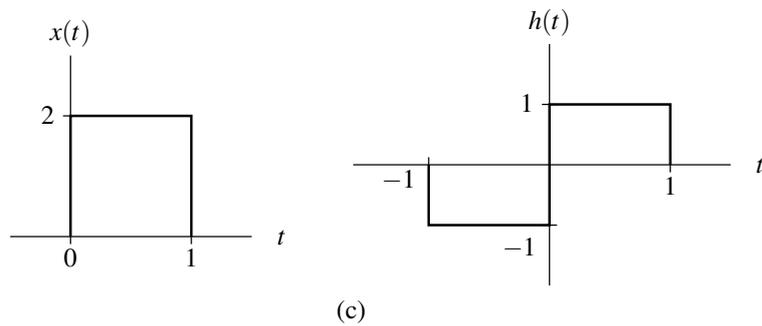
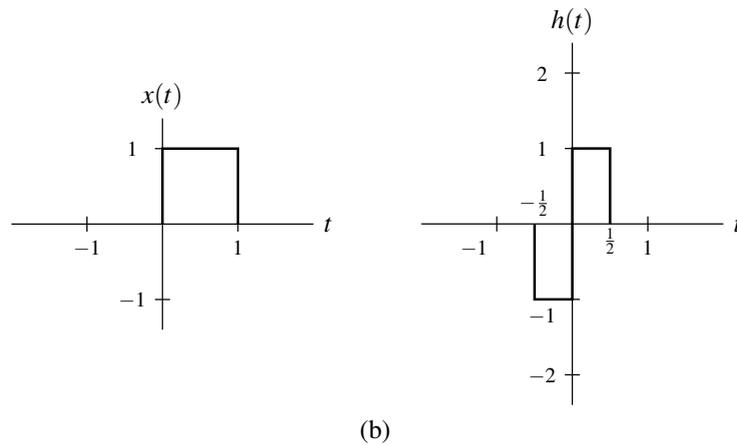
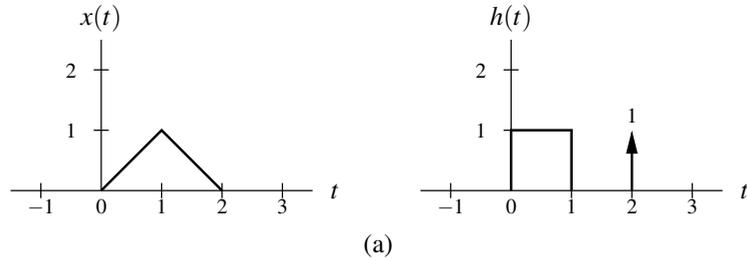
$$\begin{aligned} y(t) &= \sum_k a_k H(s_k) e^{s_k t} \\ &= a_0 H(s_0) e^{s_0 t} + a_1 H(s_1) e^{s_1 t} \\ &= \frac{1}{2} H(1 + j\pi) e^{(1+j\pi)t} + \frac{1}{2} H(1 - j\pi) e^{(1-j\pi)t} \\ &= \frac{1}{2} e^{-(1+j\pi)} e^{(1+j\pi)t} + \frac{1}{2} e^{-(1-j\pi)} e^{(1-j\pi)t} \\ &= \frac{1}{2} e^{t-1+j\pi t-j\pi} + \frac{1}{2} e^{t-1-j\pi t+j\pi} \\ &= \frac{1}{2} e^{t-1} e^{j\pi(t-1)} + \frac{1}{2} e^{t-1} e^{-j\pi(t-1)} \\ &= e^{t-1} \left[ \frac{1}{2} \left( e^{j\pi(t-1)} + e^{-j\pi(t-1)} \right) \right] \\ &= e^{t-1} \cos \pi(t-1). \end{aligned}$$

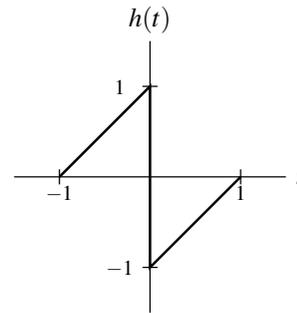
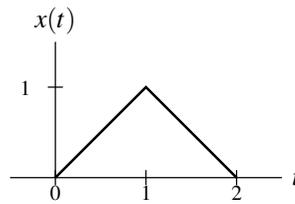
Observe that the output  $y$  is just the input  $x$  time shifted by 1. This is not a coincidence because, as it turns out, a LTI system with the system function  $H(s) = e^{-s}$  is an ideal unit delay (i.e., a system that performs a time shift of 1). ■

## 4.11 Exercises

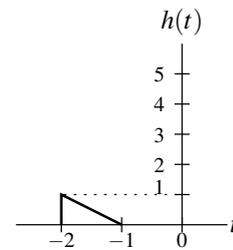
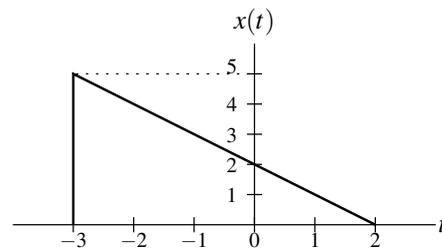
### 4.11.1 Exercises Without Answer Key

4.1 Using the graphical method, for each pair of signals  $x$  and  $h$  given in the figures below, compute  $x * h$ .





(e)



(f)

**4.2** For each pair of functions  $x$  and  $h$  given below, compute  $x * h$ .

- (a)  $x(t) = e^{at}u(t)$  and  $h(t) = e^{-at}u(t)$  where  $a$  is a nonzero real constant;  
 (b)  $x(t) = e^{-j\omega_0 t}u(t)$  and  $h(t) = e^{j\omega_0 t}u(t)$  where  $\omega_0$  is a strictly positive real constant;  
 (c)  $x(t) = u(t-2)$  and  $h(t) = u(t+3)$ ;  
 (d)  $x(t) = u(t)$  and  $h(t) = e^{-2t}u(t-1)$ ;  
 (e)  $x(t) = u(t-1) - u(t-2)$  and  $h(t) = e^t u(-t)$ .

**4.3** Using the graphical method, compute  $x * h$  for each pair of functions  $x$  and  $h$  given below.

- (a)  $x(t) = e^t u(-t)$  and  $h(t) = \begin{cases} t-1 & 1 \leq t < 2 \\ 0 & \text{otherwise;} \end{cases}$   
 (b)  $x(t) = e^{-|t|}$  and  $h(t) = \text{rect}(\frac{1}{3}[t - \frac{1}{2}])$ ;  
 (c)  $x(t) = e^{-t}u(t)$  and  $h(t) = \begin{cases} t-1 & 1 \leq t < 2 \\ 0 & \text{otherwise;} \end{cases}$   
 (d)  $x(t) = \text{rect}(t/2)$  and  $h(t) = e^{2-t}u(t-2)$ ;  
 (e)  $x(t) = e^{-|t|}$  and  $h(t) = \begin{cases} t+2 & -2 \leq t < -1 \\ 0 & \text{otherwise;} \end{cases}$   
 (f)  $x(t) = e^{-|t|}$  and  $h(t) = \begin{cases} t-1 & 1 \leq t < 2 \\ 0 & \text{otherwise;} \end{cases}$   
 (g)  $x(t) = \begin{cases} 1-t/4 & 0 \leq t < 4 \\ 0 & \text{otherwise} \end{cases}$  and  $h(t) = \begin{cases} t-1 & 1 \leq t < 2 \\ 0 & \text{otherwise;} \end{cases}$   
 (h)  $x(t) = \text{rect}(t/4)$  and  $h(t) = \begin{cases} 2-t & 1 \leq t < 2 \\ 0 & \text{otherwise;} \end{cases}$  and  
 (i)  $x(t) = e^{-t}u(t)$  and  $h(t) = \begin{cases} t-2 & 2 \leq t < 4 \\ 0 & \text{otherwise.} \end{cases}$

4.4 Let  $y = x * h$ . Given that

$$v(t) = \int_{-\infty}^{\infty} x(-\tau - b)h(\tau + at)d\tau,$$

where  $a$  and  $b$  are constants, express  $v$  in terms of  $y$ .

4.5 Consider the convolution  $y = x * h$ . Assuming that the convolution  $y$  exists, prove that each of the following assertions is true:

- (a) If  $x$  is periodic then  $y$  is periodic.
- (b) If  $x$  is even and  $h$  is odd, then  $y$  is odd.

4.6 From the definition of convolution, show that if  $y = x * h$ , then  $y'(t) = x * h'(t)$ , where  $y'$  and  $h'$  denote the derivatives of  $y$  and  $h$ , respectively.

4.7 Let  $x$  and  $h$  be functions satisfying

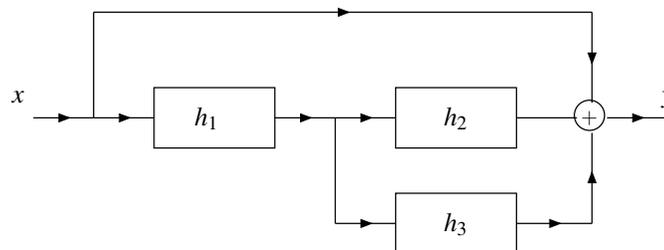
$$\begin{aligned} x(t) &= 0 \quad \text{for } t < A_1 \text{ or } t > A_2, \quad \text{and} \\ h(t) &= 0 \quad \text{for } t < B_1 \text{ or } t > B_2 \end{aligned}$$

(i.e.,  $x$  and  $h$  are finite duration). Determine for which values of  $t$  the quantity  $x * h(t)$  must be zero.

4.8 Find the impulse response of the LTI system  $\mathcal{H}$  characterized by each of the equations below.

- (a)  $\mathcal{H}x(t) = \int_{-\infty}^{t+1} x(\tau)d\tau$ ;
- (b)  $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau+5)e^{\tau-t+1}u(t-\tau-2)d\tau$ ;
- (c)  $\mathcal{H}x(t) = \int_{-\infty}^t x(\tau)v(t-\tau)d\tau$  and
- (d)  $\mathcal{H}x(t) = \int_{t-1}^t x(\tau)d\tau$ .

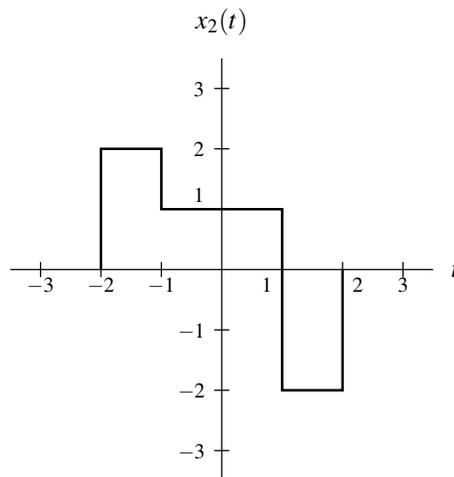
4.9 Consider the system with input  $x$  and output  $y$  as shown in the figure below. Each system in the block diagram is LTI and labelled with its impulse response.



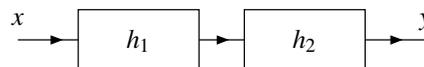
- (a) Find the impulse response  $h$  of the overall system in terms of  $h_1$ ,  $h_2$ , and  $h_3$ .
- (b) Determine the impulse response  $h$  in the specific case that

$$h_1(t) = \delta(t+1), \quad h_2(t) = \delta(t), \quad \text{and} \quad h_3(t) = \delta(t).$$

4.10 Consider a LTI system whose response to the signal  $x_1(t) = u(t) - u(t-1)$  is the signal  $y_1$ . Determine the response  $y_2$  of the system to the input  $x_2$  shown in the figure below in terms of  $y_1$ .



- 4.11** Suppose that we have the system shown in the figure below with input  $x$  and output  $y$ . This system is formed by the series interconnection of two LTI systems with the impulse responses  $h_1$  and  $h_2$ .



For each pair of  $h_1$  and  $h_2$  given below, find the output  $y$  if the input  $x(t) = u(t)$ .

- (a)  $h_1(t) = \delta(t)$  and  $h_2(t) = \delta(t)$ ;
- (b)  $h_1(t) = \delta(t+1)$  and  $h_2(t) = \delta(t+1)$ ;
- (c)  $h_1(t) = e^{-3t}u(t)$  and  $h_2(t) = \delta(t)$ .

- 4.12** Consider the LTI systems with the impulse responses given below. Determine whether each of these systems is causal and/or memoryless.

- (a)  $h(t) = (t+1)u(t-1)$ ;
- (b)  $h(t) = 2\delta(t+1)$ ;
- (c)  $h(t) = \frac{\omega_c}{\pi} \text{sinc } \omega_c t$ ;
- (d)  $h(t) = e^{-4t}u(t-1)$ ;
- (e)  $h(t) = e^t u(-1-t)$ ;
- (f)  $h(t) = e^{-3|t|}$ ; and
- (g)  $h(t) = 3\delta(t)$ .

- 4.13** Consider the LTI systems with the impulse responses given below. Determine whether each of these systems is BIBO stable.

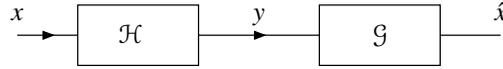
- (a)  $h(t) = e^{at}u(-t)$  where  $a$  is a strictly positive real constant;
- (b)  $h(t) = (1/t)u(t-1)$ ;
- (c)  $h(t) = e^t u(t)$ ;
- (d)  $h(t) = \delta(t-10)$ ;
- (e)  $h(t) = \text{rect } t$ ; and
- (f)  $h(t) = e^{-|t|}$ .

- 4.14** Suppose that we have two LTI systems with impulse responses

$$h_1(t) = \frac{1}{2}\delta(t-1) \quad \text{and} \quad h_2(t) = 2\delta(t+1).$$

Determine whether these systems are inverses of one another.

- 4.15 Consider the system shown in the figure below, where  $\mathcal{H}$  is a LTI system and  $\mathcal{G}$  is known to be the inverse system of  $\mathcal{H}$ . Let  $y_1 = \mathcal{H}x_1$  and  $y_2 = \mathcal{H}x_2$ .



- (a) Determine the response of the system  $\mathcal{G}$  to the input  $y'(t) = a_1y_1 + a_2y_2$  where  $a_1$  and  $a_2$  are complex constants.
- (b) Determine the response of the system  $\mathcal{G}$  to the input  $y'_1(t) = y_1(t - t_0)$  where  $t_0$  is a real constant.
- (c) Using the results of the previous parts of this question, determine whether the system  $\mathcal{G}$  is linear and/or time invariant.
- 4.16 Suppose that we have the systems  $\mathcal{H}_1$ ,  $\mathcal{H}_2$ ,  $\mathcal{H}_3$ , and  $\mathcal{H}_4$ , whose responses to a complex exponential input  $e^{j2t}$  are given by
- $$\mathcal{H}_1(e^{j2t}) = 2e^{j2t}, \quad \mathcal{H}_2(e^{j2t}) = te^{j2t}, \quad \mathcal{H}_3(e^{j2t}) = e^{j2t+\pi/3}, \quad \text{and} \quad \mathcal{H}_4(e^{j2t}) = \cos 2t.$$
- Indicate which of these systems cannot be LTI.
- 4.17 Show that, for any function  $x$ ,  $x * v(t) = x(t - t_0)$ , where  $v(t) = \delta(t - t_0)$  and  $t_0$  is an arbitrary real constant.

### 4.11.2 Exercises With Answer Key

- 4.18 Using the graphical method, compute  $x * h$  for each pair of functions  $x$  and  $h$  given below.

- (a)  $x(t) = 2\text{rect}(t - \frac{1}{2})$  and  $h(t) = \begin{cases} -1 & -1 \leq t < 0 \\ 1 & 0 \leq t < 1 \\ 0 & \text{otherwise;} \end{cases}$
- (b)  $x(t) = u(t - 1)$  and  $h(t) = \begin{cases} t + 1 & -1 \leq t < 0 \\ t - 1 & 0 \leq t < 1 \\ 0 & \text{otherwise;} \end{cases}$
- (c)  $x(t) = \begin{cases} t - 2 & 1 \leq t < 3 \\ 0 & \text{otherwise} \end{cases}$  and  $h(t) = \text{rect}[\frac{1}{2}(t + 2)]$ ;
- (d)  $x(t) = \text{rect}[\frac{1}{3}(t - \frac{3}{2})]$  and  $h(t) = \begin{cases} t - 1 & 1 \leq t < 2 \\ 0 & \text{otherwise;} \end{cases}$
- (e)  $x(t) = \begin{cases} \frac{1}{4}(t - 1)^2 & 1 \leq t < 3 \\ 0 & \text{otherwise} \end{cases}$  and  $h(t) = \begin{cases} t - 1 & 1 \leq t < 2 \\ 0 & \text{otherwise;} \end{cases}$
- (f)  $x(t) = \begin{cases} 2\cos(\frac{\pi}{4}t) & 0 \leq t < 2 \\ 0 & \text{otherwise} \end{cases}$  and  $h(t) = \begin{cases} 2 - t & 1 \leq t < 2 \\ 0 & \text{otherwise;} \end{cases}$
- (g)  $x(t) = e^{-|t|}$  and  $h(t) = \text{rect}[\frac{1}{2}(t - 2)]$ ;
- (h)  $x(t) = \begin{cases} \frac{1}{2}t - \frac{1}{2} & 1 \leq t < 3 \\ 0 & \text{otherwise} \end{cases}$  and  $h(t) = \begin{cases} -t - 1 & -2 \leq t < -1 \\ 0 & \text{otherwise;} \end{cases}$
- (i)  $x(t) = e^{-|t|}$  and  $h(t) = \text{tri}[\frac{1}{2}(t - 3)]$ ;
- (j)  $x(t) = \begin{cases} \frac{1}{4}t - \frac{1}{4} & 1 \leq t < 5 \\ 0 & \text{otherwise} \end{cases}$  and  $h(t) = \begin{cases} \frac{3}{2} - \frac{1}{2}t & 1 \leq t < 3 \\ 0 & \text{otherwise;} \end{cases}$

- (k)  $x(t) = \text{rect}(t/20)$  and  $h(t) = \begin{cases} t-1 & 1 \leq t < 2 \\ 0 & \text{otherwise;} \end{cases}$
- (l)  $x(t) = \begin{cases} 1 - \frac{1}{100}t & 0 \leq t < 100 \\ 0 & \text{otherwise} \end{cases}$  and  $h(t) = e^{-t}u(t-1)$ ;
- (m)  $x(t) = \text{rect}(t/20)$  and  $h(t) = \begin{cases} 1 - (t-2)^2 & 1 \leq t < 3 \\ 0 & \text{otherwise;} \end{cases}$
- (n)  $x(t) = e^{-t}u(t)$  and  $h(t) = e^{-3t}u(t-2)$ ;
- (o)  $x(t) = e^{-|t|}$  and  $h(t) = \text{rect}(t - \frac{3}{2})$ ;
- (p)  $x(t) = e^{-2t}u(t)$  and  $h(t) = \text{rect}(t - \frac{5}{2})$ ;
- (q)  $x(t) = u(t-1)$  and  $h(t) = \begin{cases} \sin[\pi(t-1)] & 1 \leq t < 2 \\ 0 & \text{otherwise;} \end{cases}$
- (r)  $x(t) = u(t)$  and  $h(t) = \text{rect}(\frac{1}{4}[t-4])$ ;
- (s)  $x(t) = e^{-t}u(t)$  and  $h(t) = e^{2-2t}u(t-1)$ ;
- (t)  $x(t) = e^{-3t}$  and  $h(t) = u(t+1)$ ; and
- (u)  $x(t) = \begin{cases} 2-t & 1 \leq t < 2 \\ 0 & \text{otherwise} \end{cases}$  and  $h(t) = \begin{cases} -t-2 & -3 \leq t < -2 \\ 0 & \text{otherwise.} \end{cases}$

**Short Answer.** (a)  $x * h(t) = \begin{cases} \int_0^{t+1} -2d\tau & -1 \leq t < 0 \\ \int_0^t 2d\tau + \int_t^1 -2d\tau & 0 \leq t < 1 \\ \int_{t-1}^1 2d\tau & 1 \leq t < 2 \\ 0 & \text{otherwise} \end{cases}$

(b)  $x * h(t) = \begin{cases} \int_1^{t+1} (-\tau + t + 1)d\tau & 0 \leq t < 1 \\ \int_1^t (-\tau + t - 1)d\tau + \int_t^{t+1} (-\tau + t + 1)d\tau & 1 \leq t < 2 \\ \int_{t-1}^t (-\tau + t - 1)d\tau + \int_t^{t+1} (-\tau + t + 1)d\tau & t \geq 2 \\ 0 & \text{otherwise} \end{cases}$

(c)  $x * h(t) = \begin{cases} \int_1^{t+3} (\tau - 2)d\tau & -2 \leq t < 0 \\ \int_{t+1}^3 (\tau - 2)d\tau & 0 \leq t < 2 \\ 0 & \text{otherwise} \end{cases}$

(d)  $x * h(t) = \begin{cases} \int_0^{t-1} (t - \tau - 1)d\tau & 1 \leq t < 2 \\ \int_{t-2}^{t-1} (t - \tau - 1)d\tau & 2 \leq t < 4 \\ \int_{t-2}^3 (t - \tau - 1)d\tau & 4 \leq t < 5 \\ 0 & \text{otherwise} \end{cases}$

(e)  $x * h(t) = \begin{cases} \int_1^{t-1} \frac{1}{4}(\tau - 1)^2(t - \tau - 1)d\tau & 2 \leq t < 3 \\ \int_{t-2}^{t-1} \frac{1}{4}(\tau - 1)^2(t - \tau - 1)d\tau & 3 \leq t < 4 \\ \int_{t-2}^3 \frac{1}{4}(\tau - 1)^2(t - \tau - 1)d\tau & 4 \leq t < 5 \\ 0 & \text{otherwise} \end{cases}$

(f)  $x * h(t) = \begin{cases} \int_0^{t-1} 2 \cos(\frac{\pi}{4}\tau) (\tau - t + 2)d\tau & 1 \leq t < 2 \\ \int_{t-2}^{t-1} 2 \cos(\frac{\pi}{4}\tau) (\tau - t + 2)d\tau & 2 \leq t < 3 \\ \int_{t-2}^2 2 \cos(\frac{\pi}{4}\tau) (\tau - t + 2)d\tau & 3 \leq t < 4 \\ 0 & \text{otherwise} \end{cases}$

(g)  $x * h(t) = \begin{cases} \int_{t-1}^{t-3} e^\tau d\tau & t < 1 \\ \int_{t-3}^0 e^\tau d\tau + \int_0^{t-1} e^{-\tau} d\tau & 1 \leq t < 3 \\ \int_{t-3}^{t-1} e^{-\tau} d\tau & t \geq 3 \end{cases}$

$$\begin{aligned}
\text{(h) } x * h(t) &= \begin{cases} \int_1^{t+2} \left(\frac{1}{2}\tau - \frac{1}{2}\right) (\tau - t - 1) d\tau & -1 \leq t < 0 \\ \int_{t+1}^{t+2} \left(\frac{1}{2}\tau - \frac{1}{2}\right) (\tau - t - 1) d\tau & 0 \leq t < 1 \\ \int_{t+1}^3 \left(\frac{1}{2}\tau - \frac{1}{2}\right) (\tau - t - 1) d\tau & 1 \leq t < 2 \\ 0 & \text{otherwise} \end{cases} \\
\text{(i) } x * h(t) &= \begin{cases} \int_{t-4}^{t-3} e^\tau (\tau - t + 4) d\tau + \int_{t-3}^{t-2} e^\tau (t - \tau - 2) d\tau & t < 2 \\ \int_{t-4}^{t-3} e^\tau (\tau - t + 4) d\tau + \int_{t-3}^0 e^\tau (t - \tau - 2) d\tau + \int_0^{t-2} e^{-\tau} (t - \tau - 2) d\tau & 2 \leq t < 3 \\ \int_{t-4}^0 e^\tau (\tau - t + 4) d\tau + \int_0^{t-3} e^{-\tau} (t - \tau + 4) d\tau + \int_{t-3}^{t-2} e^{-\tau} (t - \tau - 2) d\tau & 3 \leq t < 4 \\ \int_{t-4}^{t-3} e^{-\tau} (\tau - t + 4) d\tau + \int_{t-3}^{t-2} e^{-\tau} (t - \tau - 2) d\tau & t \geq 4 \end{cases} \\
\text{(j) } x * h(t) &= \begin{cases} \int_1^{t+1} \left(\frac{1}{4}\tau - \frac{1}{4}\right) \left(\frac{1}{2}\tau - \frac{1}{2}t + \frac{3}{2}\right) d\tau & 2 \leq t < 4 \\ \int_{t-3}^{t-1} \left(\frac{1}{4}\tau - \frac{1}{4}\right) \left(\frac{1}{2}\tau - \frac{1}{2}t + \frac{3}{2}\right) d\tau & 4 \leq t < 6 \\ \int_{t-3}^5 \left(\frac{1}{4}\tau - \frac{1}{4}\right) \left(\frac{1}{2}\tau - \frac{1}{2}t + \frac{3}{2}\right) d\tau & 6 \leq t < 8 \\ 0 & \text{otherwise} \end{cases} \\
\text{(k) } x * h(t) &= \begin{cases} \int_{-10}^{t-1} (t - \tau - 1) d\tau & -9 \leq t < -8 \\ \int_{t-2}^{t-1} (t - \tau - 1) d\tau & -8 \leq t < 11 \\ \int_{10}^{t-1} (t - \tau - 1) d\tau & 11 \leq t < 12 \\ 0 & \text{otherwise} \end{cases} \\
\text{(l) } x * h(t) &= \begin{cases} 0 & t < 1 \\ \int_0^{t-1} \left(1 - \frac{1}{100}\tau\right) e^{\tau-t} d\tau & 1 \leq t < 101 \\ \int_0^{100} \left(1 - \frac{1}{100}\tau\right) e^{\tau-t} d\tau & t \geq 101 \end{cases} \\
\text{(m) } x * h(t) &= \begin{cases} \int_{-10}^{t-1} [1 - (t - \tau - 2)^2] d\tau & -9 \leq t < -7 \\ \int_{t-3}^{t-1} [1 - (t - \tau - 2)^2] d\tau & -7 \leq t < 11 \\ \int_{t-3}^{10} [1 - (t - \tau - 2)^2] d\tau & 11 \leq t < 13 \\ 0 & \text{otherwise} \end{cases} \\
\text{(n) } x * h(t) &= \begin{cases} \int_0^{t-2} e^{-\tau} e^{3\tau-3t} d\tau & t \geq 2 \\ 0 & \text{otherwise} \end{cases} \\
\text{(o) } x * h(t) &= \begin{cases} \int_{t-2}^{t-1} e^\tau d\tau & t < 1 \\ \int_{t-2}^0 e^\tau d\tau + \int_0^{t-1} e^{-\tau} d\tau & 1 \leq t < 2 \\ \int_{t-2}^{t-1} e^{-\tau} d\tau & t \geq 2 \end{cases} \\
\text{(p) } x * h(t) &= \begin{cases} 0 & t < 2 \\ \int_0^{t-2} e^{-2\tau} d\tau & 2 \leq t < 3 \\ \int_{t-3}^{t-2} e^{-2\tau} d\tau & t \geq 3 \end{cases} \\
\text{(q) } x * h(t) &= \begin{cases} 0 & t < 2 \\ \int_1^{t-1} \sin(\pi t - \pi\tau - \pi) d\tau & 2 \leq t < 3 \\ \int_{t-2}^{t-1} \sin(\pi t - \pi\tau - \pi) d\tau & t \geq 3 \end{cases} \\
\text{(r) } x * h(t) &= \begin{cases} 0 & t < 2 \\ \int_0^{t-2} 1 d\tau & 2 \leq t < 6 \\ \int_{t-6}^{t-2} 1 d\tau & t \geq 6 \end{cases} \\
\text{(s) } x * h(t) &= \begin{cases} \int_0^{t-1} e^{\tau-2t+2} d\tau & t \geq 1 \\ 0 & \text{otherwise} \end{cases} \\
\text{(t) } x * h(t) &= \begin{cases} \int_0^{t+1} e^{-3\tau} d\tau & t \geq -1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

$$(u) \ x * h(t) = \begin{cases} \frac{1}{6}t^3 - t - \frac{2}{3} & -2 \leq t < -1 \\ -\frac{1}{6}t^3 & -1 \leq t < 0 \\ 0 & \text{otherwise;} \end{cases}$$

**4.19** Using the graphical method, compute  $x * h$  for each pair of functions  $x$  and  $h$  given below.

(a)  $x(t) = \text{rect} \frac{t}{2a}$  and  $h(t) = \text{rect} \frac{t}{2a}$ ; and

(b)  $x(t) = \text{rect} \frac{t}{a}$  and  $h(t) = \text{rect} \frac{t}{a}$ .

**Short Answer.** (a)  $x * h(t) = 2a \text{tri} \left( \frac{t}{4a} \right)$ ; (b)  $x * h(t) = a \text{tri} \left( \frac{t}{2a} \right)$ .

**4.20** Determine whether the LTI system with each impulse response  $h$  given below is causal.

(a)  $h(t) = u(t+1) - u(t-1)$ ; and

(b)  $h(t) = e^{-5t}u(t-1)$ .

**Short Answer.** (a) not causal (b) causal

**4.21** Determine whether the LTI system with each impulse response  $h$  given below is BIBO stable.

(a)  $h(t) = u(t-1) - u(t-2)$ .

**Short Answer.** (a) stable

**4.22** Compute the convolution  $x * h$ , where

$$x(t) = \cos 3t \quad \text{and} \quad h(t) = u(t).$$

**Short Answer.**  $x * h(t) = \frac{1}{3} \sin 3t$



## Chapter 5

# Continuous-Time Fourier Series

### 5.1 Introduction

One very important tool in the study of signals and systems is the Fourier series. A very large class of functions can be represented using Fourier series, namely most practically useful periodic functions. The Fourier series represents a periodic function as a (possibly infinite) linear combination of complex sinusoids. This is often desirable since complex sinusoids are easy functions with which to work. For example, complex sinusoids are easy to integrate and differentiate. Also, complex sinusoids have important properties in relation to LTI systems. In particular, complex sinusoids are eigenfunctions of LTI systems. Therefore, the response of a LTI system to a complex sinusoid is the same complex sinusoid multiplied by a complex scaling factor.

### 5.2 Definition of Continuous-Time Fourier Series

Suppose that we have a set of **harmonically-related** complex sinusoids of the form

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t} \quad k = 0, \pm 1, \pm 2, \dots$$

The fundamental frequency of the  $k$ th complex sinusoid  $\phi_k$  is  $k\omega_0$ , an integer multiple of  $\omega_0$ . Since the fundamental frequency of each of the harmonically-related complex sinusoids is an integer multiple of  $\omega_0$ , a linear combination of these complex sinusoids must be periodic. More specifically, a linear combination of these complex sinusoids is periodic with period  $T = \frac{2\pi}{\omega_0}$ .

Suppose that we can represent a periodic complex function  $x$  as a linear combination of harmonically-related complex sinusoids as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \quad (5.1)$$

Such a representation is known as a **Fourier series**. More specifically, this is the **complex exponential form** of the Fourier series. As a matter of terminology, we refer to (5.1) as the **Fourier series synthesis equation**. The terms in the summation for  $k = 1$  and  $k = -1$  are known as the fundamental frequency components or **first harmonic components**, and have the fundamental frequency  $\omega_0$ . More generally, the terms in the summation for  $k = K$  and  $k = -K$  are called the  $K$ th **harmonic components**, and have the fundamental frequency  $K\omega_0$ . Since the complex sinusoids are harmonically related, the function  $x$  is periodic with period  $T = \frac{2\pi}{\omega_0}$  (and frequency  $\omega_0$ ).

Since we often work with Fourier series, it is sometimes convenient to have an abbreviated notation to indicate that a function is associated with particular Fourier series coefficients. If a function  $x$  has the Fourier series coefficient sequence  $c$ , we sometimes indicate this using the notation

$$x(t) \xleftrightarrow{\text{CTFS}} c_k.$$

Consider the Fourier series representation of the periodic function  $x$  given by (5.1). In the most general case,  $x$  is a complex function, but let us now suppose that  $x$  is real. In the case of real functions, an important relationship exists between the Fourier series coefficients  $c_k$  and  $c_{-k}$  as given by the theorem below.

**Theorem 5.1** (Fourier series of real function). *Let  $x$  be a periodic function with Fourier series coefficient sequence  $c$ . The function  $x$  is real if and only if*

$$c_k = c_{-k}^* \text{ for all } k \quad (5.2)$$

(i.e.,  $c$  is conjugate symmetric).

*Proof.* Suppose that we can represent  $x$  in the form of a Fourier series, as given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \quad (5.3)$$

Taking the complex conjugate of both sides of the preceding equation, we obtain

$$\begin{aligned} x^*(t) &= \left( \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \right)^* \\ &= \sum_{k=-\infty}^{\infty} \left( c_k e^{jk\omega_0 t} \right)^* \\ &= \sum_{k=-\infty}^{\infty} c_k^* e^{-jk\omega_0 t}. \end{aligned}$$

Replacing  $k$  by  $-k$  in the summation of the preceding equation, we obtain

$$x^*(t) = \sum_{k=-\infty}^{\infty} c_{-k}^* e^{jk\omega_0 t}. \quad (5.4)$$

Suppose now that  $x$  is real. Then,  $x^* = x$  and the right-hand sides of (5.3) and (5.4) must be equal, implying that  $c_k = c_{-k}^*$  for all  $k$ .

Suppose now that  $c_k = c_{-k}^*$  for all  $k$ . Then, the right-hand sides of (5.3) and (5.4) must be equal, implying that  $x^* = x$  (i.e.,  $x$  is real). ■

Using the relationship in (5.2), we can derive two alternative forms of the Fourier series for the case of real signals. We begin by rewriting (5.1) in a slightly different form. In particular, we rearrange the summation to obtain

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left[ c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t} \right].$$

Substituting  $c_k = c_{-k}^*$  from (5.2), we obtain

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left[ c_k e^{jk\omega_0 t} + c_k^* e^{-jk\omega_0 t} \right].$$

Now, we observe that the two terms inside the summation are complex conjugates of each other. So, we can rewrite the equation as

$$x(t) = c_0 + \sum_{k=1}^{\infty} 2 \operatorname{Re}(c_k e^{jk\omega_0 t}). \quad (5.5)$$

Let us now rewrite  $c_k$  in polar form as

$$c_k = |c_k| e^{j\theta_k},$$

where  $\theta_k$  is real (i.e.,  $\theta_k = \arg c_k$ ). Substituting this expression for  $c_k$  into (5.5) yields

$$\begin{aligned} x(t) &= c_0 + \sum_{k=1}^{\infty} 2 \operatorname{Re} \left[ |c_k| e^{j(k\omega_0 t + \theta_k)} \right] \\ &= c_0 + \sum_{k=1}^{\infty} 2 \operatorname{Re} (|c_k| [\cos(k\omega_0 t + \theta_k) + j \sin(k\omega_0 t + \theta_k)]) \\ &= c_0 + \sum_{k=1}^{\infty} 2 \operatorname{Re} [|c_k| \cos(k\omega_0 t + \theta_k) + j |c_k| \sin(k\omega_0 t + \theta_k)]. \end{aligned}$$

Finally, further simplification yields

$$x(t) = c_0 + 2 \sum_{k=1}^{\infty} |c_k| \cos(k\omega_0 t + \theta_k)$$

(where  $\theta_k = \arg c_k$ ). This is known as the **combined trigonometric form** of a Fourier series.

A second alternative form of the Fourier series can be obtained by expressing  $c_k$  in Cartesian form as

$$c_k = \frac{1}{2}(a_k - jb_k).$$

where  $a_k$  and  $b_k$  are real. Substituting this expression for  $c_k$  into (5.5) from earlier yields

$$\begin{aligned} x(t) &= c_0 + \sum_{k=1}^{\infty} 2 \operatorname{Re} \left[ \frac{1}{2}(a_k - jb_k) e^{jk\omega_0 t} \right] \\ &= c_0 + \sum_{k=1}^{\infty} \operatorname{Re} [(a_k - jb_k) (\cos k\omega_0 t + j \sin k\omega_0 t)] \\ &= c_0 + \sum_{k=1}^{\infty} \operatorname{Re} (a_k \cos k\omega_0 t + ja_k \sin k\omega_0 t - jb_k \cos k\omega_0 t + b_k \sin k\omega_0 t). \end{aligned}$$

Further simplification yields

$$x(t) = c_0 + \sum_{k=1}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t)$$

(where  $a_k = \operatorname{Re} 2c_k$  and  $b_k = -\operatorname{Im} 2c_k$ ). This is known as the **trigonometric form** of a Fourier series.

By comparing the various forms of the Fourier series introduced above, we can see that the quantities  $c_k$ ,  $a_k$ ,  $b_k$ , and  $\theta_k$  are related as

$$2c_k = a_k - jb_k \quad \text{and} \quad c_k = |c_k| e^{j\theta_k}.$$

(Recall that  $a_k$ ,  $b_k$ , and  $\theta_k$  are real and  $c_k$  is complex.) Note that each of the trigonometric and combined-trigonometric forms of Fourier series only involve real quantities, whereas the exponential form involves some complex quantities. For this reason, the trigonometric and combined-trigonometric forms may sometimes be preferred when dealing with Fourier series of real functions.

### 5.3 Determining the Fourier Series Representation of a Continuous-Time Periodic Function

Given an arbitrary periodic function  $x$ , we need some means for finding its corresponding Fourier series representation. In other words, we need a method for calculating the Fourier series coefficient sequence  $c$ . Such a method is given by the theorem below.

**Theorem 5.2** (Fourier series analysis equation). *The Fourier series coefficient sequence  $c$  of a periodic function  $x$  with fundamental period  $T$  is given by*

$$c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt, \quad (5.6)$$

where  $\int_T$  denotes integration over any interval of length  $T$ .

*Proof.* We begin with the definition of the Fourier series in (5.1). Multiplying both sides of this equation by  $e^{-jn\omega_0 t}$  yields

$$\begin{aligned} x(t) e^{-jn\omega_0 t} &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} e^{-jn\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} c_k e^{j(k-n)\omega_0 t}. \end{aligned}$$

As a matter of notation, we use  $\int_T$  to denote the integral over an arbitrary interval of length  $T$  (i.e., the interval  $(t_0, t_0 + T)$  for arbitrary  $t_0$ ). Integrating both sides of this equation over one period  $T$  of  $x$ , we obtain

$$\int_T x(t) e^{-jn\omega_0 t} dt = \int_T \sum_{k=-\infty}^{\infty} c_k e^{j(k-n)\omega_0 t} dt.$$

Reversing the order of integration and summation yields

$$\int_T x(t) e^{-jn\omega_0 t} dt = \sum_{k=-\infty}^{\infty} c_k \left( \int_T e^{j(k-n)\omega_0 t} dt \right). \quad (5.7)$$

Now, we note that the following identity holds:

$$\int_T e^{j(k-n)\omega_0 t} dt = \begin{cases} T & k = n \\ 0 & \text{otherwise.} \end{cases} \quad (5.8)$$

(The proof of this fact is left as an exercise for the reader in Exercise A.11.) Substituting (5.8) into (5.7), we obtain

$$\int_T x(t) e^{-jn\omega_0 t} dt = c_n T. \quad (5.9)$$

Rearranging, we obtain

$$c_n = \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t} dt. \quad \blacksquare$$

As a matter of terminology, we refer to (5.6) as the **Fourier series analysis equation**.

Suppose that we have a complex periodic function  $x$  with period  $T$  and Fourier series coefficient sequence  $c$ . One can easily show that the coefficient  $c_0$  is the average value of  $x$  over a single period  $T$ . The proof is trivial. Consider the Fourier series analysis equation given by (5.6). Substituting  $k = 0$  into this equation, we obtain

$$\begin{aligned} c_0 &= \left[ \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \right] \Big|_{k=0} \\ &= \frac{1}{T} \int_T x(t) e^0 dt \\ &= \frac{1}{T} \int_T x(t) dt. \end{aligned}$$

Thus,  $c_0$  is simply the average value of  $x$  over a single period.

**Example 5.1** (Fourier series of a periodic square wave). Find the Fourier series representation of the periodic square wave  $x$  shown in Figure 5.1.

*Solution.* Let us consider the single period of  $x(t)$  for  $0 \leq t < T$ . For this range of  $t$ , we have

$$x(t) = \begin{cases} A & 0 \leq t < \frac{T}{2} \\ -A & \frac{T}{2} \leq t < T. \end{cases}$$

We use the Fourier series analysis equation (5.6) to write (and subsequently assume that  $k \neq 0$ ):

$$\begin{aligned} c_k &= \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\ &= \frac{1}{T} \left( \int_0^{T/2} A e^{-jk\omega_0 t} dt + \int_{T/2}^T (-A) e^{-jk\omega_0 t} dt \right). \end{aligned}$$

Assuming that  $k \neq 0$ , we have (after integration)

$$\begin{aligned} c_k &= \frac{1}{T} \left( \left[ \frac{-A}{jk\omega_0} e^{-jk\omega_0 t} \right]_0^{T/2} + \left[ \frac{A}{jk\omega_0} e^{-jk\omega_0 t} \right]_{T/2}^T \right) \\ &= \frac{-A}{j2\pi k} \left( \left[ e^{-jk\omega_0 t} \right]_0^{T/2} - \left[ e^{-jk\omega_0 t} \right]_{T/2}^T \right) \\ &= \frac{jA}{2\pi k} \left( \left[ e^{-j\pi k} - 1 \right] - \left[ e^{-j2\pi k} - e^{-j\pi k} \right] \right) \\ &= \frac{jA}{2\pi k} \left[ 2e^{-j\pi k} - e^{-j2\pi k} - 1 \right] \\ &= \frac{jA}{2\pi k} \left[ 2(e^{-j\pi})^k - (e^{-j2\pi})^k - 1 \right]. \end{aligned}$$

Now, we observe that  $e^{-j\pi} = -1$  and  $e^{-j2\pi} = 1$ . So, we have

$$\begin{aligned} c_k &= \frac{jA}{2\pi k} [2(-1)^k - 1^k - 1] \\ &= \frac{jA}{2\pi k} [2(-1)^k - 2] \\ &= \frac{jA}{\pi k} [(-1)^k - 1] \\ &= \begin{cases} \frac{-j2A}{\pi k} & k \text{ odd} \\ 0 & k \text{ even, } k \neq 0. \end{cases} \end{aligned}$$

Since we have not yet considered the case of  $k = 0$ , we now consider this case. We have

$$\begin{aligned} c_0 &= \frac{1}{T} \int_T x(t) dt \\ &= \frac{1}{T} \left[ \int_0^{T/2} A dt - \int_{T/2}^T A dt \right] \\ &= \frac{1}{T} \left[ \frac{AT}{2} - \frac{AT}{2} \right] \\ &= 0. \end{aligned}$$

Thus, the Fourier series of  $x$  is given by (5.1) where the coefficient sequence  $c$  is

$$c_k = \begin{cases} \frac{-j2A}{\pi k} & k \text{ odd} \\ 0 & k \text{ even.} \end{cases} \quad \blacksquare$$

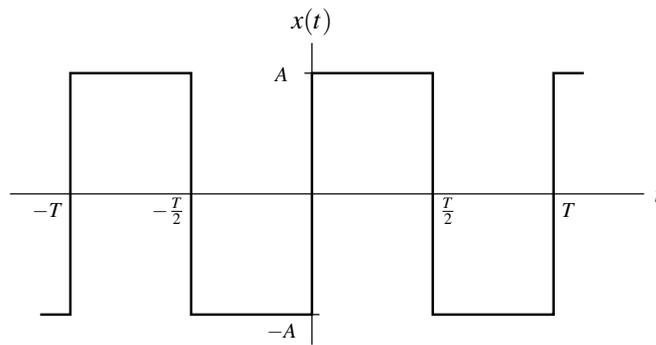


Figure 5.1: Periodic square wave.

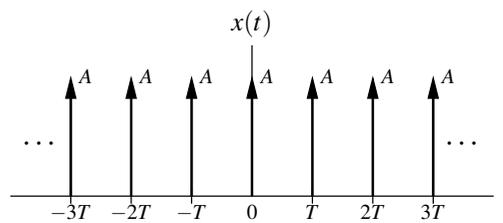


Figure 5.2: Periodic impulse train.

**Example 5.2** (Fourier series of a periodic impulse train). Consider the periodic impulse train  $x$  shown in Figure 5.2. Find the Fourier series representation of  $x$ .

*Solution.* Let us consider the single period of  $x(t)$  for  $-\frac{T}{2} \leq t < \frac{T}{2}$ . From the Fourier series analysis equation (5.6), we have

$$\begin{aligned} c_k &= \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\ &= \frac{1}{T} \int_{-T/2}^{T/2} A \delta(t) e^{-jk\omega_0 t} dt \\ &= \frac{A}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt. \end{aligned}$$

Using the sifting property of the unit-impulse function, we can simplify the above result to obtain

$$c_k = \frac{A}{T}.$$

Thus, the Fourier series for  $x$  is given by (5.1) where the coefficient sequence  $c_k$  is  $c_k = \frac{A}{T}$ . ■

**Example 5.3.** Consider the periodic function  $x$  with fundamental period  $T = 3$  as shown in Figure 5.3. Find the Fourier series representation of  $x$ .

*Solution.* The function  $x$  has the fundamental frequency  $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{3}$ . Let us consider the single period of  $x(t)$  for

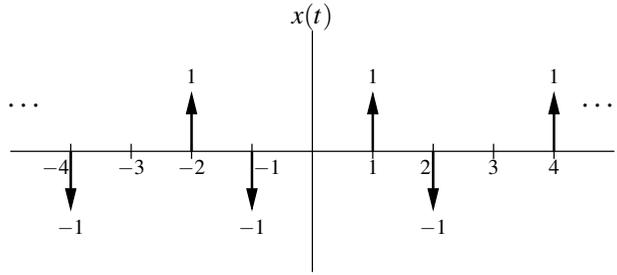


Figure 5.3: Periodic impulse train.

$-\frac{T}{2} \leq t < \frac{T}{2}$  (i.e.,  $-\frac{3}{2} \leq t < \frac{3}{2}$ ). Computing the Fourier series coefficients  $c_k$ , we have

$$\begin{aligned}
 c_k &= \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\
 &= \frac{1}{T} \int_{-3/2}^{3/2} x(t) e^{-jk\omega_0 t} dt \\
 &= \frac{1}{T} \int_{-3/2}^{3/2} [-\delta(t+1) + \delta(t-1)] e^{-jk\omega_0 t} dt \\
 &= \frac{1}{T} \left[ \int_{-3/2}^{3/2} -\delta(t+1) e^{-jk\omega_0 t} dt + \int_{-3/2}^{3/2} \delta(t-1) e^{-jk\omega_0 t} dt \right] \\
 &= \frac{1}{T} \left[ -e^{-jk\omega_0(-1)} + e^{-jk\omega_0(1)} \right] \\
 &= \frac{1}{T} \left[ e^{-jk\omega_0} - e^{jk\omega_0} \right] \\
 &= \frac{1}{T} [2j \sin(-k\omega_0)] \\
 &= \frac{2j}{T} \sin(-k\omega_0) \\
 &= -\frac{2j}{T} \sin k\omega_0 \\
 &= -\frac{2j}{3} \sin \frac{2\pi k}{3}.
 \end{aligned}$$

Thus,  $x$  has the Fourier series representation

$$\begin{aligned}
 x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \\
 &= \sum_{k=-\infty}^{\infty} -\frac{2j}{3} (\sin \frac{2\pi k}{3}) e^{j2\pi k t / 3}.
 \end{aligned}$$

**Example 5.4** (Fourier series of an even real function). Let  $x$  be an arbitrary periodic real function that is even. Let  $c$  denote the Fourier series coefficient sequence for  $x$ . Show that

- $c$  is real (i.e.,  $\text{Im}\{c_k\} = 0$  for all  $k$ );
- $c$  is even (i.e.,  $c_k = c_{-k}$  for all  $k$ ); and
- $c_0 = \frac{1}{T} \int_0^T x(t) dt$ .

*Solution.* From the Fourier series analysis equation (5.6) and using Euler's relation, we can write

$$\begin{aligned}
 c_k &= \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\
 &= \frac{1}{T} \int_T (x(t) [\cos(-k\omega_0 t) + j \sin(-k\omega_0 t)]) dt.
 \end{aligned}$$

Since  $\cos$  and  $\sin$  are even and odd functions, respectively, we can rewrite the above equation as

$$\begin{aligned} c_k &= \frac{1}{T} \int_T [x(t) (\cos k\omega_0 t - j \sin k\omega_0 t)] dt \\ &= \frac{1}{T} \left[ \int_T x(t) \cos k\omega_0 t dt - j \int_T x(t) \sin k\omega_0 t dt \right]. \end{aligned}$$

Consider the first integral above (i.e., the one involving the  $\cos$  function). Since  $x$  is even and  $\cos k\omega_0 t$  is even, we have that  $x(t) \cos k\omega_0 t$  is even. Thus,  $\int_T x(t) \cos k\omega_0 t dt = 2 \int_0^{T/2} x(t) \cos k\omega_0 t dt$ . Consider the second integral above (i.e., the one involving the  $\sin$  function). Since  $x$  is even and  $\sin k\omega_0 t$  is odd, we have that  $x(t) \sin k\omega_0 t$  is odd. If we integrate an odd periodic function over one period (or an integer multiple thereof), the result is zero. Therefore, the second integral is zero. Combining these results, we can write

$$\begin{aligned} c_k &= \frac{1}{T} \left[ 2 \int_0^{T/2} x(t) \cos k\omega_0 t dt \right] \\ &= \frac{2}{T} \int_0^{T/2} x(t) \cos k\omega_0 t dt. \end{aligned} \tag{5.10}$$

Since  $x$  is real, the quantity  $c_k$  must also be real. Thus, we have that  $\text{Im}(c_k) = 0$ .

Consider now the expression for  $c_{-k}$ . We substitute  $-k$  for  $k$  in (5.10) to obtain

$$c_{-k} = \frac{2}{T} \int_0^{T/2} x(t) \cos(-k\omega_0 t) dt.$$

Since  $\cos$  is an even function, we can simplify this expression to obtain

$$\begin{aligned} c_{-k} &= \frac{2}{T} \int_0^{T/2} x(t) \cos(k\omega_0 t) dt \\ &= c_k. \end{aligned}$$

Thus,  $c_k = c_{-k}$ .

Consider now the quantity  $c_0$ . Substituting  $k = 0$  into (5.10), we can write

$$\begin{aligned} c_0 &= \frac{2}{T} \int_0^{T/2} x(t) \cos(0) dt \\ &= \frac{2}{T} \int_0^{T/2} x(t) dt \\ &= \frac{2}{T} \left[ \frac{1}{2} \int_0^T x(t) dt \right] \\ &= \frac{1}{T} \int_0^T x(t) dt. \end{aligned}$$

Thus,  $c_0 = \frac{1}{T} \int_0^T x(t) dt$ . ■

**Example 5.5** (Fourier series of an odd real function). Let  $x$  be a periodic real function that is odd. Let  $c$  denote the Fourier series coefficient sequence for  $x$ . Show that

- $c$  is purely imaginary (i.e.,  $\text{Re}\{c_k\} = 0$  for all  $k$ );
- $c$  is odd (i.e.,  $c_k = -c_{-k}$  for all  $k$ ); and
- $c_0 = 0$ .

*Solution.* From the Fourier series analysis equation (5.6) and Euler's formula, we can write

$$\begin{aligned} c_k &= \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\ &= \frac{1}{T} \left[ \int_T x(t) [\cos(-k\omega_0 t) + j \sin(-k\omega_0 t)] dt \right]. \end{aligned}$$

Since cos and sin are even and odd functions, respectively, we can rewrite the above equation as

$$\begin{aligned} c_k &= \frac{1}{T} \left[ \int_T x(t) [\cos k\omega_0 t - j \sin k\omega_0 t] dt \right] \\ &= \frac{1}{T} \left[ \int_T x(t) \cos k\omega_0 t dt - j \int_T x(t) \sin k\omega_0 t dt \right]. \end{aligned}$$

Consider the first integral above (i.e., the one involving the cos function). Since  $x$  is odd and  $\cos k\omega_0 t$  is even, we have that  $x(t)\cos k\omega_0 t$  is odd. If we integrate an odd periodic function over a single period (or an integer multiple thereof), the result is zero. Therefore, the first integral is zero. Consider the second integral above (i.e., the one involving the sin function). Since  $x$  is odd and  $\sin k\omega_0 t$  is odd, we have that  $x(t)\sin k\omega_0 t$  is even. Thus,  $\int_T x(t)\sin k\omega_0 t dt = 2 \int_0^{T/2} x(t)\sin k\omega_0 t dt$ . Combining these results, we can write

$$\begin{aligned} c_k &= \frac{-j}{T} \int_T x(t) \sin k\omega_0 t dt \\ &= \frac{-j2}{T} \int_0^{T/2} x(t) \sin k\omega_0 t dt. \end{aligned} \tag{5.11}$$

Since  $x$  is real, the result of the integration is real, and consequently  $c_k$  is purely imaginary. Thus,  $\text{Re}(c_k) = 0$ .

Consider the quantity  $c_{-k}$ . Substituting  $-k$  for  $k$  in (5.11), we obtain

$$\begin{aligned} c_{-k} &= \frac{-j2}{T} \int_0^{T/2} x(t) \sin(-k\omega_0 t) dt \\ &= \frac{-j2}{T} \int_0^{T/2} x(t) [-\sin(k\omega_0 t)] dt \\ &= \frac{j2}{T} \int_0^{T/2} x(t) \sin k\omega_0 t dt \\ &= -c_k. \end{aligned}$$

Thus,  $c_k = -c_{-k}$ .

Consider now the quantity  $c_0$ . Substituting  $k = 0$  in the expression (5.11), we have

$$\begin{aligned} c_0 &= \frac{-j2}{T} \int_0^{T/2} x(t) \sin(0) dt \\ &= 0. \end{aligned}$$

Thus,  $c_0 = 0$ . ■

## 5.4 Convergence of Continuous-Time Fourier Series

So far we have assumed that a given periodic function  $x$  can be represented by a Fourier series. Since a Fourier series consists of an infinite number of terms (infinitely many of which may be nonzero), we need to more carefully consider the issue of convergence. That is, we want to know under what circumstances the Fourier series of  $x$  converges (in some sense) to  $x$ .

Suppose that we have an arbitrary periodic function  $x$ . This function has the Fourier series representation given by (5.1) and (5.6). Let  $x_N$  denote the finite series

$$x_N(t) = \sum_{k=-N}^N c_k e^{jk\omega_0 t}.$$

(i.e.,  $x_N$  is a Fourier series truncated after the  $N$ th harmonic components). The approximation error is given by

$$e_N(t) = x(t) - x_N(t).$$

Let us also define the mean-squared error (MSE) as

$$E_N = \frac{1}{T} \int_T |e_N(t)|^2 dt.$$

Before we can proceed further, we need to more precisely specify what we mean by convergence. This is necessary because convergence can be defined in more than one way. For example, two common types of convergence are: pointwise and MSE. In the case of pointwise convergence, the error goes to zero at every point. If convergence is pointwise and the rate of convergence is the same everywhere, we call this uniform convergence. In the case of MSE convergence, the MSE goes to zero, which does not necessarily imply that the error goes to zero at every point.

Now, we introduce a few important results regarding the convergence of Fourier series for various types of periodic functions. The first result that we consider is for the case of continuous functions as given below.

**Theorem 5.3** (Convergence of Fourier series (continuous case)). *If a periodic function  $x$  is continuous and its Fourier series coefficient sequence  $c$  is absolutely summable (i.e.,  $\sum_{k=-\infty}^{\infty} |c_k| < \infty$ ), then the Fourier series representation of  $x$  converges uniformly (i.e., converges pointwise and at the same rate everywhere).*

*Proof.* A rigorous proof of this theorem is somewhat involved and omitted here. ■

In other words, in the above theorem, we have that if  $x$  is continuous, then as  $N \rightarrow \infty$ ,  $e_N(t) \rightarrow 0$  for all  $t$ . Often, however, we must work with functions that are not continuous. For example, many useful periodic functions are not continuous (e.g., a square wave). Consequently, we must consider the matter of convergence for functions with discontinuities.

Another important result regarding convergence applies to functions that have finite energy over a single period. Mathematically, a function  $x$  has finite energy over a single period if it satisfies

$$\int_T |x(t)|^2 dt < \infty.$$

In the case of such a function, we have the following important result.

**Theorem 5.4** (Convergence of Fourier series (finite-energy case)). *If the periodic function  $x$  has finite energy in a single period (i.e.,  $\int_T |x(t)|^2 dt < \infty$ ), the Fourier series converges in the MSE sense.*

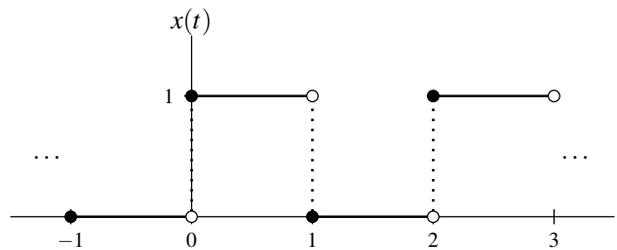
*Proof.* A rigorous proof of this theorem is somewhat involved and omitted here. ■

In other words, in the above theorem, we have that if  $x$  is of finite energy, then as  $N \rightarrow \infty$ ,  $E_N \rightarrow 0$ .

The last important result regarding convergence that we shall consider relates to what are known as the Dirichlet conditions. The Dirichlet<sup>1</sup> conditions for the periodic function  $x$  are as follows:

1. Over a single period,  $x$  is absolutely integrable (i.e.,  $\int_T |x(t)| dt < \infty$ ).
2. Over a single period,  $x$  has a finite number of maxima and minima (i.e.,  $x$  is of bounded variation).
3. Over any finite interval,  $x$  has a finite number of discontinuities, each of which is finite.

<sup>1</sup>Pronounced Dee-ree-klay.

Figure 5.4: Periodic function  $x$ .

**Theorem 5.5** (Convergence of Fourier series (Dirichlet case)). *If  $x$  is a periodic function satisfying the Dirichlet conditions, then:*

1. *The Fourier series converges pointwise everywhere to  $x$ , except at the points of discontinuity of  $x$ .*
2. *At each point  $t_a$  of discontinuity of  $x$ , the Fourier series converges to  $\frac{1}{2}(x(t_a^-) + x(t_a^+))$ , where  $x(t_a^-)$  and  $x(t_a^+)$  denote the values of the function on the left- and right-hand sides of the discontinuity, respectively.*

In other words, if the Dirichlet conditions are satisfied, then as  $N \rightarrow \infty$ ,  $e_N(t) \rightarrow 0$  for all  $t$  except at discontinuities. Furthermore, at each discontinuity, the Fourier series converges to the average of the function values on the left- and right-hand side of the discontinuity.

**Example 5.6.** Consider the periodic function  $x$  with period  $T = 2$  as shown in Figure 5.4. Let  $\hat{x}$  denote the Fourier series representation of  $x$  (i.e.,  $\hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$ , where  $\omega_0 = \pi$ ). Determine the values  $\hat{x}(0)$  and  $\hat{x}(1)$ .

*Solution.* We begin by observing that  $x$  satisfies the Dirichlet conditions. Consequently, Theorem 5.5 applies. Thus, we have that

$$\begin{aligned}\hat{x}(0) &= \frac{1}{2} [x(0^-) + x(0^+)] = \frac{1}{2}(0 + 1) = \frac{1}{2} \quad \text{and} \\ \hat{x}(1) &= \frac{1}{2} [x(1^-) + x(1^+)] = \frac{1}{2}(1 + 0) = \frac{1}{2}. \quad \blacksquare\end{aligned}$$

Although many functions of practical interest satisfy the Dirichlet conditions, not all functions satisfy these conditions. In what follows, some examples of functions that violate these conditions are given.

Consider the 1-periodic function  $x$  defined by

$$x(t) = 1/t \quad \text{for } 0 \leq t < 1.$$

A plot of  $x$  is shown in Figure 5.5(a). The function  $x$  violates the first Dirichlet condition, since  $x$  is not absolutely integrable over a single period.

Consider the 1-periodic function  $x$  defined by

$$x(t) = \sin\left(\frac{2\pi}{t}\right) \quad \text{for } 0 < t \leq 1.$$

A plot of  $x$  is shown in Figure 5.5(b). Since  $x$  has an infinite number of minima and maxima over a single period,  $x$  violates the second Dirichlet condition.

Consider the 1-periodic function  $x$  shown in Figure 5.5(c). As  $t$  goes from 0 to 1,  $x(t)$  traces out a sequence of steps in a staircase, where each step is half the size of the previous step. As it turns out, the number of steps in this staircase between 0 and 1 is infinite. Thus,  $x$  violates the third Dirichlet condition, as  $x$  has an infinite number of discontinuities over a single period.

One might wonder how the Fourier series converges for periodic functions with discontinuities. Let us consider the periodic square wave from Example 5.1. In Figure 5.6, we have plotted the truncated Fourier series  $x_N$  for the

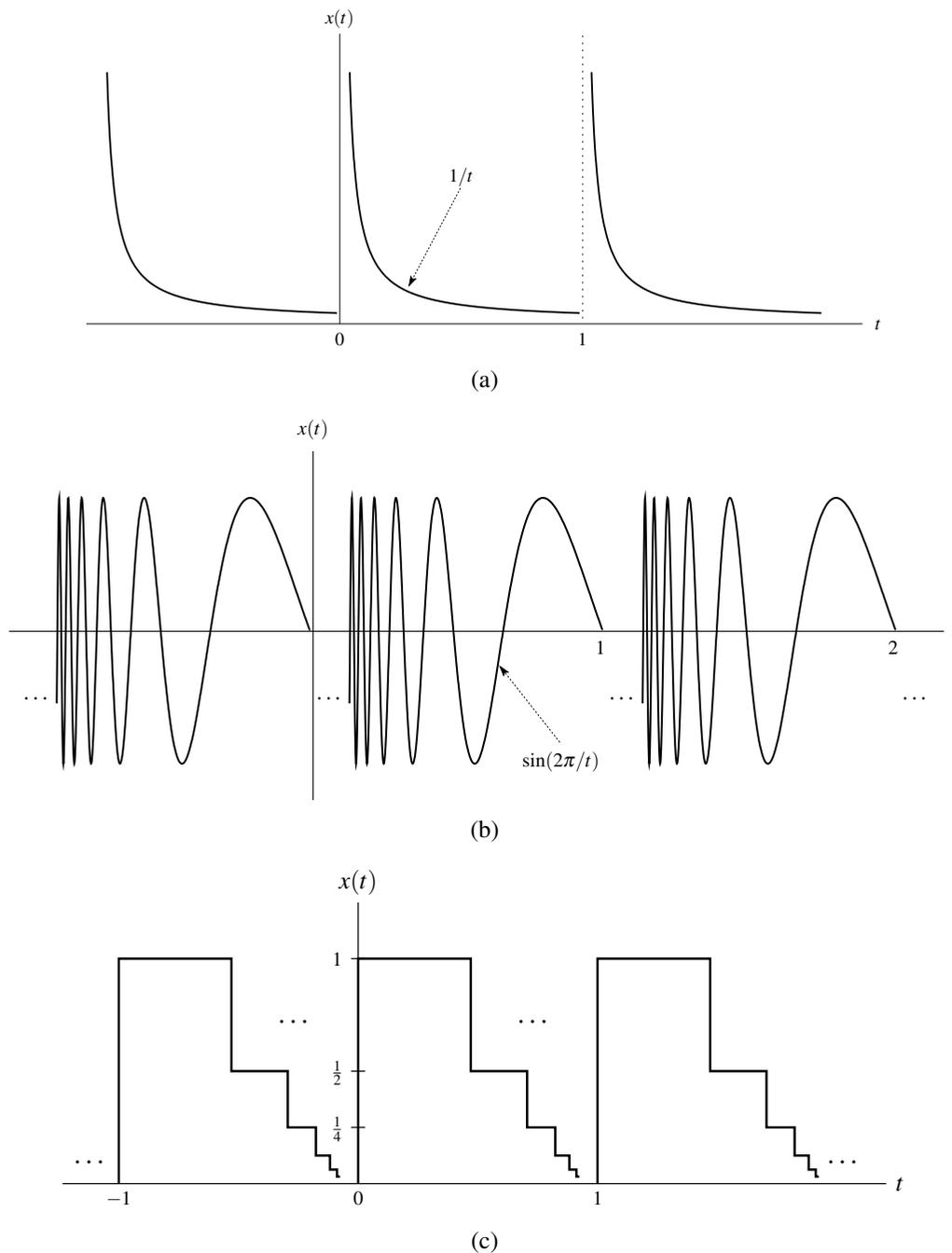


Figure 5.5: Examples of functions that violate the Dirichlet conditions. (a) A function that is not absolutely integrable over a single period. (b) A function that has an infinite number of maxima and minima over a single period. (c) A function that has an infinite number of discontinuities over a single period.

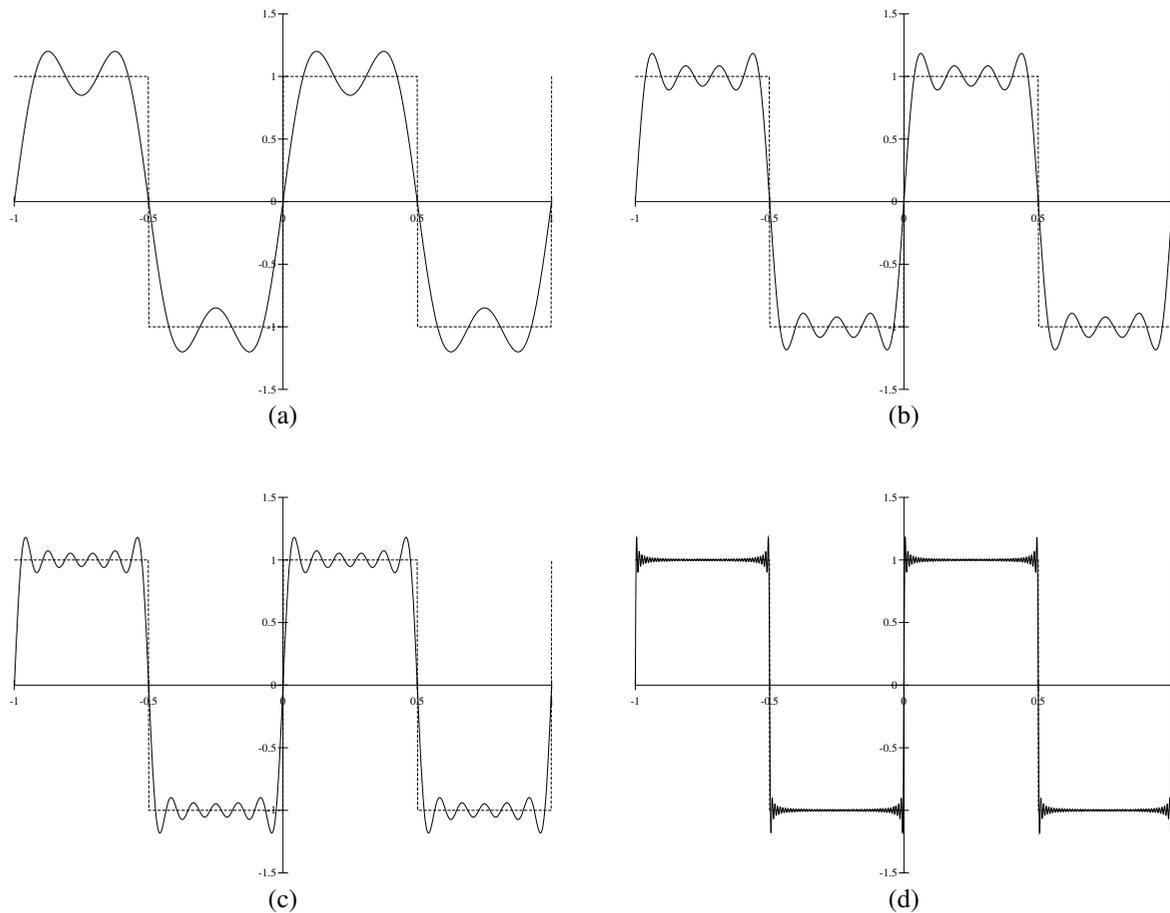


Figure 5.6: Gibbs phenomenon. The Fourier series for the periodic square wave truncated after the  $N$ th harmonic components for (a)  $N = 3$ , (b)  $N = 7$ , (c)  $N = 11$ , and (d)  $N = 101$ .

square wave (with period  $T = 1$  and amplitude  $A = 1$ ) for several values of  $N$ . At the discontinuities of  $x$ , we can see that the series appears to converge to the average of the function values on either side of the discontinuity. In the vicinity of a discontinuity, however, the truncated series  $x_N$  exhibits ripples and the peak amplitude of the ripples does not seem to decrease with increasing  $N$ . As it turns out, as  $N$  increases, the ripples get compressed towards the discontinuity, but, for any finite  $N$ , the peak amplitude of the ripples remains constant. This behavior is known as **Gibbs phenomenon**.

## 5.5 Properties of Continuous-Time Fourier Series

Fourier series representations possess a number of important properties. In the sections that follow, we introduce a few of these properties. For convenience, these properties are also summarized later in Table 5.1 (on page 120).

### 5.5.1 Linearity

Arguably, the most important property of Fourier series is linearity, as introduced below.

**Theorem 5.6** (Linearity). *Let  $x$  and  $y$  denote two periodic functions with period  $T$  and frequency  $\omega_0 = \frac{2\pi}{T}$ . If*

$$x(t) \xleftrightarrow{\text{CTFS}} a_k \quad \text{and} \quad y(t) \xleftrightarrow{\text{CTFS}} b_k,$$

then

$$Ax(t) + By(t) \xleftrightarrow{\text{CTFS}} Aa_k + Bb_k,$$

where  $A$  and  $B$  are complex constants. In other words, a linear combination of functions produces the same linear combination of their Fourier series coefficients.

*Proof.* To prove the above property, we proceed as follows. First, we express  $x$  and  $y$  in terms of their corresponding Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{and} \quad y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}.$$

Now, we determine the Fourier series of  $Ax + By$ . We have

$$\begin{aligned} Ax(t) + By(t) &= A \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} + B \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} Aa_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} Bb_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} (Aa_k + Bb_k) e^{jk\omega_0 t}. \end{aligned}$$

Therefore, we have that  $Ax(t) + By(t) \xleftrightarrow{\text{CTFS}} Aa_k + Bb_k$ . ■

### 5.5.2 Time Shifting (Translation)

The next property of Fourier series to be introduced is the time-shifting (i.e., translation) property, as given below.

**Theorem 5.7** (Time shifting (i.e., translation)). *Let  $x$  denote a periodic function with period  $T$  and frequency  $\omega_0 = \frac{2\pi}{T}$ . If*

$$x(t) \xleftrightarrow{\text{CTFS}} a_k,$$

then

$$x(t - t_0) \xleftrightarrow{\text{CTFS}} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k,$$

where  $t_0$  is a real constant.

*Proof.* To prove the time-shifting property, we proceed as follows. The Fourier series of  $x$  is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}. \tag{5.12}$$

We express  $x(t - t_0)$  in terms of its Fourier series, and then use algebraic manipulation to obtain

$$\begin{aligned} x(t - t_0) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(t-t_0)} \\ &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jk\omega_0 t_0} \\ &= \sum_{k=-\infty}^{\infty} (a_k e^{-jk\omega_0 t_0}) e^{jk\omega_0 t}. \end{aligned} \tag{5.13}$$

Comparing (5.12) and (5.13), we have that  $x(t - t_0) \xleftrightarrow{\text{CTFS}} e^{-jk\omega_0 t_0} a_k$ . ■

From the above theorem, we can see that time shifting a periodic function does not change the magnitude of its Fourier series coefficients (since  $|e^{j\theta}| = 1$  for any real  $\theta$ ).

### 5.5.3 Frequency Shifting (Modulation)

The next property of Fourier series to be introduced is the frequency-shifting (i.e., modulation) property, as given below.

**Theorem 5.8** (Frequency shifting (i.e., modulation)). *Let  $x$  denote a periodic function with period  $T$  and frequency  $\omega_0 = \frac{2\pi}{T}$ . If*

$$x(t) \xleftrightarrow{\text{CTFS}} a_k,$$

then

$$e^{jM(2\pi/T)t}x(t) = e^{jM\omega_0 t}x(t) \xleftrightarrow{\text{CTFS}} a_{k-M},$$

where  $M$  is an integer constant.

*Proof.* To prove the frequency-shifting property, we proceed as follows. We have

$$\begin{aligned} e^{j(2\pi/T)Mt}x(t) &= e^{j(2\pi/T)Mt} \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T)kt} \\ &= \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T)Mt} e^{j(2\pi/T)kt} \\ &= \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T)(k+M)t}. \end{aligned}$$

Now, we employ a change of variable. Let  $k' = k + M$  so that  $k = k' - M$ . Applying the change of variable and dropping the primes, we obtain

$$e^{j(2\pi/T)Mt}x(t) = \sum_{k=-\infty}^{\infty} a_{k-M} e^{j(2\pi/T)kt}.$$

Now, we observe that the right-hand side of this equation is a Fourier series. Therefore, the Fourier series coefficient sequence for  $e^{j(2\pi/T)Mt}x(t)$  (i.e., the left-hand side of the equation) is  $a'(k) = a_{k-M}$ . ■

### 5.5.4 Time Reversal (Reflection)

The next property of Fourier series to be introduced is the time-reversal (i.e., reflection) property, as given below.

**Theorem 5.9** (Time reversal (i.e., reflection)). *Let  $x$  denote a periodic function with period  $T$  and frequency  $\omega_0 = \frac{2\pi}{T}$ . If*

$$x(t) \xleftrightarrow{\text{CTFS}} a_k,$$

then

$$x(-t) \xleftrightarrow{\text{CTFS}} a_{-k}.$$

*Proof.* To prove the time-reversal property, we proceed in the following manner. The Fourier series of  $x$  is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}. \quad (5.14)$$

Now, we consider the Fourier series expansion of  $x(-t)$ . The Fourier series in this case is given by

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(-t)}. \quad (5.15)$$

Now, we employ a change of variable. Let  $l = -k$  so that  $k = -l$ . Performing this change of variable, we can rewrite (5.15) to obtain

$$\begin{aligned} x(-t) &= \sum_{l=-\infty}^{\infty} a_{-l} e^{j(-l)\omega_0(-t)} \\ &= \sum_{l=-\infty}^{\infty} a_{-l} e^{jl\omega_0 t}. \end{aligned} \quad (5.16)$$

Comparing (5.14) and (5.16), we have that  $x(-t) \xleftrightarrow{\text{CTFS}} a_{-k}$ . ■

In other words, the above theorem states that time reversing a function time reverses the corresponding sequence of Fourier series coefficients. Furthermore, if  $x$  is a real, we have from (5.2) that  $a_{-k} = a_k^*$  and obtain  $x(-t) \xleftrightarrow{\text{CTFS}} a_k^*$ .

### 5.5.5 Conjugation

The next property of Fourier series to be introduced is the conjugation property, as given below.

**Theorem 5.10** (Conjugation). *For a  $T$ -periodic function  $x$  with Fourier series coefficient sequence  $c$ ,*

$$x^*(t) \xleftrightarrow{\text{CTFS}} c_{-k}^*.$$

*Proof.* From the definition of a Fourier series, we have

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

Taking the complex conjugate of both sides of the preceding equation, we obtain

$$\begin{aligned} x^*(t) &= \left( \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \right)^* \\ &= \sum_{k=-\infty}^{\infty} \left( c_k e^{jk\omega_0 t} \right)^* \\ &= \sum_{k=-\infty}^{\infty} c_k^* e^{-jk\omega_0 t}. \end{aligned}$$

Replacing  $k$  by  $-k$  in the summation of the preceding equation, we obtain

$$x^*(t) = \sum_{k=-\infty}^{\infty} c_{-k}^* e^{jk\omega_0 t}.$$

Thus, the Fourier series coefficient sequence  $c'$  of  $x^*$  is  $c'_k = c_{-k}^*$ . ■

In other words, the above theorem states that conjugating a function has the effect of time reversing and conjugating the corresponding Fourier series coefficient sequence.

### 5.5.6 Periodic Convolution

The next property of Fourier series to be introduced is the periodic-convolution property, as given below.

**Theorem 5.11** (Periodic convolution). *Let  $x$  and  $y$  be  $T$ -periodic functions given by*

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{and} \quad y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t},$$

where  $\omega_0 = \frac{2\pi}{T}$ . Let  $z(t) = x \otimes y(t)$ , where

$$z(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

The sequences  $a$ ,  $b$ , and  $c$  are related as

$$c_k = T a_k b_k.$$

*Proof.* In what follows, let  $\delta$  denote the delta sequence (i.e.,  $\delta(n)$  is 1 for  $n = 0$  and 0 elsewhere). From the definition of periodic convolution, we have

$$\begin{aligned} x \otimes y(t) &= \int_T x(\tau) y(t - \tau) d\tau \\ &= \int_T \left( \sum_{\ell=-\infty}^{\infty} a_\ell e^{j\omega_0 \ell \tau} \right) \left( \sum_{k=-\infty}^{\infty} b_k e^{j\omega_0 k(t-\tau)} \right) d\tau \\ &= \int_T \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_\ell b_k e^{j\omega_0 \ell \tau} e^{j\omega_0 k(t-\tau)} d\tau \\ &= \int_T \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_\ell b_k e^{j\omega_0 k t} e^{j\omega_0 (\ell-k)\tau} d\tau \\ &= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_\ell b_k e^{j\omega_0 k t} \int_T e^{j\omega_0 (\ell-k)\tau} d\tau \\ &= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_\ell b_k e^{j\omega_0 k t} T \delta(\ell - k) \\ &= \sum_{k=-\infty}^{\infty} a_k b_k e^{j\omega_0 k t} T \\ &= \sum_{k=-\infty}^{\infty} T a_k b_k e^{j\omega_0 k t}. \end{aligned} \tag{5.17}$$

In the above simplification, we used the fact that

$$\begin{aligned} \int_T e^{j(2\pi/T)kt} dt &= \begin{cases} T & k = 0 \\ 0 & \text{otherwise} \end{cases} \\ &= T \delta(k). \end{aligned}$$

Now, we simply observe that the right-hand side of (5.17) is a Fourier series. Therefore, the Fourier series coefficient sequence  $c$  of  $x \otimes y$  is given by  $c_k = T a_k b_k$ . ■

### 5.5.7 Multiplication

The next property of Fourier series to be considered is the multiplication property, as given below.

**Theorem 5.12** (Multiplication). *Let  $x$  and  $y$  be  $T$ -periodic functions given by*

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{and} \quad y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t},$$

where  $\omega_0 = \frac{2\pi}{T}$ . Let  $z(t) = x(t)y(t)$ , where

$$z(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

The sequences  $a$ ,  $b$ , and  $c$  are related as

$$c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n}$$

(i.e.,  $c$  is the DT convolution of  $a$  and  $b$ ).

*Proof.* From the Fourier series analysis equation, we can write

$$c_k = \frac{1}{T} \int_T x(t)y(t)e^{-j(2\pi/T)kt} dt.$$

Replacing  $x$  by its Fourier series representation, we obtain

$$\begin{aligned} c_k &= \frac{1}{T} \int_T \left( \sum_{n=-\infty}^{\infty} a_n e^{j(2\pi/T)nt} \right) y(t) e^{-j(2\pi/T)kt} dt \\ &= \frac{1}{T} \int_T \sum_{n=-\infty}^{\infty} a_n e^{j(2\pi/T)nt} y(t) e^{-j(2\pi/T)kt} dt. \end{aligned}$$

Reversing the order of the summation and integration, we have

$$\begin{aligned} c_k &= \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_T a_n e^{j(2\pi/T)nt} y(t) e^{-j(2\pi/T)kt} dt \\ &= \sum_{n=-\infty}^{\infty} a_n \left( \frac{1}{T} \int_T y(t) e^{-j(2\pi/T)(k-n)t} dt \right). \end{aligned}$$

Observing that the expression on the preceding line in the large pair of parenthesis is simply the formula for computing the  $(k-n)$ th Fourier series coefficient of  $y$ , we conclude

$$c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n}. \quad \blacksquare$$

### 5.5.8 Even and Odd Symmetry

Fourier series preserves signal symmetry. In other words, we have the result below.

**Theorem 5.13** (Even/odd symmetry). *For a  $T$ -periodic function  $x$  with Fourier series coefficient sequence  $c$ , the following properties hold:*

$$\begin{aligned} x \text{ is even if and only if } c \text{ is even; and} \\ x \text{ is odd if and only if } c \text{ is odd.} \end{aligned}$$

*Proof.* The proof is left as an exercise for the reader in Exercise 5.3. ■

In other words, the above theorem states that the even/odd symmetry properties of  $x$  and  $c$  always match (i.e., Fourier series preserve symmetry).

### 5.5.9 Real Functions

As noted earlier in Theorem 5.1, the Fourier series of a real function has a special structure. In particular, a function  $x$  is real if and only if its Fourier series coefficient sequence  $c$  satisfies  $c_k = c_{-k}^*$  for all  $k$  (i.e.,  $c$  is conjugate symmetric). Thus, for a real-valued function, the negative-indexed Fourier series coefficients are *redundant*, as they are completely determined by the nonnegative-indexed coefficients. From properties of complex numbers, one can show that

$$c_k = c_{-k}^* \text{ for all } k$$

is equivalent to

$$|c_k| = |c_{-k}| \text{ for all } k \quad \text{and} \quad \arg c_k = -\arg c_{-k} \text{ for all } k$$

(i.e.,  $|c_k|$  is even and  $\arg c_k$  is odd). Note that  $x$  being real does *not* necessarily imply that  $c$  is real.

### 5.5.10 Parseval's Relation

Another important property of Fourier series relates to the energy of functions and sequences, as given by the theorem below.

**Theorem 5.14** (Parseval's relation). *A periodic function  $x$  and its Fourier series coefficient sequence  $c$  satisfy the relationship*

$$\frac{1}{T} \int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

(i.e., the energy in  $x$  and the energy in  $c$  are equal).

*Proof.* Let  $x$ ,  $y$ , and  $z$  denote  $T$ -periodic functions with the Fourier series given by

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \\ y(t) &= \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}, \quad \text{and} \\ z(t) &= x(t)y(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \end{aligned}$$

From the multiplication property of Fourier series (i.e., Theorem 5.12), we have

$$c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n}. \quad (5.18)$$

Now, let  $y(t) = x^*(t)$  so that  $z(t) = x(t)x^*(t) = |x(t)|^2$ . From the conjugation property of Fourier series (i.e., Theorem 5.10), since  $y(t) = x^*(t)$ , we know

$$b_k = a_{-k}^*.$$

So, we can rewrite (5.18) as

$$\begin{aligned} c_k &= \sum_{n=-\infty}^{\infty} a_n a_{-(k-n)}^* \\ &= \sum_{n=-\infty}^{\infty} a_n a_{n-k}^*. \end{aligned} \quad (5.19)$$

From the Fourier series analysis equation, we have

$$c_k = \frac{1}{T} \int_T |x(t)|^2 e^{-jk\omega_0 t} dt. \quad (5.20)$$

Equating (5.19) and (5.20), we obtain

$$\frac{1}{T} \int_T |x(t)|^2 e^{-jk\omega_0 t} dt = \sum_{n=-\infty}^{\infty} a_n a_{n-k}^*.$$

Letting  $k = 0$  in the preceding equation yields

$$\frac{1}{T} \int_T |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} a_n a_n^* = \sum_{n=-\infty}^{\infty} |a_n|^2. \quad \blacksquare$$

The above theorem is simply stating that the amount of energy in  $x$  (i.e.,  $\frac{1}{T} \int_T |x(t)|^2 dt$ ) and the amount of energy in the Fourier series coefficient sequence  $c$  (i.e.,  $\sum_{k=-\infty}^{\infty} |c_k|^2$ ) are equal. In other words, the transformation between a function and its Fourier series coefficient sequence preserves energy.

Table 5.1: Properties of CT Fourier series

Property	Time Domain	Fourier Domain
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha a_k + \beta b_k$
Translation	$x(t - t_0)$	$e^{-jk(2\pi/T)t_0} a_k$
Modulation	$e^{jM(2\pi/T)t} x(t)$	$a_{k-M}$
Reflection	$x(-t)$	$a_{-k}$
Conjugation	$x^*(t)$	$a_{-k}^*$
Periodic Convolution	$x \otimes y(t)$	$T a_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{n=-\infty}^{\infty} a_n b_{k-n}$

---

Property	
Parseval's relation	$\frac{1}{T} \int_T  x(t) ^2 dt = \sum_{k=-\infty}^{\infty}  a_k ^2$
Even symmetry	$x$ is even $\Leftrightarrow a$ even
Odd symmetry	$x$ is odd $\Leftrightarrow a$ odd
Real	$x$ is real $\Leftrightarrow a$ is conjugate symmetric

## 5.6 Fourier Series and Frequency Spectra

The Fourier series represents a function in terms of harmonically-related complex sinusoids. In this sense, the Fourier series captures information about the frequency content of a function. Each complex sinusoid is associated with a particular frequency (which is some integer multiple of the fundamental frequency). So, these coefficients indicate at which frequencies the information/energy in a function is concentrated. For example, if only the Fourier series coefficients for the low-order harmonics have large magnitudes, then the function is mostly associated with low frequencies. On the other hand, if a function has many large magnitude coefficients for high-order harmonics, then the function has a considerable amount of information/energy associated with high frequencies. In this way, the Fourier series representation provides a means for measuring the frequency content of a function. The distribution of the energy/information in a function over different frequencies is referred to as the **frequency spectrum** of the function.

To gain further insight into the role played by the Fourier series coefficients  $c_k$  in the context of the frequency spectrum of the function  $x$ , it is helpful to write the Fourier series with the  $c_k$  expressed in polar form as follows:

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} |c_k| e^{j \arg c_k} e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} |c_k| e^{j(k\omega_0 t + \arg c_k)}. \end{aligned}$$

Clearly (from the last line of the above equation), the  $k$ th term in the summation corresponds to a complex sinusoid with fundamental frequency  $k\omega_0$  that has had its amplitude scaled by a factor of  $|c_k|$  and has been time-shifted by an amount that depends on  $\arg c_k$ . For a given  $k$ , the larger  $|c_k|$  is, the larger the amplitude of its corresponding complex sinusoid  $e^{jk\omega_0 t}$ , and therefore the larger the contribution the  $k$ th term (which is associated with frequency  $k\omega_0$ ) will make to the overall summation. In this way, we can use  $|c_k|$  as a measure of how much information a function  $x$  has at the frequency  $k\omega_0$ .

Various ways exist to illustrate the frequency spectrum of a function. Typically, we plot the Fourier series coefficients as a function of frequency. Since, in general, the Fourier series coefficients are complex valued, we usually display this information using two plots. One plot shows the magnitude of the coefficients as a function of frequency. This is called the **magnitude spectrum**. The other plot shows the arguments of the coefficients as a function of frequency. In this context, the argument is referred to as the phase, and the plot is called the **phase spectrum**.

Since the Fourier series only has frequency components at integer multiples of the fundamental frequency, we only have values to plot for these particular frequencies. In other words, the frequency spectrum is discrete in the independent variable (i.e., frequency). For this reason, we use a stem graph to plot such functions. Due to the general appearance of the graph (i.e., a number of vertical lines at various frequencies) we refer to such spectra as **line spectra**.

Recall that, for a real function  $x$ , the Fourier series coefficient sequence  $c$  is conjugate symmetric (i.e.,  $c_k = c_{-k}^*$  for all  $k$ ). This, however, implies that  $|c_k| = |c_{-k}|$  and  $\arg c_k = -\arg c_{-k}$ . Since  $|c_k| = |c_{-k}|$ , the magnitude spectrum of a real function is always even. Similarly, since  $\arg c_k = -\arg c_{-k}$ , the phase spectrum of a real function is always odd.

**Example 5.7.** The periodic square wave  $x$  in Example 5.1 has fundamental period  $T$ , fundamental frequency  $\omega_0$ , and the Fourier series coefficient sequence given by

$$c_k = \begin{cases} \frac{-j2A}{\pi k} & k \text{ odd} \\ 0 & k \text{ even,} \end{cases}$$

where  $A$  is a positive constant. Find and plot the magnitude and phase spectra of  $x$ . Determine at what frequency (or frequencies)  $x$  has the most information.

*Solution.* First, we compute the magnitude spectrum of  $x$ , which is given by  $|c_k|$ . We have

$$\begin{aligned} |c_k| &= \begin{cases} \left| \frac{-j2A}{\pi k} \right| & k \text{ odd} \\ 0 & k \text{ even} \end{cases} \\ &= \begin{cases} \frac{2A}{\pi|k|} & k \text{ odd} \\ 0 & k \text{ even.} \end{cases} \end{aligned}$$

Next, we compute the phase spectrum of  $x$ , which is given by  $\arg c_k$ . Using the fact that  $\arg 0 = 0$  and  $\arg \frac{-j2A}{\pi k} = -\frac{\pi}{2} \operatorname{sgn} k$ , we have

$$\begin{aligned} \arg c_k &= \begin{cases} \arg \frac{-j2A}{\pi k} & k \text{ odd} \\ \arg 0 & k \text{ even} \end{cases} \\ &= \begin{cases} \frac{\pi}{2} & k \text{ odd, } k < 0 \\ -\frac{\pi}{2} & k \text{ odd, } k > 0 \\ 0 & k \text{ even} \end{cases} \\ &= \begin{cases} -\frac{\pi}{2} \operatorname{sgn} k & k \text{ odd} \\ 0 & k \text{ even.} \end{cases} \end{aligned}$$

The magnitude and phase spectra of  $x$  are plotted in Figures 5.7(a) and (b), respectively. Note that the magnitude spectrum is an even function, while the phase spectrum is an odd function. This is what we should expect, since  $x$  is real. Since  $|c_k|$  is largest for  $k = -1$  and  $k = 1$ , the function  $x$  has the most information at frequencies  $-\omega_0$  and  $\omega_0$ . ■

**Example 5.8.** The periodic impulse train  $x$  in Example 5.2 has fundamental period  $T$ , fundamental frequency  $\omega_0$ , and the Fourier series coefficient sequence  $c$  given by

$$c_k = \frac{A}{T},$$

where  $A$  is a positive real constant. Find and plot the magnitude and phase spectra of  $x$ .

*Solution.* We have  $|c_k| = \frac{A}{T}$  and  $\arg c_k = 0$ . The magnitude and phase spectra of  $x$  are plotted in Figures 5.8(a) and (b), respectively. ■

## 5.7 Fourier Series and LTI Systems

From earlier, in Theorem 4.12, we know that complex exponentials are eigenfunctions of LTI systems. Since complex sinusoids are a special case of complex exponentials, it follows that complex sinusoids are also eigenfunctions of LTI systems. In other words, we have the result below.

**Corollary 5.1.** For an arbitrary LTI system  $\mathcal{H}$  with impulse response  $h$  and a function of the form  $x(t) = e^{j\omega t}$ , where  $\omega$  is an arbitrary real constant (i.e.,  $x$  is an arbitrary complex sinusoid), the following holds:

$$\mathcal{H}x(t) = H(j\omega)e^{j\omega t},$$

where

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} dt. \quad (5.21)$$

That is,  $x$  is an eigenfunction of  $\mathcal{H}$  with the corresponding eigenvalue  $H(j\omega)$ .

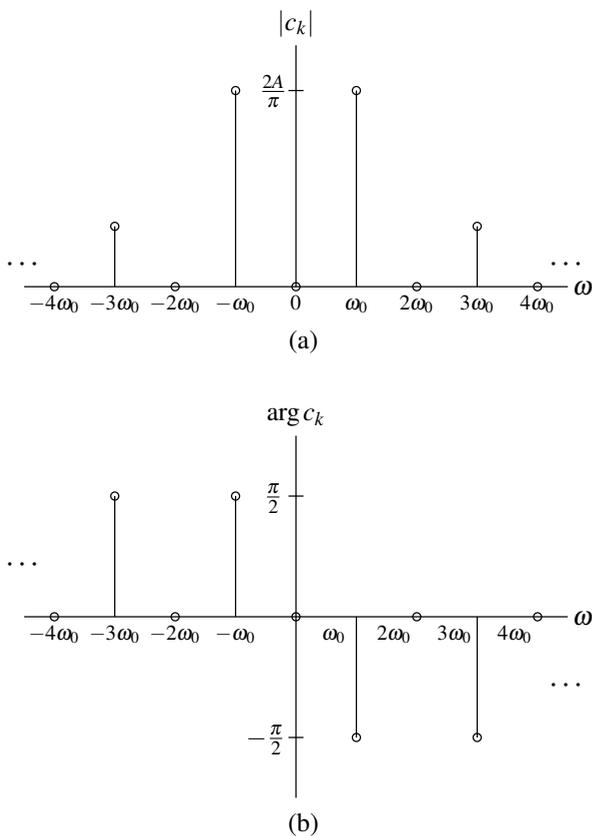


Figure 5.7: Frequency spectrum of the periodic square wave. (a) Magnitude spectrum and (b) phase spectrum.

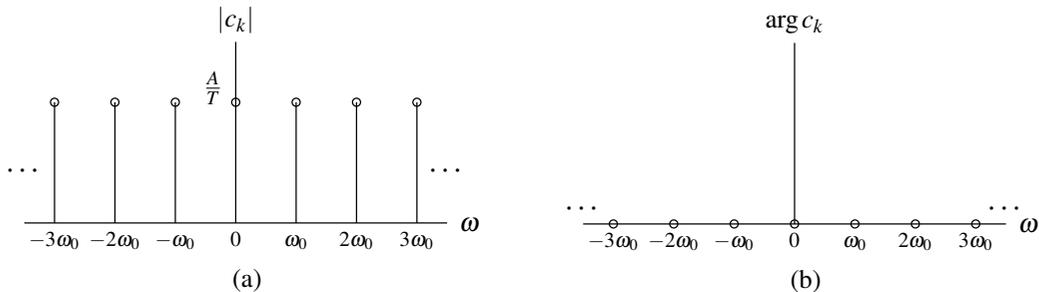


Figure 5.8: Frequency spectrum for the periodic impulse train. (a) Magnitude spectrum and (b) phase spectrum.

The preceding result (i.e., Corollary 5.1) is simply a special case of Theorem 4.12 for  $s = j\omega$ . As a matter of terminology, the quantity  $H(j\omega)$  in (5.21) is referred to as the **frequency response** of the system  $\mathcal{H}$ . The frequency response completely characterizes the behavior of a LTI system. Consequently, the frequency response is often useful when working with LTI systems. As it turns out, an integral of the form appearing on the right-hand side of (5.21) is of great importance, as it defines what is called the (CT) Fourier transform. We will study the (CT) Fourier transform in great depth later in Chapter 6.

Let us now consider an application of eigenfunctions. Since convolution can often be quite painful to handle at the best of times, let us exploit eigenfunctions in order to devise a means to avoid having to deal with convolution directly in certain circumstances. Suppose now that we have a periodic function  $x$  represented in terms of a Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

Using (5.21) and the superposition property, we can determine the system response  $y$  to the input  $x$  as follows:

$$\begin{aligned} y(t) &= \mathcal{H}x(t) \\ &= \mathcal{H} \left\{ \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \right\} (t) \\ &= \sum_{k=-\infty}^{\infty} \mathcal{H} \left\{ c_k e^{jk\omega_0 t} \right\} (t) \\ &= \sum_{k=-\infty}^{\infty} c_k \mathcal{H} \left\{ e^{jk\omega_0 t} \right\} (t) \\ &= \sum_{k=-\infty}^{\infty} c_k H(jk\omega_0) e^{jk\omega_0 t}. \end{aligned}$$

Therefore, we can view a LTI system as an entity that operates on the individual coefficients of a Fourier series. In particular, the system forms its output by multiplying each Fourier series coefficient by the value of the frequency response function at the frequency to which the Fourier series coefficient corresponds. In other words, if

$$x(t) \xleftrightarrow{\text{CTFS}} c_k$$

then

$$y(t) \xleftrightarrow{\text{CTFS}} H(jk\omega_0) c_k.$$

**Example 5.9.** Consider a LTI system with the frequency response

$$H(j\omega) = e^{-j\omega/4}.$$

Find the response  $y$  of the system to the input  $x$ , where

$$x(t) = \frac{1}{2} \cos(2\pi t).$$

*Solution.* To begin, we rewrite  $x$  as

$$x(t) = \frac{1}{4} (e^{j2\pi t} + e^{-j2\pi t}).$$

Thus, the Fourier series for  $x$  is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t},$$

where  $\omega_0 = 2\pi$  and

$$c_k = \begin{cases} \frac{1}{4} & k \in \{-1, 1\} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we can write

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{\infty} c_k H(jk\omega_0) e^{jk\omega_0 t} \\ &= c_{-1} H(-j\omega_0) e^{-j\omega_0 t} + c_1 H(j\omega_0) e^{j\omega_0 t} \\ &= \frac{1}{4} H(-j2\pi) e^{-j2\pi t} + \frac{1}{4} H(j2\pi) e^{j2\pi t} \\ &= \frac{1}{4} e^{j\pi/2} e^{-j2\pi t} + \frac{1}{4} e^{-j\pi/2} e^{j2\pi t} \\ &= \frac{1}{4} [e^{-j(2\pi t - \pi/2)} + e^{j(2\pi t - \pi/2)}] \\ &= \frac{1}{4} (2 \cos(2\pi t - \frac{\pi}{2})) \\ &= \frac{1}{2} \cos(2\pi t - \frac{\pi}{2}) \\ &= \frac{1}{2} \cos(2\pi [t - \frac{1}{4}]). \end{aligned}$$

Observe that  $y(t) = x(t - \frac{1}{4})$ . This is not a coincidence because, as it turns out, a LTI system with the frequency response  $H(j\omega) = e^{-j\omega/4}$  is an ideal delay of  $\frac{1}{4}$  (i.e., a system that performs a time shift of  $\frac{1}{4}$ ). ■

## 5.8 Filtering

In some applications, we want to change the relative amplitude of the frequency components of a function or possibly eliminate some frequency components altogether. This process of modifying the frequency components of a function is referred to as **filtering**. Various types of filters exist. Frequency-selective filters pass some frequencies with little or no distortion, while significantly attenuating other frequencies. Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

An **ideal lowpass filter** eliminates all frequency components with a frequency greater in magnitude than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where  $\omega_c$  is the cutoff frequency. A plot of this frequency response is shown in Figure 5.9(a).

The **ideal highpass filter** eliminates all frequency components with a frequency lesser in magnitude than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(j\omega) = \begin{cases} 1 & |\omega| \geq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where  $\omega_c$  is the cutoff frequency. A plot of this frequency response is shown in Figure 5.9(b).

An **ideal bandpass filter** eliminates all frequency components with a frequency whose magnitude does not lie in a particular range, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(j\omega) = \begin{cases} 1 & \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ 0 & \text{otherwise,} \end{cases}$$

where the limits of the passband are  $\omega_{c1}$  and  $\omega_{c2}$ . A plot of this frequency response is shown in Figure 5.9(c).

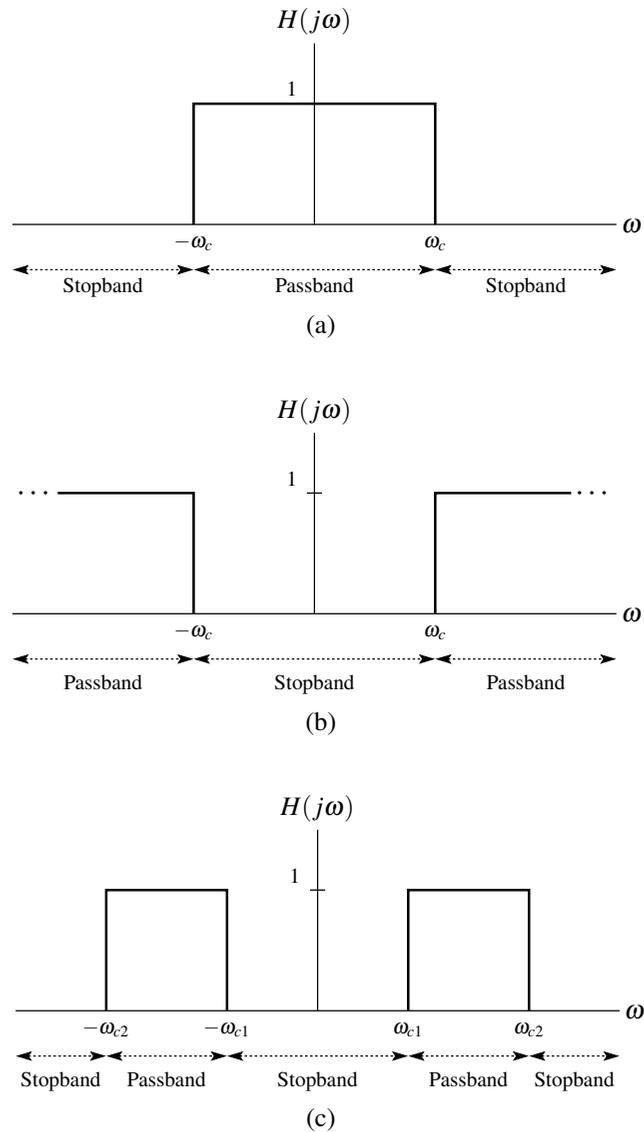


Figure 5.9: Frequency responses of (a) ideal lowpass, (b) ideal highpass, and (c) ideal bandpass filters.

**Example 5.10** (Lowpass filtering). Suppose that we have a LTI system with input  $x$ , output  $y$ , and frequency response  $H(j\omega)$ , where

$$H(j\omega) = \begin{cases} 1 & |\omega| \leq 3\pi \\ 0 & \text{otherwise.} \end{cases}$$

Further, suppose that the input  $x$  is the periodic function

$$x(t) = 1 + 2\cos 2\pi t + \cos 4\pi t + \frac{1}{2}\cos 6\pi t.$$

(a) Find the Fourier series representation of  $x$ . (b) Use this representation in order to find the response  $y$  of the system to the input  $x$ . (c) Plot the frequency spectra of  $x$  and  $y$ .

*Solution.* (a) We begin by finding the Fourier series representation of  $x$ . Using Euler's formula, we can re-express  $x$  as

$$\begin{aligned} x(t) &= 1 + 2\cos 2\pi t + \cos 4\pi t + \frac{1}{2}\cos 6\pi t \\ &= 1 + 2\left[\frac{1}{2}(e^{j2\pi t} + e^{-j2\pi t})\right] + \left[\frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t})\right] + \frac{1}{2}\left[\frac{1}{2}(e^{j6\pi t} + e^{-j6\pi t})\right] \\ &= 1 + e^{j2\pi t} + e^{-j2\pi t} + \frac{1}{2}[e^{j4\pi t} + e^{-j4\pi t}] + \frac{1}{4}[e^{j6\pi t} + e^{-j6\pi t}] \\ &= \frac{1}{4}e^{-j6\pi t} + \frac{1}{2}e^{-j4\pi t} + e^{-j2\pi t} + 1 + e^{j2\pi t} + \frac{1}{2}e^{j4\pi t} + \frac{1}{4}e^{j6\pi t} \\ &= \frac{1}{4}e^{j(-3)(2\pi)t} + \frac{1}{2}e^{j(-2)(2\pi)t} + e^{j(-1)(2\pi)t} + e^{j(0)(2\pi)t} + e^{j(1)(2\pi)t} + \frac{1}{2}e^{j(2)(2\pi)t} + \frac{1}{4}e^{j(3)(2\pi)t}. \end{aligned}$$

From the last line of the preceding equation, we deduce that  $\omega_0 = 2\pi$ , since a larger value for  $\omega_0$  would imply that some Fourier series coefficient indices are noninteger, which clearly makes no sense. Thus, we have that the Fourier series of  $x$  is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

where  $\omega_0 = 2\pi$  and

$$a_k = \begin{cases} 1 & k = 0 \\ 1 & k \in \{-1, 1\} \\ \frac{1}{2} & k \in \{-2, 2\} \\ \frac{1}{4} & k \in \{-3, 3\} \\ 0 & \text{otherwise.} \end{cases}$$

(b) Since the system is LTI, we know that the output  $y$  has the form

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t},$$

where

$$b_k = a_k H(jk\omega_0).$$

Using the results from above, we can calculate the  $b_k$  as follows:

$$\begin{aligned} b_0 &= a_0 H(j[0][2\pi]) = 1(1) = 1, \\ b_1 &= a_1 H(j[1][2\pi]) = 1(1) = 1, \\ b_{-1} &= a_{-1} H(j[-1][2\pi]) = 1(1) = 1, \\ b_2 &= a_2 H(j[2][2\pi]) = \frac{1}{2}(0) = 0, \\ b_{-2} &= a_{-2} H(j[-2][2\pi]) = \frac{1}{2}(0) = 0, \\ b_3 &= a_3 H(j[3][2\pi]) = \frac{1}{4}(0) = 0, \quad \text{and} \\ b_{-3} &= a_{-3} H(j[-3][2\pi]) = \frac{1}{4}(0) = 0. \end{aligned}$$

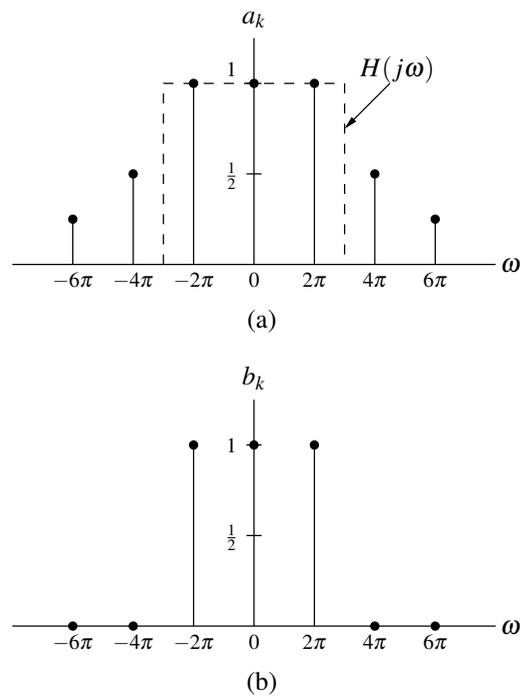


Figure 5.10: Frequency spectra of the (a) input function  $x$  and (b) output function  $y$ .

Thus, we have

$$b_k = \begin{cases} 1 & k \in \{-1, 0, 1\} \\ 0 & \text{otherwise.} \end{cases}$$

(c) Lastly, we plot the frequency spectra of  $x$  and  $y$  in Figures 5.10(a) and (b), respectively. The frequency response  $H(j\omega)$  is superimposed on the plot of the frequency spectrum of  $x$  for illustrative purposes. ■

## 5.9 Exercises

### 5.9.1 Exercises Without Answer Key

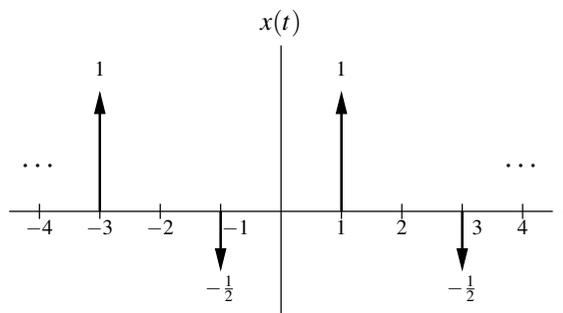
5.1 For each case below, find the Fourier series representation (in complex exponential form) of the function  $x$ , explicitly identifying the fundamental period of  $x$  and the Fourier series coefficient sequence  $c$ .

(a)  $x(t) = 1 + \cos \pi t + \sin^2 \pi t$ ;

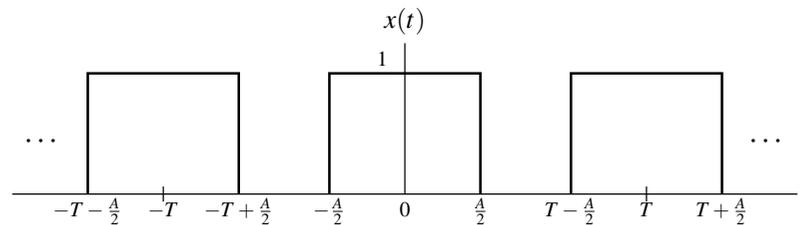
(b)  $x(t) = [\cos 4t][\sin t]$ ; and

(c)  $x(t) = |\sin 2\pi t|$ . [Hint:  $\int e^{ax} \sin bx dx = \frac{e^{ax}[a \sin bx - b \cos bx]}{a^2 + b^2} + C$ , where  $a$  and  $b$  are arbitrary complex and nonzero real constants, respectively.]

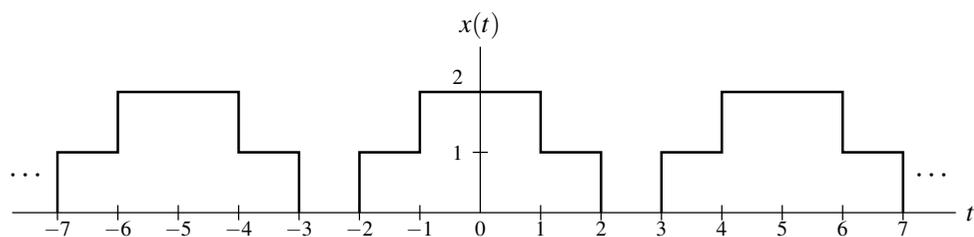
5.2 For each of the periodic functions shown in the figures below, find the corresponding Fourier series coefficient sequence.



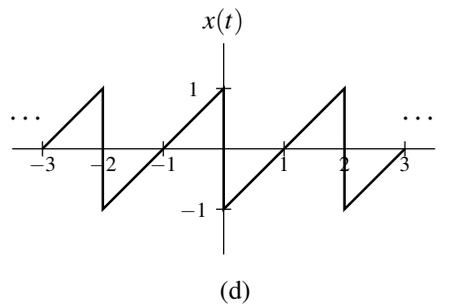
(a)



(b)



(c)

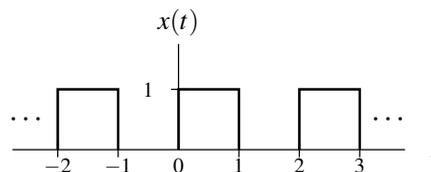


- 5.3** Show that, for a complex periodic signal  $x$  with the Fourier series coefficient sequence  $c$ :
- $x$  is even if and only if  $c$  is even; and
  - $x$  is odd if and only if  $c$  is odd.
- 5.4** Suppose that the periodic function  $x$  has the Fourier series coefficient sequence  $c$ . Determine the Fourier series coefficient sequence  $d$  of the function  $y(t) = \mathcal{D}x(t)$ , where  $\mathcal{D}$  denotes the derivative operator.
- 5.5** Let  $x$  be a periodic function with the Fourier series coefficient sequence  $c$  given by

$$c_k = \begin{cases} 1 & k = 0 \\ j\left(\frac{1}{2}\right)^{|k|} & \text{otherwise.} \end{cases}$$

Using Fourier series properties as appropriate, determine if each of the following assertions is true:

- $x$  is real;
  - $x$  is even;
  - $\frac{d}{dt}x(t)$  is even. [Hint: Consider Exercise 5.4 first.]
- 5.6** A periodic function  $x$  with period  $T$  and Fourier series coefficient sequence  $c$  is said to be odd harmonic if  $c_k = 0$  for all even  $k$ .
- Show that if  $x$  is odd harmonic, then  $x(t) = -x(t - \frac{T}{2})$  for all  $t$ .
  - Show that if  $x(t) = -x(t - \frac{T}{2})$  for all  $t$ , then  $x$  is odd harmonic.
- 5.7** Let  $x$  be a periodic function with fundamental period  $T$  and Fourier series coefficient sequence  $c$ . Find the Fourier series coefficient sequence  $c'$  of each of the following functions  $x'$  in terms of  $c$ :
- $x' = \text{Even}(x)$
  - $x' = \text{Re}(x)$ .
- 5.8** Find the Fourier series coefficient sequence  $c$  of the periodic function  $x$  shown in the figure below. Plot the frequency spectrum of  $x$ , including the first five harmonics.



5.9 Consider a LTI system with frequency response

$$H(j\omega) = \begin{cases} 1 & |\omega| \geq 5 \\ 0 & \text{otherwise.} \end{cases}$$

Using frequency-domain methods, find the output  $y$  of the system if the input  $x$  is given by

$$x(t) = 1 + 2 \cos 2t + 2 \cos 4t + \frac{1}{2} \cos 6t.$$

### 5.9.2 Exercises With Answer Key

5.10 Find the Fourier series coefficient sequence  $c$  of each periodic function  $x$  given below with fundamental period  $T$ .

(a)  $x(t) = e^{-t}$  for  $-1 \leq t < 1$  and  $T = 2$ ; and

(b)  $x(t) = \text{rect}(t - \frac{3}{2}) - \text{rect}(t + \frac{3}{2})$  for  $-\frac{5}{2} \leq t < \frac{5}{2}$  and  $T = 5$ .

**Short Answer.** (a)  $c_k = \frac{(-1)^k(e - e^{-1})}{j2\pi k + 2}$ ; (b)  $c_k = \begin{cases} \frac{1}{j\pi k} (\cos \frac{2\pi k}{5} - \cos \frac{4\pi k}{5}) & k \neq 0 \\ 0 & k = 0 \end{cases}$

## 5.10 MATLAB Exercises

5.101 Consider the periodic function  $x$  shown in Figure B of Exercise 5.2, where  $T = 1$  and  $A = \frac{1}{2}$ . We can show that  $x$  has the Fourier series representation

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

where  $c_k = \frac{1}{2} \text{sinc} \frac{\pi k}{2}$  and  $\omega_0 = 2\pi$ . Let  $\hat{x}_N(t)$  denote the above infinite series truncated after the  $N$ th harmonic component. That is,

$$\hat{x}_N(t) = \sum_{k=-N}^N c_k e^{jk\omega_0 t}.$$

(a) Use MATLAB to plot  $\hat{x}_N(t)$  for  $N = 1, 5, 10, 50, 100$ . You should see that as  $N$  increases,  $\hat{x}_N$  converges to  $x$ . [Hint: You may find the `sym`, `symsum`, `subs`, and `ezplot` functions useful for this problem. Note that the MATLAB `sinc` function does not compute the sinc function as defined herein. Instead, the MATLAB `sinc` function computes the normalized sinc function as defined by (3.25).]

(b) By examining the graphs obtained in part (a), answer the following: As  $N \rightarrow \infty$ , does  $\hat{x}_N$  converge to  $x$  uniformly (i.e., at the same rate everywhere)? If not, where is the rate of convergence slower?

(c) The function  $x$  is not continuous everywhere. For example,  $x$  has a discontinuity at  $\frac{1}{4}$ . As  $N \rightarrow \infty$ , to what value does  $\hat{x}_N$  appear to converge at this point? Again, deduce your answer from the graphs obtained in part (a).



## Chapter 6

# Continuous-Time Fourier Transform

### 6.1 Introduction

The (CT) Fourier series provides an extremely useful representation for periodic functions. Often, however, we need to deal with functions that are not periodic. A more general tool than the Fourier series is needed in this case. In this chapter, we will introduce a tool for representing arbitrary (i.e., possibly aperiodic) functions, known as the Fourier transform.

### 6.2 Development of the Continuous-Time Fourier Transform for Aperiodic Functions

As demonstrated earlier, the Fourier series is an extremely useful function representation. Unfortunately, this representation can only be used for periodic functions, since a Fourier series is inherently periodic. Many functions, however, are not periodic. Therefore, one might wonder if we can somehow use the Fourier series to develop a representation for aperiodic functions. As it turns out, this is possible. In order to understand why, we must make the following key observation. An aperiodic function can be viewed as a periodic function with a period of infinity. By viewing an aperiodic function as this limiting case of a periodic function where the period is infinite, we can use the Fourier series to develop a more general function representation that can be used in the aperiodic case. (In what follows, our development of the Fourier transform is not completely rigorous, as we assume that various integrals, summations, and limits converge. Such assumptions are not valid in all cases. Our development is mathematically sound, however, provided that the Fourier transform of the function being considered exists.)

Suppose that we have an aperiodic function  $x$ . From  $x$ , let us define the function  $x_T$  as

$$x_T(t) = \begin{cases} x(t) & -\frac{T}{2} \leq t < \frac{T}{2} \\ 0 & \text{otherwise.} \end{cases} \quad (6.1)$$

In other words,  $x_T(t)$  is identical to  $x(t)$  over the interval  $-\frac{T}{2} \leq t < \frac{T}{2}$  and zero otherwise. Let us now repeat the portion of  $x_T(t)$  for  $-\frac{T}{2} \leq t < \frac{T}{2}$  to form a  $T$ -periodic function  $\tilde{x}$ . That is, we define  $\tilde{x}$  as

$$\tilde{x}(t) = x_T(t) \text{ for } -\frac{T}{2} \leq t < \frac{T}{2} \quad \text{and} \quad \tilde{x}(t) = \tilde{x}(t + T).$$

In Figures 6.1 and 6.2, we provide illustrative examples of the functions  $x$ ,  $x_T$ , and  $\tilde{x}$ .

Before proceeding further, we make two important observations that we will use later. First, from the definition of  $x_T$ , we have

$$\lim_{T \rightarrow \infty} x_T(t) = x(t). \quad (6.2)$$

Second, from the definition of  $x_T$  and  $\tilde{x}$ , we have

$$\lim_{T \rightarrow \infty} \tilde{x}(t) = x(t). \quad (6.3)$$

Now, let us consider the function  $\tilde{x}$ . Since  $\tilde{x}$  is periodic, we can represent it using a Fourier series as

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad (6.4)$$

where

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt \quad (6.5)$$

and  $\omega_0 = \frac{2\pi}{T}$ . Since  $x_T(t) = \tilde{x}(t)$  for  $-\frac{T}{2} \leq t < \frac{T}{2}$ , we can rewrite (6.5) as

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x_T(t) e^{-jk\omega_0 t} dt.$$

Furthermore, since  $x_T(t) = 0$  for  $t < -\frac{T}{2}$  and  $t \geq \frac{T}{2}$ , we can rewrite the preceding expression for  $a_k$  as

$$a_k = \frac{1}{T} \int_{-\infty}^{\infty} x_T(t) e^{-jk\omega_0 t} dt.$$

Substituting this expression for  $a_k$  into (6.4) and rearranging, we obtain the following Fourier series representation for  $\tilde{x}$ :

$$\begin{aligned} \tilde{x}(t) &= \sum_{k=-\infty}^{\infty} \left[ \frac{1}{T} \int_{-\infty}^{\infty} x_T(\tau) e^{-jk\omega_0 \tau} d\tau \right] e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} \left[ \frac{\omega_0}{2\pi} \int_{-\infty}^{\infty} x_T(\tau) e^{-jk\omega_0 \tau} d\tau \right] e^{jk\omega_0 t} \\ &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x_T(\tau) e^{-jk\omega_0 \tau} d\tau \right] e^{jk\omega_0 t} \omega_0. \end{aligned}$$

Substituting the above expression for  $\tilde{x}$  into (6.3), we obtain

$$x(t) = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x_T(\tau) e^{-jk\omega_0 \tau} d\tau \right] e^{jk\omega_0 t} \omega_0. \quad (6.6)$$

Now, we must evaluate the above limit. As  $T \rightarrow \infty$ , we have that  $\omega_0 \rightarrow 0$ . Thus, in the limit above,  $k\omega_0$  becomes a continuous variable which we denote as  $\omega$ ,  $\omega_0$  becomes the infinitesimal  $d\omega$ , and the summation becomes an integral. This is illustrated in Figure 6.3. Also, as  $T \rightarrow \infty$ , we have that  $x_T \rightarrow x$ . Combining these results, we can rewrite (6.6) to obtain

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega,$$

where

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt.$$

Thus, we have found a representation of the aperiodic function  $x$  in terms of complex sinusoids at all frequencies. We call this the Fourier transform representation of the function  $x$ .

### 6.3 Generalized Fourier Transform

In the previous section, we used a limiting process involving the analysis and synthesis equations for Fourier series in order to develop a new mathematical tool known as the Fourier transform. As it turns out, many functions of

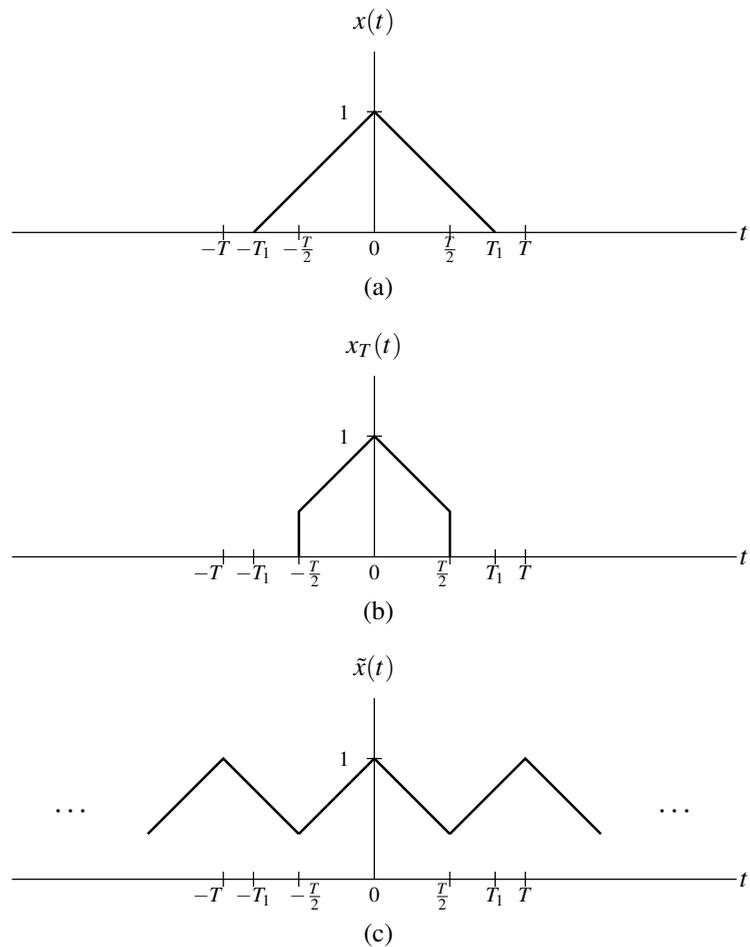


Figure 6.1: An example of the functions used in the derivation of the Fourier transform representation, where  $T_1 > \frac{T}{2}$ . (a) An aperiodic function  $x$ ; (b) the function  $x_T$ ; and (c) the  $T$ -periodic function  $\tilde{x}$ .

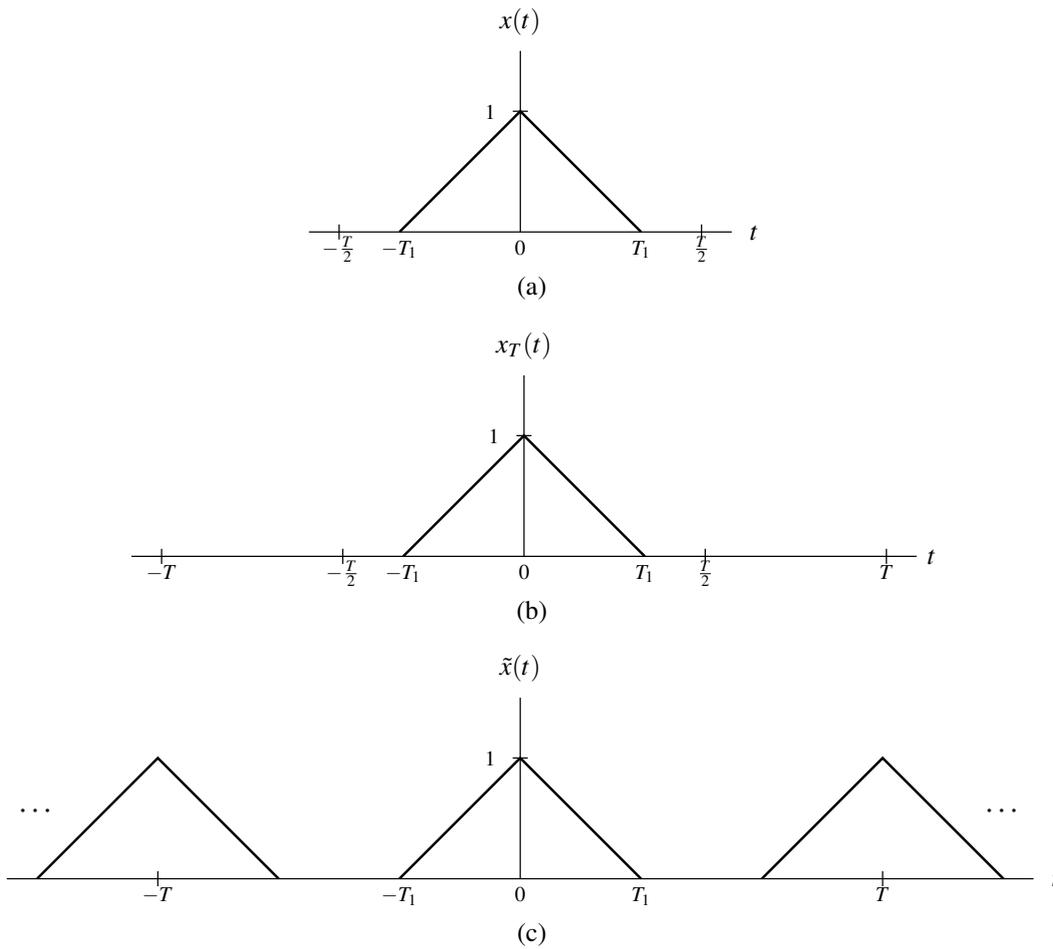


Figure 6.2: An example of the functions used in the derivation of the Fourier transform representation, where  $T_1 < \frac{T}{2}$ . (a) An aperiodic function  $x$ ; (b) the function  $x_T$ ; and (c) the  $T$ -periodic function  $\tilde{x}$ .

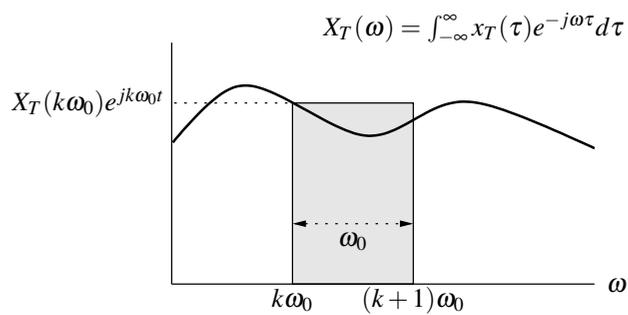


Figure 6.3: Integral obtained in the derivation of the Fourier transform representation.

practical interest do not have a Fourier transform in the sense of the definition developed previously. That is, for a given function  $x$ , the Fourier transform integral

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

may fail to converge, in which case the Fourier transform  $X$  of  $x$  does not exist. For example, the preceding integral does not converge if  $x$  is any of the following (as well as many other possibilities):

- a nonzero constant function;
- a periodic function (e.g., a real or complex sinusoid);
- the unit-step function (i.e.,  $u$ ); or
- the signum function (i.e.,  $\text{sgn}$ ).

Functions such these are of great practical interest, however. Therefore, it is highly desirable to have a mathematical tool that can handle such functions. This motivates the development of what is called the **generalized Fourier transform**. The generalized Fourier transform exists for periodic functions, nonzero constant functions, and many other types of functions as well. The underlying math associated with the generalized Fourier transform is quite complicated. So, we will not attempt to formally develop the generalized Fourier transform here. Although not entirely correct, one can think of the generalized Fourier transform as being defined by the same formulas as the classical Fourier transform. So, for this and other reasons, we can mostly ignore the distinction between the generalized Fourier transform and classical Fourier transform, and think of them as being one and the same. In what follows, we will avoid making a distinction between the classical Fourier transform and generalized Fourier transform, except in a very small number of places where it is beneficial to do so. The main disadvantage of not formally introducing the generalized Fourier transform is that some results presented later (which actually rely on the use of the generalized Fourier transform) must be accepted on faith since their proof would require formal knowledge of the generalized Fourier transform, which is not introduced herein. As long as the generalized Fourier transform is used, both periodic and aperiodic functions can be handled, and in this sense we have a more general tool than Fourier series (which require periodic functions). Later, when we discuss the Fourier transform of periodic functions, we will implicitly be using the generalized Fourier transform in that context. In fact, in much of what follows, when we speak of the Fourier transform, we are often referring to the generalized Fourier transform.

## 6.4 Definition of the Continuous-Time Fourier Transform

Earlier, we developed the Fourier transform representation of a function. This representation expresses a function in terms of complex sinusoids at all frequencies. More formally, the **Fourier transform** of the function  $x$ , denoted as  $\mathcal{F}x$  or  $X$ , is defined as

$$\mathcal{F}x(\omega) = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt. \quad (6.7)$$

Similarly, the inverse Fourier transform of  $X$ , denoted as  $\mathcal{F}^{-1}X$  or  $x$ , is given by

$$\mathcal{F}^{-1}X(t) = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega. \quad (6.8)$$

We refer to (6.7) as the **Fourier transform analysis equation** and (6.8) as the **Fourier transform synthesis equation**. If a function  $x$  has the Fourier transform  $X$ , we often denote this as

$$x(t) \xleftrightarrow{\text{CTFT}} X(\omega).$$

As a matter of terminology,  $x$  and  $X$  are said to be a **Fourier transform pair**.

**Example 6.1** (Fourier transform of the unit-impulse function). Find the Fourier transform  $X$  of the function

$$x(t) = A\delta(t - t_0),$$

where  $A$  and  $t_0$  are real constants. Then, from this result, write the Fourier transform representation of  $x$ .

*Solution.* From the definition of the Fourier transform, we can write

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} A\delta(t - t_0)e^{-j\omega t} dt \\ &= A \int_{-\infty}^{\infty} \delta(t - t_0)e^{-j\omega t} dt. \end{aligned}$$

Using the sifting property of the unit-impulse function, we can simplify the above result to obtain

$$X(\omega) = Ae^{-j\omega t_0}.$$

Thus, we have shown that

$$A\delta(t - t_0) \xleftrightarrow{\text{CTFT}} Ae^{-j\omega t_0}.$$

From the Fourier transform analysis and synthesis equations, we have that the Fourier transform representation of  $x$  is given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega, \quad \text{where } X(\omega) = Ae^{-j\omega t_0}. \quad \blacksquare$$

**Example 6.2** (Inverse Fourier transform of the unit-impulse function). Find the inverse Fourier transform  $x$  of the function

$$X(\omega) = 2\pi A\delta(\omega - \omega_0),$$

where  $A$  and  $\omega_0$  are real constants.

*Solution.* From the definition of the inverse Fourier transform, we can write

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi A\delta(\omega - \omega_0)e^{j\omega t} d\omega \\ &= A \int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{j\omega t} d\omega. \end{aligned}$$

Using the sifting property of the unit-impulse function, we can simplify the preceding equation to obtain

$$x(t) = Ae^{j\omega_0 t}.$$

Thus, we have that

$$Ae^{j\omega_0 t} \xleftrightarrow{\text{CTFT}} 2\pi A\delta(\omega - \omega_0). \quad \blacksquare$$

**Example 6.3** (Fourier transform of the rectangular function). Find the Fourier transform  $X$  of the function

$$x(t) = \text{rect}t.$$

*Solution.* From the definition of the Fourier transform, we can write

$$X(\omega) = \int_{-\infty}^{\infty} (\text{rect}t) e^{-j\omega t} dt.$$

From the definition of the rectangular function, we can simplify this equation to obtain

$$\begin{aligned} X(\omega) &= \int_{-1/2}^{1/2} (\text{rect}t) e^{-j\omega t} dt \\ &= \int_{-1/2}^{1/2} e^{-j\omega t} dt. \end{aligned}$$

Evaluating the integral and simplifying, we have

$$\begin{aligned} X(\omega) &= \left[ -\frac{1}{j\omega} e^{-j\omega t} \right]_{-1/2}^{1/2} \\ &= \frac{1}{j\omega} \left( e^{j\omega/2} - e^{-j\omega/2} \right) \\ &= \frac{1}{j\omega} \left( 2j \sin \frac{\omega}{2} \right) \\ &= \frac{2}{\omega} \sin \frac{\omega}{2} \\ &= \left( \sin \frac{\omega}{2} \right) / \left( \frac{\omega}{2} \right) \\ &= \text{sinc} \frac{\omega}{2}. \end{aligned}$$

Thus, we have shown that

$$\text{rect}t \xleftrightarrow{\text{CFT}} \text{sinc} \frac{\omega}{2}. \quad \blacksquare$$

## 6.5 Remarks on Notation Involving the Fourier Transform

Each of the Fourier transform operator  $\mathcal{F}$  and inverse Fourier transform operator  $\mathcal{F}^{-1}$  map a function to a function. Consequently, the operand for each of these operators must be a function (and not a number). Consider the unnamed function that maps  $t$  to  $e^{-|t|}$  as shown in Figure 6.4. Suppose that we would like to write an expression that denotes the Fourier transform of this function. At first, we might be inclined to write “ $\mathcal{F}\{e^{-|t|}\}$ ”. Strictly speaking, however, this notation is not correct, since the Fourier transform operator requires a function as an operand and “ $e^{-|t|}$ ” (strictly speaking) denotes a number (i.e., the value of the function in the figure evaluated at  $t$ ). Essentially, the cause of our problems here is that the function in question does not have a name (such as “ $x$ ”) by which it can be referred. To resolve this problem, we could define a function  $x$  using the equation  $x(t) = e^{-|t|}$  and then write the Fourier transform as “ $\mathcal{F}x$ ”. Unfortunately, introducing a new function name just for the sake of strictly correct notation is often undesirable as it frequently leads to overly verbose writing.

One way to avoid overly verbose writing when referring to functions without names is offered by dot notation, introduced earlier in Section 2.1. Again, consider the function from Figure 6.4 that maps  $t$  to  $e^{-|t|}$ . Using strictly correct notation, we could write the Fourier transform of this function as “ $\mathcal{F}\{e^{-|\cdot|}\}$ ”. In other words, we can indicate that an expression refers to a function (as opposed to the value of function) by using the interpunct symbol (as discussed in Section 2.1). Some examples of the use of dot notation can be found below in Example 6.4. Dot notation is often extremely beneficial when one wants to employ precise (i.e., strictly correct) notation without being overly verbose.

**Example 6.4** (Dot notation). Several examples of the use of dot notation are as follows:

1. To denote the Fourier transform of the function  $x$  defined by the equation  $x(t) = e^{t^2}$  (without the need to introduce the named function  $x$ ), we can write:  $\mathcal{F}\{e^{(\cdot)^2}\}$ .

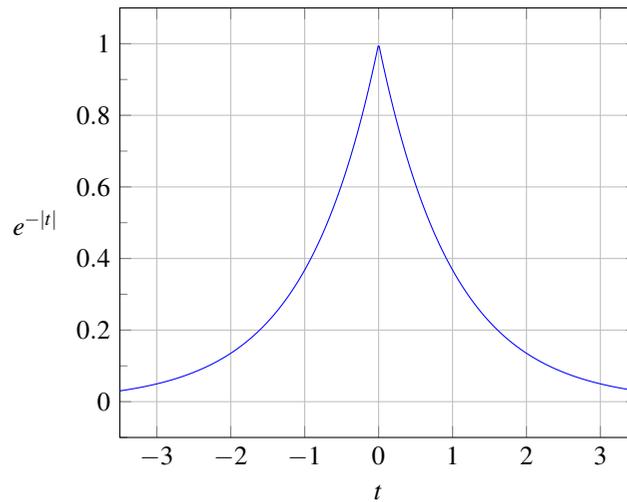


Figure 6.4: A plot of  $e^{-|t|}$  versus  $t$ .

2. To denote the Fourier transform of the function  $x$  defined by the equation  $x(t) = e^{t^2}$  evaluated at  $2\omega - 3$  (without the need to introduce the named function  $x$ ), we can write:  $\mathcal{F}\{e^{(\cdot)^2}\}(2\omega - 3)$ .
3. To denote the inverse Fourier transform of the function  $X$  defined by the equation  $X(\omega) = \frac{1}{j\omega}$  (without the need to introduce the named function  $X$ ), we can write:  $\mathcal{F}^{-1}\left\{\frac{1}{j(\cdot)}\right\}$ .
4. To denote the inverse Fourier transform of the function  $X$  defined by the equation  $X(\omega) = \frac{1}{j\omega}$  evaluated at  $t - 3$  (without the need to introduce the named function  $X$ ), we can write:  $\mathcal{F}^{-1}\left\{\frac{1}{j(\cdot)}\right\}(t - 3)$ . ■

If the reader is comfortable with dot notation, the author would encourage the reader to use it when appropriate. Since some readers may find the dot notation to be confusing, however, this book (for the most part) attempts to minimize the use of dot notation. Instead, as a compromise solution, this book adopts the following notational conventions in order to achieve conciseness and a reasonable level of clarity without the need to use dot notation pervasively:

- unless indicated otherwise, in an expression for the operand of the Fourier transform operator  $\mathcal{F}$ , the variable “ $t$ ” is assumed to be the independent variable for the function to which the Fourier transform is being applied (i.e., in terms of dot notation, the expression is treated as if each “ $t$ ” were a “ $\cdot$ ”);
- unless indicated otherwise, in an expression for the operand of the inverse Fourier transform operator  $\mathcal{F}^{-1}$ , the variable “ $\omega$ ” is assumed to be the independent variable for the function to which the inverse Fourier transform is being applied (i.e., in terms of dot notation, the expression is treated as if each “ $\omega$ ” were a “ $\cdot$ ”)

Some examples of using these book-sanctioned notational conventions can be found below in Example 6.5. Admittedly, these book-sanctioned conventions are not ideal, as they abuse mathematical notation somewhat, but they seem to be the best compromise in order to accommodate those who may prefer not to use dot notation.

**Example 6.5** (Book-sanctioned notation). Several examples of using the notational conventions that are employed throughout most of this book (as described above) are as follows:

1. To denote the Fourier transform of the function  $x$  defined by the equation  $x(t) = e^{t^2}$  (without the need to introduce the named function  $x$ ), we can write:  $\mathcal{F}\{e^{t^2}\}$ .
2. To denote the Fourier transform of the function  $x$  defined by the equation  $x(t) = e^{t^2}$  evaluated at  $2\omega - 3$  (without the need to introduce the named function  $x$ ), we can write:  $\mathcal{F}\{e^{t^2}\}(2\omega - 3)$ .

3. To denote the inverse Fourier transform of the function  $X$  defined by the equation  $X(\omega) = \frac{1}{j\omega}$  (without the need to introduce the named function  $X$ ), we can write:  $\mathcal{F}^{-1}\left\{\frac{1}{j\omega}\right\}$ .
4. To denote the inverse Fourier transform of the function  $X$  defined by the equation  $X(\omega) = \frac{1}{j\omega}$  evaluated at  $t - 3$  (without the need to introduce the named function  $X$ ), we can write:  $\mathcal{F}^{-1}\left\{\frac{1}{j\omega}\right\}(t - 3)$ . ■

Since applying the Fourier transform operator or inverse Fourier transform operator to a function yields another function, we can evaluate this other function at some value. Again, consider the function from Figure 6.4 that maps  $t$  to  $e^{-|t|}$ . To denote the value of the Fourier transform of this function evaluated at  $\omega - 1$ , we would write “ $\mathcal{F}\{e^{-|t|}\}(\omega - 1)$ ” using dot notation or “ $\mathcal{F}\{e^{-|t|}\}(\omega - 1)$ ” using the book-sanctioned notational conventions described above.

## 6.6 Convergence of the Continuous-Time Fourier Transform

When deriving the Fourier transform representation earlier, we implicitly made some assumptions about the convergence of the integrals and other expressions involved. These assumptions are not always valid. For this reason, a more careful examination of the convergence properties of the Fourier transform is in order.

Suppose that we have an arbitrary function  $x$ . This function has the Fourier transform representation  $\hat{x}$  given by

$$\hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega, \quad \text{where} \quad X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt.$$

Now, we need to concern ourselves with the convergence properties of this representation. In other words, we want to know when  $\hat{x}$  is a valid representation of  $x$ . In our earlier derivation of the Fourier transform, we relied heavily on the Fourier series. Therefore, one might expect that the convergence of the Fourier transform representation is closely related to the convergence properties of Fourier series. This is, in fact, the case. The convergence properties of the Fourier transform are very similar to the convergence properties of the Fourier series (as studied in Section 5.4).

The first important result concerning convergence relates to continuous functions as stated by the theorem below.

**Theorem 6.1** (Convergence of the Fourier transform (continuous case)). *If a function  $x$  is continuous and absolutely integrable (i.e.,  $\int_{-\infty}^{\infty} |x(t)| dt < \infty$ ) and the Fourier transform  $X$  of  $x$  is absolutely integrable (i.e.,  $\int_{-\infty}^{\infty} |X(\omega)| d\omega < \infty$ ), then the Fourier transform representation of  $x$  converges pointwise (i.e.,  $x(t) = \hat{x}(t)$  for all  $t$ ).*

*Proof.* A rigorous proof of this result is beyond the scope of this book and is therefore omitted here. ■

Since, in practice, we often encounter functions with discontinuities (e.g., a rectangular pulse), the above result is sometimes of limited value. This motivates us to consider additional results concerning convergence.

The next important result concerning convergence relates to finite-energy functions as stated by the theorem below.

**Theorem 6.2** (Convergence of Fourier transform (finite-energy case)). *If a function  $x$  is of finite energy (i.e.,  $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$ ), then its Fourier transform representation  $\hat{x}$  converges in the MSE sense.*

*Proof.* A rigorous proof of this result is beyond the scope of this book and is therefore omitted here. ■

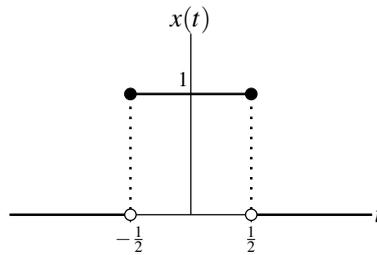
In other words, the preceding theorem states that, if  $x$  is of finite energy, then

$$E = \int_{-\infty}^{\infty} |\hat{x}(t) - x(t)|^2 dt = 0.$$

Although  $x$  and  $\hat{x}$  may differ at individual points, the energy  $E$  in the difference is zero.

The last result concerning convergence that we shall consider relates to what are known as the Dirichlet conditions. The Dirichlet conditions for the function  $x$  are as follows:

1. The function  $x$  is absolutely integrable (i.e.,  $\int_{-\infty}^{\infty} |x(t)| dt < \infty$ ).
2. The function  $x$  has a finite number of maxima and minima on any finite interval (i.e., is of bounded variation).

Figure 6.5: Function  $x$ .

3. The function  $x$  has a finite number of discontinuities on any finite interval, and each discontinuity is itself finite.

For a function satisfying the Dirichlet conditions, we have the important convergence result stated below.

**Theorem 6.3** (Convergence of Fourier transform (Dirichlet case)). *If a function  $x$  satisfies the Dirichlet conditions, then its Fourier transform representation  $\hat{x}$  converges pointwise everywhere except at points of discontinuity. Furthermore, at each discontinuity point  $t_a$ , we have that*

$$\hat{x}(t_a) = \frac{1}{2} [x(t_a^+) + x(t_a^-)],$$

where  $x(t_a^-)$  and  $x(t_a^+)$  denote the values of the function  $x$  on the left- and right-hand sides of the discontinuity, respectively.

*Proof.* A rigorous proof of this result is beyond the scope of this book and is therefore omitted here. ■

In other words, the preceding theorem states that, if a function  $x$  satisfies the Dirichlet conditions, then the Fourier transform representation  $\hat{x}$  is such that  $\hat{x}(t) = x(t)$  for all  $t$ , except at points of discontinuity where  $\hat{x}(t)$  equals the average value of  $x$  on the two sides of the discontinuity.

The finite-energy and Dirichlet conditions mentioned above are only sufficient conditions for the convergence of the Fourier transform representation. They are not necessary. In other words, a function may violate these conditions and still have a valid Fourier transform representation.

**Example 6.6.** Consider the function  $x$  shown in Figure 6.5. Let  $\hat{x}$  denote the Fourier transform representation of  $x$  (i.e.,  $\hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$ , where  $X$  denotes the Fourier transform of  $x$ ). Determine the values  $\hat{x}(-\frac{1}{2})$  and  $\hat{x}(\frac{1}{2})$ .

*Solution.* We begin by observing that  $x$  satisfies the Dirichlet conditions. Consequently, Theorem 6.3 applies. Thus, we have that

$$\begin{aligned} \hat{x}\left(-\frac{1}{2}\right) &= \frac{1}{2} \left[ x\left(-\frac{1}{2}^-\right) + x\left(-\frac{1}{2}^+\right) \right] \\ &= \frac{1}{2} (0 + 1) \\ &= \frac{1}{2} \quad \text{and} \end{aligned}$$

$$\begin{aligned} \hat{x}\left(\frac{1}{2}\right) &= \frac{1}{2} \left[ x\left(\frac{1}{2}^-\right) + x\left(\frac{1}{2}^+\right) \right] \\ &= \frac{1}{2} (1 + 0) \\ &= \frac{1}{2}. \end{aligned} \quad \blacksquare$$

## 6.7 Properties of the Continuous-Time Fourier Transform

The Fourier transform has a number of important properties. In the sections that follow, we introduce several of these properties. For convenience, these properties are also later summarized in Table 6.1 (on page 159). Also, for convenience, several Fourier-transform pairs are given later in Table 6.2 (on page 163). In what follows, we will sometimes refer to transform pairs in this table.

### 6.7.1 Linearity

Arguably, the most important property of the Fourier transform is linearity, as introduced below.

**Theorem 6.4** (Linearity). *If  $x_1(t) \xleftrightarrow{\text{CTFT}} X_1(\omega)$  and  $x_2(t) \xleftrightarrow{\text{CTFT}} X_2(\omega)$ , then*

$$a_1x_1(t) + a_2x_2(t) \xleftrightarrow{\text{CTFT}} a_1X_1(\omega) + a_2X_2(\omega),$$

where  $a_1$  and  $a_2$  are arbitrary complex constants.

*Proof.* To prove the above property, we proceed as follows. Let  $y(t) = a_1x_1(t) + a_2x_2(t)$  and let  $Y = \mathcal{F}y$ . We have

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} [a_1x_1(t) + a_2x_2(t)]e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} a_1x_1(t)e^{-j\omega t} dt + \int_{-\infty}^{\infty} a_2x_2(t)e^{-j\omega t} dt \\ &= a_1 \int_{-\infty}^{\infty} x_1(t)e^{-j\omega t} dt + a_2 \int_{-\infty}^{\infty} x_2(t)e^{-j\omega t} dt \\ &= a_1X_1(\omega) + a_2X_2(\omega). \end{aligned}$$

Thus, we have shown that the linearity property holds. ■

**Example 6.7** (Linearity property of the Fourier transform). Using properties of the Fourier transform and the transform pair

$$e^{j\omega_0 t} \xleftrightarrow{\text{CTFT}} 2\pi\delta(\omega - \omega_0),$$

find the Fourier transform  $X$  of the function

$$x(t) = A \cos \omega_0 t,$$

where  $A$  and  $\omega_0$  are real constants.

*Solution.* We recall that  $\cos \alpha = \frac{1}{2}[e^{j\alpha} + e^{-j\alpha}]$  for any real  $\alpha$ . Thus, we can write

$$\begin{aligned} X(\omega) &= (\mathcal{F}\{A \cos \omega_0 t\})(\omega) \\ &= (\mathcal{F}\{\frac{A}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})\})(\omega). \end{aligned}$$

Then, we use the linearity property of the Fourier transform to obtain

$$X(\omega) = \frac{A}{2}\mathcal{F}\{e^{j\omega_0 t}\}(\omega) + \frac{A}{2}\mathcal{F}\{e^{-j\omega_0 t}\}(\omega).$$

Using the given Fourier transform pair, we can further simplify the above expression for  $X(\omega)$  as follows:

$$\begin{aligned} X(\omega) &= \frac{A}{2}[2\pi\delta(\omega + \omega_0)] + \frac{A}{2}[2\pi\delta(\omega - \omega_0)] \\ &= A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]. \end{aligned}$$

Thus, we have shown that

$$A \cos \omega_0 t \xleftrightarrow{\text{CTFT}} A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)].$$
■

**Example 6.8** (Fourier transform of the unit-step function). Using properties of the Fourier transform and the transform pairs

$$1 \xleftrightarrow{\text{CTFT}} 2\pi\delta(\omega) \quad \text{and} \quad \text{sgn } t \xleftrightarrow{\text{CTFT}} \frac{2}{j\omega},$$

find the Fourier transform  $X$  of the function  $x = u$ .

*Solution.* First, we observe that  $x$  can be expressed in terms of the signum function as

$$x(t) = u(t) = \frac{1}{2} + \frac{1}{2} \text{sgn } t.$$

Taking the Fourier transform of both sides of this equation yields

$$X(\omega) = (\mathcal{F}\{\frac{1}{2} + \frac{1}{2} \text{sgn } t\})(\omega).$$

Using the linearity property of the Fourier transform, we can rewrite this as

$$X(\omega) = \frac{1}{2}\mathcal{F}\{1\}(\omega) + \frac{1}{2}\mathcal{F}\{\text{sgn } t\}(\omega).$$

Evaluating the two Fourier transforms using the given transform pairs, we obtain

$$\begin{aligned} X(\omega) &= \frac{1}{2}[2\pi\delta(\omega)] + \frac{1}{2}\left(\frac{2}{j\omega}\right) \\ &= \pi\delta(\omega) + \frac{1}{j\omega}. \end{aligned}$$

Thus, we have shown that

$$u(t) \xleftrightarrow{\text{CTFT}} \pi\delta(\omega) + \frac{1}{j\omega}. \quad \blacksquare$$

## 6.7.2 Time-Domain Shifting (Translation)

The next property of the Fourier transform to be introduced is the time-domain shifting (i.e., translation) property, as given below.

**Theorem 6.5** (Time-domain shifting (i.e., translation)). *If  $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$ , then*

$$x(t - t_0) \xleftrightarrow{\text{CTFT}} e^{-j\omega t_0} X(\omega),$$

where  $t_0$  is an arbitrary real constant.

*Proof.* To prove the above property, we proceed as follows. Let  $y(t) = x(t - t_0)$  and let  $Y = \mathcal{F}y$ . From the definition of the Fourier transform, we have

$$Y(\omega) = \int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t} dt.$$

Now, we use a change of variable. Let  $\lambda = t - t_0$  so that  $t = \lambda + t_0$  and  $dt = d\lambda$ . Performing the change of variable and simplifying, we obtain

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega(\lambda + t_0)} d\lambda \\ &= \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\lambda} e^{-j\omega t_0} d\lambda \\ &= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\lambda} d\lambda \\ &= e^{-j\omega t_0} X(\omega). \end{aligned}$$

Thus, we have proven that the time-shifting property holds. \blacksquare

**Example 6.9** (Time-domain shifting property of the Fourier transform). Find the Fourier transform  $X$  of the function

$$x(t) = A \cos(\omega_0 t + \theta),$$

where  $A$ ,  $\omega_0$ , and  $\theta$  are real constants.

*Solution.* Let  $v(t) = A \cos \omega_0 t$  so that  $x(t) = v(t + \frac{\theta}{\omega_0})$ . Also, let  $V = \mathcal{F}v$ . From Table 6.2, we have that

$$\cos \omega_0 t \xleftrightarrow{\text{CFT}} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)].$$

Using this transform pair and the linearity property of the Fourier transform, we have that

$$\begin{aligned} V(\omega) &= \mathcal{F}\{A \cos \omega_0 t\}(\omega) \\ &= A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]. \end{aligned}$$

From the definition of  $v$  and the time-shifting property of the Fourier transform, we have

$$\begin{aligned} X(\omega) &= e^{j\omega\theta/\omega_0} V(\omega) \\ &= e^{j\omega\theta/\omega_0} A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]. \end{aligned}$$

Thus, we have shown that

$$A \cos(\omega_0 t + \theta) \xleftrightarrow{\text{CFT}} A\pi e^{j\omega\theta/\omega_0} [\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]. \quad \blacksquare$$

### 6.7.3 Frequency-Domain Shifting (Modulation)

The next property of the Fourier transform to be introduced is the frequency-domain shifting (i.e., modulation) property, as given below.

**Theorem 6.6** (Frequency-domain shifting (i.e., modulation)). *If  $x(t) \xleftrightarrow{\text{CFT}} X(\omega)$ , then*

$$e^{j\omega_0 t} x(t) \xleftrightarrow{\text{CFT}} X(\omega - \omega_0),$$

where  $\omega_0$  is an arbitrary real constant.

*Proof.* To prove the above property, we proceed as follows. Let  $y(t) = e^{j\omega_0 t} x(t)$  and let  $Y = \mathcal{F}y$ . From the definition of the Fourier transform and straightforward algebraic manipulation, we can write

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} e^{j\omega_0 t} x(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} x(t) e^{-j(\omega - \omega_0)t} dt \\ &= X(\omega - \omega_0). \end{aligned}$$

Thus, we have shown that the frequency-domain shifting property holds. \blacksquare

**Example 6.10** (Frequency-domain shifting property of the Fourier transform). Find the Fourier transform  $X$  of the function

$$x(t) = (\cos \omega_0 t)(\cos 20\pi t),$$

where  $\omega_0$  is a real constant.

*Solution.* Recall that  $\cos \alpha = \frac{1}{2}[e^{j\alpha} + e^{-j\alpha}]$  for any real  $\alpha$ . Using this relationship and the linearity property of the Fourier transform, we can write

$$\begin{aligned} X(\omega) &= (\mathcal{F}\{(\cos \omega_0 t)(\frac{1}{2})(e^{j20\pi t} + e^{-j20\pi t})\})(\omega) \\ &= (\mathcal{F}\{\frac{1}{2}e^{j20\pi t} \cos \omega_0 t + \frac{1}{2}e^{-j20\pi t} \cos \omega_0 t\})(\omega) \\ &= \frac{1}{2}(\mathcal{F}\{e^{j20\pi t} \cos \omega_0 t\})(\omega) + \frac{1}{2}(\mathcal{F}\{e^{-j20\pi t} \cos \omega_0 t\})(\omega). \end{aligned}$$

From Table 6.2, we have that

$$\cos \omega_0 t \xleftrightarrow{\text{CTFT}} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)].$$

From this transform pair and the frequency-domain shifting property of the Fourier transform, we have

$$\begin{aligned} X(\omega) &= \frac{1}{2}(\mathcal{F}\{\cos \omega_0 t\})(\omega - 20\pi) + \frac{1}{2}(\mathcal{F}\{\cos \omega_0 t\})(\omega + 20\pi) \\ &= \frac{1}{2}[\pi[\delta(v - \omega_0) + \delta(v + \omega_0)]]|_{v=\omega-20\pi} + \frac{1}{2}[\pi[\delta(v - \omega_0) + \delta(v + \omega_0)]]|_{v=\omega+20\pi} \\ &= \frac{1}{2}[\pi[\delta(\omega + \omega_0 - 20\pi) + \delta(\omega - \omega_0 - 20\pi)]] + \frac{1}{2}[\pi[\delta(\omega + \omega_0 + 20\pi) + \delta(\omega - \omega_0 + 20\pi)]] \\ &= \frac{\pi}{2}[\delta(\omega + \omega_0 - 20\pi) + \delta(\omega - \omega_0 - 20\pi) + \delta(\omega + \omega_0 + 20\pi) + \delta(\omega - \omega_0 + 20\pi)]. \quad \blacksquare \end{aligned}$$

#### 6.7.4 Time- and Frequency-Domain Scaling (Dilation)

The next property of the Fourier transform to be introduced is the time/frequency-scaling (i.e., dilation) property, as given below.

**Theorem 6.7** (Time/frequency-domain scaling (i.e., dilation)). *If  $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$ , then*

$$x(at) \xleftrightarrow{\text{CTFT}} \frac{1}{|a|} X\left(\frac{\omega}{a}\right),$$

where  $a$  is an arbitrary nonzero real constant.

*Proof.* To prove the above property, we proceed as follows. Let  $y(t) = x(at)$  and let  $Y = \mathcal{F}y$ . From the definition of the Fourier transform, we can write

$$Y(\omega) = \int_{-\infty}^{\infty} x(at)e^{-j\omega t} dt.$$

Now, we use a change of variable. Let  $\lambda = at$  so that  $t = \lambda/a$  and  $dt = d\lambda/a$ . Performing the change of variable (and being mindful of the change in the limits of integration), we obtain

$$\begin{aligned} Y(\omega) &= \begin{cases} \int_{-\infty}^{\infty} x(\lambda)e^{-j(\omega/a)\lambda}(\frac{1}{a})d\lambda & a > 0 \\ \int_{\infty}^{-\infty} x(\lambda)e^{-j(\omega/a)\lambda}(\frac{1}{a})d\lambda & a < 0 \end{cases} \\ &= \begin{cases} \frac{1}{a} \int_{-\infty}^{\infty} x(\lambda)e^{-j(\omega/a)\lambda} d\lambda & a > 0 \\ -\frac{1}{a} \int_{-\infty}^{\infty} x(\lambda)e^{-j(\omega/a)\lambda} d\lambda & a < 0. \end{cases} \end{aligned}$$

Combining the two cases (i.e., for  $a < 0$  and  $a > 0$ ), we obtain

$$\begin{aligned} Y(\omega) &= \frac{1}{|a|} \int_{-\infty}^{\infty} x(\lambda)e^{-j(\omega/a)\lambda} d\lambda \\ &= \frac{1}{|a|} X\left(\frac{\omega}{a}\right). \end{aligned}$$

Thus, we have shown that the time/frequency-scaling property holds. ■

**Example 6.11** (Time scaling property of the Fourier transform). Using the Fourier transform pair

$$\text{rect } t \xleftrightarrow{\text{CTFT}} \text{sinc } \frac{\omega}{2},$$

find the Fourier transform  $X$  of the function

$$x(t) = \text{rect } at,$$

where  $a$  is a nonzero real constant.

*Solution.* Let  $v(t) = \text{rect } t$  so that  $x(t) = v(at)$ . Also, let  $V = \mathcal{F}v$ . From the given transform pair, we know that

$$V(\omega) = (\mathcal{F}\{\text{rect } t\})(\omega) = \text{sinc } \frac{\omega}{2}. \quad (6.9)$$

From the definition of  $v$  and the time-scaling property of the Fourier transform, we have

$$X(\omega) = \frac{1}{|a|} V\left(\frac{\omega}{a}\right).$$

Substituting the expression for  $V$  in (6.9) into the preceding equation, we have

$$X(\omega) = \frac{1}{|a|} \text{sinc } \frac{\omega}{2a}.$$

Thus, we have shown that

$$\text{rect } at \xleftrightarrow{\text{CTFT}} \frac{1}{|a|} \text{sinc } \frac{\omega}{2a}. \quad \blacksquare$$

### 6.7.5 Conjugation

The next property of the Fourier transform to be introduced is the conjugation property, as given below.

**Theorem 6.8** (Conjugation). *If  $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$ , then*

$$x^*(t) \xleftrightarrow{\text{CTFT}} X^*(-\omega).$$

*Proof.* To prove the above property, we proceed as follows. Let  $y(t) = x^*(t)$  and let  $Y = \mathcal{F}y$ . From the definition of the Fourier transform, we have

$$Y(\omega) = \int_{-\infty}^{\infty} x^*(t) e^{-j\omega t} dt.$$

From the properties of conjugation, we can rewrite this equation as

$$\begin{aligned} Y(\omega) &= \left[ \left( \int_{-\infty}^{\infty} x^*(t) e^{-j\omega t} dt \right)^* \right]^* \\ &= \left[ \int_{-\infty}^{\infty} [x(t)]^* (e^{-j\omega t})^* dt \right]^* \\ &= \left[ \int_{-\infty}^{\infty} x(t) e^{-j(-\omega)t} dt \right]^* \\ &= X^*(-\omega). \end{aligned}$$

Thus, we have shown that the conjugation property holds. \blacksquare

**Example 6.12** (Fourier transform of a real function). Let  $X$  denote the Fourier transform of the function  $x$ . Show that, if  $x$  is real, then  $X$  is conjugate symmetric (i.e.,  $X(\omega) = X^*(-\omega)$  for all  $\omega$ ).

*Solution.* From the conjugation property of the Fourier transform, we have

$$\mathcal{F}\{x^*(t)\}(\omega) = X^*(-\omega).$$

Since  $x$  is real, we can replace  $x^*$  with  $x$  to yield

$$\mathcal{F}x(\omega) = X^*(-\omega),$$

or equivalently

$$X(\omega) = X^*(-\omega). \quad \blacksquare$$

## 6.7.6 Duality

The next property of the Fourier transform to be introduced is the duality property, as given below.

**Theorem 6.9** (Duality). If  $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$ , then

$$X(t) \xleftrightarrow{\text{CTFT}} 2\pi x(-\omega).$$

*Proof.* To prove the above property, we proceed as follows. From the Fourier transform synthesis equation, we have

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) e^{j\lambda t} d\lambda.$$

Substituting  $-\omega$  for  $t$ , we obtain

$$x(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) e^{-j\lambda\omega} d\lambda.$$

Now, we multiply both sides of the equation by  $2\pi$  to yield

$$\begin{aligned} 2\pi x(-\omega) &= \int_{-\infty}^{\infty} X(\lambda) e^{-j\lambda\omega} d\lambda \\ &= \mathcal{F}X(\omega). \end{aligned}$$

Thus, we have shown that the duality property holds. \blacksquare

The duality property stated in the preceding theorem follows from the high degree of similarity in the equations for the forward and inverse Fourier transforms, given by (6.7) and (6.8), respectively. To make this similarity more obvious, we can rewrite the forward and inverse Fourier transform equations, respectively, as

$$X(\lambda) = \int_{-\infty}^{\infty} x(\theta) e^{-j\theta\lambda} d\theta \quad \text{and} \quad x(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\theta) e^{j\theta\lambda} d\theta.$$

Observe that these two equations are identical except for: 1) a factor of  $2\pi$ ; and 2) a different sign in the parameter for the exponential function. Consequently, if we were to accidentally use one equation in place of the other, we would obtain an almost correct result. In fact, this almost correct result could be made to be correct by compensating for the above two differences (i.e., the factor of  $2\pi$  and the sign difference in the exponential function). This is, in effect, what the duality property states.

Although the relationship  $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$  only directly provides us with the Fourier transform  $X$  of  $x$ , the duality property allows us to indirectly infer the Fourier transform of  $X$ . Consequently, the duality property can be used to effectively double the number of Fourier transform pairs that we know.

**Example 6.13** (Fourier transform of the sinc function). Using the transform pair

$$\text{rect } t \xleftrightarrow{\text{CTFT}} \text{sinc } \frac{\omega}{2},$$

find the Fourier transform  $X$  of the function

$$x(t) = \text{sinc } \frac{t}{2}.$$

*Solution.* From the given Fourier transform pair, we have

$$v(t) = \text{rect } t \xleftrightarrow{\text{CTFT}} V(\omega) = \text{sinc } \frac{\omega}{2}.$$

By duality, we have

$$V(t) = \text{sinc } \frac{t}{2} \xleftrightarrow{\text{CTFT}} \mathcal{F}V(\omega) = 2\pi v(-\omega) = 2\pi \text{rect}(-\omega) = 2\pi \text{rect } \omega.$$

Thus, we have

$$V(t) = \text{sinc } \frac{t}{2} \xleftrightarrow{\text{CTFT}} \mathcal{F}V(\omega) = 2\pi \text{rect } \omega.$$

Observing that  $V = x$  and  $\mathcal{F}V = X$ , we can rewrite the preceding relationship as

$$x(t) = \text{sinc } \frac{t}{2} \xleftrightarrow{\text{CTFT}} X(\omega) = 2\pi \text{rect } \omega.$$

Thus, we have shown that

$$X(\omega) = 2\pi \text{rect } \omega. \quad \blacksquare$$

### 6.7.7 Time-Domain Convolution

The next property of the Fourier transform to be introduced is the time-domain convolution property, as given below.

**Theorem 6.10** (Time-domain convolution). *If  $x_1(t) \xleftrightarrow{\text{CTFT}} X_1(\omega)$  and  $x_2(t) \xleftrightarrow{\text{CTFT}} X_2(\omega)$ , then*

$$x_1 * x_2(t) \xleftrightarrow{\text{CTFT}} X_1(\omega)X_2(\omega).$$

*Proof.* The proof of this property is as follows. Let  $y(t) = x_1 * x_2(t)$  and let  $Y = \mathcal{F}y$ . From the definition of the Fourier transform and convolution, we have

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} [x_1 * x_2(t)] e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x_1(\tau) x_2(t - \tau) d\tau \right] e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau) x_2(t - \tau) e^{-j\omega t} d\tau dt. \end{aligned}$$

Changing the order of integration, we obtain

$$Y(\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau) x_2(t - \tau) e^{-j\omega t} dt d\tau.$$

Now, we use a change of variable. Let  $\lambda = t - \tau$  so that  $t = \lambda + \tau$  and  $d\lambda = d\tau$ . Applying the change of variable and simplifying, we obtain

$$\begin{aligned}
 Y(\omega) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau)x_2(\lambda)e^{-j\omega(\lambda+\tau)}d\lambda d\tau \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau)x_2(\lambda)e^{-j\omega\lambda}e^{-j\omega\tau}d\lambda d\tau \\
 &= \int_{-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau} \left[ \int_{-\infty}^{\infty} x_2(\lambda)e^{-j\omega\lambda}d\lambda \right] d\tau \\
 &= \left[ \int_{-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau}d\tau \right] \left[ \int_{-\infty}^{\infty} x_2(\lambda)e^{-j\omega\lambda}d\lambda \right] \\
 &= X_1(\omega)X_2(\omega).
 \end{aligned}$$

Thus, we have shown that the time-domain convolution property holds. ■

The time-domain convolution property of the Fourier transform has important practical implications. Since the Fourier transform effectively converts a convolution into a multiplication, the Fourier transform can be used as a means to avoid directly dealing with convolution operations. This is often extremely helpful when solving problems involving LTI systems, for example, since such problems almost inevitably involve convolution (due to the fact that a LTI system computes a convolution).

**Example 6.14** (Time-domain convolution property of the Fourier transform). With the aid of Table 6.2, find the Fourier transform  $X$  of the function

$$x(t) = x_1 * x_2(t),$$

where

$$x_1(t) = e^{-2t}u(t) \quad \text{and} \quad x_2(t) = u(t).$$

*Solution.* Let  $X_1$  and  $X_2$  denote the Fourier transforms of  $x_1$  and  $x_2$ , respectively. From the time-domain convolution property of the Fourier transform, we know that

$$\begin{aligned}
 X(\omega) &= (\mathcal{F}\{x_1 * x_2\})(\omega) \\
 &= X_1(\omega)X_2(\omega).
 \end{aligned} \tag{6.10}$$

From Table 6.2, we know that

$$\begin{aligned}
 X_1(\omega) &= (\mathcal{F}\{e^{-2t}u(t)\})(\omega) \\
 &= \frac{1}{2+j\omega} \quad \text{and}
 \end{aligned}$$

$$\begin{aligned}
 X_2(\omega) &= \mathcal{F}u(\omega) \\
 &= \pi\delta(\omega) + \frac{1}{j\omega}.
 \end{aligned}$$

Substituting these expressions for  $X_1(\omega)$  and  $X_2(\omega)$  into (6.10), we obtain

$$\begin{aligned}
 X(\omega) &= \left[ \frac{1}{2+j\omega} \right] \left( \pi\delta(\omega) + \frac{1}{j\omega} \right) \\
 &= \frac{\pi}{2+j\omega} \delta(\omega) + \frac{1}{j\omega} \left( \frac{1}{2+j\omega} \right) \\
 &= \frac{\pi}{2+j\omega} \delta(\omega) + \frac{1}{j2\omega-\omega^2} \\
 &= \frac{\pi}{2} \delta(\omega) + \frac{1}{j2\omega-\omega^2}.
 \end{aligned} \quad \blacksquare$$

### 6.7.8 Time-Domain Multiplication

The next property of the Fourier transform to be introduced is the time-domain multiplication (or frequency-domain convolution) property, as given below.

**Theorem 6.11** (Time-domain multiplication). *If  $x_1(t) \xleftrightarrow{\text{CFT}} X_1(\omega)$  and  $x_2(t) \xleftrightarrow{\text{CFT}} X_2(\omega)$ , then*

$$x_1(t)x_2(t) \xleftrightarrow{\text{CFT}} \frac{1}{2\pi} X_1 * X_2(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\theta)X_2(\omega - \theta)d\theta.$$

*Proof.* To prove the above property, we proceed as follows. Let  $Y(\omega) = \frac{1}{2\pi} X_1 * X_2(\omega)$  and let  $y = \mathcal{F}^{-1}Y$ . From the definition of the inverse Fourier transform, we have

$$\begin{aligned} y(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{1}{2\pi} X_1 * X_2(\omega) \right] e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \frac{1}{2\pi} X_1(\lambda)X_2(\omega - \lambda)d\lambda \right] e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} X_1(\lambda)X_2(\omega - \lambda)e^{j\omega t} d\lambda d\omega. \end{aligned}$$

Reversing the order of integration, we obtain

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} X_1(\lambda)X_2(\omega - \lambda)e^{j\omega t} d\omega d\lambda.$$

Now, we employ a change of variable. Let  $v = \omega - \lambda$  so that  $\omega = v + \lambda$  and  $dv = d\omega$ . Applying the change of variable and simplifying yields

$$\begin{aligned} y(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} X_1(\lambda)X_2(v)e^{j(v+\lambda)t} dv d\lambda \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} X_1(\lambda)X_2(v)e^{jvt} e^{j\lambda t} dv d\lambda \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\lambda)e^{j\lambda t} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(v)e^{jvt} dv \right] d\lambda \\ &= \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\lambda)e^{j\lambda t} d\lambda \right] \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(v)e^{jvt} dv \right] \\ &= \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega)e^{j\omega t} d\omega \right] \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(\omega)e^{j\omega t} d\omega \right] \\ &= x_1(t)x_2(t). \end{aligned}$$

Thus, we have shown that the frequency-domain convolution property holds. ■

From the time-domain multiplication property in the preceding theorem, we can see that the Fourier transform effectively converts a multiplication operation into a convolution operation (with a scale factor of  $\frac{1}{2\pi}$ ). Since convolution is significantly more complicated than multiplication, we normally prefer to avoid using this property in a manner that would result in the introduction of additional convolution operations into our work.

**Example 6.15** (Frequency-domain convolution property). Let  $x$  and  $y$  be functions related as

$$y(t) = x(t) \cos \omega_c t,$$

where  $\omega_c$  is a nonzero real constant. Let  $Y = \mathcal{F}y$  and  $X = \mathcal{F}x$ . Find an expression for  $Y$  in terms of  $X$ .

*Solution.* To allow for simpler notation in what follows, we define

$$v(t) = \cos \omega_c t$$

and let  $V$  denote the Fourier transform of  $v$ . From Table 6.2, we have that

$$V(\omega) = \pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)].$$

From the definition of  $v$ , we have

$$y(t) = x(t)v(t).$$

Taking the Fourier transform of both sides of this equation, we have

$$Y(\omega) = \mathcal{F}\{x(t)v(t)\}(\omega).$$

Using the frequency-domain convolution property of the Fourier transform, we obtain

$$\begin{aligned} Y(\omega) &= \frac{1}{2\pi} X * V(\omega) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) V(\omega - \lambda) d\lambda. \end{aligned}$$

Substituting the above expression for  $V$ , we obtain

$$\begin{aligned} Y(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) (\pi[\delta(\omega - \lambda - \omega_c) + \delta(\omega - \lambda + \omega_c)]) d\lambda \\ &= \frac{1}{2} \int_{-\infty}^{\infty} X(\lambda) [\delta(\omega - \lambda - \omega_c) + \delta(\omega - \lambda + \omega_c)] d\lambda \\ &= \frac{1}{2} \left[ \int_{-\infty}^{\infty} X(\lambda) \delta(\omega - \lambda - \omega_c) d\lambda + \int_{-\infty}^{\infty} X(\lambda) \delta(\omega - \lambda + \omega_c) d\lambda \right] \\ &= \frac{1}{2} \left[ \int_{-\infty}^{\infty} X(\lambda) \delta(\lambda - \omega + \omega_c) d\lambda + \int_{-\infty}^{\infty} X(\lambda) \delta(\lambda - \omega - \omega_c) d\lambda \right] \\ &= \frac{1}{2} [X(\omega - \omega_c) + X(\omega + \omega_c)] \\ &= \frac{1}{2} X(\omega - \omega_c) + \frac{1}{2} X(\omega + \omega_c). \end{aligned} \quad \blacksquare$$

### 6.7.9 Time-Domain Differentiation

The next property of the Fourier transform to be introduced is the time-domain differentiation property, as given below.

**Theorem 6.12** (Time-domain differentiation). *If  $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$ , then*

$$\frac{dx(t)}{dt} \xleftrightarrow{\text{CTFT}} j\omega X(\omega).$$

*Proof.* To prove the above property, we proceed as follows. Let  $Y(\omega) = j\omega X(\omega)$  and let  $y = \mathcal{F}^{-1}Y$ . We begin by using the definition of the inverse Fourier transform to write

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega.$$

Now, we differentiate both sides of the preceding equation with respect to  $t$  and simplify to obtain

$$\begin{aligned} \frac{dx(t)}{dt} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega X(\omega) e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{j\omega t} d\omega \\ &= y(t). \end{aligned}$$

Thus, we have shown that the time-differentiation property holds. \blacksquare

By repeated application of the preceding theorem, we can obtain the more general result that

$$\left(\frac{d}{dt}\right)^n x(t) \xleftrightarrow{\text{CTFT}} (j\omega)^n X(\omega).$$

The time-domain differentiation property of the Fourier transform has important practical implications. Since the Fourier transform effectively converts differentiation into multiplication (by  $j\omega$ ), the Fourier transform can be used as a means to avoid directly dealing with differentiation operations. This can often be beneficial when working with differential and integro-differential equations.

**Example 6.16** (Time-domain differentiation property). Find the Fourier transform  $X$  of the function

$$x(t) = \frac{d}{dt} \delta(t).$$

*Solution.* Taking the Fourier transform of both sides of the given equation for  $x$  yields

$$X(\omega) = (\mathcal{F}\{\frac{d}{dt} \delta(t)\})(\omega).$$

Using the time-domain differentiation property of the Fourier transform, we can write

$$\begin{aligned} X(\omega) &= (\mathcal{F}\{\frac{d}{dt} \delta(t)\})(\omega) \\ &= j\omega \mathcal{F}\delta(\omega). \end{aligned}$$

Evaluating the Fourier transform of  $\delta$  using Table 6.2, we obtain

$$\begin{aligned} X(\omega) &= j\omega(1) \\ &= j\omega. \end{aligned} \quad \blacksquare$$

### 6.7.10 Frequency-Domain Differentiation

The next property of the Fourier transform to be introduced is the frequency-domain differentiation property, as given below.

**Theorem 6.13** (Frequency-domain differentiation). If  $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$ , then

$$tx(t) \xleftrightarrow{\text{CTFT}} j \frac{d}{d\omega} X(\omega).$$

*Proof.* To prove the above property, we proceed as follows. Let  $y(t) = tx(t)$  and let  $Y = \mathcal{F}y$ . From the definition of the Fourier transform, we can write

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt.$$

Now, we differentiate both sides of this equation with respect to  $\omega$  and simplify to obtain

$$\begin{aligned} \frac{d}{d\omega} X(\omega) &= \int_{-\infty}^{\infty} x(t) (-jt) e^{-j\omega t} dt \\ &= -j \int_{-\infty}^{\infty} tx(t) e^{-j\omega t} dt \\ &= -jY(\omega). \end{aligned}$$

Multiplying both sides of the preceding equation by  $j$  yields

$$j \frac{d}{d\omega} X(\omega) = Y(\omega).$$

Thus, we have shown that the frequency-domain differentiation property holds. \blacksquare

**Example 6.17** (Frequency-domain differentiation property). Find the Fourier transform  $X$  of the function

$$x(t) = t \cos \omega_0 t,$$

where  $\omega_0$  is a nonzero real constant.

*Solution.* Taking the Fourier transform of both sides of the equation for  $x$  yields

$$X(\omega) = \mathcal{F}\{t \cos \omega_0 t\}(\omega).$$

From the frequency-domain differentiation property of the Fourier transform, we can write

$$\begin{aligned} X(\omega) &= \mathcal{F}\{t \cos \omega_0 t\}(\omega) \\ &= j(\mathcal{D}\mathcal{F}\{\cos \omega_0 t\})(\omega), \end{aligned}$$

where  $\mathcal{D}$  denotes the derivative operator. Evaluating the Fourier transform on the right-hand side using Table 6.2, we obtain

$$\begin{aligned} X(\omega) &= j \frac{d}{d\omega} [\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]] \\ &= j\pi \frac{d}{d\omega} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \\ &= j\pi \frac{d}{d\omega} \delta(\omega - \omega_0) + j\pi \frac{d}{d\omega} \delta(\omega + \omega_0). \end{aligned} \quad \blacksquare$$

### 6.7.11 Time-Domain Integration

The next property of the Fourier transform to be introduced is the time-domain integration property, as given below.

**Theorem 6.14** (Time-domain integration). If  $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$ , then

$$\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{\text{CTFT}} \frac{1}{j\omega} X(\omega) + \pi X(0) \delta(\omega).$$

*Proof.* The above property can be proven as follows. Let  $y(t) = \int_{-\infty}^t x(\tau) d\tau$ ; let  $Y = \mathcal{F}y$  and  $U = \mathcal{F}u$ . First, we observe that

$$y(t) = x * u(t).$$

Taking the Fourier transform of both sides of the preceding equation and using the time-domain convolution property of the Fourier transform, we have

$$Y(\omega) = X(\omega)U(\omega). \quad (6.11)$$

From Example 6.8, we know that  $u(t) \xleftrightarrow{\text{CTFT}} \pi\delta(\omega) + \frac{1}{j\omega}$ . Using this fact, we can rewrite (6.11) as

$$\begin{aligned} Y(\omega) &= X(\omega) \left[ \pi\delta(\omega) + \frac{1}{j\omega} \right] \\ &= \frac{1}{j\omega} X(\omega) + \pi X(\omega) \delta(\omega). \end{aligned}$$

From the equivalence property of the unit-impulse function, we have

$$Y(\omega) = \frac{1}{j\omega} X(\omega) + \pi X(0) \delta(\omega).$$

Thus, we have shown that the time-domain integration property holds. \blacksquare

The time-domain integration property of the Fourier transform has important practical implications. Since the Fourier transform effectively converts integration into an operation involving division (by  $j\omega$ ), the Fourier transform can be used as a means to avoid directly dealing with integration operations. This can often be beneficial when working with integral and integro-differential equations.

**Example 6.18** (Time-domain integration property of the Fourier transform). Use the time-domain integration property of the Fourier transform in order to find the Fourier transform  $X$  of the function  $x = u$ .

*Solution.* We begin by observing that  $x$  can be expressed in terms of an integral as

$$x(t) = u(t) = \int_{-\infty}^t \delta(\tau) d\tau.$$

Now, we consider the Fourier transform of  $x$ . We have

$$X(\omega) = \left( \mathcal{F} \left\{ \int_{-\infty}^t \delta(\tau) d\tau \right\} \right) (\omega).$$

From the time-domain integration property, we can write

$$X(\omega) = \frac{1}{j\omega} \mathcal{F}\delta(\omega) + \pi \mathcal{F}\delta(0)\delta(\omega).$$

Evaluating the two Fourier transforms on the right-hand side using Table 6.2, we obtain

$$\begin{aligned} X(\omega) &= \frac{1}{j\omega}(1) + \pi(1)\delta(\omega) \\ &= \frac{1}{j\omega} + \pi\delta(\omega). \end{aligned}$$

Thus, we have shown that  $u(t) \xleftrightarrow{\text{CTFT}} \frac{1}{j\omega} + \pi\delta(\omega)$ . ■

### 6.7.12 Parseval's Relation

The next property of the Fourier transform to be introduced, given below, relates to signal energy and is known as Parseval's relation.

**Theorem 6.15** (Parseval's relation). *If  $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$ , then*

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega. \quad (6.12)$$

*That is, the energy of  $x$  and energy of  $X$  are equal within a scaling factor of  $2\pi$ . (Recall that the energy of a function  $x$  is given by  $\int_{-\infty}^{\infty} |x(t)|^2 dt$ .)*

*Proof.* To prove the above relationship, we proceed as follows. Consider the left-hand side of (6.12) which we can write as

$$\begin{aligned} \int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t)x^*(t) dt \\ &= \int_{-\infty}^{\infty} x(t) \mathcal{F}^{-1}\{\mathcal{F}(x^*)\}(t) dt. \end{aligned}$$

From the conjugation property of the Fourier transform, we have that  $x^*(t) \xleftrightarrow{\text{CTFT}} X^*(-\omega)$ . So, we can rewrite the above equation as

$$\begin{aligned} \int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t) (\mathcal{F}^{-1}\{X^*(-\omega)\})(t) dt \\ &= \int_{-\infty}^{\infty} x(t) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(-\omega) e^{j\omega t} d\omega \right] dt. \end{aligned}$$

Now, we employ a change of variable (i.e., replace  $\omega$  by  $-\omega$ ) to obtain

$$\begin{aligned}\int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} d\omega \right] dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) X^*(\omega) e^{-j\omega t} d\omega dt.\end{aligned}$$

Reversing the order of integration and simplifying, we have

$$\begin{aligned}\int_{-\infty}^{\infty} |x(t)|^2 dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) X^*(\omega) e^{-j\omega t} dt d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \left( \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) X(\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega.\end{aligned}$$

Thus, Parseval's relation holds. ■

Since energy is often a quantity of great significance in engineering applications, it is extremely helpful to know that the Fourier transform preserves energy (up to a scale factor). For example, if we are solving a problem in the Fourier domain, we do not have to return to the time domain to compute energy, since we can do this directly in the Fourier domain by using Parseval's relation.

**Example 6.19** (Energy of the sinc function). Consider the function  $x(t) = \text{sinc } \frac{t}{2}$ , which has the Fourier transform  $X$  given by  $X(\omega) = 2\pi \text{rect } \omega$ . Compute the energy of  $x$ .

*Solution.* We could directly compute the energy of  $x$  as

$$\begin{aligned}E &= \int_{-\infty}^{\infty} |x(t)|^2 dt \\ &= \int_{-\infty}^{\infty} \left| \text{sinc } \frac{t}{2} \right|^2 dt.\end{aligned}$$

This integral is not so easy to compute, however. Instead, we use Parseval's relation to write

$$\begin{aligned}E &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |2\pi \text{rect } \omega|^2 d\omega \\ &= \frac{1}{2\pi} \int_{-1/2}^{1/2} (2\pi)^2 d\omega \\ &= 2\pi \int_{-1/2}^{1/2} d\omega \\ &= 2\pi [\omega]_{-1/2}^{1/2} \\ &= 2\pi \left[ \frac{1}{2} + \frac{1}{2} \right] \\ &= 2\pi.\end{aligned}$$

Thus, we have

$$E = \int_{-\infty}^{\infty} \left| \text{sinc } \frac{t}{2} \right|^2 dt = 2\pi. \quad \blacksquare$$

### 6.7.13 Even/Odd Symmetry

The Fourier transform preserves symmetry. In other words, we have the result below.

**Theorem 6.16** (Even/odd symmetry). *For a function  $x$  with Fourier transform  $X$ , the following assertions hold:*

- $x$  is even if and only if  $X$  is even; and
- $x$  is odd if and only if  $X$  is odd.

*Proof.* First, we show that, if a function  $x$  is even/odd, then its Fourier transform  $X$  is even/odd. From the definition of the Fourier transform, we have

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt.$$

Since  $x$  is even/odd, we have that  $x(t) = \pm x(-t)$ , where the plus case and minus case in the “ $\pm$ ” correspond to  $x$  being even and odd, respectively. Using this, we can rewrite the above expression for  $X(\omega)$  as

$$X(\omega) = \int_{-\infty}^{\infty} \pm x(-t)e^{-j\omega t} dt.$$

Now, we employ a change of variable. Let  $\lambda = -t$  so that  $d\lambda = -dt$ . Applying this change of variable, we obtain

$$\begin{aligned} X(\omega) &= \int_{\infty}^{-\infty} \pm x(\lambda)e^{-j\omega(-\lambda)}(-1)d\lambda \\ &= \mp \int_{\infty}^{-\infty} x(\lambda)e^{j\omega\lambda} d\lambda \\ &= \pm \int_{-\infty}^{\infty} x(\lambda)e^{j\omega\lambda} d\lambda \\ &= \pm \int_{-\infty}^{\infty} x(\lambda)e^{-j(-\omega)\lambda} d\lambda \\ &= \pm X(-\omega). \end{aligned}$$

Therefore,  $X$  is even/odd.

Next, we show that if  $X$  is even/odd, then  $x$  is even/odd. From the definition of the inverse Fourier transform, we have

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega.$$

Since  $X$  is even/odd, we have that  $X(\omega) = \pm X(-\omega)$ , where the plus case and minus case in the “ $\pm$ ” correspond to  $X$  being even and odd, respectively. Using this, we can rewrite the above expression for  $x(t)$  as

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \pm X(-\omega)e^{j\omega t} d\omega.$$

Now, we employ a change of variable. Let  $\lambda = -\omega$  so that  $d\lambda = -d\omega$ . Applying this change of variable, we obtain

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{\infty}^{-\infty} \pm X(\lambda)e^{-j\lambda t}(-1)d\lambda \\ &= \pm \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda)e^{-j\lambda t} d\lambda \\ &= \pm \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda)e^{j\lambda(-t)} d\lambda \\ &= \pm x(-t). \end{aligned}$$

Therefore,  $x$  is even/odd. This completes the proof. ■

In other words, the preceding theorem simply states that the forward and inverse Fourier transforms preserve even/odd symmetry.

### 6.7.14 Real Functions

As it turns out, the Fourier transform of a real function has a special structure, as given by the theorem below.

**Theorem 6.17.** *A function  $x$  is real if and only if its Fourier transform  $X$  satisfies*

$$X(\omega) = X^*(-\omega) \text{ for all } \omega$$

(i.e.,  $X$  is conjugate symmetric).

*Proof.* From the definition of the Fourier transform, we have

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt. \quad (6.13)$$

Substituting  $-\omega$  for  $\omega$  in the preceding equation, we have

$$X(-\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t} dt.$$

Conjugating both sides of this equation, we obtain

$$X^*(-\omega) = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t} dt. \quad (6.14)$$

First, we show that  $x$  being real implies that  $X$  is conjugate symmetric. Suppose that  $x$  is real. Since  $x$  is real, we can replace  $x^*$  with  $x$  in (6.14) to yield

$$X^*(-\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt.$$

Observing that the right-hand side is simply  $X(\omega)$ , we have

$$X^*(-\omega) = X(\omega).$$

Thus,  $x$  being real implies that  $X$  is conjugate symmetric.

Next, we show that  $X$  being conjugate symmetric implies that  $x$  is real. Suppose that  $X$  is conjugate symmetric. Since  $X$  is conjugate symmetric, the expressions for  $X(\omega)$  in (6.13) and  $X^*(-\omega)$  in (6.14) must be equal. Thus, we can write

$$\begin{aligned} X(\omega) - X^*(-\omega) &= 0 \\ \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt - \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t} dt &= 0 \\ \int_{-\infty}^{\infty} [x(t) - x^*(t)]e^{-j\omega t} dt &= 0. \end{aligned}$$

This implies that  $x^* = x$ . Therefore,  $x$  is real. Thus,  $X$  being conjugate symmetric implies that  $x$  is real. This completes the proof. ■

Suppose that  $X$  is the Fourier transform of a real function  $x$  so that  $X$  is conjugate symmetric. From properties of complex numbers, we can show that that  $X$  being conjugate symmetric is equivalent to

$$|X(\omega)| = |X(-\omega)| \text{ for all } \omega \quad \text{and} \quad (6.15a)$$

$$\arg X(\omega) = -\arg X(-\omega) \text{ for all } \omega \quad (6.15b)$$

(i.e., the magnitude and argument of  $X$  are even and odd, respectively).

Since the Fourier transform  $X$  of a real function  $x$  is conjugate symmetric, the graph of  $X$  for negative values is completely redundant and can be determined from the graph of  $X$  for nonnegative values. Lastly, note that  $x$  being real does not necessarily imply that  $X$  is real, since a conjugate-symmetric function need not be real.

Table 6.1: Properties of the CT Fourier transform

Property	Time Domain	Frequency Domain
Linearity	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time-Domain Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(\omega)$
Frequency-Domain Shifting	$e^{j\omega_0 t}x(t)$	$X(\omega - \omega_0)$
Time/Frequency-Domain Scaling	$x(at)$	$\frac{1}{ a }X\left(\frac{\omega}{a}\right)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time-Domain Convolution	$x_1 * x_2(t)$	$X_1(\omega)X_2(\omega)$
Time-Domain Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1 * X_2(\omega)$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$j\omega X(\omega)$
Frequency-Domain Differentiation	$tx(t)$	$j\frac{d}{d\omega}X(\omega)$
Time-Domain Integration	$\int_{-\infty}^t x(\tau)d\tau$	$\frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$

---

Property

---

Parseval's Relation  $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$

Even  $x$  is even  $\Leftrightarrow X$  is even

Odd  $x$  is odd  $\Leftrightarrow X$  is odd

Real  $x$  is real  $\Leftrightarrow X$  is conjugate symmetric

---

## 6.8 Continuous-Time Fourier Transform of Periodic Functions

By making use of the generalized Fourier transform briefly discussed in Section 6.3, the Fourier transform can also be applied to periodic functions. In particular, the Fourier transform of a periodic function can be computed using the result below.

**Theorem 6.18** (Fourier transform of a periodic function). *Let  $x$  be a  $T$ -periodic function with frequency  $\omega_0 = \frac{2\pi}{T}$  and Fourier series coefficient sequence  $a$ . Let  $x_T$  denote the function*

$$x_T(t) = \begin{cases} x(t) & -\frac{T}{2} \leq t < \frac{T}{2} \\ 0 & \text{otherwise.} \end{cases}$$

(i.e.,  $x_T$  is a truncated/windowed version of the function  $x$ ). (Note that  $x_T$  is a function equal to  $x$  over a single period and zero elsewhere.) Let  $X_T$  denote the Fourier transform of  $x_T$ . The Fourier transform  $X$  of  $x$  is given by

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0), \quad (6.16a)$$

or equivalently,

$$X(\omega) = \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0) \delta(\omega - k\omega_0). \quad (6.16b)$$

Furthermore,  $a$  and  $X_T$  are related by

$$a_k = \frac{1}{T} X_T(k\omega_0). \quad (6.17)$$

*Proof.* Since  $x$  is  $T$ -periodic, we can express it using a Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad (6.18a)$$

where

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt. \quad (6.18b)$$

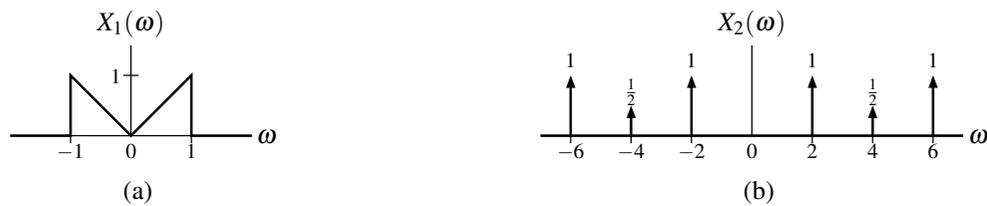
Consider the expression for  $a_k$  in (6.18b). Since  $x_T(t) = x(t)$  for a single period of  $x$  and is zero otherwise, we can rewrite (6.18b) as

$$\begin{aligned} a_k &= \frac{1}{T} \int_{-\infty}^{\infty} x_T(t) e^{-jk\omega_0 t} dt \\ &= \frac{1}{T} X_T(k\omega_0). \end{aligned} \quad (6.19)$$

Thus, we have shown (6.17) to be correct.

Now, let us consider the Fourier transform  $X$  of  $x$ . By taking the Fourier transform of both sides of (6.18a), we obtain

$$\begin{aligned} X(\omega) &= \left( \mathcal{F} \left\{ \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right\} \right) (\omega) \\ &= \int_{-\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right) e^{-j\omega t} dt. \end{aligned}$$

Figure 6.6: Frequency spectra. The frequency spectra (a)  $X_1$  and (b)  $X_2$ .

Reversing the order of summation and integration, we have

$$\begin{aligned} X(\omega) &= \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t} dt \\ &= \sum_{k=-\infty}^{\infty} a_k \left( \mathcal{F}\{e^{jk\omega_0 t}\} \right) (\omega). \end{aligned} \quad (6.20)$$

From Table 6.2, we know that  $e^{j\lambda t} \xleftrightarrow{\text{CTFT}} 2\pi\delta(\omega - \lambda)$ . So, we can simplify (6.20) to obtain

$$\begin{aligned} X(\omega) &= \sum_{k=-\infty}^{\infty} a_k [2\pi\delta(\omega - k\omega_0)] \\ &= \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0). \end{aligned} \quad (6.21)$$

Thus, we have shown (6.16a) to be correct. Furthermore, by substituting (6.19) into (6.21), we have

$$\begin{aligned} X(\omega) &= \sum_{k=-\infty}^{\infty} 2\pi \left[ \frac{1}{T} X_T(k\omega_0) \right] \delta(\omega - k\omega_0) \\ &= \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0) \delta(\omega - k\omega_0). \end{aligned}$$

Thus, we have shown (6.16b) to be correct. This completes the proof.  $\blacksquare$

Theorem 6.18 above provides two formulas for computing the Fourier transform  $X$  of a periodic function  $x$ . One formula is in written terms of the Fourier series coefficient sequence  $a$  of  $x$ , while the other formula is in written terms of the Fourier transform  $X_T$  of a function consisting of a single period of  $x$ . The choice of which formula to use would be driven by what information is available or most easily determined. For example, if the Fourier series coefficients of  $x$  were known, the use of (6.16b) would likely be preferred.

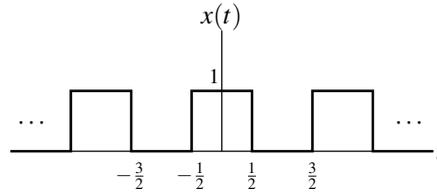
From Theorem 6.18, we can also make a few important observations. First, the Fourier transform of a periodic function is a series of impulse functions located at integer multiples of the fundamental frequency  $\omega_0$ . The weight of each impulse is  $2\pi$  times the corresponding Fourier series coefficient. Second, the Fourier series coefficient sequence  $a$  of the periodic function  $x$  is produced by sampling the Fourier transform of  $x_T$  at integer multiples of the fundamental frequency  $\omega_0$  and scaling the resulting sequence by  $\frac{1}{T}$ .

**Example 6.20.** Let  $X_1$  and  $X_2$  denote the Fourier transforms of  $x_1$  and  $x_2$ , respectively. Suppose that  $X_1$  and  $X_2$  are as shown in Figures 6.6(a) and (b). Determine whether  $x_1$  and  $x_2$  are periodic.

*Solution.* We know that the Fourier transform  $X$  of a periodic function  $x$  must be of the form

$$X(\omega) = \sum_{k=-\infty}^{\infty} \alpha_k \delta(\omega - k\omega_0),$$

where  $\omega_0$  is a nonnegative real constant and the  $\{\alpha_k\}$  are complex constants. The spectrum  $X_1$  does not have this form. Therefore,  $x_1$  must not be periodic. The spectrum  $X_2$  does have this form, with  $\omega_0 = 2$ . Therefore,  $x_2$  must be periodic.  $\blacksquare$

Figure 6.7: Periodic function  $x$ .

**Example 6.21.** Consider the periodic function  $x$  with fundamental period  $T = 2$  as shown in Figure 6.7. Using the Fourier transform, find the Fourier series representation of  $x$ .

*Solution.* Let  $\omega_0$  denote the fundamental frequency of  $x$ . We have that  $\omega_0 = \frac{2\pi}{T} = \pi$ . Let  $y(t) = \text{rect}t$  (i.e.,  $y$  corresponds to a single period of the periodic function  $x$ ). Thus, we have that

$$x(t) = \sum_{k=-\infty}^{\infty} y(t - 2k).$$

Let  $Y$  denote the Fourier transform of  $y$ . Taking the Fourier transform of  $y$ , we obtain

$$Y(\omega) = (\mathcal{F}\{\text{rect}t\})(\omega) = \text{sinc} \frac{\omega}{2}.$$

Now, we seek to find the Fourier series representation of  $x$ , which has the form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

Using the Fourier transform, we have

$$\begin{aligned} c_k &= \frac{1}{T} Y(k\omega_0) \\ &= \frac{1}{2} \text{sinc} \frac{k\omega_0}{2} \\ &= \frac{1}{2} \text{sinc} \frac{k\pi}{2}. \end{aligned}$$

■

## 6.9 More Fourier Transforms

Throughout this chapter, we have derived a number of Fourier transform pairs. Some of these and other important transform pairs are listed in Table 6.2. Using the various Fourier transform properties listed in Table 6.1 and the Fourier transform pairs listed in Table 6.2, we can determine (more easily) the Fourier transform of more complicated functions.

**Example 6.22.** Suppose that  $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$ ,  $x_1(t) \xleftrightarrow{\text{CTFT}} X_1(\omega)$ , and

$$x_1(t) = \mathcal{D}^2 x(t - 2),$$

where  $\mathcal{D}$  denotes the derivative operator. Express  $X_1$  in terms of  $X$ .

*Solution.* Let  $v_1(t) = x(t - 2)$ . From the definition of  $v_1$  and the time-shifting property of the Fourier transform, we have

$$V_1(\omega) = e^{-j2\omega} X(\omega). \quad (6.22)$$

Table 6.2: Transform pairs for the CT Fourier transform

Pair	$x(t)$	$X(\omega)$
1	$\delta(t)$	1
2	$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
3	1	$2\pi\delta(\omega)$
4	$\text{sgn}t$	$\frac{2}{j\omega}$
5	$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
6	$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
7	$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
8	$\text{rect} \frac{t}{T}$	$ T  \text{sinc} \frac{T\omega}{2}$
9	$\frac{ B }{\pi} \text{sinc} Bt$	$\text{rect} \frac{\omega}{2B}$
10	$e^{-at}u(t), \text{Re}\{a\} > 0$	$\frac{1}{a + j\omega}$
11	$t^{n-1}e^{-at}u(t), \text{Re}\{a\} > 0$	$\frac{(n-1)!}{(a + j\omega)^n}$
12	$\text{tri} \frac{t}{T}$	$\frac{ T }{2} \text{sinc}^2 \frac{T\omega}{4}$

From the definition of  $v_1$ , we have

$$x_1(t) = \mathcal{D}^2 v_1(t).$$

Thus, from the time-differentiation property of the Fourier transform, we can write

$$\begin{aligned} X_1(\omega) &= (j\omega)^2 V_1(\omega) \\ &= -\omega^2 V_1(\omega). \end{aligned} \tag{6.23}$$

Combining (6.22) and (6.23), we obtain

$$X_1(\omega) = -\omega^2 e^{-j2\omega} X(\omega). \quad \blacksquare$$

**Example 6.23.** Suppose that  $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$ ,  $x_1(t) \xleftrightarrow{\text{CTFT}} X_1(\omega)$ , and

$$x_1(t) = x(at - b),$$

where  $a$  is a nonzero real constant and  $b$  is a real constant. Express  $X_1$  in terms of  $X$ .

*Solution.* We rewrite  $x_1$  as

$$x_1(t) = v_1(at)$$

where

$$v_1(t) = x(t - b).$$

We now take the Fourier transform of both sides of each of the preceding equations. Using the time-shifting property of the Fourier transform, we can write

$$V_1(\omega) = e^{-jb\omega} X(\omega). \tag{6.24}$$

Using the time-scaling property of the Fourier transform, we can write

$$X_1(\omega) = \frac{1}{|a|} V_1\left(\frac{\omega}{a}\right). \tag{6.25}$$

Substituting the expression for  $V_1(\omega)$  in (6.24) into (6.25), we obtain

$$X_1(\omega) = \frac{1}{|a|} e^{-j(b/a)\omega} X\left(\frac{\omega}{a}\right). \quad \blacksquare$$

**Example 6.24.** Consider the periodic function  $x$  given by

$$x(t) = \sum_{k=-\infty}^{\infty} x_0(t - kT),$$

where a single period of  $x$  is given by

$$x_0(t) = A \operatorname{rect} \frac{2t}{T}$$

and  $A$  is a real constant. Find the Fourier transform  $X$  of the function  $x$ .

*Solution.* From (6.16b), we know that

$$\begin{aligned} X(\omega) &= \left( \mathcal{F} \left\{ \sum_{k=-\infty}^{\infty} x_0(t - kT) \right\} \right) (\omega) \\ &= \sum_{k=-\infty}^{\infty} \omega_0 X_0(k\omega_0) \delta(\omega - k\omega_0). \end{aligned}$$

So, we need to find  $X_0$ . Using the linearity property of the Fourier transform and Table 6.2, we have

$$\begin{aligned} X_0(\omega) &= (\mathcal{F}\{A \operatorname{rect} \frac{2t}{T}\}) (\omega) \\ &= A (\mathcal{F}\{\operatorname{rect} \frac{2t}{T}\}) (\omega) \\ &= \frac{AT}{2} \operatorname{sinc} \frac{\omega T}{4}. \end{aligned}$$

Thus, we have that

$$\begin{aligned} X(\omega) &= \sum_{k=-\infty}^{\infty} \omega_0 \frac{AT}{2} \left( \operatorname{sinc} \frac{k\omega_0 T}{4} \right) \delta(\omega - k\omega_0) \\ &= \sum_{k=-\infty}^{\infty} \pi A \left( \operatorname{sinc} \frac{\pi k}{2} \right) \delta(\omega - k\omega_0). \end{aligned} \quad \blacksquare$$

**Example 6.25.** Find the Fourier transform  $X$  of the function  $x$  given by

$$x(t) = \int_{-\infty}^t e^{-(3+j2)\tau} u(\tau) d\tau.$$

*Solution.* We can rewrite  $x$  as

$$x(t) = \int_{-\infty}^t v_1(\tau) d\tau, \quad (6.26)$$

where

$$v_1(t) = e^{-j2t} v_2(t) \quad \text{and} \quad (6.27)$$

$$v_2(t) = e^{-3t} u(t). \quad (6.28)$$

Taking the Fourier transform of (6.26) (using the time-domain integration property of the Fourier transform), we have

$$X(\omega) = \frac{1}{j\omega} V_1(\omega) + \pi V_1(0) \delta(\omega). \quad (6.29)$$

Taking the Fourier transform of (6.27) (using the frequency-domain shifting property of the Fourier transform), we have

$$V_1(\omega) = V_2(\omega + 2). \quad (6.30)$$

Taking the Fourier transform of (6.28) by using Table 6.2 (i.e., the entry for  $\mathcal{F}\{e^{-at} u(t)\}$ ), we have

$$V_2(\omega) = \frac{1}{3 + j\omega}. \quad (6.31)$$

Combining (6.29), (6.30), and (6.31), we obtain

$$\begin{aligned} X(\omega) &= \frac{1}{j\omega} V_1(\omega) + \pi V_1(0) \delta(\omega) \\ &= \frac{1}{j\omega} V_2(\omega + 2) + \pi V_2(2) \delta(\omega) \\ &= \frac{1}{j\omega} \left( \frac{1}{3 + j(\omega + 2)} \right) + \pi \left( \frac{1}{3 + j2} \right) \delta(\omega). \end{aligned} \quad \blacksquare$$

**Example 6.26.** Let  $X$  and  $Y$  denote the Fourier transforms of  $x$  and  $y$ , respectively. Suppose that  $y(t) = x(t) \cos at$ , where  $a$  is a nonzero real constant. Find an expression for  $Y$  in terms of  $X$ .

*Solution.* Essentially, we need to take the Fourier transform of both sides of the given equation. There are two obvious ways in which to do this. One is to use the time-domain multiplication property of the Fourier transform, and another is to use the frequency-domain shifting property. We will solve this problem using each method in turn in order to show that the two approaches do not involve an equal amount of effort.

FIRST SOLUTION (USING AN UNENLIGHTENED APPROACH). We use the time-domain multiplication property. To allow for simpler notation in what follows, we define

$$v(t) = \cos at$$

and let  $V$  denote the Fourier transform of  $v$ . From Table 6.2, we have that

$$V(\omega) = \pi[\delta(\omega - a) + \delta(\omega + a)].$$

Taking the Fourier transform of both sides of the given equation, we obtain

$$\begin{aligned} Y(\omega) &= (\mathcal{F}\{x(t)v(t)\})(\omega) \\ &= \frac{1}{2\pi} X * V(\omega) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) V(\omega - \lambda) d\lambda. \end{aligned}$$

Substituting the above expression for  $V$ , we obtain

$$\begin{aligned} Y(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) (\pi[\delta(\omega - \lambda - a) + \delta(\omega - \lambda + a)]) d\lambda \\ &= \frac{1}{2} \int_{-\infty}^{\infty} X(\lambda) [\delta(\omega - \lambda - a) + \delta(\omega - \lambda + a)] d\lambda \\ &= \frac{1}{2} \left[ \int_{-\infty}^{\infty} X(\lambda) \delta(\omega - \lambda - a) d\lambda + \int_{-\infty}^{\infty} X(\lambda) \delta(\omega - \lambda + a) d\lambda \right] \\ &= \frac{1}{2} \left[ \int_{-\infty}^{\infty} X(\lambda) \delta(\lambda - \omega + a) d\lambda + \int_{-\infty}^{\infty} X(\lambda) \delta(\lambda - \omega - a) d\lambda \right] \\ &= \frac{1}{2} [X(\omega - a) + X(\omega + a)] \\ &= \frac{1}{2} X(\omega - a) + \frac{1}{2} X(\omega + a). \end{aligned}$$

Note that the above solution is essentially identical to the one appearing earlier in Example 6.15 on page 151.

SECOND SOLUTION (USING AN ENLIGHTENED APPROACH). We use the frequency-domain shifting property. Taking the Fourier transform of both sides of the given equation, we obtain

$$\begin{aligned} Y(\omega) &= (\mathcal{F}\{x(t) \cos at\})(\omega) \\ &= (\mathcal{F}\{\frac{1}{2}(e^{jat} + e^{-jat})x(t)\})(\omega) \\ &= \frac{1}{2} (\mathcal{F}\{e^{jat}x(t)\})(\omega) + \frac{1}{2} (\mathcal{F}\{e^{-jat}x(t)\})(\omega) \\ &= \frac{1}{2} X(\omega - a) + \frac{1}{2} X(\omega + a). \end{aligned}$$

COMMENTARY. Clearly, of the above two solution methods, the second approach is simpler and much less error prone. Generally, the use of the time-domain multiplication property tends to lead to less clean solutions, as this forces a convolution to be performed in the frequency domain and convolution is often best avoided if possible. ■

## 6.10 Frequency Spectra of Functions

The Fourier transform representation expresses a function in terms of complex sinusoids at all frequencies. In this sense, the Fourier transform representation captures information about the frequency content of a function. For example, suppose that we have a function  $x$  with Fourier transform  $X$ . If  $X$  is nonzero at some frequency  $\omega_0$ , then the function  $x$  contains some information at the frequency  $\omega_0$ . On the other hand, if  $X$  is zero at the frequency  $\omega_0$ , then the function  $x$  has no information at that frequency. In this way, the Fourier transform representation provides a means for measuring the frequency content of a function. This distribution of information in a function over different frequencies is referred to as the **frequency spectrum** of the function. That is,  $X$  is the frequency spectrum of  $x$ .

To gain further insight into the role played by the Fourier transform  $X$  in the context of the frequency spectrum of  $x$ , it is helpful to write the Fourier transform representation of  $x$  with  $X$  expressed in polar form as follows:

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)| e^{j\arg X(\omega)} e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)| e^{j[\omega t + \arg X(\omega)]} d\omega. \end{aligned}$$

In effect, the quantity  $|X(\omega)|$  is a weight that determines how much the complex sinusoid at frequency  $\omega$  contributes to the integration result  $x(t)$ . Perhaps, this can be more easily seen if we express the above integral as the limit of a sum, derived from an approximation of the integral using the area of rectangles (i.e.,  $\int_{-\infty}^{\infty} f(x) dx = \lim_{\Delta x \rightarrow 0} \sum_{k=-\infty}^{\infty} \Delta x f(k\Delta x)$ ). Expressing  $x$  in this way, we obtain

$$\begin{aligned} x(t) &= \lim_{\Delta\omega \rightarrow 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Delta\omega |X(k\Delta\omega)| e^{j[k\Delta\omega t + \arg X(k\Delta\omega)]} \\ &= \lim_{\Delta\omega \rightarrow 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Delta\omega |X(\omega')| e^{j[\omega' t + \arg X(\omega')]}, \end{aligned}$$

where  $\omega' = k\Delta\omega$ . From the last line of the above equation, the  $k$ th term in the summation (associated with the frequency  $\omega' = k\Delta\omega$ ) corresponds to a complex sinusoid with fundamental frequency  $\omega'$  that has had its amplitude scaled by a factor of  $|X(\omega')|$  and has been time-shifted by an amount that depends on  $\arg X(\omega')$ . For a given  $\omega' = k\Delta\omega$  (which is associated with the  $k$ th term in the summation), the larger  $|X(\omega')|$  is, the larger the amplitude of its corresponding complex sinusoid  $e^{j\omega' t}$  will be, and therefore the larger the contribution the  $k$ th term will make to the overall summation. In this way, we can use  $|X(\omega')|$  as a measure of how much information a function  $x$  has at the frequency  $\omega'$ .

Note that, since the Fourier transform  $X$  is a function of a real variable, a function can, in the most general case, have information at *any arbitrary* real frequency. This is different from the case of frequency spectra in the Fourier series context (which deals only with periodic functions), where a function can only have information at certain specific frequencies (namely, at integer multiples of the fundamental frequency). There is no inconsistency here, however. As we saw in Section 6.8, in the case of periodic functions the Fourier transform will also be zero, except possibly at integer multiples of the fundamental frequency.

Since the frequency spectrum is complex (in the general case), it is usually represented using two plots, one showing the magnitude of  $X$  and one showing the argument. We refer to  $|X(\omega)|$  as the **magnitude spectrum** of the function  $x$ . Similarly, we refer to  $\arg X(\omega)$  as the **phase spectrum** of the function  $x$ . In the special case that  $X$  is a real (or purely imaginary) function, we usually plot the frequency spectrum directly on a single graph.

Suppose that we have a function  $x$  with Fourier transform  $X$ . If  $x$  is real, then

$$\begin{aligned} |X(\omega)| &= |X(-\omega)| \text{ for all } \omega \quad \text{and} \\ \arg X(\omega) &= -\arg X(-\omega) \text{ for all } \omega \end{aligned}$$

(i.e., the magnitude and argument of  $X$  are even and odd, respectively). (See (6.15a) and (6.15b).) Due to the symmetry in the frequency spectra of real functions, we typically ignore negative frequencies when dealing with such functions.

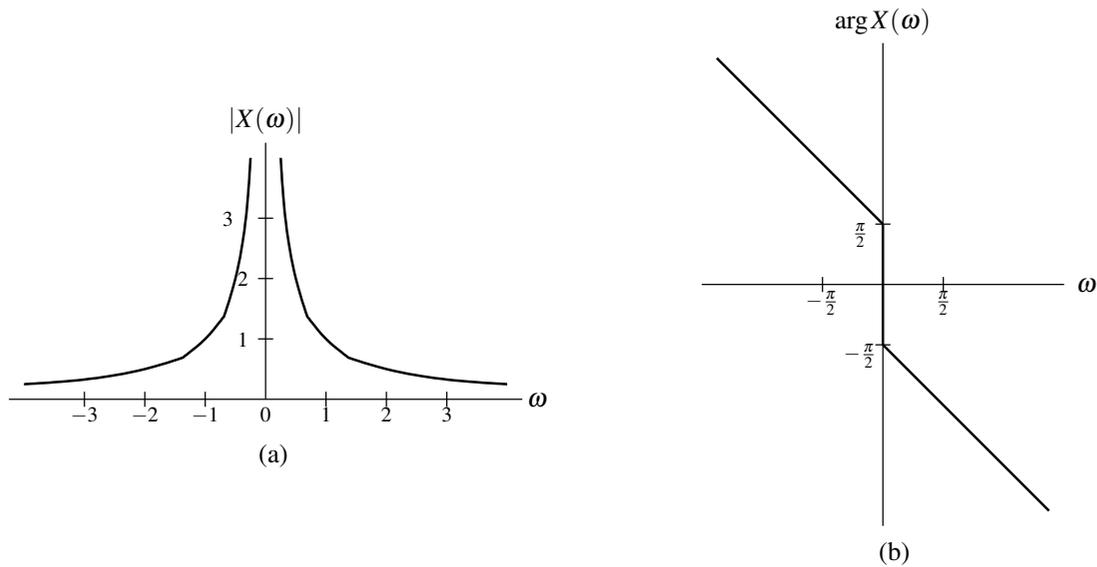


Figure 6.8: Frequency spectrum of the amplitude-scaled time-shifted signum function  $x$ . (a) Magnitude spectrum and (b) phase spectrum of  $x$ .

In the case of functions that are complex but not real, frequency spectra do not possess the above symmetry, and negative frequencies become important.

**Example 6.27** (Frequency spectrum of an amplitude-scaled time-shifted signum function). Consider the function

$$x(t) = \frac{1}{2} \operatorname{sgn}(t - 1).$$

We can show that this function has the Fourier transform

$$X(\omega) = \frac{1}{j\omega} e^{-j\omega}.$$

In this case,  $X$  is neither purely real nor purely imaginary, so we use two separate graphs to present the frequency spectrum  $X$ . We plot the magnitude spectrum and phase spectrum as shown in Figures 6.8(a) and (b), respectively. ■

**Example 6.28** (Frequency spectrum of the delta function). Consider the function

$$x(t) = \delta(t).$$

We can show that this function has the Fourier transform

$$X(\omega) = 1.$$

Since, in this case,  $X$  is real, we can plot the frequency spectrum  $X$  on a single graph, as shown in Figure 6.9. ■

**Example 6.29** (Frequency spectrum of a time-scaled sinc function). Consider the function

$$x(t) = \operatorname{sinc} \frac{t}{2}.$$

We can show that this function has the Fourier transform

$$X(\omega) = 2\pi \operatorname{rect} \omega.$$

Since, in this case,  $X$  is real, we can plot the frequency spectrum  $X$  on a single graph, as shown in Figure 6.10. ■

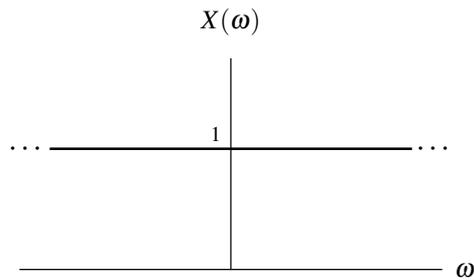
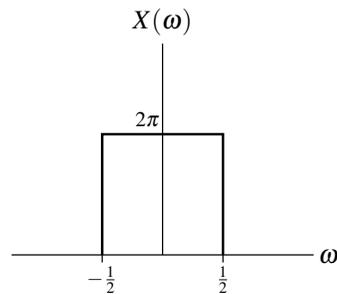


Figure 6.9: Frequency spectrum of the unit-impulse function.

Figure 6.10: Frequency spectrum of the time-scaled sinc function  $x$ .

**Example 6.30** (Frequency spectrum of a time-shifted signum function). The function

$$x(t) = \operatorname{sgn}(t - 1)$$

has the Fourier transform

$$X(\omega) = \frac{2}{j\omega} e^{-j\omega}.$$

(a) Find and plot the magnitude and phase spectra of  $x$ . (b) Determine at what frequency (or frequencies)  $x$  has the most information.

*Solution.* (a) First, we find the magnitude spectrum  $|X(\omega)|$ . From the expression for  $X(\omega)$ , we can write

$$\begin{aligned} |X(\omega)| &= \left| \frac{2}{j\omega} e^{-j\omega} \right| \\ &= \left| \frac{2}{j\omega} \right| |e^{-j\omega}| \\ &= \left| \frac{2}{j\omega} \right| \\ &= \frac{2}{|\omega|}. \end{aligned}$$

Next, we find the phase spectrum  $\arg X(\omega)$ . First, we observe that  $\arg X(\omega)$  is not well defined if  $\omega = 0$ . So, we

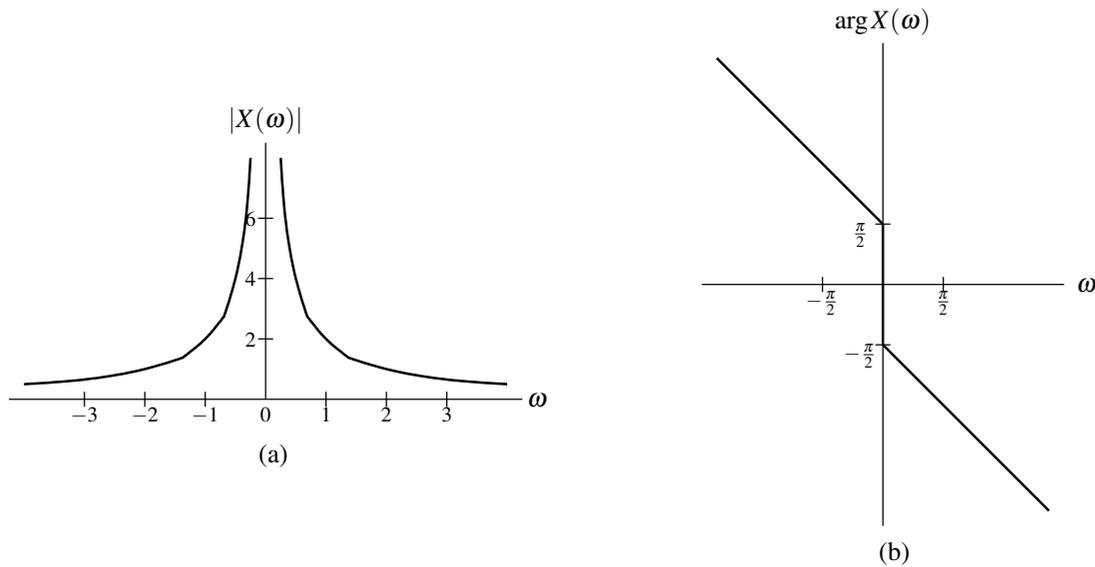


Figure 6.11: Frequency spectrum of the time-shifted signum function. (a) Magnitude spectrum and (b) phase spectrum of  $x$ .

assume that  $\omega \neq 0$ . From the expression for  $X(\omega)$ , we can write (for  $\omega \neq 0$ )

$$\begin{aligned}
 \arg X(\omega) &= \arg \left\{ \frac{2}{j\omega} e^{-j\omega} \right\} \\
 &= \arg e^{-j\omega} + \arg \frac{2}{j\omega} \\
 &= -\omega + \arg \frac{2}{j\omega} \\
 &= -\omega + \arg(-j\frac{2}{\omega}) \\
 &= \begin{cases} -\frac{\pi}{2} - \omega & \omega > 0 \\ \frac{\pi}{2} - \omega & \omega < 0 \end{cases} \\
 &= -\frac{\pi}{2} \operatorname{sgn} \omega - \omega.
 \end{aligned}$$

In the above simplification, we used the fact that

$$\arg \frac{2}{j\omega} = \arg(-j\frac{2}{\omega}) = \begin{cases} -\frac{\pi}{2} & \omega > 0 \\ \frac{\pi}{2} & \omega < 0. \end{cases}$$

Finally, using numerical calculation, we can plot the graphs of  $|X(\omega)|$  and  $\arg X(\omega)$  to obtain the results shown in Figures 6.11(a) and (b).

(b) Since  $|X(\omega)|$  is largest for  $\omega = 0$ ,  $x$  has the most information at the frequency 0. ■

**Example 6.31** (Frequency spectra of images). The human visual system is more sensitive to the phase spectrum of an image than its magnitude spectrum. This can be aptly demonstrated by separately modifying the magnitude and phase spectra of an image, and observing the effect. Below, we consider two variations on this theme.

Consider the potatohead and hongkong images shown in Figures 6.12(a) and (b), respectively. Replacing the magnitude spectrum of the potatohead image with the magnitude spectrum of the hongkong image (and leaving the phase spectrum unmodified), we obtain the new image shown in Figure 6.12(c). Although changing the magnitude spectrum has led to distortion, the basic essence of the original image has not been lost. On the other hand, replacing

the phase spectrum of the potatohead image with the phase spectrum of the hongkong image (and leaving the magnitude spectrum unmodified), we obtain the image shown in Figure 6.12(d). Clearly, by changing the phase spectrum of the image, the fundamental nature of the image has been altered, with the new image more closely resembling the hongkong image than the original potatohead image.

A more extreme scenario is considered in Figure 6.13. In this case, we replace each of the magnitude and phase spectra of the potatohead image with random data, with this data being taken from the image consisting of random noise shown in Figure 6.13(b). When we completely replace the magnitude spectrum of the potatohead image with random values, we can still recognize the resulting image in Figure 6.13(c) as a very grainy version of the original potatohead image. On the other hand, when the phase spectrum of the potatohead image is replaced with random values, all visible traces of the original potatohead image are lost in the resulting image in Figure 6.13(d). ■

## 6.11 Bandwidth of Functions

A function  $x$  with Fourier transform  $X$  is said to be **bandlimited** if, for some nonnegative real constant  $B$ ,

$$X(\omega) = 0 \text{ for all } |\omega| > B.$$

We sometimes refer to  $B$  as the **bandwidth** of  $x$ . An illustrative example is provided in Figure 6.14.

One can show that a function cannot be both time limited and bandlimited. To help understand why this is so we recall the time/frequency scaling property of the Fourier transform. From this property, we know that as we compress a function  $x$  (by time scaling), its Fourier transform  $X$  will expand (by time scaling). Similarly, as we compress the Fourier transform  $X$  (by time scaling),  $x$  will expand (by time scaling). So, clearly, there is an inverse relationship between the time-extent and bandwidth of a function.

## 6.12 Energy Spectral Density

Suppose that we have a function  $x$  with finite energy  $E$  and Fourier transform  $X$ . By definition, the energy contained in  $x$  is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt.$$

We can use Parseval's relation (6.12) to express  $E$  in terms of  $X$  as

$$\begin{aligned} E &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi} |X(\omega)|^2 d\omega. \end{aligned}$$

Thus, the energy  $E$  is given by

$$E = \int_{-\infty}^{\infty} E_x(\omega) d\omega,$$

where

$$E_x(\omega) = \frac{1}{2\pi} |X(\omega)|^2.$$

We refer to  $E_x$  as the **energy spectral density** of the function  $x$ . The function  $E_x$  indicates how the energy in  $x$  is distributed with respect to frequency (in units of energy per rad/unit-time). For example, the energy contributed by frequencies  $\omega$  in the range  $\omega_1 \leq \omega \leq \omega_2$  is simply given by

$$\int_{\omega_1}^{\omega_2} E_x(\omega) d\omega.$$

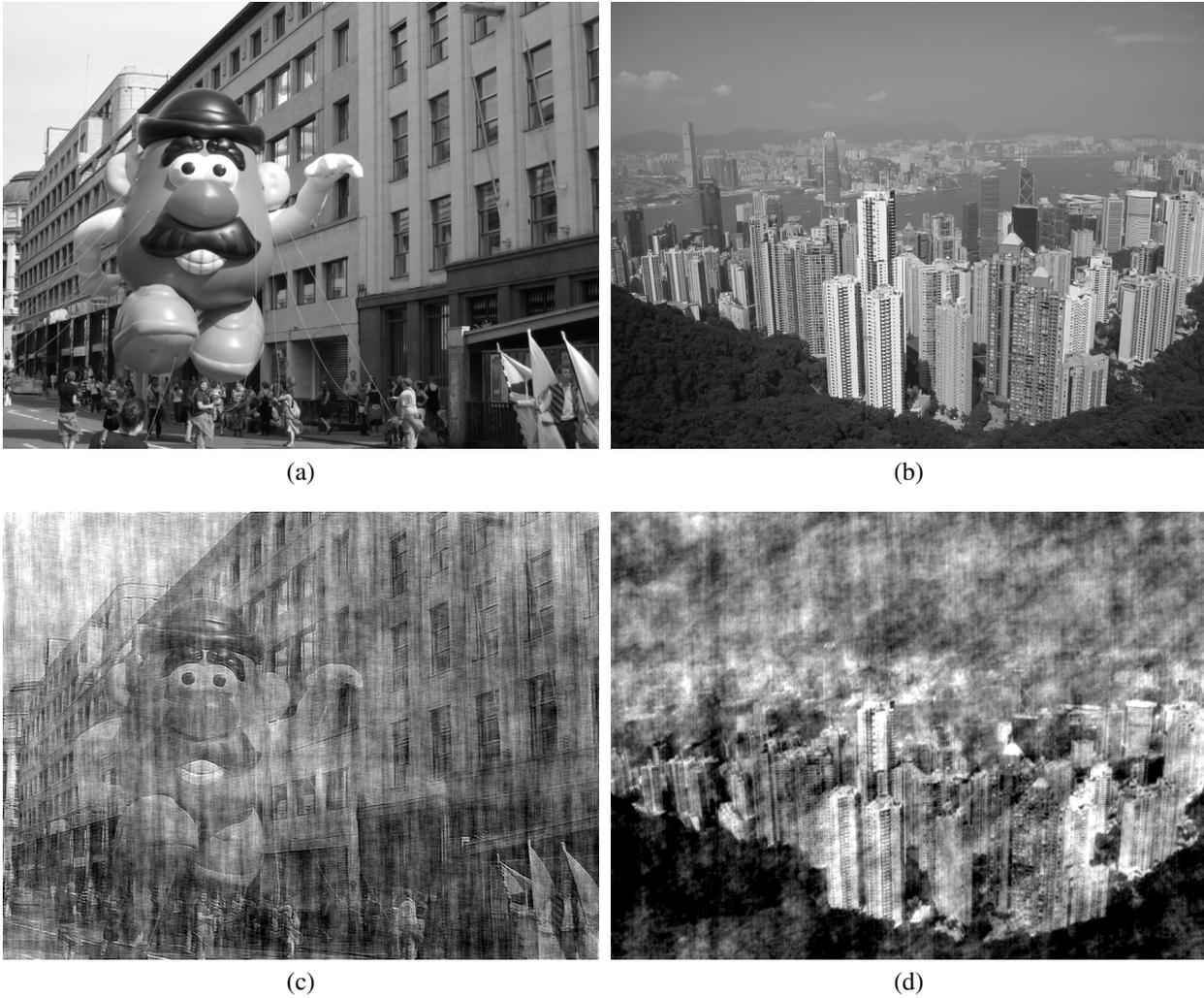


Figure 6.12: Importance of phase information in images. The (a) potatohead and (b) hongkong images. (c) The potatohead image after having its magnitude spectrum replaced with the magnitude spectrum of the hongkong image. (d) The potatohead image after having its phase spectrum replaced with the phase spectrum of the hongkong image.

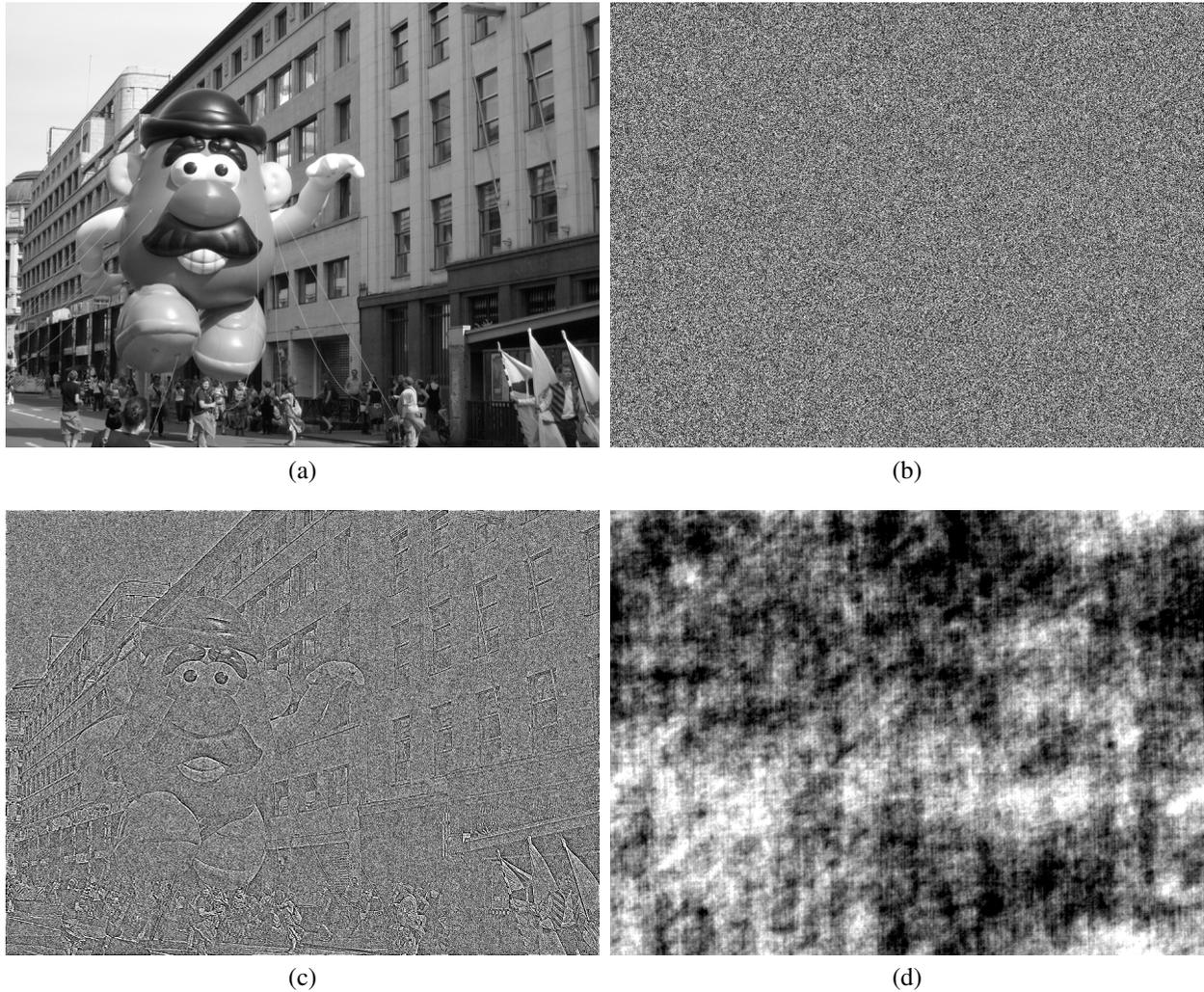


Figure 6.13: Importance of phase information in images. The (a) potatohead and (b) random images. (c) The potatohead image after having its magnitude spectrum replaced with the magnitude spectrum of the random image. (d) The potatohead image after having its phase spectrum replaced with the phase spectrum of the random image.

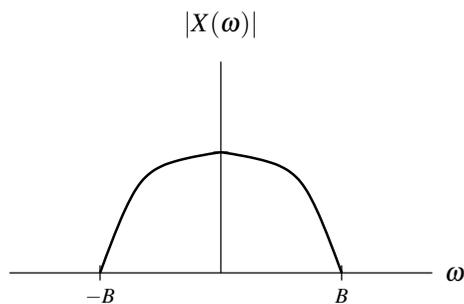


Figure 6.14: Bandwidth of a function  $x$  with the Fourier transform  $X$ .

**Example 6.32.** Compute the energy spectral density  $E_x$  of the function  $x(t) = \text{sinc} \frac{t}{2}$ . Determine the amount of energy contained in the frequency components in the range  $|\omega| \leq \frac{1}{4}$ . Also, determine the total amount of energy in the function.

*Solution.* First, we compute the Fourier transform  $X$  of  $x$ . We obtain  $X(\omega) = 2\pi \text{rect } \omega$ . Computing the energy spectral density  $E_x$ , we have

$$\begin{aligned} E_x(\omega) &= \frac{1}{2\pi} |X(\omega)|^2 \\ &= \frac{1}{2\pi} |2\pi \text{rect } \omega|^2 \\ &= 2\pi \text{rect}^2 \omega \\ &= 2\pi \text{rect } \omega. \end{aligned}$$

Let  $E_1$  denote the energy contained in  $x$  for frequencies  $|\omega| \leq \frac{1}{4}$ . Then, we have

$$\begin{aligned} E_1 &= \int_{-1/4}^{1/4} E_x(\omega) d\omega \\ &= \int_{-1/4}^{1/4} 2\pi \text{rect } \omega d\omega \\ &= \int_{-1/4}^{1/4} 2\pi d\omega \\ &= \pi. \end{aligned}$$

Let  $E$  denote the total amount of energy in  $x$ . Then, we have

$$\begin{aligned} E &= \int_{-\infty}^{\infty} E_x(\omega) d\omega \\ &= \int_{-\infty}^{\infty} 2\pi \text{rect } \omega d\omega \\ &= \int_{-1/2}^{1/2} 2\pi d\omega \\ &= 2\pi. \end{aligned} \quad \blacksquare$$

### 6.13 Power Spectral Density

Suppose that we have a function  $x$  with finite power  $P$  and Fourier transform  $X$ . Define  $x_T$  as the truncated (or “windowed”) version of  $x$  given by

$$x_T(t) = \left( \text{rect} \frac{t}{T} \right) x(t).$$

Also, let  $X_T$  denote the Fourier transform of  $x_T$ . From this definition of  $x_T$ , we have that

$$\lim_{T \rightarrow \infty} x_T(t) = x(t).$$

The average power of the function  $x_T$  is given by

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{\infty} |x_T(t)|^2 dt.$$

Using Parseval's relation (6.12), we can rewrite this expression for  $P$  in terms of  $X$  to obtain

$$\begin{aligned} P &= \lim_{T \rightarrow \infty} \frac{1}{T} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega \right] \\ &= \lim_{T \rightarrow \infty} \left[ \frac{1}{2\pi T} \int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega \right]. \end{aligned}$$

Interchanging the order of the limit and integration, we obtain

$$P = \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{2\pi T} |X_T(\omega)|^2 d\omega.$$

Thus, we have that

$$P = \int_{-\infty}^{\infty} S_x(\omega) d\omega,$$

where

$$S_x(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2\pi T} |X_T(\omega)|^2.$$

We refer to  $S_x$  as the **power spectral density** of the function  $x$ . This function indicates how the power in  $x$  is distributed with respect to frequency (in units of power per rad/unit-time). We can, therefore, determine the amount of power contributed by frequencies  $\omega$  in the range  $\omega_1 \leq \omega \leq \omega_2$  as

$$\int_{\omega_1}^{\omega_2} S_x(\omega) d\omega.$$

## 6.14 Characterizing LTI Systems Using the Fourier Transform

Consider a LTI system with input  $x$ , output  $y$ , and impulse response  $h$ . Such a system is depicted in Figure 6.15. The behavior of such a system is governed by the equation

$$y(t) = x * h(t). \quad (6.32)$$

Let  $X$ ,  $Y$ , and  $H$  denote the Fourier transforms of  $x$ ,  $y$ , and  $h$ , respectively. Taking the Fourier transform of both sides of (6.32) and using the the time-domain convolution property of the Fourier transform, we obtain

$$Y(\omega) = X(\omega)H(\omega). \quad (6.33)$$

This result provides an alternative way of viewing the behavior of a LTI system. That is, we can view the system as operating in the frequency domain on the Fourier transforms of the input and output functions. In other words, we have a system resembling that in Figure 6.16. In this case, however, the convolution operation from the time domain is replaced by multiplication in the frequency domain. The frequency spectrum (i.e., Fourier transform) of the output is the product of the frequency spectrum (i.e., Fourier transform) of the input and the frequency spectrum (i.e., Fourier transform) of the impulse response. As a matter of terminology, we refer to  $H$  as the **frequency response** of the system. The system behavior is completely characterized by the frequency response  $H$ . If we know the input, we can compute its Fourier transform  $X$ , and then determine the Fourier transform  $Y$  of the output. Using the inverse Fourier transform, we can then determine the output  $y$ .

In the most general case, the frequency response  $H$  is a complex-valued function. Thus, we can represent  $H$  in terms of its magnitude and argument. We refer to the magnitude of  $H$  as the **magnitude response** of the system. Similarly, we refer to the argument of  $H$  as the **phase response** of the system.

From (6.33), we can write

$$\begin{aligned} |Y(\omega)| &= |X(\omega)H(\omega)| \\ &= |X(\omega)||H(\omega)| \quad \text{and} \end{aligned} \quad (6.34a)$$

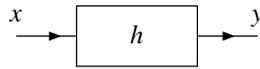


Figure 6.15: Time-domain view of a LTI system with input  $x$ , output  $y$ , and impulse response  $h$ .

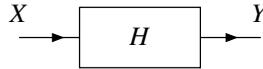


Figure 6.16: Frequency-domain view of a LTI system with input spectrum  $X$ , output spectrum  $Y$ , and frequency response  $H$ .

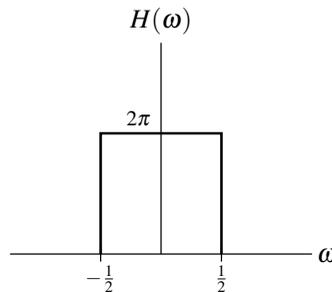


Figure 6.17: Frequency response of example system.

$$\begin{aligned}\arg Y(\omega) &= \arg(X(\omega)H(\omega)) \\ &= \arg X(\omega) + \arg H(\omega).\end{aligned}\tag{6.34b}$$

From (6.34a), we can see that the magnitude spectrum of the output equals the magnitude spectrum of the input times the magnitude spectrum of the impulse response. From (6.34b), we have that the phase spectrum of the output equals the phase spectrum of the input plus the phase spectrum of the impulse response.

Since the frequency response  $H$  is simply the frequency spectrum of the impulse response  $h$ , for the reasons explained in Section 6.10, if  $h$  is real, then

$$\begin{aligned}|H(\omega)| &= |H(-\omega)| \text{ for all } \omega \text{ and} \\ \arg H(\omega) &= -\arg H(-\omega) \text{ for all } \omega\end{aligned}$$

(i.e., the magnitude and phase responses are even and odd, respectively).

**Example 6.33.** Consider a LTI system with impulse response

$$h(t) = \text{sinc } \frac{t}{2}.$$

This system has the frequency response

$$H(\omega) = 2\pi \text{rect } \omega.$$

In this particular case,  $H$  is real. So, we can plot the frequency response  $H$  on a single graph, as shown in Figure 6.17. ■

## 6.15 Interconnection of LTI Systems

From the properties of the Fourier transform and the definition of the frequency response, we can derive a number of equivalences involving the frequency response and series- and parallel-interconnected systems.

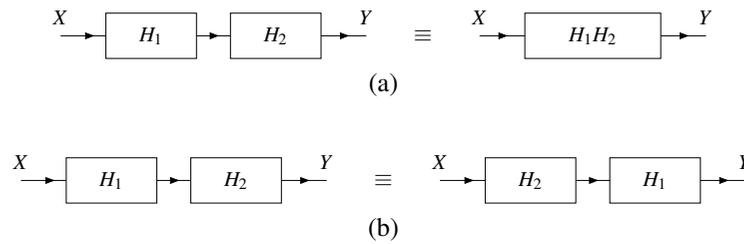


Figure 6.18: Equivalences involving frequency responses and the series interconnection of LTI systems. The (a) first and (b) second equivalences.

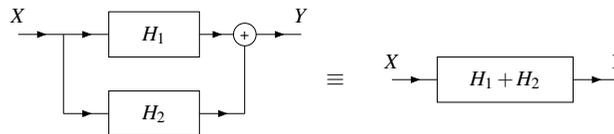


Figure 6.19: Equivalence involving frequency responses and the parallel interconnection of LTI systems.

Suppose that we have two LTI systems  $\mathcal{H}_1$  and  $\mathcal{H}_2$  with frequency responses  $H_1$  and  $H_2$ , respectively, that are connected in a series configuration as shown in the left-hand side of Figure 6.18(a). Let  $h_1$  and  $h_2$  denote the impulse responses of  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively. The impulse response  $h$  of the overall system is given by

$$h(t) = h_1 * h_2(t).$$

Taking the Fourier transform of both sides of this equation yields

$$\begin{aligned} H(\omega) &= \mathcal{F}\{h_1 * h_2\}(\omega) \\ &= \mathcal{F}h_1(\omega)\mathcal{F}h_2(\omega) \\ &= H_1(\omega)H_2(\omega). \end{aligned}$$

Thus, we have the equivalence shown in Figure 6.18(a). Also, since multiplication commutes, we also have the equivalence shown in Figure 6.18(b).

Suppose that we have two LTI systems  $\mathcal{H}_1$  and  $\mathcal{H}_2$  with frequency responses  $H_1$  and  $H_2$  that are connected in a parallel configuration as shown on the left-hand side of Figure 6.19. Let  $h_1$  and  $h_2$  denote the impulse responses of  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively. The impulse response  $h$  of the overall system is given by

$$h(t) = h_1(t) + h_2(t).$$

Taking the Fourier transform of both sides of this equation yields

$$\begin{aligned} H(\omega) &= \mathcal{F}\{h_1 + h_2\}(\omega) \\ &= \mathcal{F}h_1(\omega) + \mathcal{F}h_2(\omega) \\ &= H_1(\omega) + H_2(\omega). \end{aligned}$$

Thus, we have the equivalence shown in Figure 6.19.

## 6.16 LTI Systems and Differential Equations

Many LTI systems of practical interest can be represented using an  $N$ th-order linear differential equation with constant coefficients. Suppose that we have such a system with input  $x$  and output  $y$ . Then, the input-output behavior of the system is given by an equation of the form

$$\sum_{k=0}^N b_k \left(\frac{d}{dt}\right)^k y(t) = \sum_{k=0}^M a_k \left(\frac{d}{dt}\right)^k x(t)$$

(where  $M \leq N$ ). Let  $X$  and  $Y$  denote the Fourier transforms of  $x$  and  $y$ , respectively. Taking the Fourier transform of both sides of the above equation yields

$$\mathcal{F} \left\{ \sum_{k=0}^N b_k \left( \frac{d}{dt} \right)^k y(t) \right\} (\omega) = \mathcal{F} \left\{ \sum_{k=0}^M a_k \left( \frac{d}{dt} \right)^k x(t) \right\} (\omega).$$

Using the linearity property of the Fourier transform, we can rewrite this as

$$\sum_{k=0}^N b_k \mathcal{F} \left\{ \left( \frac{d}{dt} \right)^k y(t) \right\} (\omega) = \sum_{k=0}^M a_k \mathcal{F} \left\{ \left( \frac{d}{dt} \right)^k x(t) \right\} (\omega).$$

Using the time-differentiation property of the Fourier transform, we can re-express this as

$$\sum_{k=0}^N b_k (j\omega)^k Y(\omega) = \sum_{k=0}^M a_k (j\omega)^k X(\omega).$$

Then, factoring we have

$$Y(\omega) \sum_{k=0}^N b_k (j\omega)^k = X(\omega) \sum_{k=0}^M a_k (j\omega)^k.$$

Rearranging this equation, we obtain

$$\frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^M a_k (j\omega)^k}{\sum_{k=0}^N b_k (j\omega)^k} = \frac{\sum_{k=0}^M a_k j^k \omega^k}{\sum_{k=0}^N b_k j^k \omega^k}.$$

Thus, the frequency response  $H$  is given by

$$H(\omega) = \frac{\sum_{k=0}^M a_k j^k \omega^k}{\sum_{k=0}^N b_k j^k \omega^k}.$$

Observe that, for a system of the form considered above, the frequency response is a rational function—hence, our interest in rational functions.

**Example 6.34** (Differential equation to frequency response). A LTI system with input  $x$  and output  $y$  is characterized by the differential equation

$$7y''(t) + 11y'(t) + 13y(t) = 5x'(t) + 3x(t),$$

where  $x'$ ,  $y'$ , and  $y''$  denote the first derivative of  $x$ , the first derivative of  $y$ , and the second derivative of  $y$ , respectively. Find the frequency response  $H$  of this system.

*Solution.* Taking the Fourier transform of the given differential equation, we obtain

$$7(j\omega)^2 Y(\omega) + 11j\omega Y(\omega) + 13Y(\omega) = 5j\omega X(\omega) + 3X(\omega).$$

Rearranging the terms and factoring, we have

$$(-7\omega^2 + 11j\omega + 13)Y(\omega) = (5j\omega + 3)X(\omega).$$

Thus,  $H$  is given by

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{5j\omega + 3}{-7\omega^2 + 11j\omega + 13}. \quad \blacksquare$$

**Example 6.35** (Frequency response to differential equation). A LTI system with input  $x$  and output  $y$  has the frequency response

$$H(\omega) = \frac{-7\omega^2 + 11j\omega + 3}{-5\omega^2 + 2}.$$

Find the differential equation that characterizes this system.

*Solution.* From the given frequency response  $H$ , we have

$$\frac{Y(\omega)}{X(\omega)} = \frac{-7\omega^2 + 11j\omega + 3}{-5\omega^2 + 2}.$$

Multiplying both sides by  $(-5\omega^2 + 2)X(\omega)$ , we have

$$-5\omega^2 Y(\omega) + 2Y(\omega) = -7\omega^2 X(\omega) + 11j\omega X(\omega) + 3X(\omega).$$

Applying some simple algebraic manipulation yields

$$5(j\omega)^2 Y(\omega) + 2Y(\omega) = 7(j\omega)^2 X(\omega) + 11(j\omega)X(\omega) + 3X(\omega).$$

Taking the inverse Fourier transform of the preceding equation, we obtain

$$5y''(t) + 2y(t) = 7x''(t) + 11x'(t) + 3x(t). \quad \blacksquare$$

## 6.17 Filtering

In some applications, we want to change the relative amplitude of the frequency components of a function or possibly eliminate some frequency components altogether. This process of modifying the frequency components of a function is referred to as **filtering**. Various types of filters exist. One type is frequency-selective filters. Frequency selective filters pass some frequencies with little or no distortion, while significantly attenuating other frequencies. Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

An **ideal lowpass filter** eliminates all frequency components with a frequency whose magnitude is greater than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where  $\omega_c$  is the cutoff frequency. A plot of this frequency response is given in Figure 6.20(a).

An **ideal highpass filter** eliminates all frequency components with a frequency whose magnitude is less than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\omega) = \begin{cases} 1 & |\omega| \geq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where  $\omega_c$  is the cutoff frequency. A plot of this frequency response is given in Figure 6.20(b).

An **ideal bandpass filter** eliminates all frequency components with a frequency whose magnitude does not lie between two cutoff frequencies, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\omega) = \begin{cases} 1 & \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ 0 & \text{otherwise,} \end{cases}$$

where  $\omega_{c1}$  and  $\omega_{c2}$  are the cutoff frequencies. A plot of this frequency response is given in Figure 6.20(c).

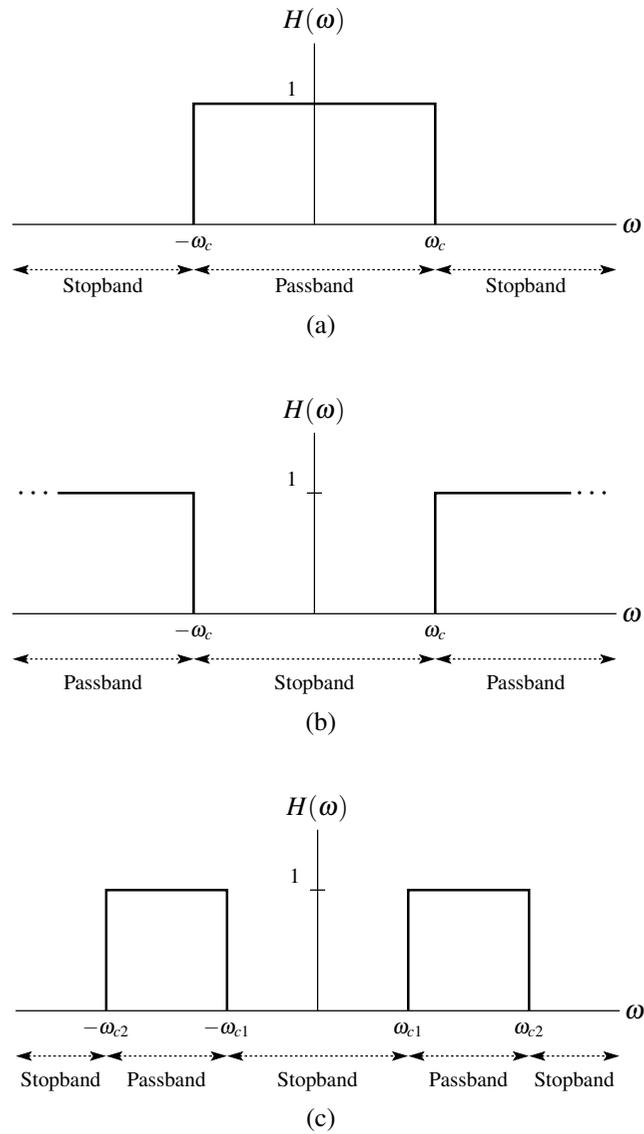


Figure 6.20: Frequency responses of (a) ideal lowpass, (b) ideal highpass, and (c) ideal bandpass filters.

**Example 6.36** (Ideal filters). For each LTI system whose impulse response  $h$  is given below, find and plot the frequency response  $H$  of the system, and identify the type of frequency-selective filter to which the system corresponds.

- (a)  $h(t) = \frac{\omega_c}{\pi} \text{sinc } \omega_c t$ , where  $\omega_c$  is a positive real constant;  
 (b)  $h(t) = \delta(t) - \frac{\omega_c}{\pi} \text{sinc } \omega_c t$ , where  $\omega_c$  is a positive real constant; and  
 (c)  $h(t) = \frac{2\omega_b}{\pi} (\text{sinc } \omega_b t) \cos \omega_a t$ , where  $\omega_a$  and  $\omega_b$  are positive real constants.

*Solution.* In what follows, let us denote the input and output of the system as  $x$  and  $y$ , respectively. Also, let  $X$  and  $Y$  denote the Fourier transforms of  $x$  and  $y$ , respectively.

(a) The frequency response  $H$  of the system is simply the Fourier transform of the impulse response  $h$ . Thus, we have

$$\begin{aligned} H(\omega) &= \mathcal{F}\left\{\frac{\omega_c}{\pi} \text{sinc } \omega_c t\right\}(\omega) \\ &= \frac{\omega_c}{\pi} \mathcal{F}\{\text{sinc } \omega_c t\}(\omega) \\ &= \frac{\omega_c}{\pi} \left(\frac{\pi}{\omega_c} \text{rect } \frac{\omega}{2\omega_c}\right) \\ &= \text{rect } \frac{\omega}{2\omega_c} \\ &= \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The frequency response  $H$  is plotted in Figure 6.21(a). Since  $Y(\omega) = H(\omega)X(\omega)$  and  $H(\omega) = 0$  for  $|\omega| > \omega_c$ ,  $Y$  will contain only those frequency components in  $X$  that lie in the frequency range  $|\omega| \leq \omega_c$ . In other words, only the lower frequency components from  $X$  are kept. Thus, the system represents a lowpass filter.

(b) The frequency response  $H$  of the system is simply the Fourier transform of the impulse response  $h$ . Thus, we have

$$\begin{aligned} H(\omega) &= \mathcal{F}\{\delta(t) - \frac{\omega_c}{\pi} \text{sinc } \omega_c t\}(\omega) \\ &= \mathcal{F}\delta(\omega) - \frac{\omega_c}{\pi} \mathcal{F}\{\text{sinc } \omega_c t\}(\omega) \\ &= 1 - \frac{\omega_c}{\pi} \left(\frac{\pi}{\omega_c} \text{rect } \frac{\omega}{2\omega_c}\right) \\ &= 1 - \text{rect } \frac{\omega}{2\omega_c} \\ &= \begin{cases} 1 & |\omega| \geq \omega_c \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The frequency response  $H$  is plotted in Figure 6.21(b). Since  $Y(\omega) = H(\omega)X(\omega)$  and  $H(\omega) = 0$  for  $|\omega| < \omega_c$ ,  $Y$  will contain only those frequency components in  $X$  that lie in the frequency range  $|\omega| \geq \omega_c$ . In other words, only the higher frequency components from  $X$  are kept. Thus, the system represents a highpass filter.

(c) The frequency response  $H$  of the system is simply the Fourier transform of the impulse response  $h$ . Thus, we have

$$\begin{aligned} H(\omega) &= \mathcal{F}\left\{\frac{2\omega_b}{\pi} [\text{sinc } \omega_b t] \cos \omega_a t\right\}(\omega) \\ &= \frac{\omega_b}{\pi} \mathcal{F}\{[\text{sinc } \omega_b t](2 \cos \omega_a t)\}(\omega) \\ &= \frac{\omega_b}{\pi} \mathcal{F}\{[\text{sinc } \omega_b t][e^{j\omega_a t} + e^{-j\omega_a t}]\}(\omega) \\ &= \frac{\omega_b}{\pi} [\mathcal{F}\{e^{j\omega_a t} \text{sinc } \omega_b t\}(\omega) + \mathcal{F}\{e^{-j\omega_a t} \text{sinc } \omega_b t\}(\omega)] \\ &= \frac{\omega_b}{\pi} \left[\frac{\pi}{\omega_b} \text{rect } \frac{\omega - \omega_a}{2\omega_b} + \frac{\pi}{\omega_b} \text{rect } \frac{\omega + \omega_a}{2\omega_b}\right] \\ &= \text{rect } \frac{\omega - \omega_a}{2\omega_b} + \text{rect } \frac{\omega + \omega_a}{2\omega_b} \\ &= \begin{cases} 1 & \omega_a - \omega_b \leq |\omega| \leq \omega_a + \omega_b \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

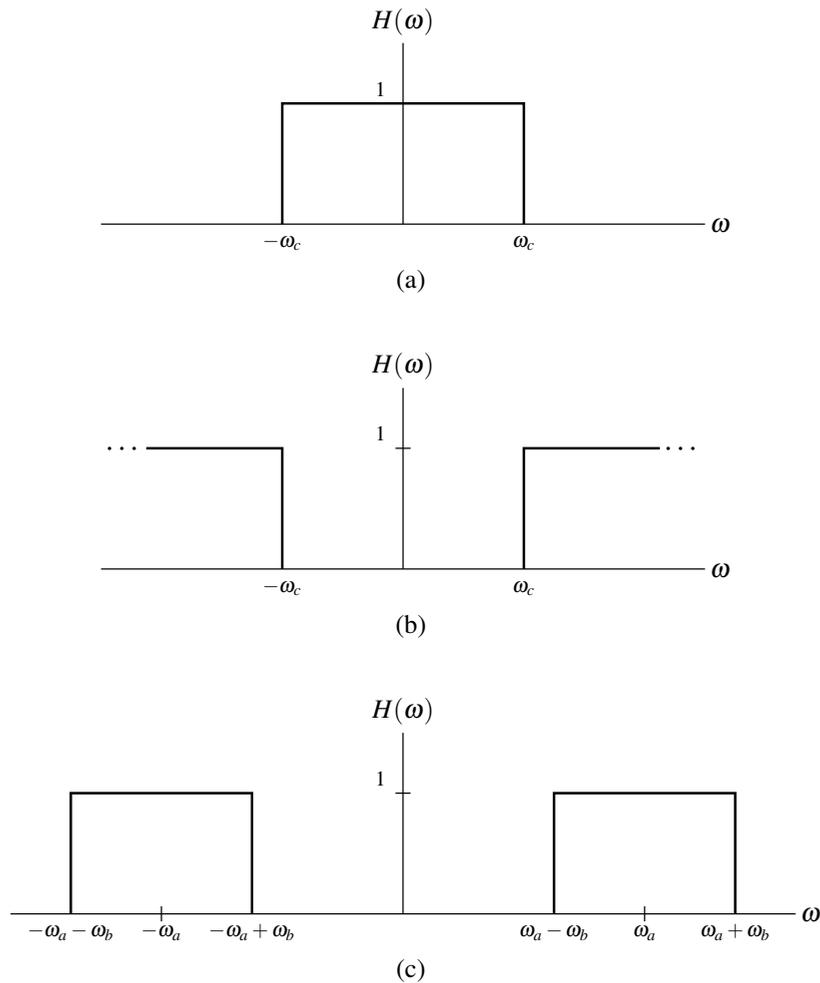


Figure 6.21: Frequency responses of each of the (a) first, (b) second, and (c) third systems from the example.

The frequency response  $H$  is plotted in Figure 6.21(c). Since  $Y(\omega) = H(\omega)X(\omega)$  and  $H(\omega) = 0$  for  $|\omega| < \omega_a - \omega_b$  or  $|\omega| > \omega_a + \omega_b$ ,  $Y$  will contain only those frequency components in  $X$  that lie in the frequency range  $\omega_a - \omega_b \leq |\omega| \leq \omega_a + \omega_b$ . In other words, only the middle frequency components of  $X$  are kept. Thus, the system represents a bandpass filter. ■

**Example 6.37** (Lowpass filtering). Consider a LTI system with impulse response

$$h(t) = 300 \operatorname{sinc} 300\pi t.$$

Using frequency-domain methods, find the response  $y$  of the system to the input

$$x(t) = \frac{1}{2} + \frac{3}{4} \cos 200\pi t + \frac{1}{2} \cos 400\pi t - \frac{1}{4} \cos 600\pi t.$$

*Solution.* To begin, we find the Fourier transform  $X$  of  $x$ . Computing  $X$ , we have

$$\begin{aligned} X(\omega) &= \mathcal{F}\left\{\frac{1}{2} + \frac{3}{4}\cos 200\pi t + \frac{1}{2}\cos 400\pi t - \frac{1}{4}\cos 600\pi t\right\}(\omega) \\ &= \frac{1}{2}\mathcal{F}\{1\}(\omega) + \frac{3}{4}\mathcal{F}\{\cos 200\pi t\}(\omega) + \frac{1}{2}\mathcal{F}\{\cos 400\pi t\}(\omega) - \frac{1}{4}\mathcal{F}\{\cos 600\pi t\}(\omega) \\ &= \frac{1}{2}[2\pi\delta(\omega)] + \frac{3\pi}{4}[\delta(\omega + 200\pi) + \delta(\omega - 200\pi)] + \frac{\pi}{2}[\delta(\omega + 400\pi) + \delta(\omega - 400\pi)] \\ &\quad - \frac{\pi}{4}[\delta(\omega + 600\pi) + \delta(\omega - 600\pi)] \\ &= -\frac{\pi}{4}\delta(\omega + 600\pi) + \frac{\pi}{2}\delta(\omega + 400\pi) + \frac{3\pi}{4}\delta(\omega + 200\pi) + \pi\delta(\omega) + \frac{3\pi}{4}\delta(\omega - 200\pi) \\ &\quad + \frac{\pi}{2}\delta(\omega - 400\pi) - \frac{\pi}{4}\delta(\omega - 600\pi). \end{aligned}$$

A plot of the frequency spectrum  $X$  is shown in Figure 6.22(a). Using the results of Example 6.36, we can determine the frequency response  $H$  of the system to be

$$\begin{aligned} H(\omega) &= \mathcal{F}\{300 \operatorname{sinc} 300\pi t\} \\ &= \operatorname{rect} \frac{\omega}{2(300\pi)} \\ &= \begin{cases} 1 & |\omega| \leq 300\pi \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The frequency response  $H$  is shown in Figure 6.22(b). The frequency spectrum  $Y$  of the output can be computed as

$$\begin{aligned} Y(\omega) &= H(\omega)X(\omega) \\ &= \frac{3\pi}{4}\delta(\omega + 200\pi) + \pi\delta(\omega) + \frac{3\pi}{4}\delta(\omega - 200\pi). \end{aligned}$$

The frequency spectrum  $Y$  is shown in Figure 6.22(c). Taking the inverse Fourier transform of  $Y$  yields

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}\left\{\frac{3\pi}{4}\delta(\omega + 200\pi) + \pi\delta(\omega) + \frac{3\pi}{4}\delta(\omega - 200\pi)\right\} \\ &= \pi\mathcal{F}^{-1}\{\delta(\omega)\} + \frac{3}{4}\mathcal{F}^{-1}\{\pi[\delta(\omega + 200\pi) + \delta(\omega - 200\pi)]\} \\ &= \pi\left(\frac{1}{2\pi}\right) + \frac{3}{4}\cos 200\pi t \\ &= \frac{1}{2} + \frac{3}{4}\cos 200\pi t. \end{aligned}$$

**Example 6.38** (Bandpass filtering). Consider a LTI system with the impulse response

$$h(t) = \left(\frac{2}{\pi} \operatorname{sinc} t\right) \cos 4t.$$

Using frequency-domain methods, find the response  $y$  of the system to the input

$$x(t) = -2 + 2\cos 2t + \cos 4t - \cos 6t.$$

*Solution.* Taking the Fourier transform of  $x$ , we have

$$\begin{aligned} X(\omega) &= -2\pi\delta(\omega) + 2(\pi[\delta(\omega - 2) + \delta(\omega + 2)]) + \pi[\delta(\omega - 4) + \delta(\omega + 4)] - \pi[\delta(\omega - 6) + \delta(\omega + 6)] \\ &= -\pi\delta(\omega + 6) + \pi\delta(\omega + 4) + 2\pi\delta(\omega + 2) - 2\pi\delta(\omega) + 2\pi\delta(\omega - 2) + \pi\delta(\omega - 4) - \pi\delta(\omega - 6). \end{aligned}$$

The frequency spectrum  $X$  is shown in Figure 6.23(a). Now, we compute the frequency response  $H$  of the system. Using the results of Example 6.36, we can determine  $H$  to be

$$\begin{aligned} H(\omega) &= \mathcal{F}\left\{\left(\frac{2}{\pi} \operatorname{sinc} t\right) \cos 4t\right\} \\ &= \operatorname{rect} \frac{\omega-4}{2} + \operatorname{rect} \frac{\omega+4}{2} \\ &= \begin{cases} 1 & 3 \leq |\omega| \leq 5 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

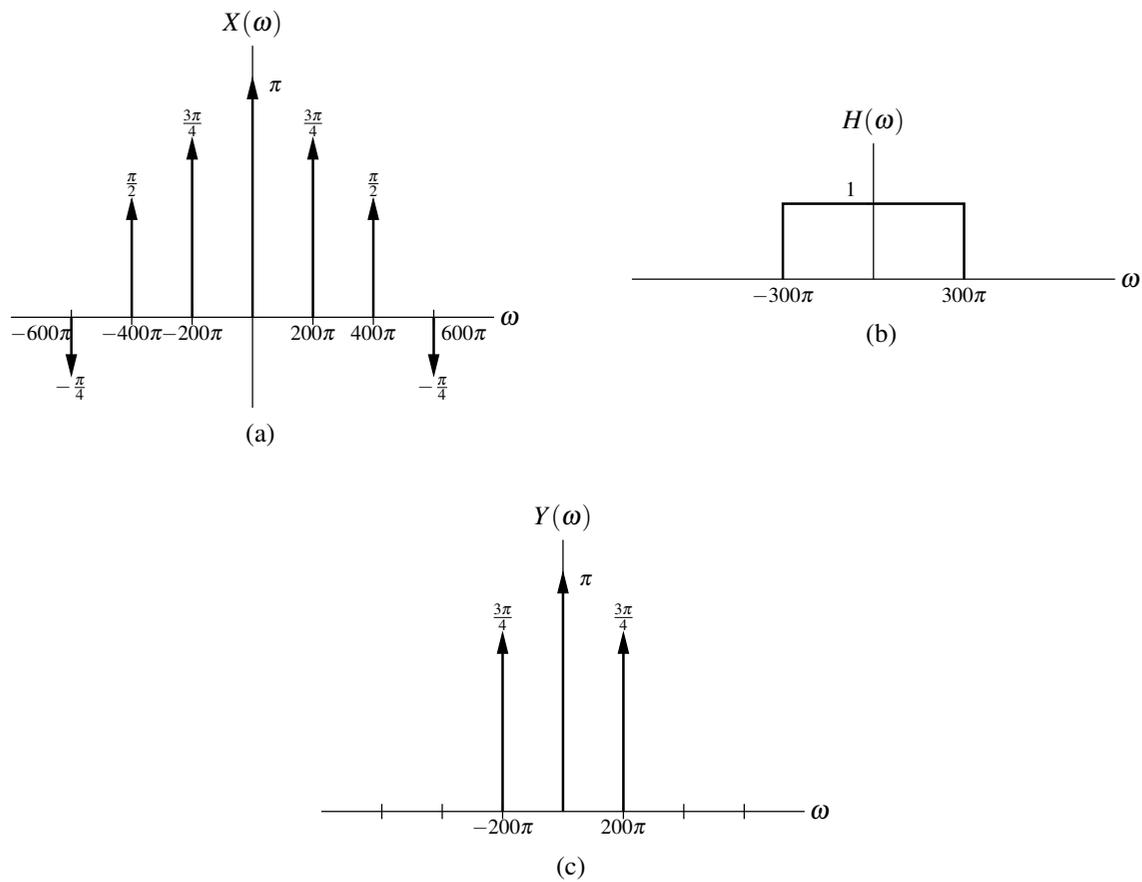


Figure 6.22: Frequency spectra for the lowpass filtering example. (a) Frequency spectrum of the input  $x$ . (b) Frequency response of the system. (c) Frequency spectrum of the output  $y$ .

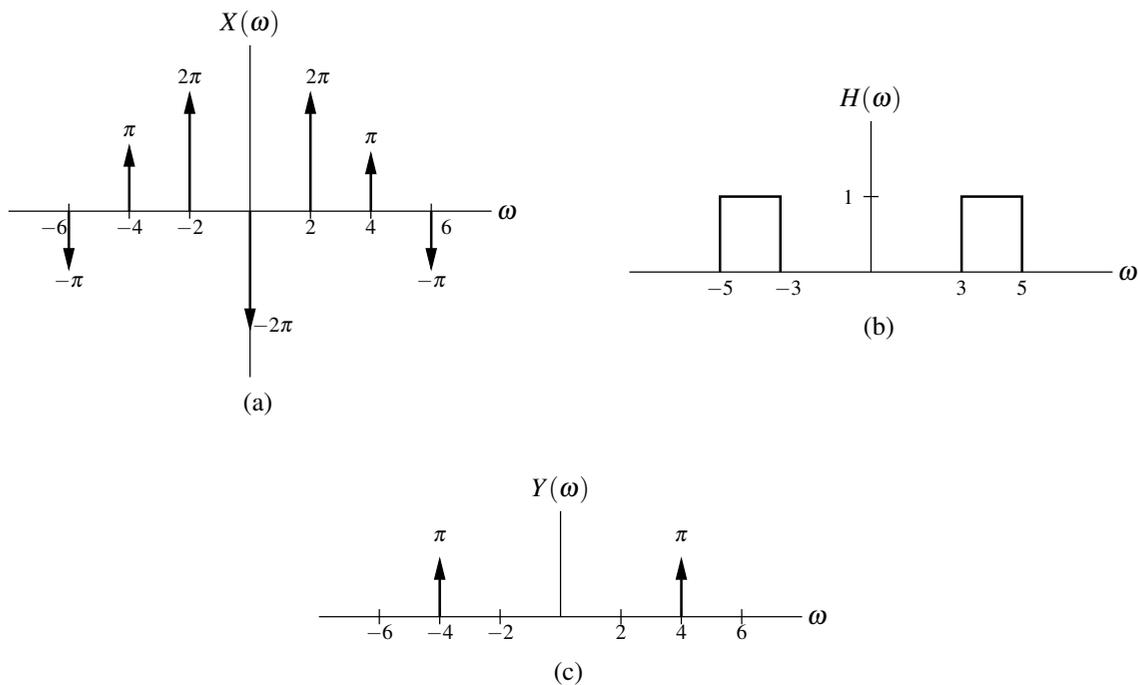


Figure 6.23: Frequency spectra for bandpass filtering example. (a) Frequency spectrum of the input  $x$ . (b) Frequency response of the system. (c) Frequency spectrum of the output  $y$ .

The frequency response  $H$  is shown in Figure 6.23(b). The frequency spectrum  $Y$  of the output is given by

$$\begin{aligned} Y(\omega) &= H(\omega)X(\omega) \\ &= \pi\delta(\omega + 4) + \pi\delta(\omega - 4). \end{aligned}$$

Taking the inverse Fourier transform, we obtain

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}\{\pi\delta(\omega + 4) + \pi\delta(\omega - 4)\}(t) \\ &= \mathcal{F}^{-1}\{\pi[\delta(\omega + 4) + \delta(\omega - 4)]\}(t) \\ &= \cos 4t. \end{aligned}$$

## 6.18 Equalization

Often, we find ourselves faced with a situation where we have a system with a particular frequency response that is undesirable for the application at hand. As a result, we would like to change the frequency response of the system to be something more desirable. This process of modifying the frequency response in this way is referred to as **equalization**. Essentially, equalization is just a filtering operation, where the filtering is applied with the specific goal of obtaining a more desirable frequency response.

Let us now examine the mathematics behind equalization. Consider the LTI system with impulse response  $h_{\text{orig}}$  as shown in Figure 6.24(a). Let  $H_{\text{orig}}$  denote the Fourier transform of  $h_{\text{orig}}$ . Suppose that the frequency response  $H_{\text{orig}}$  is undesirable for some reason (i.e., the system does not behave in a way that is good for the application at hand). Consequently, we would instead like to have a system with frequency response  $H_d$ . In effect, we would like to somehow change the frequency response  $H_{\text{orig}}$  of the original system to  $H_d$ . This can be accomplished by using another system called an **equalizer**. More specifically, consider the new system shown in Figure 6.24(b), which

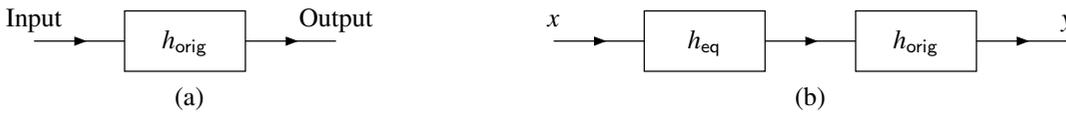


Figure 6.24: Equalization example. (a) Original system. (b) New system with equalization.



Figure 6.25: System from example that employs equalization.

consists of a LTI equalizer with impulse response  $h_{\text{eq}}$  connected in series with the original system having impulse response  $h_{\text{orig}}$ . Let  $H_{\text{eq}}$  denote the Fourier transform of  $h_{\text{eq}}$ . From the block diagram, we have

$$Y(\omega) = H(\omega)X(\omega),$$

where  $H(\omega) = H_{\text{orig}}(\omega)H_{\text{eq}}(\omega)$ . In effect, we want to force  $H$  to be equal to  $H_d$  so that the overall (i.e., series-interconnected) system has the frequency response desired. So, we choose the equalizer to be such that

$$H_{\text{eq}}(\omega) = \frac{H_d(\omega)}{H_{\text{orig}}(\omega)}.$$

Then, we have

$$\begin{aligned} H(\omega) &= H_{\text{orig}}(\omega)H_{\text{eq}}(\omega) \\ &= H_{\text{orig}}(\omega) \left[ \frac{H_d(\omega)}{H_{\text{orig}}(\omega)} \right] \\ &= H_d(\omega). \end{aligned}$$

Thus, the system in Figure 6.24(b) has the frequency response  $H_d$  as desired.

Equalization is used in many applications. In real-world communication systems, equalization is used to eliminate or minimize the distortion introduced when a signal is sent over a (nonideal) communication channel. In audio applications, equalization can be employed to emphasize or de-emphasize certain ranges of frequencies. For example, often we like to boost the bass (i.e., emphasize the low frequencies) in the audio output of a stereo.

**Example 6.39** (Communication channel equalization). Consider a LTI communication channel with frequency response

$$H(\omega) = \frac{1}{3+j\omega}.$$

Unfortunately, this channel has the undesirable effect of attenuating higher frequencies. Find the frequency response  $G$  of an equalizer that when connected in series with the communication channel yields an ideal (i.e., distortionless) channel. The new system with equalization is shown in Figure 6.25, where  $g$  and  $h$  denote the inverse Fourier transforms of  $G$  and  $H$ , respectively.

*Solution.* An ideal communication channel has a frequency response equal to one for all frequencies. Consequently, we want  $H(\omega)G(\omega) = 1$  or equivalently  $G(\omega) = 1/H(\omega)$ . Thus, we conclude that

$$\begin{aligned} G(\omega) &= \frac{1}{H(\omega)} \\ &= \frac{1}{\left(\frac{1}{3+j\omega}\right)} \\ &= 3 + j\omega. \end{aligned} \quad \blacksquare$$

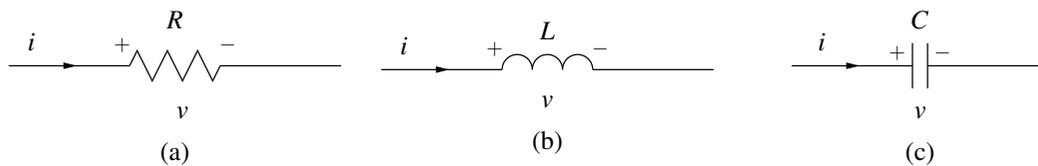


Figure 6.26: Basic electrical components. (a) Resistor, (b) inductor, and (c) capacitor.

## 6.19 Circuit Analysis

One application of the Fourier transform is circuit analysis. In this section, we consider this particular application.

The basic building blocks of many electrical networks are resistors, inductors, and capacitors. In what follows, we briefly introduce each of these circuit elements.

A **resistor** is a circuit element that opposes the flow of electric current. The resistor, shown in schematic form in Figure 6.26(a), is governed by the relationship

$$v(t) = Ri(t) \quad (\text{or equivalently, } i(t) = \frac{1}{R}v(t)),$$

where  $R$ ,  $v$  and  $i$  denote the resistance of, voltage across, and current through the resistor, respectively. Note that the resistance  $R$  is a nonnegative quantity (i.e.,  $R \geq 0$ ). In the frequency domain, the above relationship becomes

$$V(\omega) = RI(\omega) \quad (\text{or equivalently, } I(\omega) = \frac{1}{R}V(\omega)),$$

where  $V$  and  $I$  denote the Fourier transforms of  $v$  and  $i$ , respectively.

An **inductor** is a circuit element that converts an electric current into a magnetic field and vice versa. The inductor, shown in schematic form in Figure 6.26(b), is governed by the relationship

$$v(t) = L \frac{d}{dt}i(t) \quad \left( \text{or equivalently, } i(t) = \frac{1}{L} \int_{-\infty}^t v(\tau) d\tau \right),$$

where  $L$ ,  $v$ , and  $i$  denote the inductance of, voltage across, and current through the inductor, respectively. Note that the inductance  $L$  is a nonnegative quantity (i.e.,  $L \geq 0$ ). In the frequency domain, the above relationship becomes

$$V(\omega) = j\omega LI(\omega) \quad \left( \text{or equivalently, } I(\omega) = \frac{1}{j\omega L}V(\omega) \right),$$

where  $V$  and  $I$  denote the Fourier transforms of  $v$  and  $i$ , respectively.

A **capacitor** is a circuit element that stores electric charge. The capacitor, shown in schematic form in Figure 6.26(c), is governed by the relationship

$$v(t) = \frac{1}{C} \int_{-\infty}^t i(\tau) d\tau \quad (\text{or equivalently, } i(t) = C \frac{d}{dt}v(t)),$$

where  $C$ ,  $v$ , and  $i$  denote the capacitance of, voltage across, and current through the capacitor, respectively. Note that the capacitance  $C$  is a nonnegative quantity (i.e.,  $C \geq 0$ ). In the frequency domain, the above relationship becomes

$$V(\omega) = \frac{1}{j\omega C}I(\omega) \quad (\text{or equivalently, } I(\omega) = j\omega CV(\omega)),$$

where  $V$  and  $I$  denote the Fourier transforms of  $v$  and  $i$ , respectively.

Observe that, in the case of inductors and capacitors, the equations that characterize these circuit elements are arguably much simpler to express in the Fourier domain than in the time domain. Consequently, the use of the Fourier transform has the potential to greatly simplify the process of analyzing circuits containing inductors and capacitors.

**Example 6.40** (Simple RL network). Consider the resistor-inductor (RL) network shown in Figure 6.27 with input  $v_1$  and output  $v_2$ . This system is LTI, since it can be characterized by a linear differential equation with constant coefficients. (a) Find the frequency response  $H$  of the system. (b) Find the response  $v_2$  of the system to the input  $v_1(t) = \text{sgn}t$ .

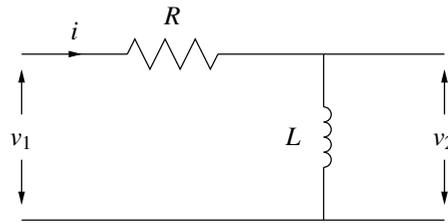


Figure 6.27: Simple RL network.

*Solution.* (a) From basic circuit analysis, we can write

$$v_1(t) = Ri(t) + L\frac{d}{dt}i(t) \quad \text{and} \quad (6.35)$$

$$v_2(t) = L\frac{d}{dt}i(t). \quad (6.36)$$

(Recall that the voltage  $v$  across an inductor  $L$  is related to the current  $i$  through the inductor as  $v(t) = L\frac{d}{dt}i(t)$ .) Taking the Fourier transform of (6.35) and (6.36) yields

$$\begin{aligned} V_1(\omega) &= RI(\omega) + j\omega LI(\omega) \\ &= (R + j\omega L)I(\omega) \quad \text{and} \end{aligned} \quad (6.37)$$

$$V_2(\omega) = j\omega LI(\omega). \quad (6.38)$$

From (6.37) and (6.38), we have

$$\begin{aligned} H(\omega) &= \frac{V_2(\omega)}{V_1(\omega)} \\ &= \frac{j\omega LI(\omega)}{(R + j\omega L)I(\omega)} \\ &= \frac{j\omega L}{R + j\omega L}. \end{aligned} \quad (6.39)$$

Thus, we have found the frequency response of the system.

(b) Now, suppose that  $v_1(t) = \text{sgn}t$  (as given). Taking the Fourier transform of the input  $v_1$  (with the aid of Table 6.2), we have

$$V_1(\omega) = \frac{2}{j\omega}. \quad (6.40)$$

From the definition of the system, we know

$$V_2(\omega) = H(\omega)V_1(\omega). \quad (6.41)$$

Substituting (6.40) and (6.39) into (6.41), we obtain

$$\begin{aligned} V_2(\omega) &= \left( \frac{j\omega L}{R + j\omega L} \right) \left( \frac{2}{j\omega} \right) \\ &= \frac{2L}{R + j\omega L}. \end{aligned}$$

Taking the inverse Fourier transform of both sides of this equation, we obtain

$$\begin{aligned} v_2(t) &= \mathcal{F}^{-1} \left\{ \frac{2L}{R + j\omega L} \right\} (t) \\ &= \mathcal{F}^{-1} \left\{ \frac{2}{R/L + j\omega} \right\} (t) \\ &= 2\mathcal{F}^{-1} \left\{ \frac{1}{R/L + j\omega} \right\} (t). \end{aligned}$$

Using Table 6.2, we can simplify to obtain

$$v_2(t) = 2e^{-(R/L)t} u(t).$$

Thus, we have found the response  $v_2$  to the input  $v_1(t) = \text{sgn } t$ . ■

## 6.20 Amplitude Modulation

In communication systems, we often need to transmit a signal using a frequency range that is different from that of the original signal. For example, voice/audio signals typically have information in the range of 0 to 22 kHz. Often, it is not practical to transmit such a signal using its original frequency range. Two potential problems with such an approach are: 1) interference and 2) constraints on antenna length. Since many signals are broadcast over the airwaves, we need to ensure that no two transmitters use the same frequency bands in order to avoid interference. Also, in the case of transmission via electromagnetic waves (e.g., radio waves), the length of antenna required becomes impractically large for the transmission of relatively low frequency signals. For the preceding reasons, we often need to change the frequency range associated with a signal before transmission. In what follows, we consider one possible scheme for accomplishing this. This scheme is known as amplitude modulation.

Amplitude modulation (AM) is used in many communication systems. Numerous variations on amplitude modulation are possible. Here, we consider two of the simplest variations: double-side-band/suppressed-carrier (DSB/SC) and single-side-band/suppressed-carrier (SSB/SC).

### 6.20.1 Modulation With a Complex Sinusoid

Consider the communication system shown in Figure 6.28. First, let us consider the transmitter in Figure 6.28(a). The transmitter is the system with the input  $x$  and output  $y$  that is characterized by the equation

$$y(t) = c_1(t)x(t), \tag{6.42}$$

where

$$c_1(t) = e^{j\omega_c t}.$$

Let  $X$ ,  $Y$ , and  $C_1$  denote the Fourier transforms of  $x$ ,  $y$ , and  $c_1$ , respectively. Taking the Fourier transform of both sides of (6.42), we obtain

$$\begin{aligned} Y(\omega) &= \mathcal{F}\{c_1 x\}(\omega) \\ &= \mathcal{F}\{e^{j\omega_c t} x(t)\}(\omega) \\ &= X(\omega - \omega_c). \end{aligned} \tag{6.43}$$

Thus, the frequency spectrum of the (transmitter) output is simply the frequency spectrum of the (transmitter) input shifted by  $\omega_c$ . The relationship between the frequency spectra of the input and output is illustrated in Figure 6.29.

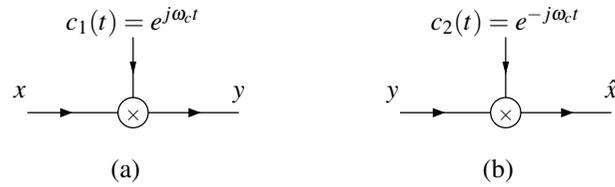


Figure 6.28: Simple communication system. (a) Transmitter and (b) receiver.

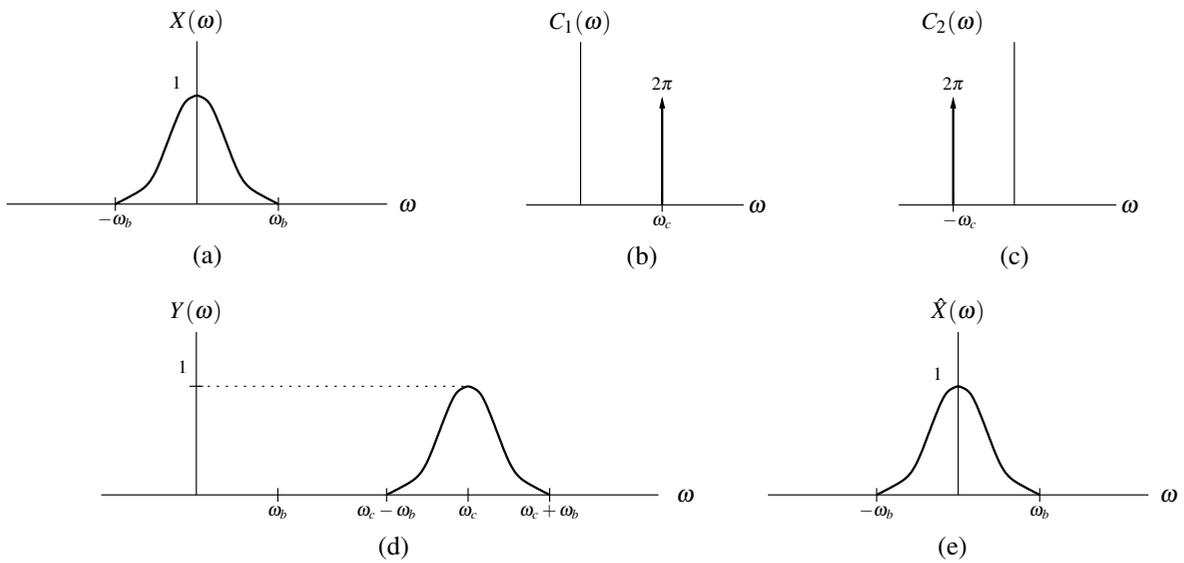


Figure 6.29: Frequency spectra for modulation with a complex sinusoid. (a) Spectrum of the transmitter input. (b) Spectrum of the complex sinusoid used in the transmitter. (c) Spectrum of the complex sinusoid used in the receiver. (d) Spectrum of the transmitted signal. (e) Spectrum of the receiver output.

Clearly, the spectrum of the output has been shifted to a different frequency range as desired. Next, we need to determine whether the receiver can recover the original signal  $x$  from the transmitted signal  $y$ .

Consider the receiver shown in Figure 6.28(b). The receiver is a system with the input  $y$  and output  $\hat{x}$  that is characterized by the equation

$$\hat{x}(t) = c_2(t)y(t), \quad (6.44)$$

where

$$c_2(t) = e^{-j\omega_c t}.$$

In order for the communication system to be useful, we need for the received signal  $\hat{x}$  to be equal to the original signal  $x$  from the transmitter. Let  $Y$ ,  $\hat{X}$ , and  $C_2$  denote the Fourier transform of  $y$ ,  $\hat{x}$ , and  $c_2$ , respectively. Taking the Fourier transform of both sides of (6.44), we obtain

$$\begin{aligned} \hat{X}(\omega) &= \mathcal{F}\{c_2 y\}(\omega) \\ &= \mathcal{F}\{e^{-j\omega_c t} y(t)\}(\omega) \\ &= Y(\omega + \omega_c). \end{aligned}$$

Substituting the expression for  $Y$  in (6.43) into this equation, we obtain

$$\begin{aligned} \hat{X}(\omega) &= X([\omega + \omega_c] - \omega_c) \\ &= X(\omega). \end{aligned}$$

Since  $\hat{X} = X$ , we have that the received signal  $\hat{x}$  is equal to the original signal  $x$  from the transmitter. Thus, the communication system has the desired behavior. The relationship between the frequency spectra of the various signals in the AM system is illustrated in Figure 6.29.

Although the above result is quite interesting mathematically, it does not have direct practical application. The difficulty here is that  $c_1$ ,  $c_2$ , and  $y$  are complex signals, and we cannot realize complex signals in the physical world. This communication system is not completely without value, however, as it inspires the development of the practically useful system that we consider next.

## 6.20.2 DSB/SC Amplitude Modulation

Now, let us consider the communication system shown in Figure 6.30. This system is known as a double-sideband/suppressed-carrier (DSB/SC) amplitude modulation (AM) system. The receiver in Figure 6.30(b) contains a LTI subsystem that is labelled with its impulse response  $h$ . The DSB/SC AM system is very similar to the one considered earlier in Figure 6.28. In the new system, however, multiplication by a complex sinusoid has been replaced by multiplication by a real sinusoid. The new system also requires that the input signal  $x$  be bandlimited to frequencies in the interval  $[-\omega_b, \omega_b]$  and that

$$\omega_b < \omega_{c0} < 2\omega_c - \omega_b. \quad (6.45)$$

The reasons for this restriction will become clear after having studied this system in more detail.

Consider the transmitter shown in Figure 6.30(a). The transmitter is a system with input  $x$  and output  $y$  that is characterized by the equation

$$y(t) = c(t)x(t),$$

where

$$c(t) = \cos \omega_c t.$$

Taking the Fourier transform of both sides of the preceding equation, we obtain

$$\begin{aligned}
Y(\omega) &= \mathcal{F}\{cx\}(\omega) \\
&= \mathcal{F}\{(\cos \omega_c t)x(t)\}(\omega) \\
&= \mathcal{F}\left\{\frac{1}{2}[e^{j\omega_c t} + e^{-j\omega_c t}]x(t)\right\}(\omega) \\
&= \frac{1}{2} [\mathcal{F}\{e^{j\omega_c t}x(t)\}(\omega) + \mathcal{F}\{e^{-j\omega_c t}x(t)\}(\omega)] \\
&= \frac{1}{2} [X(\omega - \omega_c) + X(\omega + \omega_c)].
\end{aligned} \tag{6.46}$$

(Note that, above, we used the fact that  $\cos \omega_c t = \frac{1}{2}(e^{j\omega_c t} + e^{-j\omega_c t})$ .) Thus, the frequency spectrum of the (transmitter) output is the average of two shifted versions of the frequency spectrum of the (transmitter) input. The relationship between the frequency spectra of the input and output can be seen through Figures 6.31(a) and (d). Observe that we have managed to shift the frequency spectrum of the input signal into a different range of frequencies for transmission as desired. Next, we must determine whether the receiver can recover the original signal  $x$ .

Consider the receiver shown in Figure 6.30(b). The receiver is a system with input  $y$  and output  $\hat{x}$  that is characterized by the equations

$$v(t) = c(t)y(t) \quad \text{and} \tag{6.47a}$$

$$\hat{x}(t) = v * h(t), \tag{6.47b}$$

where  $c$  is as defined earlier and

$$h(t) = \frac{2\omega_{c0}}{\pi} \text{sinc } \omega_{c0}t. \tag{6.47c}$$

Let  $H$ ,  $Y$ ,  $V$ , and  $\hat{X}$  denote the Fourier transforms of  $h$ ,  $y$ ,  $v$  and  $\hat{x}$ , respectively. Taking the Fourier transform of  $\hat{X}$  (in (6.47b)), we have

$$\hat{X}(\omega) = H(\omega)V(\omega). \tag{6.48}$$

Taking the Fourier transform of  $h$  (in (6.47c)) with the assistance of Table 6.2, we have

$$\begin{aligned}
H(\omega) &= \mathcal{F}\left\{\frac{2\omega_{c0}}{\pi} \text{sinc } \omega_{c0}t\right\}(\omega) \\
&= 2 \text{rect } \frac{\omega}{2\omega_{c0}} \\
&= \begin{cases} 2 & |\omega| \leq \omega_{c0} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Taking the Fourier transform of  $v$  (in (6.47a)) yields

$$\begin{aligned}
V(\omega) &= \mathcal{F}\{cy\}(\omega) \\
&= \mathcal{F}\{(\cos \omega_c t)y(t)\}(\omega) \\
&= \mathcal{F}\left\{\frac{1}{2}(e^{j\omega_c t} + e^{-j\omega_c t})y(t)\right\}(\omega) \\
&= \frac{1}{2} [\mathcal{F}\{e^{j\omega_c t}y(t)\}(\omega) + \mathcal{F}\{e^{-j\omega_c t}y(t)\}(\omega)] \\
&= \frac{1}{2} [Y(\omega - \omega_c) + Y(\omega + \omega_c)].
\end{aligned}$$

Substituting the expression for  $Y$  in (6.46) into this equation, we obtain

$$\begin{aligned}
V(\omega) &= \frac{1}{2} \left[ \frac{1}{2} [X([\omega - \omega_c] - \omega_c) + X([\omega - \omega_c] + \omega_c)] + \frac{1}{2} [X([\omega + \omega_c] - \omega_c) + X([\omega + \omega_c] + \omega_c)] \right] \\
&= \frac{1}{2} X(\omega) + \frac{1}{4} X(\omega - 2\omega_c) + \frac{1}{4} X(\omega + 2\omega_c).
\end{aligned} \tag{6.49}$$

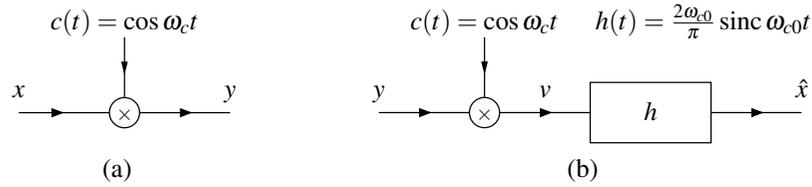


Figure 6.30: DSB/SC amplitude modulation system. (a) Transmitter and (b) receiver.

The relationship between  $V$  and  $X$  can be seen via Figures 6.31(a) and (e). Substituting the above expression for  $V$  into (6.47b) and simplifying, we obtain

$$\begin{aligned}
 \hat{X}(\omega) &= H(\omega)V(\omega) \\
 &= H(\omega) \left[ \frac{1}{2}X(\omega) + \frac{1}{4}X(\omega - 2\omega_c) + \frac{1}{4}X(\omega + 2\omega_c) \right] \\
 &= \frac{1}{2}H(\omega)X(\omega) + \frac{1}{4}H(\omega)X(\omega - 2\omega_c) + \frac{1}{4}H(\omega)X(\omega + 2\omega_c) \\
 &= \frac{1}{2}[2X(\omega)] + \frac{1}{4}(0) + \frac{1}{4}(0) \\
 &= X(\omega).
 \end{aligned}$$

In the above simplification, since  $H(\omega) = 2 \operatorname{rect} \frac{\omega}{2\omega_c}$  and condition (6.45) holds, we were able to deduce that  $H(\omega)X(\omega) = 2X(\omega)$ ,  $H(\omega)X(\omega - 2\omega_c) = 0$ , and  $H(\omega)X(\omega + 2\omega_c) = 0$ . The relationship between  $\hat{X}$  and  $X$  can be seen from Figures 6.31(a) and (f). Thus, we have that  $\hat{X} = X$ , implying  $\hat{x} = x$ . So, we have recovered the original signal  $x$  at the receiver. This system has managed to shift  $x$  into a different frequency range before transmission and then recover  $x$  at the receiver. This is exactly what we wanted to accomplish.

### 6.20.3 SSB/SC Amplitude Modulation

By making a minor modification to the DSB/SC AM system, we can reduce the bandwidth requirements of the system by half. The resulting system is referred to as a single-side-band/suppressed-carrier (SSB/SC) AM system. This modified system is illustrated in Figure 6.32. The transmitter in Figure 6.32(a) contains a LTI subsystem that is labelled with its impulse response  $g$ . Similarly, the receiver in Figure 6.32(b) contains a LTI subsystem that is labelled with its impulse response  $h$ . Let  $X, Y, Q, V, \hat{X}, C, G$ , and  $H$  denote the Fourier transforms of  $x, y, q, v, \hat{x}, c, g$ , and  $h$ , respectively.

The transmitter is a system with input  $x$  and output  $y$  that is characterized by the equations

$$\begin{aligned}
 q(t) &= c(t)x(t) \quad \text{and} \\
 y(t) &= q * g(t),
 \end{aligned}$$

where

$$\begin{aligned}
 c(t) &= \cos \omega_c t \quad \text{and} \\
 g(t) &= \delta(t) - \frac{\omega_c}{\pi} \operatorname{sinc} \omega_c t.
 \end{aligned}$$

Taking the Fourier transform of  $g$ , we obtain

$$G(\omega) = \begin{cases} 1 & |\omega| \geq \omega_c \\ 0 & \text{otherwise,} \end{cases} \quad \text{and}$$

The receiver is a system with input  $y$  and output  $\hat{x}$  that is characterized by the equations

$$\begin{aligned}
 v(t) &= c(t)y(t) \quad \text{and} \\
 \hat{x}(t) &= v * h(t)
 \end{aligned}$$

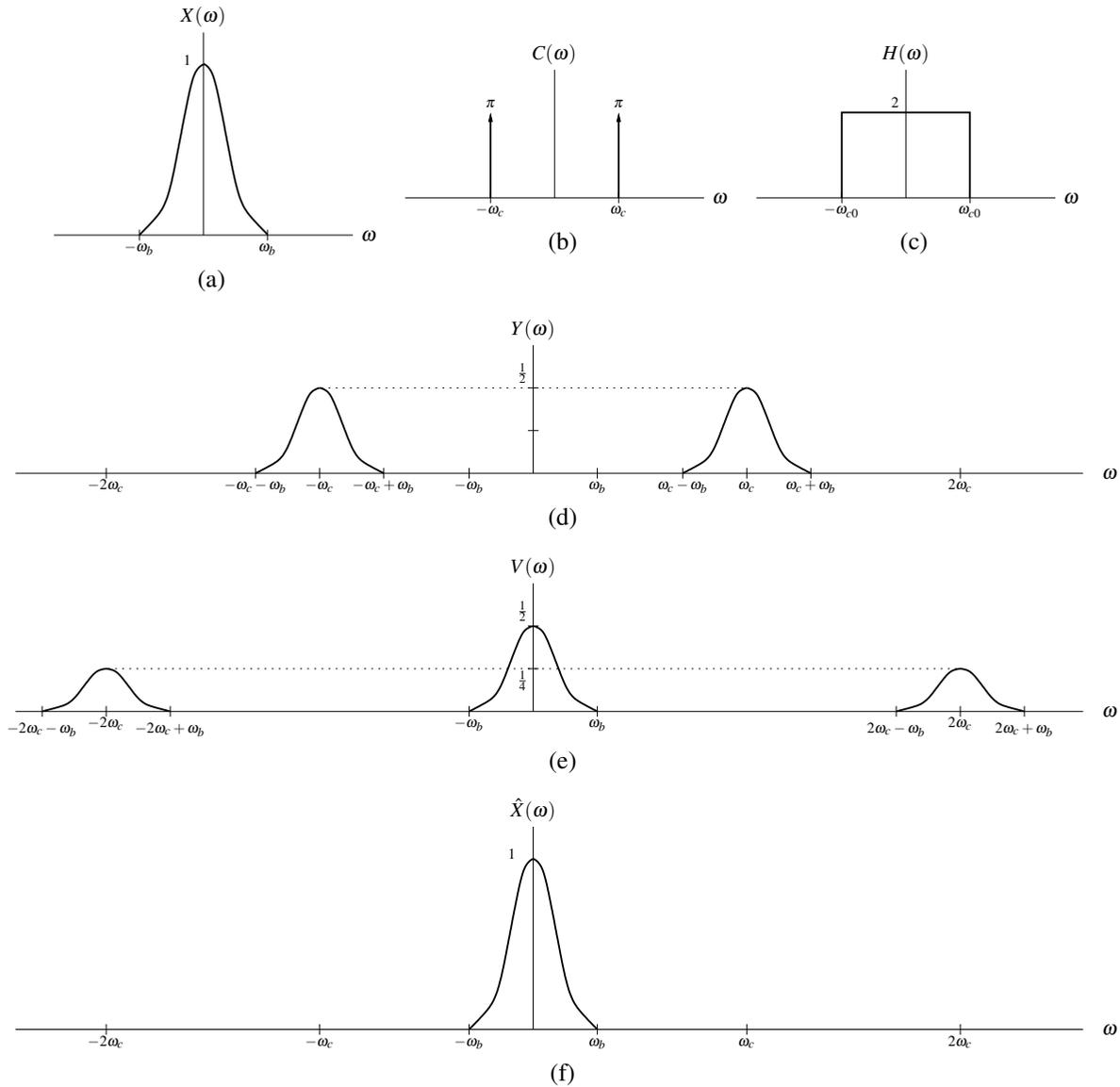


Figure 6.31: Signal spectra for DSB/SC amplitude modulation. (a) Spectrum of the transmitter input. (b) Spectrum of the sinusoidal function used in the transmitter and receiver. (c) Frequency response of the filter in the receiver. (d) Spectrum of the transmitted signal. (e) Spectrum of the multiplier output in the receiver. (f) Spectrum of the receiver output.

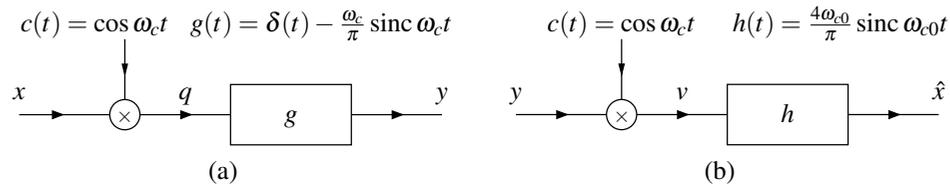


Figure 6.32: SSB/SC amplitude modulation system. (a) Transmitter and (b) receiver.

where  $c$  is as defined earlier and

$$h(t) = \frac{4\omega_{c0}}{\pi} \text{sinc } \omega_{c0}t.$$

Taking the Fourier transform of  $h$ , we obtain

$$H(\omega) = \begin{cases} 4 & |\omega| \leq \omega_{c0} \\ 0 & \text{otherwise.} \end{cases}$$

Figure 6.33 depicts the transformations that the signal undergoes as it passes through the system. Again, the output from the receiver is equal to the input to the transmitter. A detailed analysis of this communication system is left as an exercise for the reader.

## 6.21 Sampling and Interpolation

Often, we encounter situations in which we would like to process a continuous-time signal in the discrete-time domain or vice versa. For example, we might have a continuous-time audio signal that we would like to process using a digital computer (which is a discrete-time system), or we might have a discrete-time audio signal that we wish to play on a loudspeaker (which is a continuous-time system). Clearly, some means is needed to link the continuous- and discrete-time domains. This connection is established through processes known as sampling and interpolation. In what follows, we will formally introduce these processes and study them in some detail.

Sampling allows us to create sequence (i.e., a discrete-time signal) from a function (i.e., a continuous-time signal). Although sampling can be performed in many different ways, the most commonly used scheme is **periodic sampling**. With this scheme, a sequence  $y$  of samples is obtained from a function  $x$  as given by

$$y(n) = x(Tn) \quad \text{for all integer } n, \quad (6.50)$$

where  $T$  is a positive real constant. As a matter of terminology,  $T$  is referred to as the **sampling period**, and  $\omega_s = \frac{2\pi}{T}$  is referred to as the (angular) **sampling frequency**. A system such as that described by (6.50) is known as an **ideal continuous-to-discrete-time (C/D) converter**, and is shown diagrammatically in Figure 6.34. An example of periodic sampling is shown in Figure 6.35. Figure 6.35(a) shows a function  $x$  to be sampled, and Figure 6.35(b) shows the sequence  $y$  obtained by sampling  $x$  with the sampling period  $T = 10$ .

Interpolation allows us to construct a function (i.e., a continuous-time signal) from a sequence (i.e., a discrete-time signal). In effect, for a given sequence, this process constructs a function that would produce the given sequence when sampled, typically with some additional constraints imposed. More formally, for a given sequence  $y$  associated with a sampling period  $T$ , interpolation produces a function  $\hat{x}$  as given by

$$\hat{x} = \mathcal{H}y,$$

subject to the constraint that

$$\hat{x}(Tn) = y(n) \quad \text{for all integer } n,$$

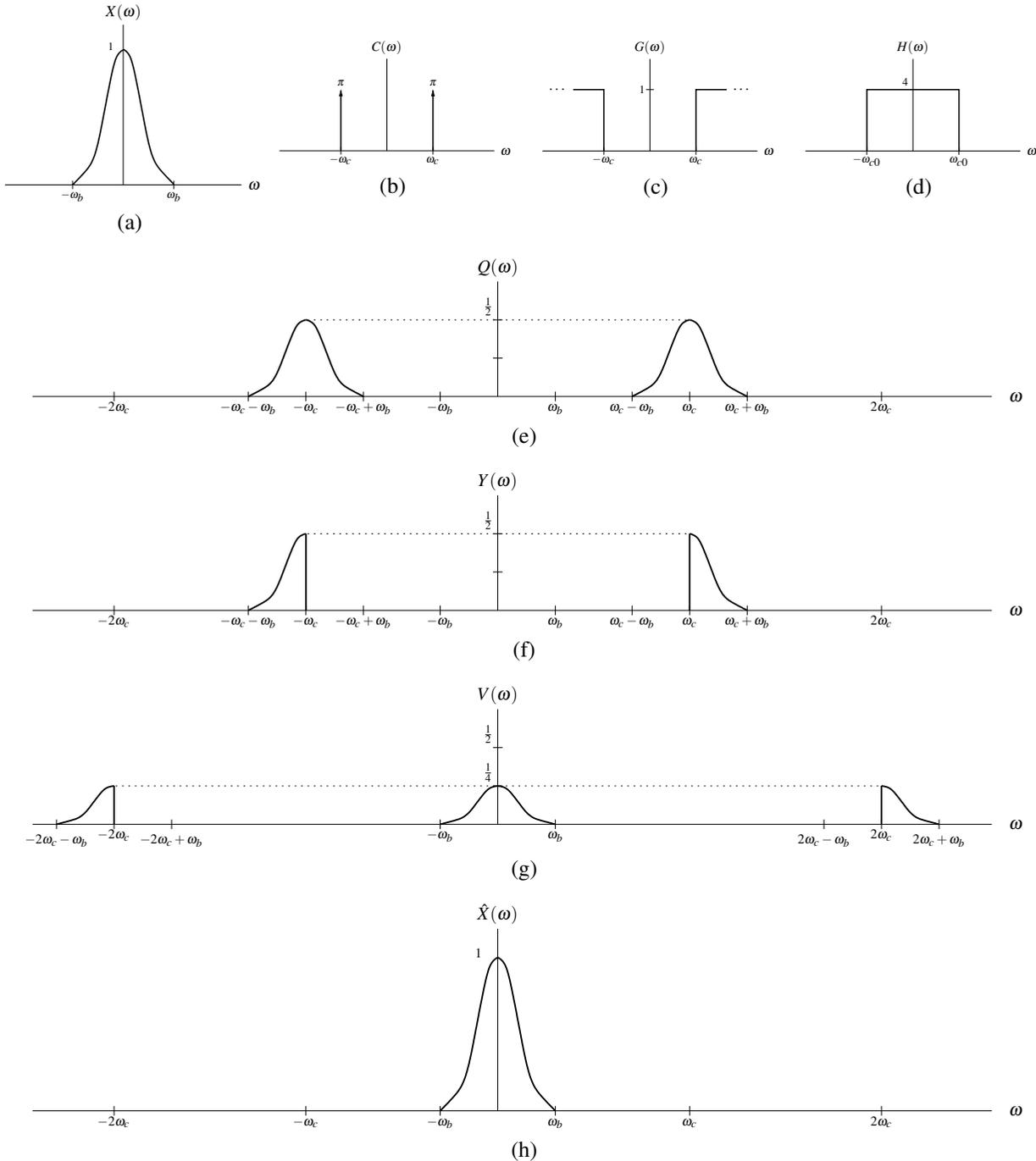
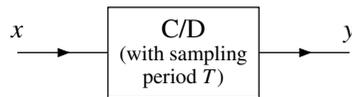
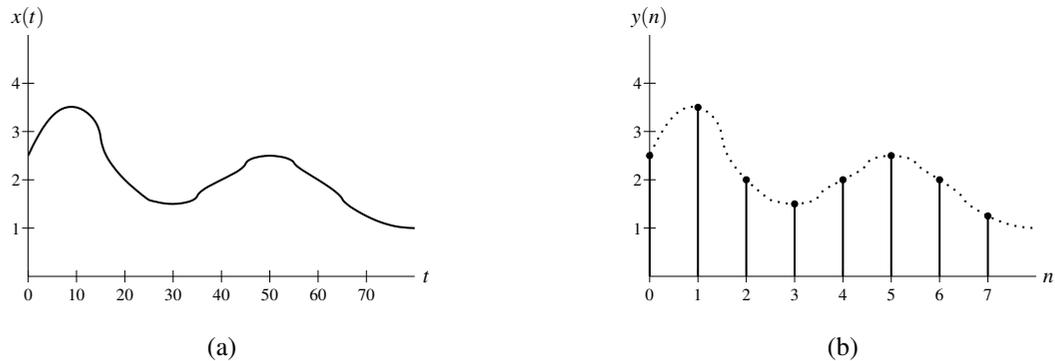
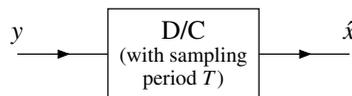


Figure 6.33: Signal spectra for SSB/SC amplitude modulation. (a) Spectrum of the transmitter input. (b) Spectrum of the sinusoid in the transmitter and receiver. (c) Frequency response of the filter in the transmitter. (d) Frequency response of the filter in the receiver. (e) Spectrum of the multiplier output in the transmitter. (f) Spectrum of the transmitted signal. (g) Spectrum of the multiplier output in the receiver. (h) Spectrum of the receiver output.

Figure 6.34: Ideal C/D converter with input function  $x$  and output sequence  $y$ .Figure 6.35: Example of periodic sampling. (a) The function  $x$  to be sampled and (b) the sequence  $y$  produced by sampling  $x$  with a sampling period of 10.Figure 6.36: Ideal D/C converter with input sequence  $y$  and output function  $\hat{x}$ .

where  $\mathcal{H}$  is an operator that maps a sequence to a function. The precise form of  $\mathcal{H}$  depends on the particular interpolation scheme employed. Although there are many different ways in which to perform interpolation, we will focus our attention in subsequent sections on one particular scheme known as bandlimited interpolation. The interpolation process is performed by a system known as an **ideal discrete-to-continuous-time (D/C) converter**, as shown in Figure 6.36.

In the absence of any constraints on the sampling process, a function cannot be uniquely determined from a sequence of its equally-spaced samples. In other words, the sampling process is not generally invertible. Consider, for example, the functions  $x_1$  and  $x_2$  given by

$$x_1(t) = 0 \quad \text{and} \quad x_2(t) = \sin(2\pi t).$$

If we sample each of these functions with the sampling period  $T = 1$ , we obtain the respective sequences

$$y_1(n) = x_1(Tn) = x_1(n) = 0 \quad \text{and} \quad y_2(n) = x_2(Tn) = \sin(2\pi n) = 0.$$

Thus,  $y_1 = y_2$  in spite of the fact that  $x_1 \neq x_2$ . This example trivially shows that if no constraints are placed upon a function, then the function cannot be uniquely determined from its samples.

Fortunately, under certain circumstances, a function can be recovered exactly from its samples. In particular, in the case that the function being sampled is bandlimited, we can show that a sequence of its equally-spaced samples uniquely determines the function if the sampling period is sufficiently small. This result, known as the sampling theorem, is of paramount importance.

### 6.21.1 Sampling

In order to gain some insight into sampling, we need a way in which to mathematically model this process. To this end, we employ the simple model for the ideal C/D converter shown in Figure 6.37. In short, we may view the process

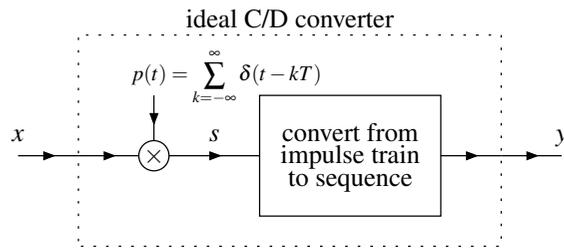


Figure 6.37: Model of ideal C/D converter with input function  $x$  and output sequence  $y$ .

of sampling as impulse train modulation followed by conversion of an impulse train to a sequence of sample values. More specifically, to sample a function  $x$  with sampling period  $T$ , we first multiply  $x$  by the periodic impulse train  $p$  to obtain

$$s(t) = x(t)p(t), \quad (6.51)$$

where

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

Then, we take the weights of successive impulses in  $s$  to form a sequence  $y$  of samples. The sampling frequency is given by  $\omega_s = \frac{2\pi}{T}$ . As a matter of terminology,  $p$  is referred to as a sampling function. From the diagram, we can see that the signals  $s$  and  $y$ , although very closely related, have some key differences. The impulse train  $s$  is a function (i.e., continuous-time signal) that is zero everywhere except at integer multiples of  $T$  (i.e., at sample points), while  $y$  is a sequence (i.e., discrete-time signal), defined only on the integers with its values corresponding to the weights of successive impulses in  $s$ . The various signals involved in sampling are illustrated in Figure 6.38.

In passing, we note that the above model of sampling is only a mathematical convenience. That is, the model provides us with a relatively simple way in which to study the mathematical behavior of sampling. The above model, however, is not directly useful as a means for actually realizing sampling in a real-world system. Obviously, the impulse train employed in the above model poses some insurmountable problems as far as implementation is concerned.

Now, let us consider the above model of sampling in more detail. In particular, we would like to find the relationship between the frequency spectra of the original function  $x$  and its impulse-train sampled version  $s$ . In what follows, let  $X$ ,  $Y$ ,  $P$ , and  $S$  denote the Fourier transforms of  $x$ ,  $y$ ,  $p$ , and  $s$ , respectively. Since  $p$  is  $T$ -periodic, it can be represented in terms of a Fourier series as

$$p(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}. \quad (6.52)$$

Using the Fourier series analysis equation, we calculate the coefficients  $c_k$  to be

$$\begin{aligned} c_k &= \frac{1}{T} \int_{-T/2}^{T/2} p(t) e^{-jk\omega_s t} dt \\ &= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_s t} dt \\ &= \frac{1}{T} \\ &= \frac{\omega_s}{2\pi}. \end{aligned} \quad (6.53)$$

Substituting (6.52) and (6.53) into (6.51), we obtain

$$\begin{aligned} s(t) &= x(t) \sum_{k=-\infty}^{\infty} \frac{\omega_s}{2\pi} e^{jk\omega_s t} \\ &= \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} x(t) e^{jk\omega_s t}. \end{aligned}$$

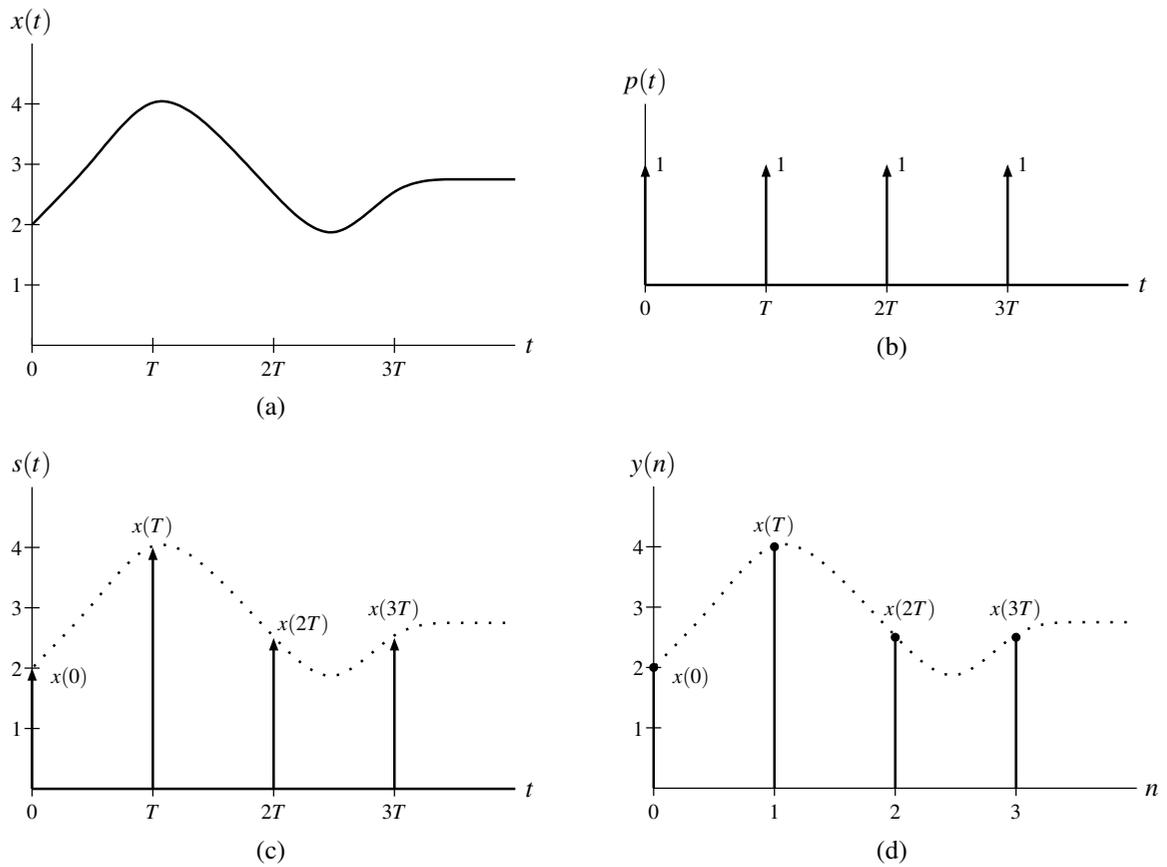


Figure 6.38: An example of the various signals involved in the sampling process for a sampling period of  $T$ . (a) The function  $x$  to be sampled. (b) The sampling function  $p$ . (c) The impulse-modulated function  $s$ . (d) The sequence  $y$  produced by sampling.

Taking the Fourier transform of  $s$  yields

$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s). \quad (6.54)$$

Thus, the spectrum of the impulse-train sampled function  $s$  is a scaled sum of an infinite number of shifted copies of the spectrum of the original function  $x$ .

Now, we consider a simple example to further illustrate the behavior of the sampling process in the frequency domain. Suppose that we have a function  $x$  with the Fourier transform  $X$  where  $|X(\omega)| = 0$  for  $|\omega| > \omega_m$  (i.e.,  $x$  is bandlimited). To simplify the visualization process, we will assume  $X$  has the particular form shown in Figure 6.39(a). In what follows, however, we only actually rely on the bandlimited nature of  $x$  and not its specific definition. So, the results that we derive in what follows generally apply to any bandlimited function. From (6.54), we know that  $S$  is formed by the superposition of an infinite number of shifted copies of  $X$ . Upon more careful consideration, we can see that two distinct situations can arise. That is, the nonzero portions of the shifted copies of  $X$  used to form  $S$  can either: 1) overlap or, 2) not overlap. These two cases are illustrated in Figures 6.39(b) and 6.39(c), respectively. From these graphs, we can see that the nonzero portions of the shifted copies of  $X$  will not overlap if

$$\omega_m < \omega_s - \omega_m \quad \text{and} \quad -\omega_m > -\omega_s + \omega_m$$

or equivalently

$$\omega_s > 2\omega_m.$$

Consider the case in which the copies of the original spectrum  $X$  in  $S$  do not overlap, as depicted in Figure 6.39(b). In this situation, the spectrum  $X$  of the original function is clearly discernible in the spectrum  $S$ . In fact, one can see that the original spectrum  $X$  can be obtained directly from  $S$  through a lowpass filtering operation. Thus, the original function  $x$  can be exactly recovered from  $s$ .

Now, consider the case in which copies of the original spectrum  $X$  in  $S$  do overlap. In this situation, multiple frequencies in the spectrum  $X$  of the original function are mapped to the same frequency in  $S$ . This phenomenon is referred to as **aliasing**. Clearly, aliasing leads to individual periods of  $S$  having a different shape than the original spectrum  $X$ . When aliasing occurs, the shape of the original spectrum  $X$  is no longer discernible from  $S$ . Consequently, we are unable to recover the original function  $x$  from  $s$  in this case.

## 6.21.2 Interpolation and Reconstruction of a Function From Its Samples

Interpolation allows us to construct a function (i.e., continuous-time signal) from a sequence (i.e., discrete-time signal). This process is essentially responsible for determining the value of a function between sample points. Except in very special circumstances, it is not generally possible to exactly reproduce a function from its samples. Although many interpolation schemes exist, we shall focus our attention shortly on one particular scheme. The interpolation process can be modeled with the simple ideal D/C converter system, shown in Figure 6.40.

Recall the ideal C/D converter of Figure 6.37. Since the process of converting an impulse train to a sequence is invertible, we can reconstruct the original function  $x$  from a sequence  $y$  of its samples if we can somehow recover  $x$  from  $s$ . Let us suppose now that  $x$  is bandlimited. As we saw in the previous section, we can recover  $x$  from  $s$  provided that  $x$  is bandlimited and sampled at a sufficiently high rate so as to avoid aliasing. In the case that aliasing does not occur, we can reconstruct the original function  $x$  from  $y$  using the ideal D/C converter shown in Figure 6.40. In what follows, we will derive a formula for computing the original function  $\hat{x}$  from its samples  $y$ . Consider the model of the D/C converter. We have a lowpass filter with impulse response

$$h(t) = \text{sinc} \frac{\pi t}{T} = \text{sinc} \frac{\omega_s t}{2}$$

and frequency response

$$H(\omega) = T \text{rect} \frac{T\omega}{2\pi} = \frac{2\pi}{\omega_s} \text{rect} \frac{\omega}{\omega_s} = \begin{cases} T & |\omega| < \frac{\omega_s}{2} \\ 0 & \text{otherwise.} \end{cases}$$

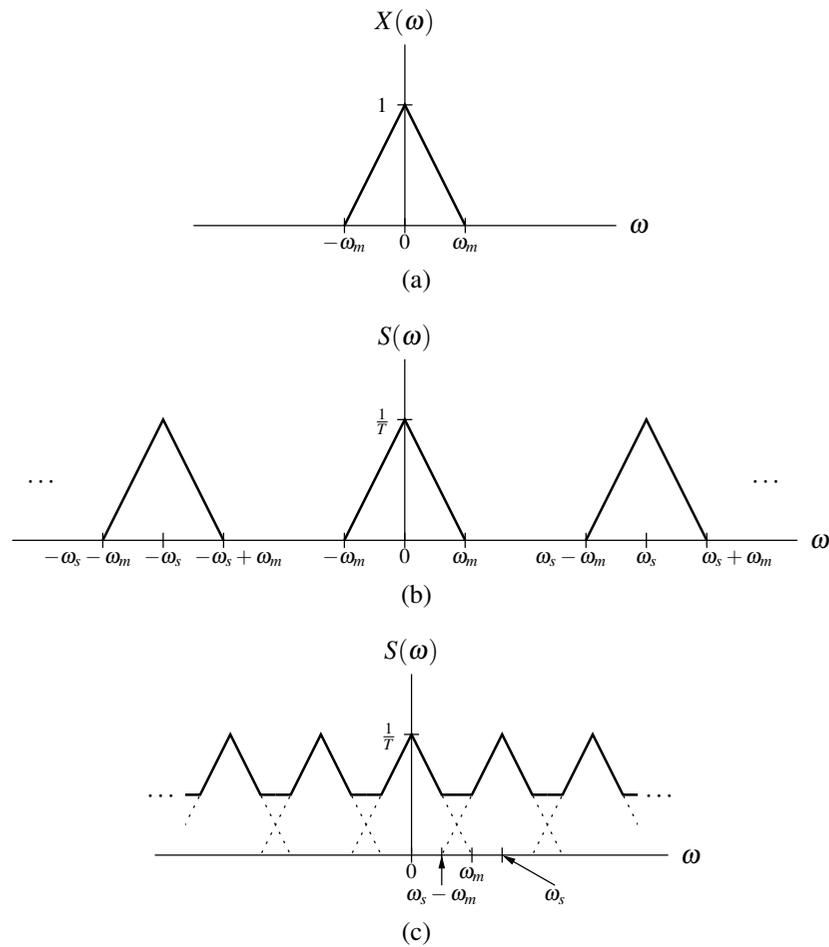


Figure 6.39: Effect of impulse-train sampling on the frequency spectrum. (a) Spectrum of the function  $x$  being sampled. (b) Spectrum of  $s$  in the absence of aliasing. (c) Spectrum of  $s$  in the presence of aliasing.

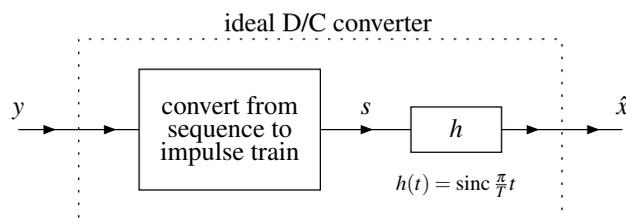


Figure 6.40: Model of ideal D/C converter with input sequence  $y$  and output function  $\hat{x}$ .

First, we convert the sequence  $y$  to the impulse train  $s$  to obtain

$$s(t) = \sum_{n=-\infty}^{\infty} y(n)\delta(t - Tn).$$

Then, we filter the resulting function  $s$  with the lowpass filter having impulse response  $h$ , yielding

$$\begin{aligned} \hat{x}(t) &= s * h(t) \\ &= \int_{-\infty}^{\infty} s(\tau)h(t - \tau)d\tau \\ &= \int_{-\infty}^{\infty} h(t - \tau) \sum_{n=-\infty}^{\infty} y(n)\delta(\tau - Tn)d\tau \\ &= \sum_{n=-\infty}^{\infty} y(n) \int_{-\infty}^{\infty} h(t - \tau)\delta(\tau - Tn)d\tau \\ &= \sum_{n=-\infty}^{\infty} y(n)h(t - Tn) \\ &= \sum_{n=-\infty}^{\infty} y(n) \operatorname{sinc} \left[ \frac{\pi}{T}(t - Tn) \right]. \end{aligned}$$

If  $x$  is bandlimited and aliasing is avoided,  $\hat{x} = x$  and we have a formula for exactly reproducing  $x$  from its samples  $y$ .

### 6.21.3 Sampling Theorem

In the preceding sections, we have established the important result given by the theorem below.

**Theorem 6.19** (Sampling Theorem). *Let  $x$  be a function with the Fourier transform  $X$ , and let  $y$  be the sequence resulting from the periodic sampling of  $x$  with the sampling period  $T$  (i.e.,  $y(n) = x(Tn)$ ). Suppose that  $|X(\omega)| = 0$  for all  $|\omega| > \omega_M$  (i.e.,  $x$  is bandlimited to the interval  $[-\omega_M, \omega_M]$ ). Then,  $x$  is uniquely determined by  $y$  if*

$$\omega_s > 2\omega_M, \tag{6.55}$$

where  $\omega_s = \frac{2\pi}{T}$ . In particular, if (6.55) is satisfied, we have that

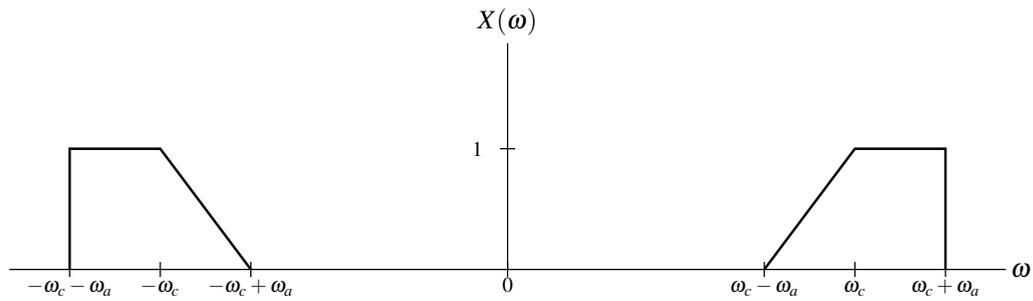
$$x(t) = \sum_{n=-\infty}^{\infty} y(n) \operatorname{sinc} \left[ \frac{\pi}{T}(t - Tn) \right],$$

or equivalently (i.e., rewritten in terms of  $\omega_s$  instead of  $T$ ),

$$x(t) = \sum_{n=-\infty}^{\infty} y(n) \operatorname{sinc} \left( \frac{\omega_s}{2}t - \pi n \right).$$

As a matter of terminology, we refer to (6.55) as the **Nyquist condition** (or Nyquist criterion). Also, we call  $\frac{\omega_s}{2}$  the **Nyquist frequency** and  $2\omega_M$  the **Nyquist rate**. It is important to note that the Nyquist condition is a strict inequality. Therefore, to ensure aliasing does not occur in the most general case, one must choose the sampling rate larger than the Nyquist rate. One can show, however, that if the frequency spectrum does not have impulses at the Nyquist frequency, it is sufficient to sample at exactly the Nyquist rate.

**Example 6.41.** Let  $x$  denote a continuous-time audio signal with Fourier transform  $X$ . Suppose that  $|X(\omega)| = 0$  for all  $|\omega| \geq 44100\pi$ . Determine the largest period  $T$  with which  $x$  can be sampled that will allow  $x$  to be exactly recovered from its samples.

Figure 6.41: Frequency spectrum of the function  $x$ .

*Solution.* The function  $x$  is bandlimited to frequencies in the range  $(-\omega_m, \omega_m)$ , where  $\omega_m = 44100\pi$ . From the sampling theorem, we know that the minimum sampling rate required is given by

$$\begin{aligned}\omega_s &= 2\omega_m \\ &= 2(44100\pi) \\ &= 88200\pi.\end{aligned}$$

Thus, the largest permissible sampling period is given by

$$\begin{aligned}T &= \frac{2\pi}{\omega_s} \\ &= \frac{2\pi}{88200\pi} \\ &= \frac{1}{44100}.\end{aligned}$$

Although the sampling theorem provides an upper bound on the sampling rate that holds in the case of arbitrary bandlimited functions, in some special cases it may be possible to employ an even smaller sampling rate. This point is further illustrated by way of the example below.

**Example 6.42.** Consider the function  $x$  with the Fourier transform  $X$  shown in Figure 6.41 (where  $\omega_c \gg \omega_a$ ). (a) Using the sampling theorem directly, determine the largest permissible sampling period  $T$  that will allow  $x$  to be exactly reconstructed from its samples. (b) Explain how one can exploit the fact that  $X(\omega) = 0$  for a large portion of the interval  $[-\omega_c - \omega_a, \omega_c + \omega_a]$  in order to reduce the rate at which  $x$  must be sampled.

*Solution.* (a) The function  $x$  is bandlimited to  $[-\omega_m, \omega_m]$ , where  $\omega_m = \omega_c + \omega_a$ . Thus, the minimum sampling rate required is given by

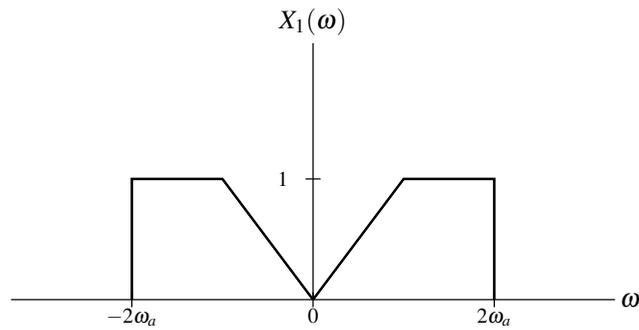
$$\begin{aligned}\omega_s &> 2\omega_m \\ &= 2(\omega_c + \omega_a) \\ &= 2\omega_c + 2\omega_a.\end{aligned}$$

and the maximum sampling period is calculated as

$$\begin{aligned}T &< \frac{2\pi}{\omega_s} \\ &= \frac{2\pi}{2\omega_c + 2\omega_a} \\ &= \frac{\pi}{\omega_c + \omega_a}.\end{aligned}$$

(b) We can modulate and lowpass filter  $x$  in order to compress all of its spectral information into the frequency range  $[-2\omega_a, 2\omega_a]$ , yielding the function  $x_1$ . That is, we have

$$x_1(t) = \{x(t) \cos[(\omega_c - \omega_a)t]\} * h(t)$$

Figure 6.42: Frequency spectrum of the function  $x_1$ .

where

$$h(t) = \frac{4\omega_a}{\pi} \operatorname{sinc}(2\omega_a t) \xleftrightarrow{\text{CTFT}} H(\omega) = 2 \operatorname{rect}\left(\frac{\omega}{4\omega_a}\right).$$

This process can be inverted (by modulation and filtering) to obtain  $x$  from  $x_1$ . In particular, we have that

$$x(t) = \{x_1(t) \cos[(\omega_c - \omega_a)t]\} * h_2(t)$$

where

$$h_2(t) = \delta(t) - \frac{2(\omega_c - \omega_a)}{\pi} \operatorname{sinc}[(\omega_c - \omega_a)t] \xleftrightarrow{\text{CTFT}} H_2(\omega) = 2 - 2 \operatorname{rect}\left(\frac{\omega}{4(\omega_c - \omega_a)}\right).$$

Let  $X_1$  denote the Fourier transform of  $x_1$ . The Fourier transform  $X_1$  is as shown in Figure 6.42. Applying the sampling theorem to  $x_1$  we find that the minimum sampling rate is given by

$$\begin{aligned} \omega_s &> 2(2\omega_a) \\ &= 4\omega_a \end{aligned}$$

and the largest sampling period is given by

$$\begin{aligned} T &< \frac{2\pi}{\omega_s} \\ &= \frac{2\pi}{4\omega_a} \\ &= \frac{\pi}{2\omega_a}. \end{aligned}$$

Since  $\omega_c \gg \omega_a$  (by assumption), this new sampling period is larger than the one computed in the first part of this problem. ■

## 6.22 Exercises

### 6.22.1 Exercises Without Answer Key

6.1 Using the Fourier transform analysis equation, find the Fourier transform  $X$  of each function  $x$  below.

- (a)  $x(t) = A\delta(t - t_0)$ , where  $t_0$  and  $A$  are real and complex constants, respectively;
- (b)  $x(t) = \text{rect}(t - t_0)$ , where  $t_0$  is a constant;
- (c)  $x(t) = e^{-4t}u(t - 1)$ ;
- (d)  $x(t) = 3[u(t) - u(t - 2)]$ ; and
- (e)  $x(t) = e^{-|t|}$ .

6.2 Use a Fourier transform table and properties of the Fourier transform to find the Fourier transform  $X$  of each function  $x$  below.

- (a)  $x(t) = \cos(t - 5)$ ;
- (b)  $x(t) = e^{-j5t}u(t + 2)$ ;
- (c)  $x(t) = [\cos t]u(t)$ ;
- (d)  $x(t) = 6[u(t) - u(t - 3)]$ ;
- (e)  $x(t) = 1/t$ ;
- (f)  $x(t) = t \text{rect}(2t)$ ;
- (g)  $x(t) = e^{-j3t} \sin(5t - 2)$ ;
- (h)  $x(t) = \cos(5t - 2)$ ;
- (i)  $x(t) = e^{-j2t} \frac{1}{3t+1}$ ;
- (j)  $x(t) = \int_{-\infty}^{5t} e^{-\tau-1}u(\tau - 1)d\tau$ ;
- (k)  $x(t) = (t + 1) \sin(5t - 3)$ ;
- (l)  $x(t) = (\sin 2\pi t)\delta(t - \frac{\pi}{2})$ ;
- (m)  $x(t) = e^{-jt} \frac{1}{3t-2}$ ;
- (n)  $x(t) = e^{j5t}(\cos 2t)u(t)$ ; and
- (o)  $x(t) = e^{-j2t} \text{sgn}(-t - 1)$ .

6.3 Compute the Fourier transform  $X$  of the function

$$x(t) = \sum_{k=0}^{\infty} a^k \delta(t - kT),$$

where  $a$  is a constant satisfying  $|a| < 1$  and  $T$  is a strictly-positive real constant. (Hint: Recall the formula for the sum of an infinite geometric series. That is,  $b + br + br^2 + \dots = \frac{b}{1-r}$  if  $|r| < 1$ .)

6.4 The ideal Hilbert transformer is a LTI system with the frequency response

$$H(\omega) = -j \text{sgn } \omega.$$

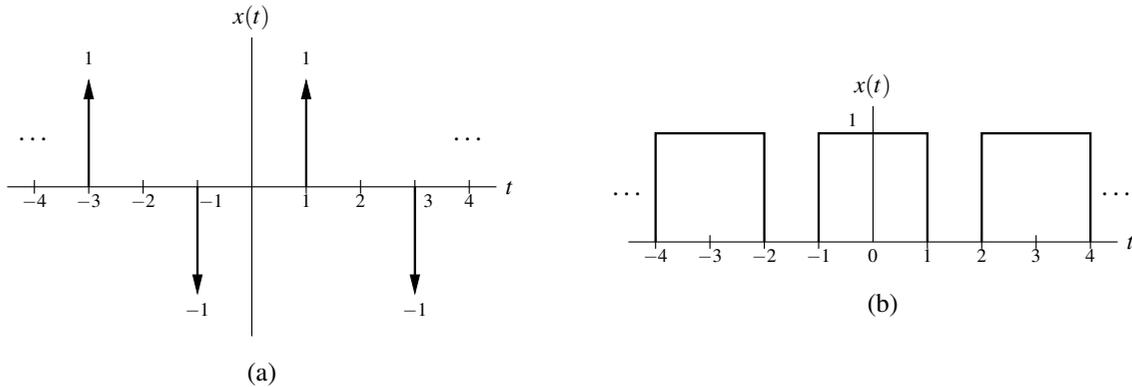
This type of system is useful in a variety of signal processing applications (e.g., SSB/SC amplitude modulation). By using the duality property of the Fourier transform, find the impulse response  $h$  of this system.

6.5 For each function  $y$  given below, find the Fourier transform  $Y$  of  $y$  (in terms of the Fourier transform  $X$  of  $x$ ).

- (a)  $y(t) = x(at - b)$ , where  $a$  and  $b$  are constants and  $a \neq 0$ ;
- (b)  $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$ ;
- (c)  $y(t) = \int_{-\infty}^t x^2(\tau)d\tau$ ;
- (d)  $y(t) = \mathcal{D}(x * x)(t)$ , where  $\mathcal{D}$  denotes the derivative operator;
- (e)  $y(t) = tx(2t - 1)$ ;

- (f)  $y(t) = e^{j2t}x(t-1)$ ;  
 (g)  $y(t) = (te^{-j5t}x(t))^*$ ;  
 (h)  $y(t) = (\mathcal{D}x) * x_1(t)$ , where  $x_1(t) = e^{-jt}x(t)$  and  $\mathcal{D}$  denotes the derivative operator;  
 (i)  $y(t) = \int_{-\infty}^{3t} x^*(\tau-1)d\tau$ ;  
 (j)  $y(t) = [\cos(3t-1)]x(t)$ ;  
 (k)  $y(t) = \left[\frac{d}{dt}x(t)\right] \sin(t-2)$ ;  
 (l)  $y(t) = tx(t) \sin 3t$ ; and  
 (m)  $y(t) = e^{j7t} [x * x(t-1)]$ .

**6.6** Find the Fourier transform  $X$  of each periodic function  $x$  shown below.



**6.7** Using the time-domain convolution property of the Fourier transform, compute the convolution  $h = h_1 * h_2$ , where

$$h_1(t) = 2000 \operatorname{sinc}(2000\pi t) \quad \text{and} \quad h_2(t) = \delta(t) - 1000 \operatorname{sinc}(1000\pi t).$$

**6.8** Compute the energy contained in the function  $x(t) = 200 \operatorname{sinc}(200\pi t)$ .

**6.9** For each function  $x$  given below, compute the frequency spectrum of  $x$ , and find and plot the corresponding magnitude and phase spectra.

- (a)  $x(t) = e^{-at}u(t)$ , where  $a$  is a positive real constant; and  
 (b)  $x(t) = \operatorname{sinc} \frac{t-1}{200}$ .

**6.10** For each differential/integral equation below that defines a LTI system with input  $x$  and output  $y$ , find the frequency response  $H$  of the system. (Note that the prime symbol denotes differentiation.)

- (a)  $y''(t) + 5y'(t) + y(t) + 3x'(t) - x(t) = 0$ ;  
 (b)  $y'(t) + 2y(t) + \int_{-\infty}^t 3y(\tau)d\tau + 5x'(t) - x(t) = 0$ ; and  
 (c)  $y''(t) + 5y'(t) + 6y(t) = x'(t) + 11x(t)$ .

**6.11** For each frequency response  $H$  given below for a LTI system with input  $x$  and output  $y$ , find the differential equation that characterizes the system.

- (a)  $H(\omega) = \frac{j\omega}{1+j\omega}$ ; and  
 (b)  $H(\omega) = \frac{j\omega + \frac{1}{2}}{-j\omega^3 - 6\omega^2 + 11j\omega + 6}$ .

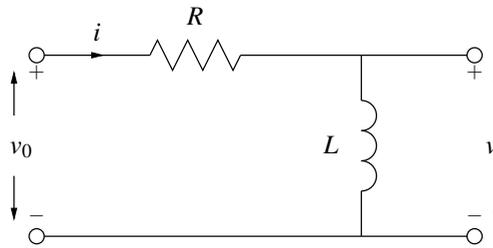
6.12 Consider the LTI system with impulse response

$$h(t) = \delta(t) - 300 \operatorname{sinc} 300\pi t.$$

Using frequency-domain methods, find the response  $y$  of the system to the input

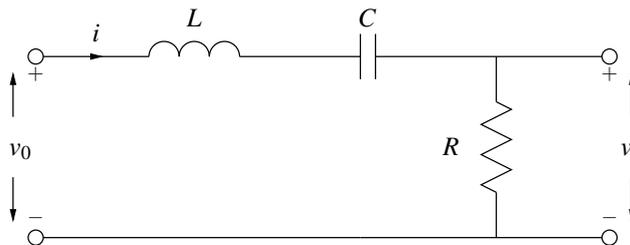
$$x(t) = \frac{1}{2} + \frac{3}{4} \cos 200\pi t + \frac{1}{2} \cos 400\pi t - \frac{1}{4} \cos 600\pi t.$$

6.13 Consider the LTI system with input  $v_0$  and output  $v_1$  as shown in the figure below, where  $R = 1$  and  $L = 1$ .



- Find the frequency response  $H$  of the system.
- Determine the magnitude and phase responses of the system.
- Find the impulse response  $h$  of the system.

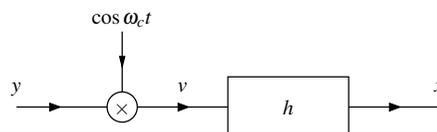
6.14 Consider the LTI system with input  $v_0$  and output  $v_1$  as shown in the figure below, where  $R = 1$ ,  $C = \frac{1}{1000}$ , and  $L = \frac{1}{1000}$ .



- Find the frequency response  $H$  of the system.
- Use a computer to plot the magnitude and phase responses of the system.
- From the plots in part (b), identify the type of ideal filter that this system approximates.

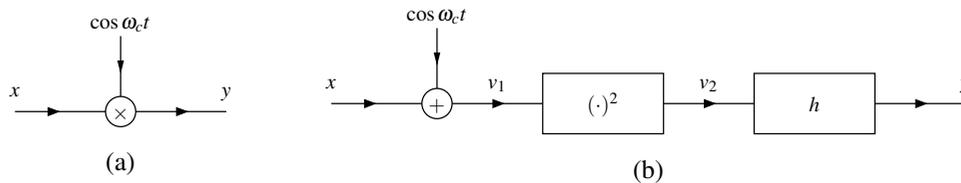
6.15 Let  $x$  be a real function with Fourier transform  $X$  satisfying  $X(\omega) = 0$  for  $|\omega| > \omega_b$ . We use amplitude modulation to produce the function  $y(t) = x(t) \sin \omega_c t$ . Note that  $\omega_c \gg \omega_b$ . In order to recover the original function  $x$ , it is proposed that the system shown in the figure below be used. This system contains a LTI subsystem that is labelled with its impulse response  $h$ . Let  $Y$ ,  $V$ ,  $\hat{X}$ , and  $H$  denote the Fourier transforms of  $y$ ,  $v$ ,  $\hat{x}$ , and  $h$ , respectively. The system is such that

$$H(\omega) = \begin{cases} 2 & |\omega| < \omega_b \\ 0 & \text{otherwise.} \end{cases}$$



- (a) Find an expression for  $Y$  in terms of  $X$ . Find an expression for  $\hat{X}$  in terms of  $V$ . Find an expression for  $\hat{X}$  in terms of  $X$ .
- (b) Compare  $\hat{x}$  and  $x$ . Comment on the utility of the proposed system.

**6.16** When discussing DSB/SC amplitude modulation, we saw that a system of the form shown below in Figure A is often useful. In practice, however, the multiplier unit needed by this system is not always easy to implement. For this reason, we sometimes employ a system like that shown below in Figure B. In this second system, we sum the sinusoidal carrier and modulating signal  $x$  and then pass the result through a nonlinear squaring device (i.e.,  $v_2(t) = v_1^2(t)$ ). This system also contains a LTI subsystem with impulse response  $h$ .



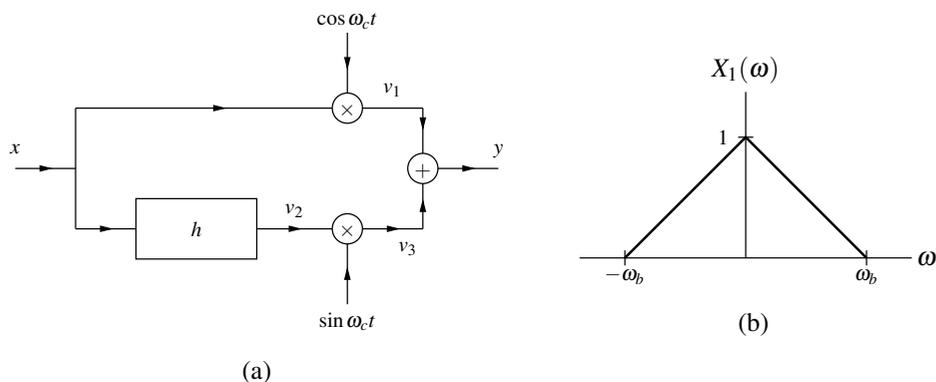
Let  $X$ ,  $V_1$ ,  $V_2$ , and  $H$  denote the Fourier transforms of  $x$ ,  $v_1$ ,  $v_2$ , and  $h$ , respectively. Suppose that  $X(\omega) = 0$  for  $|\omega| > \omega_b$  (i.e.,  $x$  is bandlimited).

- (a) Find an expression for  $v_1$ ,  $v_2$ , and  $V_2$ . (Hint: If  $X(\omega) = 0$  for  $|\omega| > \omega_b$ , then using the time-domain convolution property of the Fourier transform, we can deduce that the Fourier transform of  $x^2$  is zero for  $|\omega| > 2\omega_b$ .)
- (b) Determine the frequency response  $H$  required for the system shown in Figure B to be equivalent to the system in Figure A. State any assumptions made with regard to the relationship between  $\omega_c$  and  $\omega_b$ . (Hint: It might be helpful to sketch  $X$  and  $V_2$  for the case of some simple  $X$ . Then, compare  $V_2$  to  $X$  in order to deduce your answer.)

**6.17** Consider the system with input  $x$  and output  $y$  as shown in Figure A below. The impulse response  $h$  is that of an ideal Hilbert transformer, whose frequency response  $H$  is given by

$$H(\omega) = -j \operatorname{sgn} \omega.$$

Let  $X$ ,  $Y$ ,  $V_1$ ,  $V_2$ , and  $V_3$  denote the Fourier transforms of  $x$ ,  $y$ ,  $v_1$ ,  $v_2$ , and  $v_3$ , respectively.

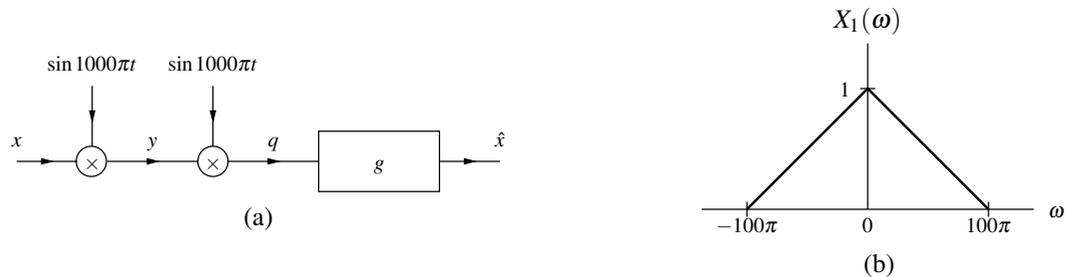


- (a) Suppose that  $X(\omega) = 0$  for  $|\omega| > \omega_b$ , where  $\omega_b \ll \omega_c$ . Find expressions for  $V_1$ ,  $V_2$ ,  $V_3$ , and  $Y$  in terms of  $X$ .
- (b) Suppose that  $X = X_1$ , where  $X_1$  is as shown in Figure B. Sketch  $V_1$ ,  $V_2$ ,  $V_3$ , and  $Y$  in this case.
- (c) Draw the block diagram of a system that could be used to recover  $x$  from  $y$ .

- 6.18 Consider the system shown below in Figure A with input  $x$  and output  $\hat{x}$ , where this system contains a LTI subsystem with impulse response  $g$ . The Fourier transform  $G$  of  $g$  is given by

$$G(\omega) = \begin{cases} 2 & |\omega| \leq 100\pi \\ 0 & \text{otherwise.} \end{cases}$$

Let  $X, \hat{X}, Y,$  and  $Q$  denote the Fourier transforms of  $x, \hat{x}, y,$  and  $q$ , respectively.

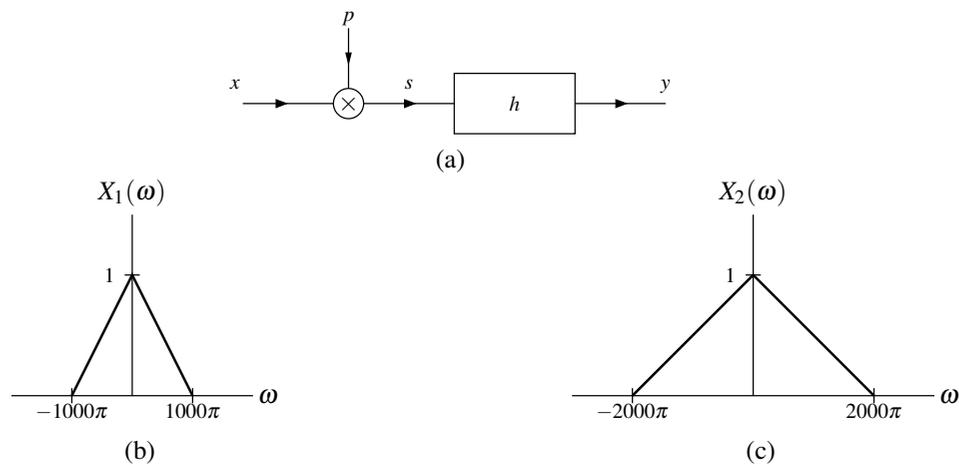


- (a) Suppose that  $X(\omega) = 0$  for  $|\omega| > 100\pi$ . Find expressions for  $Y, Q,$  and  $\hat{X}$  in terms of  $X$ .  
 (b) If  $X = X_1$  where  $X_1$  is as shown in Figure B, sketch  $Y, Q,$  and  $\hat{X}$ .

- 6.19 Consider the system shown below in Figure A with input  $x$  and output  $y$ . Let  $X, P, S, H,$  and  $Y$  denote the Fourier transforms of  $x, p, s, h,$  and  $y$ , respectively. Suppose that

$$p(t) = \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{1000}\right) \quad \text{and} \quad H(\omega) = \frac{1}{1000} \text{rect}\left(\frac{\omega}{2000\pi}\right).$$

- (a) Derive an expression for  $S$  in terms of  $X$ . Derive an expression for  $Y$  in terms of  $S$  and  $H$ .  
 (b) Suppose that  $X = X_1$ , where  $X_1$  is as shown in Figure B. Using the results of part (a), plot  $S$  and  $Y$ . Indicate the relationship (if any) between the input  $x$  and output  $y$  of the system.  
 (c) Suppose that  $X = X_2$ , where  $X_2$  is as shown in Figure C. Using the results of part (a), plot  $S$  and  $Y$ . Indicate the relationship (if any) between the input  $x$  and output  $y$  of the system.



- 6.20 Let  $X$  denote the Fourier transform of  $x$ . Show that:  
 (a) if  $x$  is even, then  $X(\omega) = 2 \int_0^{\infty} x(t) \cos \omega t dt$ ; and  
 (b) if  $x$  is odd, then  $X(\omega) = -2j \int_0^{\infty} x(t) \sin \omega t dt$ .

- 6.21** Let  $x$  be a real function with even part  $x_e$  and odd part  $x_o$ . Let  $X$ ,  $X_e$ , and  $X_o$  denote the Fourier transforms of  $x$ ,  $x_e$ , and  $x_o$ , respectively. Show that:
- $X_e(\omega) = \text{Re}[X(\omega)]$ ; and
  - $X_o(\omega) = j \text{Im}[X(\omega)]$ .

- 6.22** Let  $y(t) = x(t+a) + x(t-a)$ , where  $a$  is a real constant, and let  $X$  and  $Y$  denote the Fourier transforms of  $x$  and  $y$ , respectively. Show that  $Y(\omega) = 2X(\omega) \cos a\omega$ .

- 6.23** Show that, if a function  $x$  has bandwidth  $B$ , then  $x^n(t)$  has bandwidth  $nB$ .

- 6.24** Suppose that we want to transmit a binary sequence  $y$  (where  $y(n)$  is either zero or one) over a continuous-time communication channel. To do this, we choose to represent each binary sample of  $y$  with a continuous-time pulse. Using the sampling theorem, show that it is possible to transmit the bits of  $y$  at a rate of  $2B$  bits per second over an ideal (i.e., noise-free) channel of bandwidth  $B$ . As it turns out, this is the theoretical upper bound on the data transmission rate, assuming that each pulse is used to represent only a single bit of data. (Hint: According to the sampling theorem, a continuous-time function of bandwidth  $B$  can be constructed from  $2B$  pieces of information.)

- 6.25** Show that a function cannot be both timelimited and bandlimited. [Hint: Let  $X$  denote the Fourier transform of  $x$ . Suppose that  $x$  is timelimited and also bandlimited such that  $X(\omega) = 0$  for  $|\omega| \geq B$ . Due to the bandlimited nature of  $x$ , we have that  $X(\omega) = X(\omega) \text{rect}\left(\frac{\omega}{2B'}\right)$  for  $B' > B$ . Then, show that the inverse Fourier transform of the preceding equation leads to a contradiction. To do this, you will need to observe that the convolution of a timelimited function with a non-timelimited function must be a non-timelimited function.]

- 6.26** The bandwidth of a LTI system is most simply defined as the bandwidth of the system's frequency response  $H$ . Explain why a (LTI) communication channel with (finite) bandwidth  $B$  cannot be used to (reliably) transmit a signal with bandwidth greater than  $B$ .

- 6.27** Let  $y_1$  and  $y_2$  be functions bandlimited to frequencies in the range  $[-\omega_b, \omega_b]$ . Suppose that these functions are sampled at a frequency  $\omega_s$  satisfying the Nyquist condition to produce the sequences

$$x_1(n) = y_1(Tn) \quad \text{and} \quad x_2(n) = y_2(Tn),$$

where  $T = \frac{2\pi}{\omega_s}$ . Now, consider the function  $y = y_1 * y_2$ . Suppose that  $y$  is also sampled with period  $T$  to produce the sequence

$$x(n) = y(Tn).$$

- Show that  $y$  is bandlimited to frequencies in the range  $[-\omega_b, \omega_b]$ , meaning that it must be possible to recover  $y$  exactly from its samples.
- Show that the samples of  $y$  can be computed by

$$x(n) = \frac{2\pi}{\omega_s} \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k).$$

- Explain how we might use the above results to compute the (continuous-time) convolution of bandlimited functions using a (discrete-time) computer.

- 6.28** A function  $x$  is bandlimited to 22 kHz (i.e., only has spectral content for frequencies  $f$  in the range  $[-22000, 22000]$ ). Due to excessive noise, the portion of the spectrum that corresponds to frequencies  $f$  satisfying  $|f| > 20000$  has been badly corrupted and rendered useless. (a) Determine the minimum sampling rate for  $x$  that would allow the uncorrupted part of the spectrum to be recovered. (b) Suppose now that the corrupted part of the spectrum were eliminated by filtering prior to sampling. In this case, determine the minimum sampling rate for  $x$ .
- 6.29** A function  $x$  is bandlimited for frequencies in the range  $[-B, B]$ . Find the lowest rate at which the function  $y(t) = x^2(t)$  can be sampled such that aliasing does not occur.

### 6.22.2 Exercises With Answer Key

- 6.30** Using the Fourier transform analysis equation, find the Fourier transform  $X$  of the function

$$x(t) = \text{rect} \frac{t}{T},$$

where  $T$  is a nonzero real constant. (Hint: Be careful to correctly consider the case that  $T < 0$ .)

**Short Answer.**  $X(\omega) = |T| \text{sinc} \frac{T\omega}{2}$

- 6.31** Using the Fourier transform synthesis equation, find the inverse Fourier transform  $x$  of the function

$$X(\omega) = \text{rect} \frac{\omega}{2B},$$

where  $B$  is a nonzero real constant. (Hint: Be careful to correctly consider the case that  $B < 0$ .)

**Short Answer.**  $x(t) = \frac{|B|}{\pi} \text{sinc} Bt$

- 6.32** Find the Fourier transform  $X$  of each function  $x$  given below.

(a)  $x(t) = \frac{1}{2} \left[ \delta(t) + \frac{j}{\pi t} \right]$ ; and

(b)  $x(t) = \delta(t+a) + \delta(t-a)$ , where  $a$  is a real constant.

**Short Answer.** (a)  $X(\omega) = u(\omega)$ ; (b)  $X(\omega) = 2 \cos a\omega$ .

- 6.33** Using properties of the Fourier transform, the Fourier transform pair  $\text{rect} \frac{t}{T} \xleftrightarrow{\text{CTFT}} |T| \text{sinc} \frac{T\omega}{2}$ , and the fact that  $\text{tri} \frac{t}{2} = \text{rect} * \text{rect}(t)$ , find the Fourier transform  $X$  of the function  $x(t) = \text{tri} \frac{t}{T}$ , where  $T$  is a nonzero real constant.

**Short Answer.**  $\frac{|T|}{2} \text{sinc}^2 \frac{T\omega}{4}$

- 6.34** Given that  $e^{-a|t|} \xleftrightarrow{\text{CTFT}} \frac{2a}{\omega^2 + a^2}$ , find the Fourier transform  $X$  of the function  $x(t) = \frac{2a}{t^2 + a^2}$ .

**Short Answer.**  $X(\omega) = 2\pi e^{-a|\omega|}$

- 6.35** Using a Fourier transform table and properties of the Fourier transform, find the Fourier transform  $Y$  of each function  $y$  given below in terms of the Fourier transform  $X$  of the function  $x$ .

(a)  $y(t) = t(\cos 3t)x(t)$ ; and

(b)  $y(t) = (t-1)^{100}x^*(t-1)$ .

**Short Answer.** (a)  $Y(\omega) = \frac{j}{2}X'(\omega-3) + \frac{j}{2}X'(\omega+3)$  (where the prime symbol denotes the first derivative);  
 (b)  $Y(\omega) = e^{-j\omega} \left(\frac{d}{d\omega}\right)^{100}[X^*(-\omega)]$ .

**6.36** For each pair of functions  $M$  and  $P$  given below, find the function  $x$  having magnitude spectrum  $M$  and phase spectrum  $P$ .

(a)  $M(\omega) = 1$   $P(\omega) = \omega$ ; and

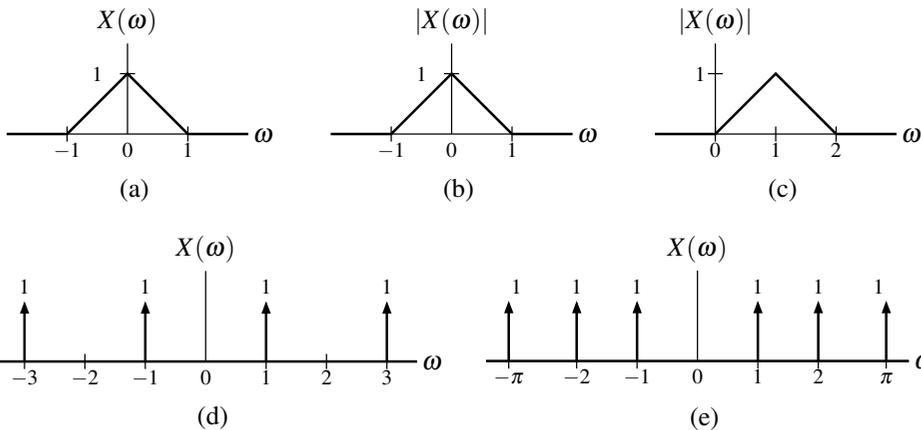
(b)  $M(\omega) = \text{rect} \frac{\omega}{3}$  and  $P(\omega) = 5\omega$ .

**Short Answer.** (a)  $x(t) = \delta(t + 1)$ ; (b)  $x(t) = \frac{3}{2\pi} \text{sinc} \left[ \frac{3}{2}(t + 5) \right]$ .

**6.37** Using Parseval's relation, evaluate the integral  $\int_{-\infty}^{\infty} \text{sinc}^2(kt) dt$ , where  $k$  is a nonzero real constant.

**Short Answer.**  $\pi/k$

**6.38** Let  $x$  denote a function with the Fourier transform  $X$ . For each frequency/magnitude spectrum given below, determine (if possible) whether  $x$  has each of the following properties: real, even/odd, periodic, finite duration, and finite energy.



**Short Answer.** (a) real, even, aperiodic, not finite duration, finite energy; (b) aperiodic, finite energy, not finite duration; (c) not real, aperiodic, not finite duration, finite energy, not even/odd; (d) real, even, periodic (fundamental period  $\frac{2\pi}{1} = 2\pi$ ), not finite duration, not finite energy; (e) real, even, aperiodic, not finite duration, not finite energy

**6.39** Using properties of the Fourier transform and a Fourier transform table, compute the convolution  $y = x_1 * x_2$ , where

$$x_1(t) = \text{sinc}(at - b_1) \quad \text{and} \quad x_2(t) = \text{sinc}(at - b_2)$$

and  $a$ ,  $b_1$ , and  $b_2$  are real constants with  $a \neq 0$ .

**Short Answer.**  $y(t) = \frac{\pi}{|a|} \text{sinc}(at - b_1 - b_2)$

**6.40** Let  $x(t) = \text{sinc}(5t - 3)$ . By direct application of the Nyquist sampling theorem, determine the lowest sampling rate  $\omega_s$  at which  $x$  can be sampled such that it can be exactly reconstructed from its samples.

**Short Answer.** 10 rad/s

**6.41** An audio signal  $x$  consists of two perfect sinusoidal tones at 440 Hz and 880 Hz. The signal  $x$  is sampled at a rate of  $f_s$  Hz and then played back on a loudspeaker. Determine how many tones will be heard on the loudspeaker and their frequencies, if  $f_s$  equals (a) 500 Hz and (b) 2000 Hz. (You should assume that a human can hear frequencies from 20 Hz to 20 kHz.)

**Short Answer.** (a) Two tones are heard with frequencies 60 Hz and 120 Hz; (b) Two tones are heard at 440 Hz and 880 Hz

**6.42** Consider a bandlimited function  $x$  that has been sampled at a frequency  $\omega_s$  (in radians) satisfying the Nyquist condition to produce the sequence  $y$ . Find an expression for the Fourier transform  $X$  of the function  $x$  in terms of the sequence  $y$  of samples and  $\omega_s$ . [Hint: Recall that, from the sampling theorem,  $x(t) = \sum_{n=-\infty}^{\infty} y(n) \text{sinc}(\frac{\omega_s}{2}t - \pi n)$ .]

**Short Answer.**  $X(\omega) = \frac{2\pi}{\omega_s} \sum_{n=-\infty}^{\infty} y(n) e^{-j2\pi n\omega/\omega_s} \text{rect} \frac{\omega}{\omega_s}$

## 6.23 MATLAB Exercises

**6.101** (a) Consider a frequency response  $H$  of the form

$$H(\omega) = \frac{\sum_{k=0}^{M-1} a_k \omega^k}{\sum_{k=0}^{N-1} b_k \omega^k},$$

where  $a_k$  and  $b_k$  are complex constants. Write a MATLAB function called `freqw` that evaluates a function of the above form at an arbitrary number of specified points. The function should take three input arguments:

- 1) a vector containing the  $a_k$  coefficients;
- 2) a vector containing the  $b_k$  coefficients; and
- 3) a vector containing the values of  $\omega$  for which to evaluate  $H(\omega)$ .

The function should generate two return values:

- 1) a vector of function values; and
- 2) a vector of points at which the function was evaluated.

If the function is called with no output arguments (i.e., the `nargout` variable is zero), then the function should plot the magnitude and phase responses before returning. (Hint: The `polyval` function may be helpful.)

(b) Use the function developed in part (a) to plot the magnitude and phase responses of the system with the frequency response

$$H(\omega) = \frac{16.0000}{1.0000\omega^4 - j5.2263\omega^3 - 13.6569\omega^2 + j20.9050\omega + 16.0000}$$

For each of the plots, use the frequency range  $[-5, 5]$ .

(c) What type of ideal frequency-selective filter does this system approximate?

**6.102** Consider the filter associated with each of the frequency responses given below. In each case, plot the magnitude and phase responses of the filter, and indicate what type of ideal frequency-selective filter it best approximates.

(a)  $H(\omega) = \frac{\omega_b^3}{(j\omega)^3 + 2\omega_b(j\omega)^2 + 2\omega_b(j\omega) + \omega_b^3}$  where  $\omega_b = 1$ ;

(b)  $H(\omega) = \frac{(j\omega)^5}{(j\omega)^5 + 17.527635(j\omega)^4 + 146.32995(j\omega)^3 + 845.73205(j\omega)^2 + 2661.6442(j\omega) + 7631.0209}$ ; and

(c)  $H(\omega) = \frac{13.104406(j\omega)^3}{(j\omega)^6 + 3.8776228(j\omega)^5 + 34.517979(j\omega)^4 + 75.146371(j\omega)^3 + 276.14383(j\omega)^2 + 248.16786(j\omega) + 512}$ .

(Hint: Use the `freqs` function with  $s = j\omega$  to compute the frequency response. The `abs`, `angle`, `linspace`, `plot`, `xlabel`, `ylabel`, and `print` functions may also prove useful for this problem.)

- 6.103** (a) Use the `butter` and `besself` functions to design a tenth-order Butterworth lowpass filter and tenth-order Bessel lowpass filter, each with a cutoff frequency of 10 rad/s.
- (b) For each of the filters designed in part (a), plot the magnitude and phase responses using a linear scale for the frequency axis. In the case of the phase response, plot the unwrapped phase (as this will be helpful later in part (d) of this problem). (Hint: The `freqs` and `unwrap` functions may be helpful.)
- (c) Consider the magnitude responses for each of the filters. Recall that an ideal lowpass filter has a magnitude response that is constant in the passband. Which of the two filters more closely approximates this ideal behavior?
- (d) Consider the phase responses for each of the filters. An ideal lowpass filter has a phase response that is a linear function. Which of the two filters has a phase response that best approximates a linear (i.e., straight line) function in the passband?

## Chapter 7

# Laplace Transform

### 7.1 Introduction

In this chapter, we introduce another important mathematical tool in the study of signals and systems known as the Laplace transform. The Laplace transform can be viewed as a generalization of the (classical) Fourier transform. Due to its more general nature, the Laplace transform has a number of advantages over the Fourier transform. First, the Laplace transform representation exists for some functions that do not have a Fourier transform representation. So, we can handle some functions with the Laplace transform that cannot be handled with the Fourier transform. Second, since the Laplace transform is a more general tool, it can provide additional insights beyond those facilitated by the Fourier transform.

### 7.2 Motivation Behind the Laplace Transform

In Section 4.10, we showed that complex exponentials are eigenfunctions of LTI systems. Suppose that we have a LTI system with impulse response  $h$ . This eigenfunction property leads to the result that the response  $y$  of the system to the complex exponential input  $x(t) = e^{st}$  (where  $s$  is a complex constant) is

$$y(t) = H(s)e^{st},$$

where

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st} dt. \quad (7.1)$$

Previously, we referred to  $H$  as the system function. In this chapter, we will learn that  $H$  is, in fact, the Laplace transform of  $h$ . That is, the integral in (7.1) is simply the definition of the Laplace transform. In the case that  $s = j\omega$  where  $\omega$  is real (i.e.,  $s$  is purely imaginary), (7.1) becomes the Fourier transform integral (studied in Chapter 6). From our earlier reading, we know that  $H(j\omega)$  is frequency response of the LTI system (i.e., the Fourier transform of  $h$ ). Since (7.1) includes the Fourier transform as a special case, the Laplace transform can be thought of as a generalization of the (classical) Fourier transform.

### 7.3 Definition of the Laplace Transform

The (bilateral) **Laplace transform** of the function  $x$  is denoted as  $\mathcal{L}x$  or  $X$  and is defined as

$$\mathcal{L}x(s) = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt. \quad (7.2)$$

Similarly, the **inverse Laplace transform** of  $X$  is denoted  $\mathcal{L}^{-1}X$  or  $x$  and is given by

$$\mathcal{L}^{-1}X(t) = x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st} ds, \quad (7.3)$$

where  $\sigma = \text{Re}(s)$ . We refer to  $x$  and  $X$  as a **Laplace transform pair** and denote this relationship as

$$x(t) \xleftrightarrow{\text{LT}} X(s).$$

As we can see from (7.3), the calculation of the inverse Laplace transform requires a contour integration (since  $s$  is a complex variable). In particular, we must integrate along the vertical line  $s = \sigma$  in the complex plane. Such a contour integration is often not so easy to compute. Therefore, in practice, we do not usually compute the inverse Laplace transform using (7.3) directly. Instead, we resort to other means (to be discussed later).

Two different versions of the Laplace transform are commonly used. The first is the bilateral version, as introduced above. The second is the unilateral version. The unilateral Laplace transform is most frequently used to solve systems of linear differential equations with nonzero initial conditions. As it turns out, the only difference between the definitions of the bilateral and unilateral Laplace transforms is in the lower limit of integration. In the bilateral case, the lower limit is  $-\infty$ , whereas in the unilateral case, the lower limit is 0. In the remainder of this chapter, we will focus our attention primarily on the bilateral Laplace transform. We will, however, briefly introduce the unilateral Laplace transform as a tool for solving differential equations. Unless otherwise noted, all subsequent references to the Laplace transform should be understood to mean bilateral Laplace transform.

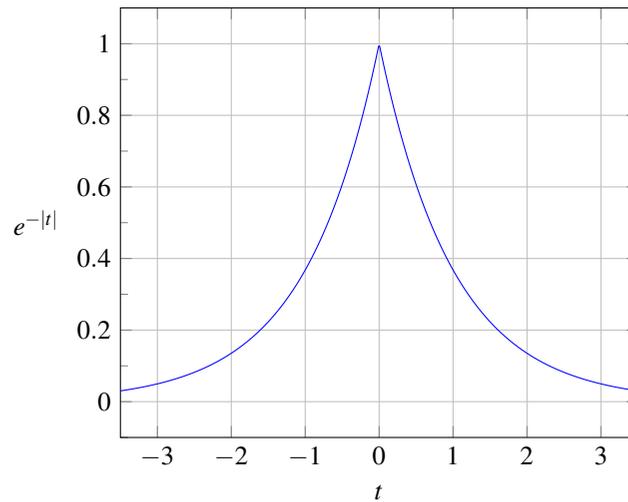
## 7.4 Remarks on Notation Involving the Laplace Transform

Each of the Laplace transform operator  $\mathcal{L}$  and inverse Laplace transform operator  $\mathcal{L}^{-1}$  map a function to a function. Consequently, the operand for each of these operators must be a function (not a number). Consider the unnamed function that maps  $t$  to  $e^{-|t|}$  as shown in Figure 7.1. Suppose that we would like to write an expression that denotes the Laplace transform of this function. At first, we might be inclined to write “ $\mathcal{L}\{e^{-|t|}\}$ ”. Strictly speaking, however, this notation is not correct, since the Laplace transform operator requires a function as an operand and “ $e^{-|t|}$ ” (strictly speaking) denotes a number (i.e., the value of the function in the figure evaluated at  $t$ ). Essentially, the cause of our problems here is that the function in question does not have a name (such as “ $x$ ”) by which it can be referred. To resolve this problem, we could define a function  $x$  using the equation  $x(t) = e^{-|t|}$  and then write the Laplace transform as “ $\mathcal{L}x$ ”. Unfortunately, introducing a new function name just for the sake of strictly correct notation is often undesirable as it frequently leads to highly verbose writing.

One way to avoid overly verbose writing when referring to functions without names is offered by dot notation, introduced earlier in Section 2.1. Again, consider the function from Figure 7.1 that maps  $t$  to  $e^{-|t|}$ . Using strictly correct notation, we could write the Laplace transform of this function as “ $\mathcal{L}\{e^{-|\cdot|}\}$ ”. In other words, we can indicate that an expression refers to a function (as opposed to the value of function) by using the interpunct symbol (as discussed in Section 2.1). Some examples of the use of dot notation can be found below in Example 7.1. Dot notation is often extremely beneficial when one wants to employ precise (i.e., strictly correct) notation without being overly verbose.

**Example 7.1** (Dot notation). Several examples of the use of dot notation are as follows:

1. To denote the Laplace transform of the function  $x$  defined by the equation  $x(t) = t^2 e^{-3t} u(t)$  (without the need to introduce the named function  $x$ ), we can write:  $\mathcal{L}\{(\cdot)^2 e^{-3(\cdot)} u(\cdot)\}$ .
2. To denote the Laplace transform of the function  $x$  defined by the equation  $x(t) = t^2 e^{-3t} u(t)$  evaluated at  $s - 5$  (without the need to introduce the named function  $x$ ), we can write:  $\mathcal{L}\{(\cdot)^2 e^{-3(\cdot)} u(\cdot)\}(s - 5)$ .
3. To denote the inverse Laplace transform of the function  $X$  defined by the equation  $X(s) = s^{-1}$  (without the need to introduce the named function  $X$ ), we can write:  $\mathcal{L}^{-1}\{(\cdot)^{-1}\}$ .

Figure 7.1: A plot of  $e^{-|t|}$  versus  $t$ .

4. To denote the inverse Laplace transform of the function  $X$  defined by the equation  $X(s) = s^{-1}$  evaluated at  $t - 3$  (without the need to introduce the named function  $X$ ), we can write:  $\mathcal{L}^{-1}\{(\cdot)^{-1}\}(t - 3)$ . ■

If the reader is comfortable with dot notation, the author would encourage the reader to use it when appropriate. Since some readers may find the dot notation to be confusing, however, this book (for the most part) attempts to minimize the use of dot notation. Instead, as a compromise solution, this book adopts the following notational conventions in order to achieve conciseness and a reasonable level of clarity without the need to use dot notation pervasively:

- unless indicated otherwise, in an expression for the operand of the Laplace transform operator  $\mathcal{L}$ , the variable “ $t$ ” is assumed to be the independent variable for the function to which the Laplace transform is being applied (i.e., in terms of dot notation, the expression is treated as if each “ $t$ ” were a “ $\cdot$ ”);
- unless indicated otherwise, in an expression for the operand of the inverse Laplace transform operator  $\mathcal{L}^{-1}$ , the variable “ $s$ ” is assumed to be the independent variable for the function to which the inverse Laplace transform is being applied (i.e., in terms of dot notation, the expression is treated as if each “ $s$ ” were a “ $\cdot$ ”).

Some examples of using these book-sanctioned notational conventions can be found below in Example 7.2. Admittedly, these book-sanctioned conventions are not ideal, as they abuse mathematical notation somewhat, but they seem to be the best compromise in order to accommodate those who may prefer not to use dot notation.

**Example 7.2** (Book-sanctioned notation). Several examples of using the notational conventions that are employed throughout most of this book (as described above) are as follows:

1. To denote the Laplace transform of the function  $x$  defined by the equation  $x(t) = t^2 e^{-3t} u(t)$  (without the need to introduce the named function  $x$ ), we can write:  $\mathcal{L}\{t^2 e^{-3t} u(t)\}$ .
2. To denote the Laplace transform of the function  $x$  defined by the equation  $x(t) = t^2 e^{-3t} u(t)$  evaluated at  $s - 5$  (without the need to introduce the named function  $x$ ), we can write:  $\mathcal{L}\{t^2 e^{-3t} u(t)\}(s - 5)$ .
3. To denote the inverse Laplace transform of the function  $X$  defined by the equation  $X(s) = s^{-1}$  (without the need to introduce the named function  $X$ ), we can write:  $\mathcal{L}^{-1}\{s^{-1}\}$ .
4. To denote the inverse Laplace transform of the function  $X$  defined by the equation  $X(s) = s^{-1}$  evaluated at  $t - 3$  (without the need to introduce the named function  $X$ ), we can write:  $\mathcal{L}^{-1}\{s^{-1}\}(t - 3)$ . ■

Since applying the Laplace transform operator or inverse Laplace transform operator to a function yields another function, we can evaluate this other function at some value. Again, consider the function from Figure 7.1 that maps  $t$  to  $e^{-|t|}$ . To denote the value of the Laplace transform of this function evaluated at  $s - 1$ , we would write “ $\mathcal{L}\{e^{-|t|}\}(s - 1)$ ” using dot notation or “ $\mathcal{L}\{e^{-|t|}\}(s - 1)$ ” using the book-sanctioned notational conventions described above.

## 7.5 Relationship Between Laplace Transform and Continuous-Time Fourier Transform

In Section 7.3 of this chapter, we introduced the Laplace transform, and in the previous chapter, we studied the (CT) Fourier transform. As it turns out, the Laplace transform and (CT) Fourier transform are very closely related. Recall the definition of the Laplace transform in (7.2). Consider now the special case of (7.2) where  $s = j\omega$  and  $\omega$  is real (i.e.,  $\text{Re}(s) = 0$ ). In this case, (7.2) becomes

$$\begin{aligned} X(j\omega) &= \left[ \int_{-\infty}^{\infty} x(t)e^{-st} dt \right] \Big|_{s=j\omega} \\ &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \\ &= \mathcal{F}x(\omega). \end{aligned}$$

Thus, the Fourier transform is simply the Laplace transform evaluated at  $s = j\omega$ , assuming that this quantity is well defined (i.e., converges). In other words,

$$X(j\omega) = \mathcal{F}x(\omega). \quad (7.4)$$

Incidentally, it is due to the preceding relationship that the Fourier transform of  $x$  is sometimes written as  $X(j\omega)$ . When this notation is used, the function  $X$  actually corresponds to the Laplace transform of  $x$  rather than its Fourier transform (i.e., the expression  $X(j\omega)$  corresponds to the Laplace transform evaluated at points on the imaginary axis).

Now, consider the general case of an arbitrary complex value for  $s$  in (7.2). Let us express  $s$  in Cartesian form as  $s = \sigma + j\omega$  where  $\sigma$  and  $\omega$  are real. Substituting  $s = \sigma + j\omega$  into (7.2), we obtain

$$\begin{aligned} X(\sigma + j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-(\sigma + j\omega)t} dt \\ &= \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t} dt \\ &= \mathcal{F}\{e^{-\sigma t}x(t)\}(\omega). \end{aligned}$$

Thus, we have shown

$$X(\sigma + j\omega) = \mathcal{F}\{e^{-\sigma t}x(t)\}(\omega). \quad (7.5)$$

Thus, the Laplace transform of  $x$  can be viewed as the (CT) Fourier transform of  $x'(t) = e^{-\sigma t}x(t)$  (i.e.,  $x$  weighted by a real exponential function). As a consequence of multiplying by the real exponential  $e^{-\sigma t}$ , the Laplace transform of a function may exist when the Fourier transform of the same function does not.

By using the above relationship, we can derive the formula for the inverse Laplace transform given in (7.3). Let  $X$  denote the Laplace transform of  $x$ , and let  $s = \sigma + j\omega$ , where  $\sigma$  and  $\omega$  are real. From the relationship between the Fourier and Laplace transforms in (7.5), we have

$$X(\sigma + j\omega) = \mathcal{F}\{e^{-\sigma t}x(t)\}(\omega),$$

where  $\sigma$  is chosen so that  $X(s)$  converges for  $s = \sigma + j\omega$ . Taking the inverse Fourier transform of both sides of the preceding equation yields

$$\mathcal{F}^{-1}\{X(\sigma + j\omega)\}(t) = e^{-\sigma t}x(t).$$

Multiplying both sides by  $e^{\sigma t}$ , we obtain

$$x(t) = e^{\sigma t} \mathcal{F}^{-1}\{X(\sigma + j\omega)\}(t).$$

From the definition of the inverse Fourier transform, we have

$$\begin{aligned} x(t) &= e^{\sigma t} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} d\omega \right] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{(\sigma + j\omega)t} d\omega. \end{aligned}$$

Since  $s = \sigma + j\omega$ , we have that  $ds = jd\omega$ , and consequently,

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} \left(\frac{1}{j}\right) ds \\ &= \frac{1}{j2\pi} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds. \end{aligned}$$

Thus, we have just derived the inverse Laplace transform formula in (7.3).

## 7.6 Laplace Transform Examples

In this section, we calculate the Laplace transform of several relatively simple functions. In the process, we gain some important insights into the Laplace transform.

**Example 7.3.** Find the Laplace transform  $X$  of the function

$$x(t) = e^{-at} u(t),$$

where  $a$  is a real constant.

*Solution.* Let  $s = \sigma + j\omega$ , where  $\sigma$  and  $\omega$  are real. From the definition of the Laplace transform, we have

$$\begin{aligned} X(s) &= \mathcal{L}\{e^{-at} u(t)\}(s) \\ &= \int_{-\infty}^{\infty} e^{-at} u(t) e^{-st} dt \\ &= \int_0^{\infty} e^{-(s+a)t} dt \\ &= \left[ \left(-\frac{1}{s+a}\right) e^{-(s+a)t} \right]_0^{\infty}. \end{aligned}$$

At this point, we substitute  $s = \sigma + j\omega$  in order to more easily determine when the above expression converges to a finite value. This yields

$$\begin{aligned} X(s) &= \left[ \left(-\frac{1}{\sigma+a+j\omega}\right) e^{-(\sigma+a+j\omega)t} \right]_0^{\infty} \\ &= \left(\frac{-1}{\sigma+a+j\omega}\right) \left[ e^{-(\sigma+a)t} e^{-j\omega t} \right]_0^{\infty} \\ &= \left(\frac{-1}{\sigma+a+j\omega}\right) \left[ e^{-(\sigma+a)\infty} e^{-j\omega\infty} - 1 \right]. \end{aligned}$$

Thus, we can see that the above expression only converges for  $\sigma + a > 0$  (i.e.,  $\text{Re}(s) > -a$ ). In this case, we have that

$$\begin{aligned} X(s) &= \left(\frac{-1}{\sigma+a+j\omega}\right) [0 - 1] \\ &= \left(\frac{-1}{s+a}\right) (-1) \\ &= \frac{1}{s+a}. \end{aligned}$$

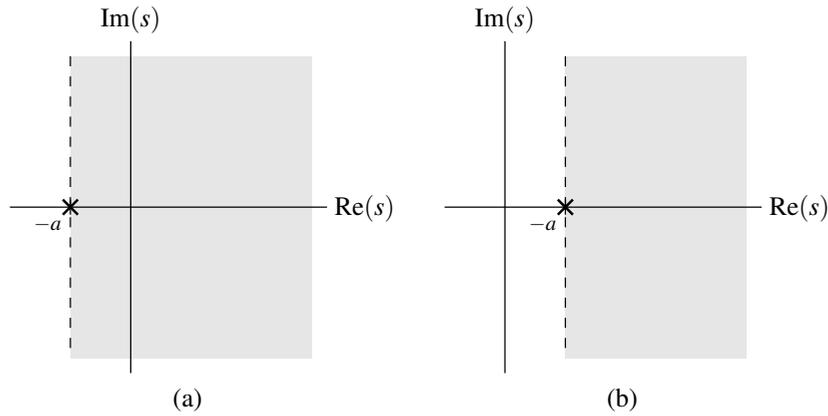


Figure 7.2: Region of convergence for the case that (a)  $a > 0$  and (b)  $a < 0$ .

Thus, we have that

$$e^{-at}u(t) \xleftrightarrow{\text{LT}} \frac{1}{s+a} \quad \text{for } \text{Re}(s) > -a.$$

The region of convergence for  $X$  is illustrated in Figures 7.2(a) and (b) for the cases of  $a > 0$  and  $a < 0$ , respectively. ■

**Example 7.4.** Find the Laplace transform  $X$  of the function

$$x(t) = -e^{-at}u(-t),$$

where  $a$  is a real constant.

*Solution.* Let  $s = \sigma + j\omega$ , where  $\sigma$  and  $\omega$  are real. From the definition of the Laplace transform, we can write

$$\begin{aligned} X(s) &= \mathcal{L}\{-e^{-at}u(-t)\}(s) \\ &= \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st} dt \\ &= \int_{-\infty}^0 -e^{-at}e^{-st} dt \\ &= \int_{-\infty}^0 -e^{-(s+a)t} dt \\ &= \left[ \left( \frac{1}{s+a} \right) e^{-(s+a)t} \right]_{-\infty}^0. \end{aligned}$$

In order to more easily determine when the above expression converges to a finite value, we substitute  $s = \sigma + j\omega$ . This yields

$$\begin{aligned} X(s) &= \left[ \left( \frac{1}{\sigma+a+j\omega} \right) e^{-(\sigma+a+j\omega)t} \right]_{-\infty}^0 \\ &= \left( \frac{1}{\sigma+a+j\omega} \right) \left[ e^{-(\sigma+a)t} e^{-j\omega t} \right]_{-\infty}^0 \\ &= \left( \frac{1}{\sigma+a+j\omega} \right) \left[ 1 - e^{(\sigma+a)\infty} e^{j\omega\infty} \right]. \end{aligned}$$

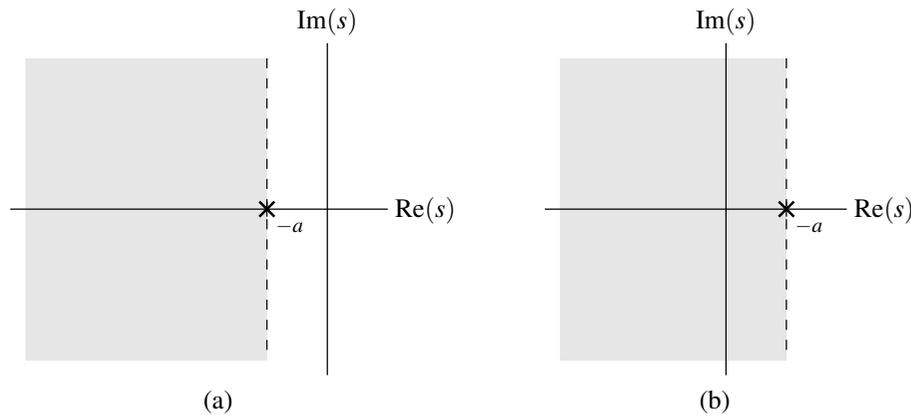


Figure 7.3: Region of convergence for the case that (a)  $a > 0$  and (b)  $a < 0$ .

Thus, we can see that the above expression only converges for  $\sigma + a < 0$  (i.e.,  $\text{Re}(s) < -a$ ). In this case, we have

$$\begin{aligned} X(s) &= \left( \frac{1}{\sigma + a + j\omega} \right) [1 - 0] \\ &= \frac{1}{s + a}. \end{aligned}$$

Thus, we have that

$$-e^{-at}u(-t) \xleftrightarrow{\text{LT}} \frac{1}{s + a} \quad \text{for } \text{Re}(s) < -a.$$

The region of convergence for  $X$  is illustrated in Figures 7.3(a) and (b) for the cases of  $a > 0$  and  $a < 0$ , respectively. ■

At this point, we compare the results of Examples 7.3 and 7.4, and make an important observation. Notice that the same algebraic expression for  $X$  was obtained in both of these examples (i.e.,  $X(s) = \frac{1}{s+a}$ ). The only difference is in the convergence properties of  $X$ . In one case,  $X(s)$  converges for  $\text{Re}(s) > -a$  while in the other it converges for  $\text{Re}(s) < -a$ . As it turns out, one must specify both the algebraic expression for  $X$  and its region of convergence in order to uniquely determine  $x = \mathcal{L}^{-1}X$  from  $X$ .

**Example 7.5** (Laplace transform of the unit-step function). Find the Laplace transform  $X$  of the function

$$x(t) = u(t).$$

*Solution.* Let  $s = \sigma + j\omega$ , where  $\sigma$  and  $\omega$  are real. From the definition of the Laplace transform, we have

$$\begin{aligned} X(s) &= \mathcal{L}u(s) \\ &= \int_{-\infty}^{\infty} u(t)e^{-st} dt \\ &= \int_0^{\infty} e^{-st} dt \\ &= \left[ \left(-\frac{1}{s}\right) e^{-st} \right]_0^{\infty}. \end{aligned}$$

At this point, we substitute  $s = \sigma + j\omega$  in order to more clearly see the region of convergence for this expression. This yields

$$\begin{aligned} X(s) &= \left[ \left(-\frac{1}{\sigma + j\omega}\right) e^{-(\sigma + j\omega)t} \right]_0^{\infty} \\ &= \left[ \left(-\frac{1}{\sigma + j\omega}\right) e^{-\sigma t} e^{-j\omega t} \right]_0^{\infty}. \end{aligned}$$

Thus, we can see that the above expression converges only for  $\sigma > 0$  (i.e.,  $\text{Re}(s) > 0$ ). In this case, we have

$$\begin{aligned} X(s) &= \left(-\frac{1}{\sigma + j\omega}\right)[0 - 1] \\ &= \left(-\frac{1}{s}\right)(-1) \\ &= \frac{1}{s}. \end{aligned}$$

Thus, we have that

$$u(t) \xleftrightarrow{\text{LT}} \frac{1}{s} \quad \text{for } \text{Re}(s) > 0. \quad \blacksquare$$

**Example 7.6** (Laplace transform of the unit-impulse function). Find the Laplace transform  $X$  of the function

$$x(t) = A\delta(t - t_0),$$

where  $A$  and  $t_0$  are arbitrary real constants.

*Solution.* From the definition of the Laplace transform, we can write

$$\begin{aligned} X(s) &= \mathcal{L}\{A\delta(t - t_0)\}(s) \\ &= \int_{-\infty}^{\infty} A\delta(t - t_0)e^{-st} dt \\ &= A \int_{-\infty}^{\infty} \delta(t - t_0)e^{-st} dt. \end{aligned}$$

Using the sifting property of the unit-impulse function, we can simplify this result to obtain

$$X(s) = Ae^{-st_0}.$$

Thus, we have shown that

$$A\delta(t - t_0) \xleftrightarrow{\text{LT}} Ae^{-st_0} \quad \text{for all } s. \quad \blacksquare$$

## 7.7 Region of Convergence for the Laplace Transform

Before discussing the region of convergence (ROC) of the Laplace transform in detail, we need to introduce some terminology involving sets in the complex plane. Let  $R$  denote a set in the complex plane. A set  $R$  comprised of all complex numbers  $s$  such that

$$\text{Re}(s) < a,$$

for some real constant  $a$  (or  $a = \infty$ ), is said to be a **left-half plane** (LHP). A set  $R$  comprised of all complex numbers  $s$  such that

$$\text{Re}(s) > a,$$

for some real constant  $a$  (or  $a = -\infty$ ), is said to be a **right-half plane** (RHP). Examples of LHPs and RHPs are given in Figure 7.4. Note that the set consisting of the entire complex plane is trivially both a LHP and RHP (i.e., the entire complex plane satisfies  $\text{Re}(s) > -\infty$  and  $\text{Re}(s) < \infty$ ).

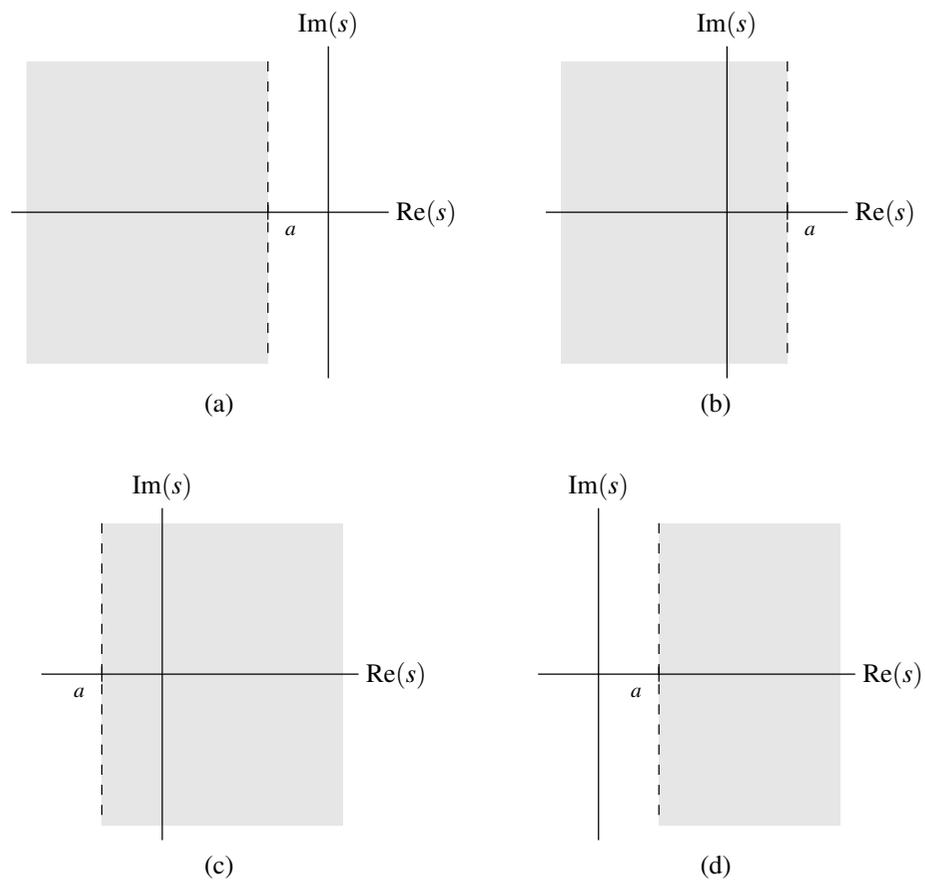


Figure 7.4: Examples of LHPs and RHPs. An example of a LHP in the case that (a)  $a < 0$  and (b)  $a > 0$ . An example of a RHP in the case that (c)  $a < 0$  and (d)  $a > 0$ .

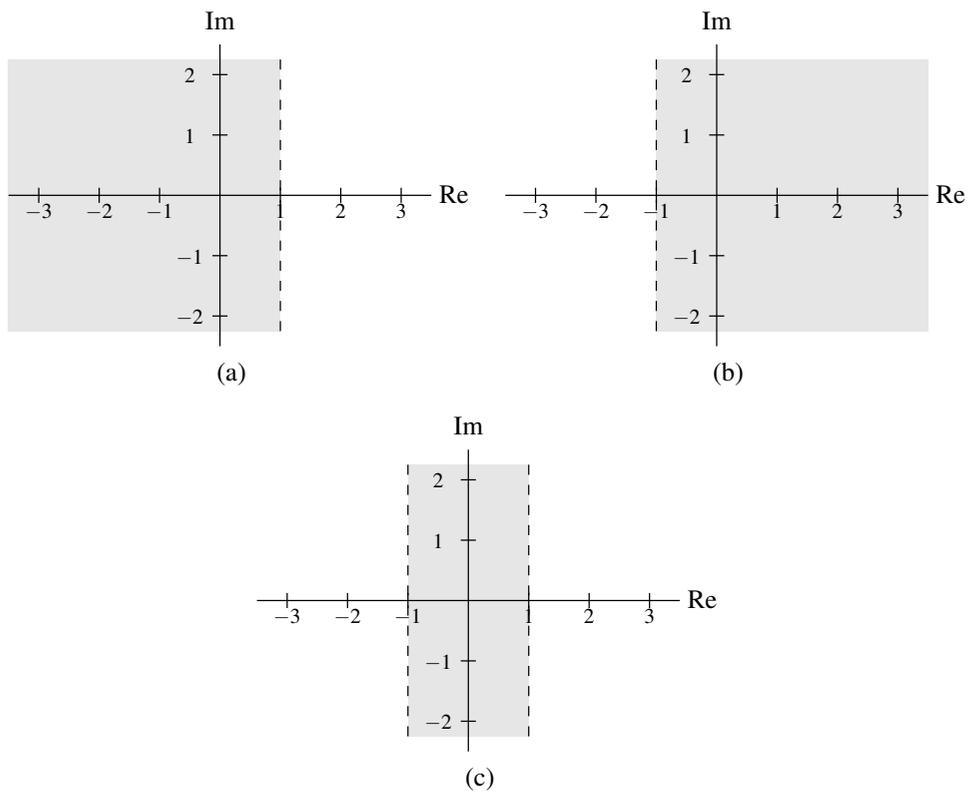


Figure 7.5: Example of set intersection. The sets (a)  $R_1$  and (b)  $R_2$ ; and (c) their intersection  $R_1 \cap R_2$ .

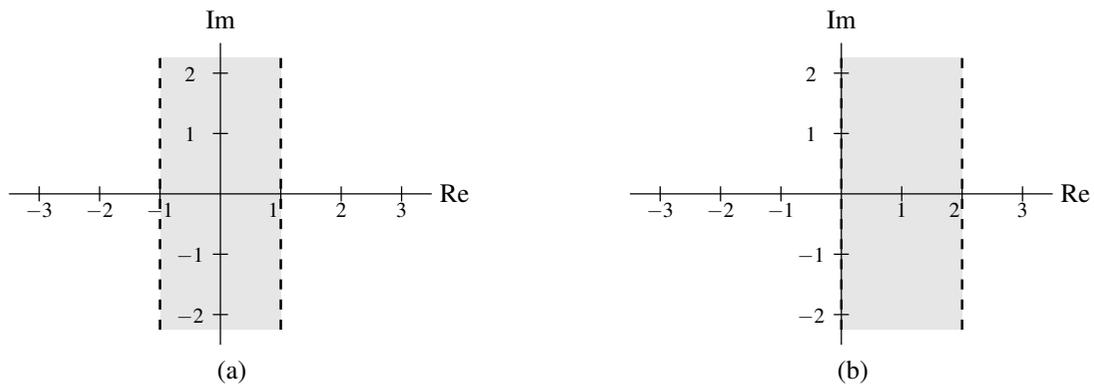


Figure 7.6: Example of adding a scalar to a set. (a) The set  $R$ . (b) The set  $R + 1$ .

Since the ROC is a set (of points in the complex plane), we often need to employ some basic set operations when dealing with ROCs. For two sets  $A$  and  $B$ , the **intersection** of  $A$  and  $B$ , denoted  $A \cap B$ , is the set of all points that are in both  $A$  and  $B$ . An illustrative example of set intersection is shown in Figure 7.5.

For a set  $S$  and a scalar constant  $a$ ,  $S + a$  denotes the set given by

$$S + a = \{z + a : z \in S\}.$$

That is,  $S + a$  denotes the set formed by adding  $a$  to each element of  $S$ . For example, suppose that  $R$  is the set of complex numbers  $s$  satisfying

$$-1 < \operatorname{Re}(s) < 1,$$

as shown in Figure 7.6(a). Then,  $R + 1$  is the set of complex numbers  $s$  satisfying

$$0 < \operatorname{Re}(s) < 2,$$

as shown in Figure 7.6(b).

For a set  $S$  and a scalar constant  $a$ ,  $aS$  denotes the set given by

$$aS = \{az : z \in S\}.$$

That is,  $aS$  denotes the set formed by multiplying each element of  $S$  by  $a$ . For example, suppose that  $R$  is the set of complex numbers  $s$  satisfying

$$-1 < \operatorname{Re}(s) < 2,$$

as shown in Figure 7.7(a). Then,  $2R$  is the set of complex numbers  $s$  satisfying

$$-2 < \operatorname{Re}(s) < 4,$$

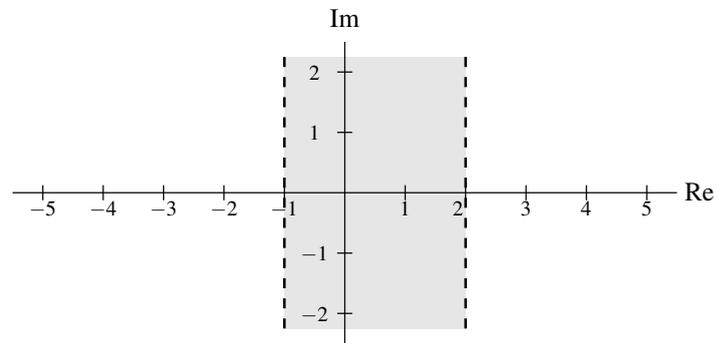
as shown in Figure 7.7(b); and  $-2R$  is the set of complex numbers  $s$  satisfying

$$-4 < \operatorname{Re}(s) < 2,$$

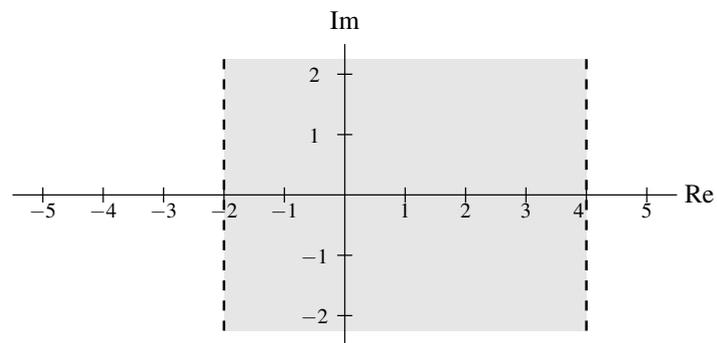
as shown in Figure 7.7(c).

As we saw earlier, for a function  $x$ , the complete specification of its Laplace transform  $X$  requires not only an algebraic expression for  $X$ , but also the ROC associated with  $X$ . Two distinct functions can have the same algebraic expression for their Laplace transform. In what follows, we examine some of the constraints on the ROC (of the Laplace transform) for various classes of functions.

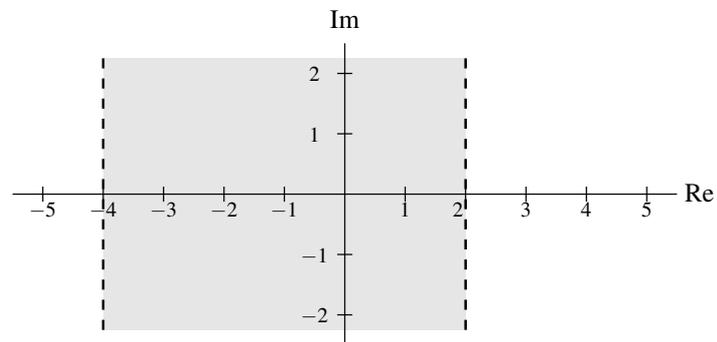
One can show that the ROC for the Laplace transform has the following properties:



(a)



(b)



(c)

Figure 7.7: Example of multiplying a set by a scalar. (a) The set  $R$ . The sets (b)  $2R$  and (c)  $-2R$ .

1. The ROC of the Laplace transform  $X$  consists of strips parallel to the imaginary axis in the complex plane. That is, if  $s$  is in the ROC, then  $s + j\omega$  is in the ROC for all real  $\omega$ .

Justification: The Laplace transform  $X$  of the function  $x$  is simply the (CT) Fourier transform of  $x'(t) = x(t)e^{-\text{Re}(s)t}$ . Thus,  $X$  converges whenever this Fourier transform converges. Since the convergence of the Fourier transform only depends on  $\text{Re}(s)$ , the convergence of the Laplace transform only depends on  $\text{Re}(s)$ .

2. If the Laplace transform  $X$  is a rational function, the ROC does not contain any poles, and the ROC is bounded by poles or extends to infinity.

Partial justification: Since  $X$  is rational, its value becomes infinite at a pole. So obviously,  $X$  does not converge at a pole. Therefore, it follows that the ROC cannot contain a pole.

3. If a function  $x$  is finite duration and its Laplace transform  $X$  converges for some value of  $s$ , then  $X$  converges for all values of  $s$  (i.e., the ROC is the entire complex plane).
4. If a function  $x$  is right sided and the (vertical) line  $\text{Re}(s) = \sigma_0$  is in the ROC of the Laplace transform  $X$  of  $x$ , then all values of  $s$  for which  $\text{Re}(s) > \sigma_0$  must also be in the ROC (i.e., the ROC is a right-half plane including  $\text{Re}(s) = \sigma_0$ ).
5. If a function  $x$  is left sided and the (vertical) line  $\text{Re}(s) = \sigma_0$  is in the ROC of the Laplace transform  $X$  of  $x$ , then all values of  $s$  for which  $\text{Re}(s) < \sigma_0$  must also be in the ROC (i.e., the ROC is a left-half plane including  $\text{Re}(s) = \sigma_0$ ).
6. If a function  $x$  is two sided and the (vertical) line  $\text{Re}(s) = \sigma_0$  is in the ROC of the Laplace transform  $X$  of  $x$ , then the ROC will consist of a strip in the complex plane that includes the line  $\text{Re}(s) = \sigma_0$ .
7. If the Laplace transform  $X$  of a function  $x$  is rational, then:
  - (a) If  $x$  is right sided, the ROC of  $X$  is to the right of the rightmost pole of  $X$  (i.e., the right-half plane to the right of the rightmost pole).
  - (b) If  $x$  is left sided, the ROC of  $X$  is to the left of the leftmost pole of  $X$  (i.e., the left-half plane to the left of the leftmost pole).

Note that some of the above properties are redundant. For example, properties 1, 2, and 4 imply property 7(a). Also, properties 1, 2, and 5 imply property 7(b). Moreover, since every function can be classified as one of left sided (but not right sided), right sided (but not left sided), two sided, or finite duration, we can infer from properties 3, 4, 5, and 6 that the ROC can only be of the form of a left-half plane, a right-half plane, a (single) vertical strip, the entire complex plane, or the empty set. Thus, the ROC must be a connected region. That is, it cannot consist of multiple (unconnected) vertical strips.

**Example 7.7.** The Laplace transform  $X$  of the function  $x$  has the algebraic expression

$$X(s) = \frac{s + \frac{1}{2}}{(s^2 + 2s + 2)(s^2 + s - 2)}.$$

Identify all of the possible ROCs of  $X$ . For each ROC, indicate whether the corresponding function  $x$  is left sided, right sided, two sided, or finite duration.

*Solution.* The possible ROCs associated with  $X$  are determined by the poles of this function. So, we must find the poles of  $X$ . Factoring the denominator of  $X$ , we obtain

$$X(s) = \frac{s + \frac{1}{2}}{(s + 1 - j)(s + 1 + j)(s + 2)(s - 1)}.$$

Thus,  $X$  has poles at  $-2$ ,  $-1 - j$ ,  $-1 + j$ , and  $1$ . Since these poles only have three distinct real parts (namely,  $-2$ ,  $-1$ , and  $1$ ), there are four possible ROCs:

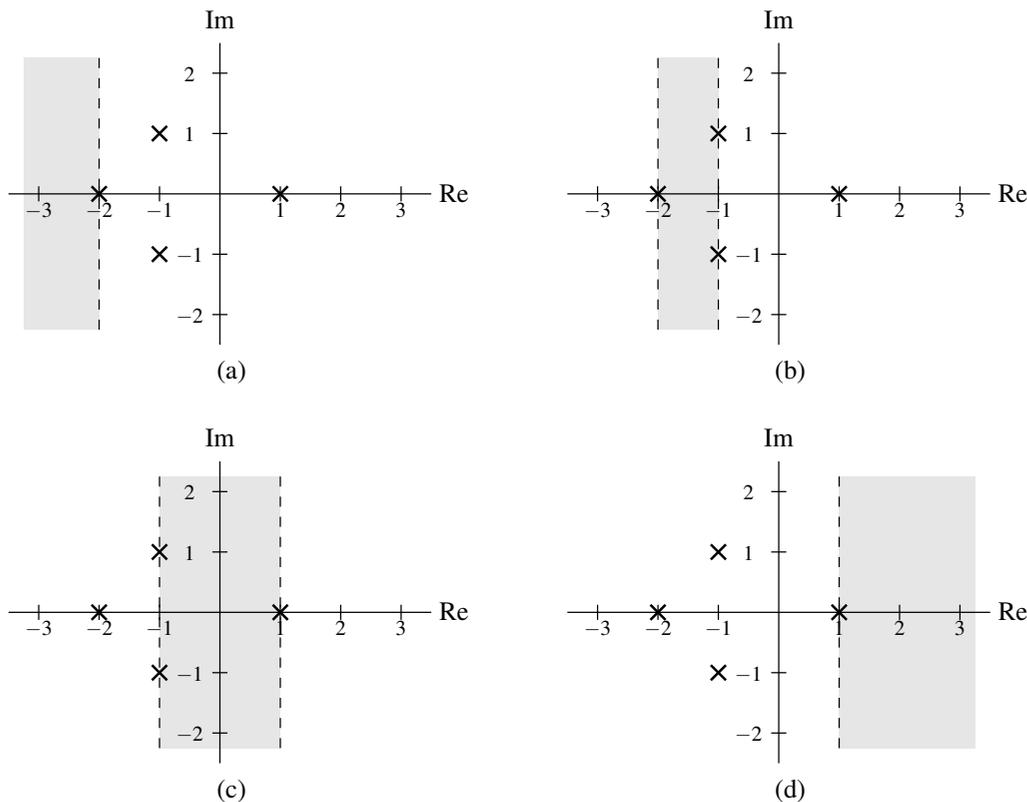


Figure 7.8: ROCs for example. The (a) first, (b) second, (c) third, and (d) fourth possible ROCs for  $X$ .

- i)  $\text{Re}(s) < -2$ ,
- ii)  $-2 < \text{Re}(s) < -1$ ,
- iii)  $-1 < \text{Re}(s) < 1$ , and
- iv)  $\text{Re}(s) > 1$ .

These ROCs are plotted in Figures 7.8(a), (b), (c), and (d), respectively. The first ROC is a left-half plane, so the corresponding  $x$  must be left sided. The second ROC is a vertical strip (i.e., neither a left- nor right-half plane), so the corresponding  $x$  must be two sided. The third ROC is a vertical strip (i.e., neither a left- nor right-half plane), so the corresponding  $x$  must be two sided. The fourth ROC is a right-half plane, so the corresponding  $x$  must be right sided. ■

## 7.8 Properties of the Laplace Transform

The Laplace transform has a number of important properties. In the sections that follow, we introduce several of these properties. For the convenience of the reader, the properties described in the subsequent sections are also summarized in Table 7.1 (on page 245). Also, for convenience, several Laplace-transform pairs are given later in Table 7.2 (on page 246). In what follows, we will sometimes refer to transform pairs in this table.

### 7.8.1 Linearity

Arguably, the most important property of the Laplace transform is linearity, as introduced below.

**Theorem 7.1** (Linearity). If  $x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s)$  with ROC  $R_1$  and  $x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s)$  with ROC  $R_2$ , then

$$a_1x_1(t) + a_2x_2(t) \xleftrightarrow{\mathcal{L}} a_1X_1(s) + a_2X_2(s) \text{ with ROC } R \text{ containing } R_1 \cap R_2,$$

where  $a_1$  and  $a_2$  are arbitrary complex constants.

*Proof.* Let  $y(t) = a_1x_1(t) + a_2x_2(t)$ , and let  $Y$  denote the Laplace transform of  $y$ . Using the definition of the Laplace transform and straightforward algebraic manipulation, we have

$$\begin{aligned} Y(s) &= \int_{-\infty}^{\infty} [a_1x_1(t) + a_2x_2(t)]e^{-st} dt \\ &= \int_{-\infty}^{\infty} a_1x_1(t)e^{-st} dt + \int_{-\infty}^{\infty} a_2x_2(t)e^{-st} dt \\ &= a_1 \int_{-\infty}^{\infty} x_1(t)e^{-st} dt + a_2 \int_{-\infty}^{\infty} x_2(t)e^{-st} dt \\ &= a_1X_1(s) + a_2X_2(s). \end{aligned}$$

The ROC  $R$  can be deduced as follows. If  $X_1$  and  $X_2$  both converge at some point  $s$ , say  $s = \lambda$ , then any linear combination of these functions must also converge at  $s = \lambda$ . Therefore, the ROC  $R$  must contain the intersection of  $R_1$  and  $R_2$ . Thus, we have shown that the linearity property holds. ■

In the preceding theorem, note that the ROC of the result can be larger than  $R_1 \cap R_2$ . When  $X_1$  and  $X_2$  are rational functions, this can only happen if pole-zero cancellation occurs in the expression  $a_1X_1(s) + a_2X_2(s)$ .

**Example 7.8** (Linearity property of the Laplace transform). Find the Laplace transform of the function

$$x = x_1 + x_2,$$

where

$$x_1(t) = e^{-t}u(t) \quad \text{and} \quad x_2(t) = e^{-t}u(t) - e^{-2t}u(t).$$

*Solution.* Using Laplace transform pairs from Table 7.2, we have

$$\begin{aligned} X_1(s) &= \mathcal{L}\{e^{-t}u(t)\}(s) \\ &= \frac{1}{s+1} \quad \text{for } \operatorname{Re}(s) > -1 \quad \text{and} \\ X_2(s) &= \mathcal{L}\{e^{-t}u(t) - e^{-2t}u(t)\}(s) \\ &= \mathcal{L}\{e^{-t}u(t)\}(s) - \mathcal{L}\{e^{-2t}u(t)\}(s) \\ &= \frac{1}{s+1} - \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -1 \\ &= \frac{1}{(s+1)(s+2)} \quad \text{for } \operatorname{Re}(s) > -1. \end{aligned}$$

So, from the definition of  $X$ , we can write

$$\begin{aligned} X(s) &= \mathcal{L}\{x_1 + x_2\}(s) \\ &= X_1(s) + X_2(s) \\ &= \frac{1}{s+1} + \frac{1}{(s+1)(s+2)} \\ &= \frac{s+2+1}{(s+1)(s+2)} \\ &= \frac{s+3}{(s+1)(s+2)}. \end{aligned}$$

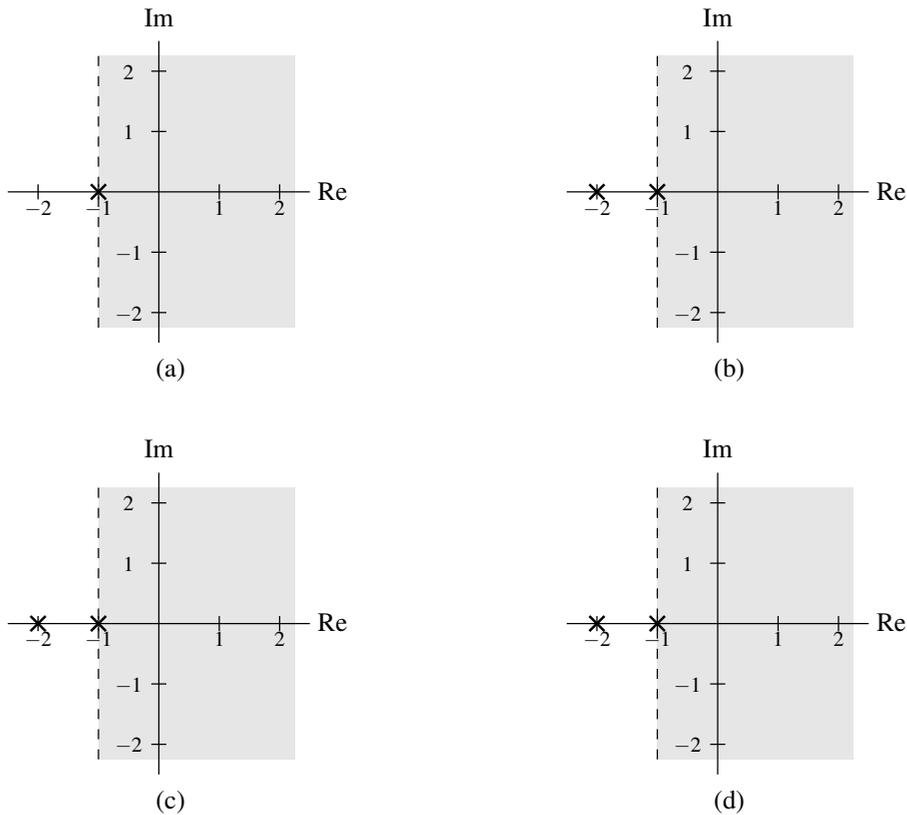


Figure 7.9: ROCs for the linearity example. The (a) ROC of  $X_1$ , (b) ROC of  $X_2$ , (c) ROC associated with the intersection of the ROCs of  $X_1$  and  $X_2$ , and (d) ROC of  $X$ .

Now, we must determine the ROC of  $X$ . We know that the ROC of  $X$  must contain the intersection of the ROCs of  $X_1$  and  $X_2$ . So, the ROC must contain  $\text{Re}(s) > -1$ . Furthermore, the ROC cannot be larger than this intersection, since  $X$  has a pole at  $-1$ . Therefore, the ROC of  $X$  is  $\text{Re}(s) > -1$ . The various ROCs are illustrated in Figure 7.9. So, in conclusion, we have

$$X(s) = \frac{s+3}{(s+1)(s+2)} \quad \text{for } \text{Re}(s) > -1. \quad \blacksquare$$

**Example 7.9** (Linearity property of the Laplace transform and pole-zero cancellation). Find the Laplace transform  $X$  of the function

$$x = x_1 - x_2,$$

where  $x_1$  and  $x_2$  are as defined in the previous example.

*Solution.* From the previous example, we know that

$$X_1(s) = \frac{1}{s+1} \quad \text{for } \text{Re}(s) > -1 \quad \text{and}$$

$$X_2(s) = \frac{1}{(s+1)(s+2)} \quad \text{for } \text{Re}(s) > -1.$$

From the definition of  $X$ , we have

$$\begin{aligned}
 X(s) &= \mathcal{L}\{x_1 - x_2\}(s) \\
 &= X_1(s) - X_2(s) \\
 &= \frac{1}{s+1} - \frac{1}{(s+1)(s+2)} \\
 &= \frac{s+2-1}{(s+1)(s+2)} \\
 &= \frac{s+1}{(s+1)(s+2)} \\
 &= \frac{1}{s+2}.
 \end{aligned}$$

Now, we must determine the ROC of  $X$ . We know that the ROC of  $X$  must at least contain the intersection of the ROCs of  $X_1$  and  $X_2$ . Therefore, the ROC must contain  $\operatorname{Re}(s) > -1$ . Since  $X$  is rational, we also know that the ROC must be bounded by poles or extend to infinity. Since  $X$  has only one pole and this pole is at  $-2$ , the ROC must also include  $-2 < \operatorname{Re}(s) < -1$ . Therefore, the ROC of  $X$  is  $\operatorname{Re}(s) > -2$ . In effect, the pole at  $-1$  has been cancelled by a zero at the same location. As a result, the ROC of  $X$  is larger than the intersection of the ROCs of  $X_1$  and  $X_2$ . The various ROCs are illustrated in Figure 7.10. So, in conclusion, we have

$$X(s) = \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2. \quad \blacksquare$$

## 7.8.2 Time-Domain Shifting

The next property of the Laplace transform to be introduced is the time-domain shifting property, as given below.

**Theorem 7.2** (Time-domain shifting). *If  $x(t) \xleftrightarrow{\mathcal{L}} X(s)$  with ROC  $R$ , then*

$$x(t - t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0} X(s) \quad \text{with ROC } R,$$

where  $t_0$  is an arbitrary real constant.

*Proof.* To prove the above property, we proceed as follows. Let  $y(t) = x(t - t_0)$ , and let  $Y$  denote the Laplace transform of  $y$ . From the definition of the Laplace transform, we have

$$Y(s) = \int_{-\infty}^{\infty} x(t - t_0) e^{-st} dt.$$

Now, we perform a change of variable. Let  $\tau = t - t_0$  so that  $t = \tau + t_0$  and  $d\tau = dt$ . Applying this change of variable, we obtain

$$\begin{aligned}
 Y(s) &= \int_{-\infty}^{\infty} x(\tau) e^{-s(\tau+t_0)} d\tau \\
 &= e^{-st_0} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau} d\tau \\
 &= e^{-st_0} X(s).
 \end{aligned}$$

The ROC of  $Y$  is the same as the ROC of  $X$ , since  $Y$  and  $X$  differ only by a finite constant factor (i.e.,  $e^{-st_0}$ ). Thus, we have proven that the time-domain shifting property holds.  $\blacksquare$

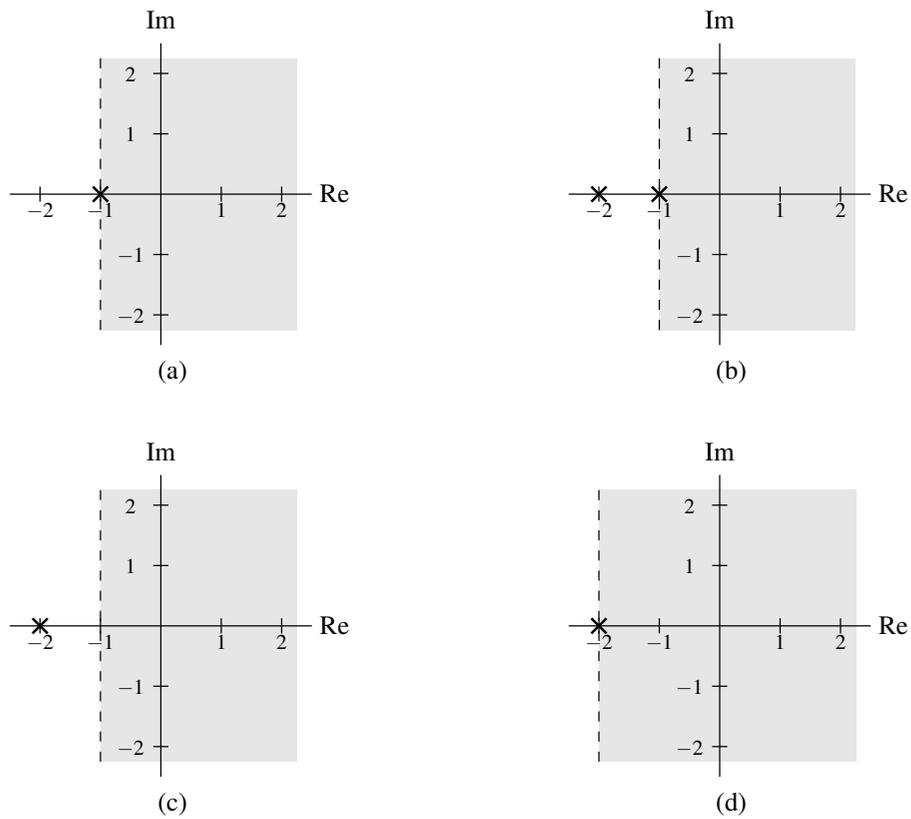


Figure 7.10: ROCs for the linearity example. The (a) ROC of  $X_1$ , (b) ROC of  $X_2$ , (c) ROC associated with the intersection of the ROCs of  $X_1$  and  $X_2$ , and (d) ROC of  $X$ .

**Example 7.10** (Time-domain shifting property). Find the Laplace transform  $X$  of

$$x(t) = u(t - 1).$$

*Solution.* From Table 7.2, we know that

$$u(t) \xleftrightarrow{\text{LT}} 1/s \text{ for } \text{Re}(s) > 0.$$

Using the time-domain shifting property, we can deduce

$$x(t) = u(t - 1) \xleftrightarrow{\text{LT}} X(s) = e^{-s} \left(\frac{1}{s}\right) \text{ for } \text{Re}(s) > 0.$$

Therefore, we have

$$X(s) = \frac{e^{-s}}{s} \text{ for } \text{Re}(s) > 0. \quad \blacksquare$$

### 7.8.3 Laplace-Domain Shifting

The next property of the Laplace transform to be introduced is the Laplace-domain shifting property, as given below.

**Theorem 7.3** (Laplace-domain shifting). *If  $x(t) \xleftrightarrow{\text{LT}} X(s)$  with ROC  $R$ , then*

$$e^{s_0 t} x(t) \xleftrightarrow{\text{LT}} X(s - s_0) \text{ with ROC } R + \text{Re}(s_0),$$

where  $s_0$  is an arbitrary complex constant. The ROCs are illustrated in Figure 7.11.

*Proof.* To prove the above property, we proceed as follows. Let  $y(t) = e^{s_0 t} x(t)$ , and let  $Y$  denote the Laplace transform of  $y$ . Using the definition of the Laplace transform and straightforward manipulation, we obtain

$$\begin{aligned} Y(s) &= \int_{-\infty}^{\infty} e^{s_0 t} x(t) e^{-st} dt \\ &= \int_{-\infty}^{\infty} x(t) e^{-(s-s_0)t} dt \\ &= X(s - s_0). \end{aligned}$$

Since  $Y(s + s_0) = X(s)$ ,  $Y$  converges at  $\lambda + s_0$  if and only if  $X$  converges at  $\lambda$ . Since the convergence properties of a Laplace transform only depend on the real part of the  $s$  parameter,  $Y$  converges at  $\lambda + \text{Re}(s_0)$  if and only if  $X$  converges at  $\lambda$ . Consequently, the ROC of  $Y$  is simply the ROC of  $X$  shifted by  $\text{Re}(s_0)$ . Thus, we have shown that the Laplace-domain shifting property holds.  $\blacksquare$

**Example 7.11** (Laplace-domain shifting property). Using only the properties of the Laplace transform and the transform pair

$$e^{-|t|} \xleftrightarrow{\text{LT}} \frac{2}{1-s^2} \text{ for } -1 < \text{Re}(s) < 1,$$

find the Laplace transform  $X$  of

$$x(t) = e^{5t} e^{-|t|}.$$

*Solution.* We are given

$$e^{-|t|} \xleftrightarrow{\text{LT}} \frac{2}{1-s^2} \text{ for } -1 < \text{Re}(s) < 1.$$

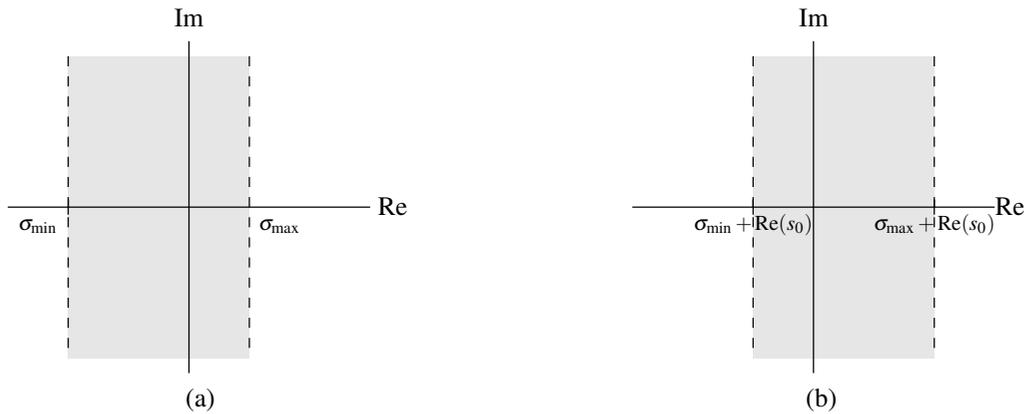


Figure 7.11: Regions of convergence for Laplace-domain shifting. (a) Before shift. (b) After shift.

Using the Laplace-domain shifting property, we can deduce

$$x(t) = e^{5t} e^{-|t|} \xleftrightarrow{\text{LT}} X(s) = \frac{2}{1 - (s-5)^2} \text{ for } -1 + 5 < \text{Re}(s) < 1 + 5,$$

Thus, we have

$$X(s) = \frac{2}{1 - (s-5)^2} \text{ for } 4 < \text{Re}(s) < 6.$$

Rewriting  $X$  in factored form, we have

$$X(s) = \frac{2}{1 - (s-5)^2} = \frac{2}{1 - (s^2 - 10s + 25)} = \frac{2}{-s^2 + 10s - 24} = \frac{-2}{s^2 - 10s + 24} = \frac{-2}{(s-6)(s-4)}.$$

Therefore, we have

$$X(s) = \frac{-2}{(s-4)(s-6)} \text{ for } 4 < \text{Re}(s) < 6. \quad \blacksquare$$

### 7.8.4 Time-Domain/Laplace-Domain Scaling

The next property of the Laplace transform to be introduced is the time-domain/Laplace-domain scaling property, as given below.

**Theorem 7.4** (Time-domain/Laplace-domain scaling). *If  $x(t) \xleftrightarrow{\text{LT}} X(s)$  with ROC  $R$ , then*

$$x(at) \xleftrightarrow{\text{LT}} \frac{1}{|a|} X\left(\frac{s}{a}\right) \text{ with ROC } R_1 = aR,$$

where  $a$  is a nonzero real constant.

*Proof.* To prove the above property, we proceed as below. Let  $y(t) = x(at)$ , and let  $Y$  denote that Laplace transform of  $y$ . From the definition of the Laplace transform, we have

$$Y(s) = \int_{-\infty}^{\infty} x(at) e^{-st} dt.$$

Now, we perform a change of variable. Let  $\tau = at$  so that  $t = \tau/a$  and  $d\tau = adt$ . Performing the change of variable (and being mindful of the change in the limits of integration), we obtain

$$\begin{aligned} Y(s) &= \begin{cases} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau/a} \left(\frac{1}{a}\right) d\tau & a > 0 \\ \int_{\infty}^{-\infty} x(\tau) e^{-s\tau/a} \left(\frac{1}{a}\right) d\tau & a < 0 \end{cases} \\ &= \begin{cases} \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau/a} d\tau & a > 0 \\ -\frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau/a} d\tau & a < 0. \end{cases} \end{aligned}$$

Combining the two cases for  $a$  (i.e.,  $a > 0$  and  $a < 0$ ), we obtain

$$\begin{aligned} Y(s) &= \frac{1}{|a|} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau/a} d\tau \\ &= \frac{1}{|a|} X\left(\frac{s}{a}\right). \end{aligned}$$

Since  $|a|Y(as) = X(s)$ ,  $Y$  converges at  $a\lambda$  if and only if  $X$  converges at  $\lambda$ . Thus, the ROC of  $Y$  is  $aR$ . Thus, we have proven that the scaling property holds. ■

The affect of time-domain scaling on the ROC of the Laplace transform is illustrated in Figure 7.12. Suppose that the ROC of the Laplace transform of a function  $x$  is as shown in Figure 7.12(a). Then, the ROC of the Laplace transform of the function  $y(t) = x(at)$  is as shown in Figure 7.12(b) for the case that  $a > 0$  and Figure 7.12(c) for the case that  $a < 0$ .

**Example 7.12** (Time-domain scaling property). Using only properties of the Laplace transform and the transform pair

$$e^{-|t|} \xleftrightarrow{\text{LT}} \frac{2}{1-s^2} \quad \text{for } -1 < \text{Re}(s) < 1,$$

find the Laplace transform of the function

$$x(t) = e^{-|3t|}.$$

*Solution.* We are given

$$e^{-|t|} \xleftrightarrow{\text{LT}} \frac{2}{1-s^2} \quad \text{for } -1 < \text{Re}(s) < 1.$$

Using the time-domain scaling property, we can deduce

$$x(t) = e^{-|3t|} \xleftrightarrow{\text{LT}} X(s) = \frac{1}{|3|} \frac{2}{1-\left(\frac{s}{3}\right)^2} \quad \text{for } 3(-1) < \text{Re}(s) < 3(1).$$

Thus, we have

$$X(s) = \frac{2}{3\left[1-\left(\frac{s}{3}\right)^2\right]} \quad \text{for } -3 < \text{Re}(s) < 3.$$

Simplifying, we have

$$X(s) = \frac{2}{3\left(1-\frac{s^2}{9}\right)} = \frac{2}{3\left(\frac{9-s^2}{9}\right)} = \frac{2(9)}{3(9-s^2)} = \frac{6}{9-s^2} = \frac{-6}{(s+3)(s-3)}.$$

Therefore, we have

$$X(s) = \frac{-6}{(s+3)(s-3)} \quad \text{for } -3 < \text{Re}(s) < 3. \quad \blacksquare$$

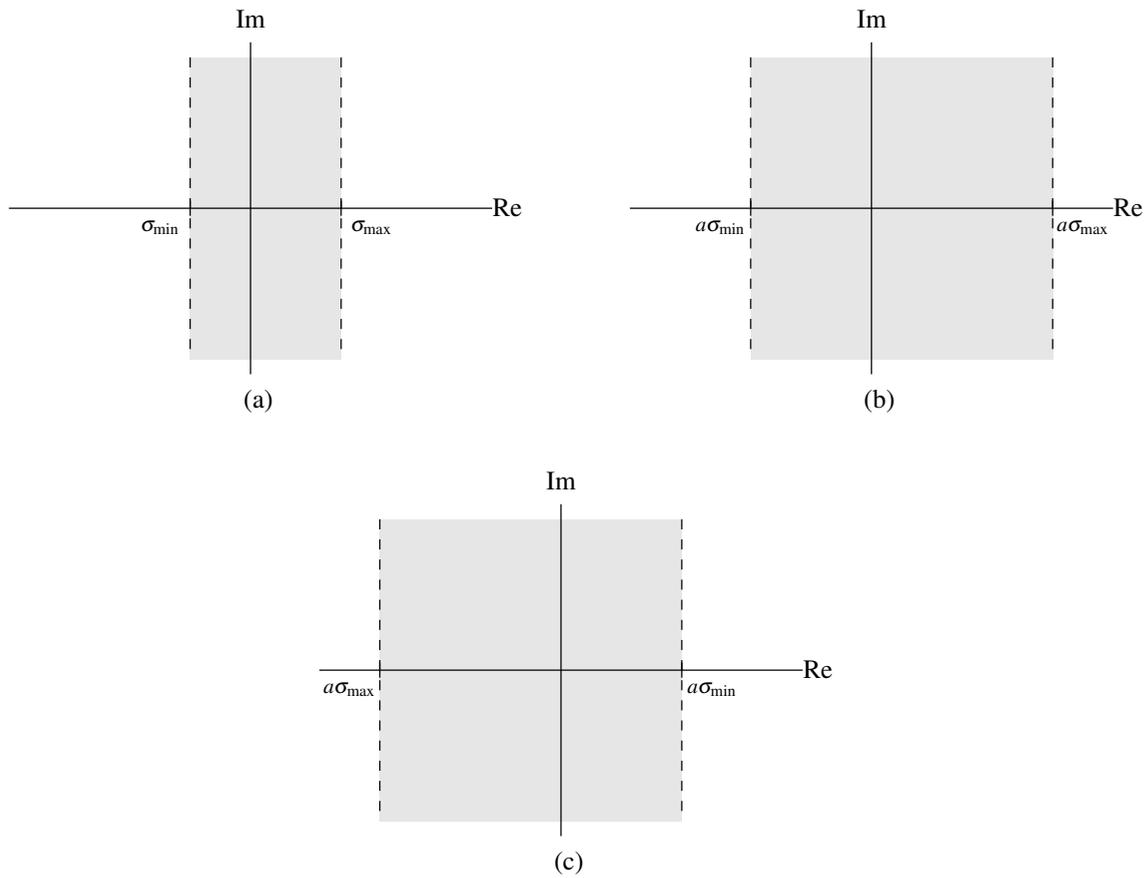


Figure 7.12: Regions of convergence for time-domain/Laplace-domain scaling. (a) Before scaling. After scaling for (b)  $a > 0$  and (c)  $a < 0$ .

### 7.8.5 Conjugation

The next property of the Laplace transform to be introduced is the conjugation property, as given below.

**Theorem 7.5** (Conjugation). *If  $x(t) \xrightarrow{\text{LT}} X(s)$  with ROC  $R$ , then*

$$x^*(t) \xrightarrow{\text{LT}} X^*(s^*) \text{ with ROC } R.$$

*Proof.* To prove the above property, we proceed as follows. Let  $y(t) = x^*(t)$ , let  $Y$  denote the Laplace transform of  $y$ , and let  $s = \sigma + j\omega$ , where  $\sigma$  and  $\omega$  are real. From the definition of the Laplace transform and the properties of conjugation, we can write

$$\begin{aligned} Y(s) &= \int_{-\infty}^{\infty} x^*(t) e^{-st} dt \\ &= \left[ \int_{-\infty}^{\infty} x^*(t) e^{-st} dt \right]^* \\ &= \left[ \int_{-\infty}^{\infty} [x^*(t)]^* (e^{-st})^* dt \right]^* \\ &= \left[ \int_{-\infty}^{\infty} x(t) (e^{-st})^* dt \right]^*. \end{aligned}$$

Now, we observe that  $(e^{-st})^* = e^{-s^*t}$ . Thus, we can write

$$\begin{aligned} Y(s) &= \left[ \int_{-\infty}^{\infty} x(t) e^{-s^*t} dt \right]^* \\ &= X^*(s^*). \end{aligned}$$

We determine the ROC of  $Y$  as follows. First, we observe that  $X(s) = Y^*(s^*)$ . Since  $Y^*(s^*) = X(s)$ ,  $Y$  converges at  $\lambda$  if and only if  $X$  converges at  $\lambda^*$ . We know, however, that convergence only depends on the real part of  $\lambda$ . So,  $Y$  converges at  $\lambda$  if and only if  $X$  converges at  $\lambda$ . From these results, we have that the ROC of  $Y$  must be identical to the ROC of  $X$ . Thus, we have shown that the conjugation property holds. ■

**Example 7.13** (Conjugation property). Using only properties of the Laplace transform and the transform pair

$$e^{(-1-j)t} u(t) \xrightarrow{\text{LT}} \frac{1}{s+1+j} \text{ for } \text{Re}(s) > -1,$$

find the Laplace transform of

$$x(t) = e^{(-1+j)t} u(t).$$

*Solution.* We are given

$$v(t) = e^{(-1-j)t} u(t) \xrightarrow{\text{LT}} V(s) = \frac{1}{s+1+j} \text{ for } \text{Re}(s) > -1.$$

We have

$$\begin{aligned} x(t) &= \left( \left( e^{(-1+j)t} u(t) \right)^* \right)^* \\ &= \left( \left( e^{(-1+j)t} \right)^* u^*(t) \right)^* \\ &= \left[ e^{(-1-j)t} u(t) \right]^*. \end{aligned}$$

Thus,  $x = v^*$ . Consequently, using the conjugation property, we can deduce

$$x(t) = e^{(-1+j)t}u(t) \xrightarrow{\text{LT}} X(s) = \left( \frac{1}{s^* + 1 + j} \right)^* \text{ for } \text{Re}(s) > -1.$$

Simplifying the algebraic expression for  $X$ , we have

$$X(s) = \left( \frac{1}{s^* + 1 + j} \right)^* = \frac{1^*}{[s^* + 1 + j]^*} = \frac{1}{s + 1 - j}.$$

Therefore, we can conclude

$$X(s) = \frac{1}{s + 1 - j} \text{ for } \text{Re}(s) > -1. \quad \blacksquare$$

### 7.8.6 Time-Domain Convolution

The next property of the Laplace transform to be introduced is the time-domain convolution property, as given below.

**Theorem 7.6** (Time-domain convolution). *If  $x_1(t) \xrightarrow{\text{LT}} X_1(s)$  with ROC  $R_1$  and  $x_2(t) \xrightarrow{\text{LT}} X_2(s)$  with ROC  $R_2$ , then*

$$x_1 * x_2(t) \xrightarrow{\text{LT}} X_1(s)X_2(s) \text{ with ROC containing } R_1 \cap R_2.$$

*Proof.* To prove the above property, we proceed as below. Let  $y(t) = x_1 * x_2(t)$ , and let  $Y$  denote the Laplace transform of  $y$ . From the definition of the Laplace transform and convolution, we have

$$\begin{aligned} Y(s) &= \mathcal{L} \left\{ \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau \right\} (s) \\ &= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau \right] e^{-st} dt \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)e^{-st} d\tau dt. \end{aligned}$$

Changing the order of integration, we have

$$\begin{aligned} Y(s) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)e^{-st} dt d\tau \\ &= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x_2(t-\tau)e^{-st} dt \right] x_1(\tau)d\tau. \end{aligned}$$

Now, we perform a change of variable. Let  $v = t - \tau$  so that  $t = v + \tau$  and  $dv = dt$ . Applying the change of variable and simplifying, we obtain

$$\begin{aligned} Y(s) &= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x_2(v)e^{-s(v+\tau)}dv \right] x_1(\tau)d\tau \\ &= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x_2(v)e^{-sv}dv \right] e^{-s\tau}x_1(\tau)d\tau \\ &= \left[ \int_{-\infty}^{\infty} x_2(v)e^{-sv}dv \right] \left[ \int_{-\infty}^{\infty} e^{-s\tau}x_1(\tau)d\tau \right] \\ &= X_1(s)X_2(s). \end{aligned}$$

Now, we consider the ROC of  $Y$ . If  $X_1$  and  $X_2$  both converge at some  $\lambda$ , then  $Y(s) = X_1(s)X_2(s)$  must also converge at  $\lambda$ . Therefore, the ROC of  $Y$  must contain the intersection of the ROCs of  $X_1$  and  $X_2$ . Thus, we have shown that the time-domain convolution property holds.  $\blacksquare$

In the preceding theorem, note that the ROC of the result can be larger than  $R_1 \cap R_2$ . When  $X_1$  and  $X_2$  are rational functions, this can only happen if pole-zero cancellation occurs in the expression  $X_1(s)X_2(s)$ .

The time-domain convolution property of the Laplace transform has important practical implications. Since the Laplace transform effectively converts a convolution into a multiplication, the Laplace transform can be used as a means to avoid directly dealing with convolution operations. This is often extremely helpful when working with (CT) LTI systems, for example, since such systems fundamentally involve convolution.

**Example 7.14** (Time-domain convolution property). Find the Laplace transform  $X$  of the function

$$x(t) = x_1 * x_2(t),$$

where

$$x_1(t) = (\sin 3t)u(t) \quad \text{and} \quad x_2(t) = tu(t).$$

*Solution.* From Table 7.2, we have that

$$\begin{aligned} x_1(t) = (\sin 3t)u(t) &\xleftrightarrow{\text{LT}} X_1(s) = \frac{3}{s^2 + 9} \quad \text{for } \text{Re}(s) > 0 \quad \text{and} \\ x_2(t) = tu(t) &\xleftrightarrow{\text{LT}} X_2(s) = \frac{1}{s^2} \quad \text{for } \text{Re}(s) > 0. \end{aligned}$$

Using the time-domain convolution property, we have

$$x(t) \xleftrightarrow{\text{LT}} X(s) = \left( \frac{3}{s^2 + 9} \right) \left( \frac{1}{s^2} \right) \quad \text{for } \{\text{Re}(s) > 0\} \cap \{\text{Re}(s) > 0\}.$$

The ROC of  $X$  is  $\{\text{Re}(s) > 0\} \cap \{\text{Re}(s) > 0\}$  (as opposed to a superset thereof), since no pole-zero cancellation occurs. Simplifying the expression for  $X$ , we conclude

$$X(s) = \frac{3}{s^2(s^2 + 9)} \quad \text{for } \text{Re}(s) > 0. \quad \blacksquare$$

### 7.8.7 Time-Domain Differentiation

The next property of the Laplace transform to be introduced is the time-domain differentiation property, as given below.

**Theorem 7.7** (Time-domain differentiation). *If  $x(t) \xleftrightarrow{\text{LT}} X(s)$  with ROC  $R$ , then*

$$\frac{dx(t)}{dt} \xleftrightarrow{\text{LT}} sX(s) \quad \text{with ROC containing } R.$$

*Proof.* To prove the above property, we proceed as follows. Let  $\mathcal{D}$  denote that derivative operator, let  $y = \mathcal{D}x$ , and let  $Y$  denote the Laplace transform of  $y$ . From the definition of the inverse Laplace transform, we have

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st} ds.$$

Differentiating both sides of this equation with respect to  $t$ , we have

$$y(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st} ds.$$

Observing that the right-hand side of the above equation is simply the inverse Laplace transform of  $sX(s)$ , we can write

$$y(t) = \mathcal{L}^{-1}\{sX(s)\}(t).$$

Taking the Laplace transform of both sides yields

$$Y(s) = sX(s).$$

Now, we consider the ROC of  $Y(s) = sX(s)$ . Clearly,  $Y$  must converge at  $\lambda$  if  $X$  converges at  $\lambda$ . Since multiplication by  $s$  has the potential to cancel a pole in  $X$ , it is possible that the ROC of  $Y$  may be larger than the ROC of  $X$ . Consequently, the ROC of  $Y$  must at least contain  $R$ . Thus, we have shown that the time-domain differentiation property holds. ■

In the preceding theorem, note that the ROC of the result can be larger than  $R$ . When  $X$  is a rational function, this can only happen if pole-zero cancellation occurs in the expression  $sX(s)$ .

The time-domain differentiation property of the Laplace transform has important practical implications. Since the Laplace transform effectively converts differentiation into multiplication (by  $s$ ), the Laplace transform can be used as a means to avoid directly dealing with differentiation operations. This can often be beneficial when working with differential and integro-differential equations, for example.

**Example 7.15** (Time-domain differentiation property). Find the Laplace transform  $X$  of the function

$$x(t) = \frac{d}{dt}\delta(t).$$

*Solution.* From Table 7.2, we have that

$$\delta(t) \xleftrightarrow{\text{LT}} 1 \text{ for all } s.$$

Using the time-domain differentiation property, we can deduce

$$x(t) = \frac{d}{dt}\delta(t) \xleftrightarrow{\text{LT}} X(s) = s(1) \text{ for all } s.$$

Therefore, we have

$$X(s) = s \text{ for all } s. \quad \blacksquare$$

### 7.8.8 Laplace-Domain Differentiation

The next property of the Laplace transform to be introduced is the Laplace-domain differentiation property, as given below.

**Theorem 7.8** (Laplace-domain differentiation). *If  $x(t) \xleftrightarrow{\text{LT}} X(s)$  with ROC  $R$ , then*

$$-tx(t) \xleftrightarrow{\text{LT}} \frac{d}{ds}X(s) \text{ with ROC } R.$$

*Proof.* To prove the above property, we proceed as follows. Let  $y(t) = -tx(t)$  and let  $Y$  denote the Laplace transform of  $y$ . From the definition of the Laplace transform, we have

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt.$$

Differentiating both sides of the above equation with respect to  $s$ , we obtain

$$\begin{aligned} \frac{d}{ds}X(s) &= \int_{-\infty}^{\infty} -tx(t)e^{-st} dt \\ &= Y(s). \end{aligned}$$

Thus, we have shown that the Laplace-domain differentiation property holds. ■

**Example 7.16** (Laplace-domain differentiation property). Using only the properties of the Laplace transform and the transform pair

$$e^{-2t}u(t) \xleftrightarrow{\text{LT}} \frac{1}{s+2} \quad \text{for } \text{Re}(s) > -2,$$

find the Laplace transform of the function

$$x(t) = te^{-2t}u(t).$$

*Solution.* We are given

$$e^{-2t}u(t) \xleftrightarrow{\text{LT}} \frac{1}{s+2} \quad \text{for } \text{Re}(s) > -2.$$

Using the Laplace-domain differentiation and linearity properties, we can deduce

$$x(t) = te^{-2t}u(t) \xleftrightarrow{\text{LT}} X(s) = -\frac{d}{ds} \left( \frac{1}{s+2} \right) \quad \text{for } \text{Re}(s) > -2.$$

Simplifying the algebraic expression for  $X$ , we have

$$X(s) = -\frac{d}{ds} \left( \frac{1}{s+2} \right) = -\frac{d}{ds} (s+2)^{-1} = (-1)(-1)(s+2)^{-2} = \frac{1}{(s+2)^2}.$$

Therefore, we conclude

$$X(s) = \frac{1}{(s+2)^2} \quad \text{for } \text{Re}(s) > -2. \quad \blacksquare$$

### 7.8.9 Time-Domain Integration

The next property of the Laplace transform to be introduced is the time-domain integration property, as given below.

**Theorem 7.9** (Time-domain integration). *If  $x(t) \xleftrightarrow{\text{LT}} X(s)$  with ROC  $R$ , then*

$$\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{\text{LT}} \frac{1}{s} X(s) \quad \text{with ROC containing } R \cap \{\text{Re}(s) > 0\}.$$

*Proof.* To prove the above property, we proceed as follows. Let  $y(t) = \int_{-\infty}^t x(\tau) d\tau$ , and let  $Y$  and  $U$  denote the Laplace transforms of  $y$  and  $u$ , respectively. First, we observe that

$$y(t) = x * u(t).$$

Taking the Laplace transform of both sides of this equation, we have

$$Y(s) = \mathcal{L}\{x * u\}(s).$$

From the time-domain convolution property of the Laplace transform, we have

$$\begin{aligned} Y(s) &= \mathcal{L}x(s)\mathcal{L}u(s) \\ &= X(s)U(s). \end{aligned}$$

From Example 7.5, we know that  $U(s) = \frac{1}{s}$  for  $\text{Re}(s) > 0$ . So,

$$Y(s) = \frac{1}{s} X(s).$$

Now, we need to consider the ROC of  $Y$ . Clearly,  $Y$  must converge at  $\lambda$  if  $X$  and  $U$  both converge at  $\lambda$ . Consequently, the ROC of  $Y$  must contain the intersection of the ROCs of  $X$  and  $U$ . Since  $U$  converges for  $\text{Re}(s) > 0$ , the ROC of  $Y$  must contain  $R \cap \{\text{Re}(s) > 0\}$ . Thus, we have shown that the time-domain integration property holds.  $\blacksquare$

In the preceding theorem, note that the ROC of the result can be larger than  $R \cap \{\operatorname{Re}(s) > 0\}$ . When  $X$  is a rational function, this can only happen if pole-zero cancellation occurs in the expression  $\frac{1}{s}X(s)$ .

The time-domain integration property of the Laplace transform has important practical implications. Since the Laplace transform effectively converts integration into division (by  $s$ ), the Laplace transform can be used as a means to avoid directly dealing with integration operations. This can often be beneficial when working with integral and integro-differential equations, for example.

**Example 7.17** (Time-domain integration property). Find the Laplace transform of the function

$$x(t) = \int_{-\infty}^t e^{-2\tau} [\sin \tau] u(\tau) d\tau.$$

*Solution.* From Table 7.2, we have that

$$e^{-2t} [\sin t] u(t) \xleftrightarrow{\text{LT}} \frac{1}{(s+2)^2 + 1} \text{ for } \operatorname{Re}(s) > -2.$$

Using the time-domain integration property, we can deduce

$$x(t) = \int_{-\infty}^t e^{-2\tau} [\sin \tau] u(\tau) d\tau \xleftrightarrow{\text{LT}} X(s) = \frac{1}{s} \left( \frac{1}{(s+2)^2 + 1} \right) \text{ for } \{\operatorname{Re}(s) > -2\} \cap \{\operatorname{Re}(s) > 0\}.$$

The ROC of  $X$  is  $\{\operatorname{Re}(s) > -2\} \cap \{\operatorname{Re}(s) > 0\}$  (as opposed to a superset thereof), since no pole-zero cancellation takes place. Simplifying the algebraic expression for  $X$ , we have

$$X(s) = \frac{1}{s} \left( \frac{1}{(s+2)^2 + 1} \right) = \frac{1}{s} \left( \frac{1}{s^2 + 4s + 4 + 1} \right) = \frac{1}{s} \left( \frac{1}{s^2 + 4s + 5} \right).$$

Therefore, we have

$$X(s) = \frac{1}{s(s^2 + 4s + 5)} \text{ for } \operatorname{Re}(s) > 0.$$

[Note:  $s^2 + 4s + 5 = (s + 2 - j)(s + 2 + j)$ .] ■

### 7.8.10 Initial and Final Value Theorems

The next properties of the Laplace transform to be introduced are known as the initial and final value theorems, as given below.

**Theorem 7.10** (Initial value theorem). *Let  $x$  be a function with the Laplace transform  $X$ . If  $x$  is causal and contains no impulses or higher order singularities at the origin, then*

$$x(0^+) = \lim_{s \rightarrow \infty} sX(s),$$

where  $x(0^+)$  denotes the limit of  $x(t)$  as  $t$  approaches zero from positive values of  $t$ .

*Proof.* To prove the above property, we proceed as below. First, we expand  $x$  as a Taylor series at  $0^+$ .

$$x(t) = \left[ x(0^+) + x^{(1)}(0^+)t + \dots + x^{(n)}(0^+) \frac{t^n}{n!} + \dots \right] u(t),$$

where  $x^{(n)}$  denotes the  $n$ th derivative of  $x$ . Taking the Laplace transform of the above equation, and using the fact that  $\mathcal{L}\{t^n u(t)\}(s) = \frac{n!}{s^{n+1}}$ , we can write

$$X(s) = x(0^+) \frac{1}{s} + x^{(1)}(0^+) \frac{1}{s^2} + \dots + x^{(n)}(0^+) \frac{1}{s^{n+1}} + \dots$$

Multiplying both sides of this equation by  $s$ , we obtain

$$sX(s) = x(0^+) + x^{(1)}(0^+) \frac{1}{s} + \dots + x^{(n)}(0^+) \frac{1}{s^n} + \dots$$

Taking the limit as  $s \rightarrow \infty$ , we have

$$\lim_{s \rightarrow \infty} sX(s) = x(0^+).$$

Thus, the initial value theorem holds. ■

**Theorem 7.11** (Final value theorem). *Let  $x$  be a function with the Laplace transform  $X$ . If  $x$  is causal and  $x(t)$  has a finite limit as  $t \rightarrow \infty$ , then*

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s).$$

*Proof.* Let  $x'$  denote the derivative of  $x$ . The differentiation property of the unilateral Laplace transform (introduced later in Section 7.17) states that  $s\mathcal{L}_u x(s) - x(0^-) = \mathcal{L}_u x'(s)$ . Furthermore, since  $x$  is causal, this property can be rewritten as  $s\mathcal{L}x(s) - x(0^-) = \mathcal{L}_u x'(s)$ . Thus, we have that

$$sX(s) - x(0^-) = \int_{0^-}^{\infty} x'(t)e^{-st} dt.$$

Taking the limit of both sides of this equation as  $s \rightarrow 0$ , we obtain

$$\begin{aligned} \lim_{s \rightarrow 0} [sX(s) - x(0^-)] &= \lim_{s \rightarrow 0} \int_{0^-}^{\infty} x'(t)e^{-st} dt \\ &= \int_{0^-}^{\infty} x'(t) dt \\ &= x(t) \Big|_{0^-}^{\infty} \\ &= \lim_{t \rightarrow \infty} x(t) - x(0^-). \end{aligned}$$

Thus, we have

$$\lim_{s \rightarrow 0} sX(s) - x(0^-) = \lim_{t \rightarrow \infty} x(t) - x(0^-).$$

Adding  $x(0^-)$  to both sides of the equation yields

$$\lim_{s \rightarrow 0} sX(s) = \lim_{t \rightarrow \infty} x(t). \quad \blacksquare$$

**Example 7.18** (Initial and final value theorems). A bounded causal function  $x$  with a (finite) limit at infinity has the Laplace transform

$$X(s) = \frac{2s^2 + 3s + 2}{s^3 + 2s^2 + 2s} \quad \text{for } \operatorname{Re}(s) > 0.$$

Determine  $x(0^+)$  and  $\lim_{t \rightarrow \infty} x(t)$ .

*Solution.* Since  $x$  is causal (i.e.,  $x(t) = 0$  for all  $t < 0$ ) and does not have any singularities at the origin, the initial value theorem can be applied. From this theorem, we have

$$\begin{aligned} x(0^+) &= \lim_{s \rightarrow \infty} sX(s) \\ &= \lim_{s \rightarrow \infty} s \left[ \frac{2s^2 + 3s + 2}{s^3 + 2s^2 + 2s} \right] \\ &= \lim_{s \rightarrow \infty} \frac{2s^2 + 3s + 2}{s^2 + 2s + 2} \\ &= 2. \end{aligned}$$

Since  $x$  is bounded and causal and has well-defined limit at infinity, we can apply the final value theorem. From this theorem, we have

$$\begin{aligned}\lim_{t \rightarrow \infty} x(t) &= \lim_{s \rightarrow 0} sX(s) \\ &= \lim_{s \rightarrow 0} s \left[ \frac{2s^2 + 3s + 2}{s^3 + 2s^2 + 2s} \right] \\ &= \left. \frac{2s^2 + 3s + 2}{s^2 + 2s + 2} \right|_{s=0} \\ &= 1.\end{aligned}$$

In passing, we note that the inverse Laplace transform  $x$  of  $X$  can be shown to be

$$x(t) = [1 + e^{-t} \cos t]u(t).$$

As we would expect, the values calculated above for  $x(0^+)$  and  $\lim_{t \rightarrow \infty} x(t)$  are consistent with this formula for  $x$ . ■

Amongst other things, the initial and final value theorems can be quite useful in checking for errors in Laplace transform calculations. For example, suppose that we are asked to compute the Laplace transform  $X$  of the function  $x$ . If we were to make a mistake in this computation, the values obtained for  $x(0)$  and  $\lim_{t \rightarrow \infty} x(t)$  using  $X$  with the initial and final value theorems and using  $x$  directly would most likely disagree. In this manner, we can relatively easily detect some types of errors in Laplace transform calculations.

## 7.9 More Laplace Transform Examples

Earlier in this chapter, we derived a number of Laplace transform pairs. Some of these and other important transform pairs are listed in Table 7.2. Using the various Laplace transform properties listed in Table 7.1 and the Laplace transform pairs listed in Table 7.2, we can more easily determine the Laplace transform of more complicated functions.

**Example 7.19.** Using properties of the Laplace transform and the Laplace transform pair

$$e^{-a|t|} \xleftrightarrow{\text{LT}} \frac{-2a}{(s+a)(s-a)} \text{ for } -a < \text{Re}(s) < a,$$

find the Laplace transform  $X$  of the function

$$x(t) = e^{-5|3t-7|}.$$

*Solution.* We begin by re-expressing  $x$  in terms of the following equations:

$$\begin{aligned}v_1(t) &= e^{-5|t|}, \\ v_2(t) &= v_1(t-7), \quad \text{and} \\ x(t) &= v_2(3t).\end{aligned}$$

In what follows, let  $R_{V_1}$ ,  $R_{V_2}$ , and  $R_X$  denote the ROCs of  $V_1$ ,  $V_2$ , and  $X$ , respectively. Taking the Laplace transform of the above three equations, we obtain

$$\begin{aligned}V_1(s) &= \frac{-10}{(s+5)(s-5)}, \quad R_{V_1} = (-5 < \text{Re}(s) < 5), \\ V_2(s) &= e^{-7s}V_1(s), \quad R_{V_2} = R_{V_1}, \\ X(s) &= \frac{1}{3}V_2(s/3), \quad \text{and} \quad R_X = 3R_{V_2}.\end{aligned}$$

Table 7.1: Properties of the (bilateral) Laplace transform

Property	Time Domain	Laplace Domain	ROC
Linearity	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(s) + a_2X_2(s)$	At least $R_1 \cap R_2$
Time-Domain Shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	$R$
Laplace-Domain Shifting	$e^{s_0t}x(t)$	$X(s - s_0)$	$R + \text{Re}(s_0)$
Time/Laplace-Domain Scaling	$x(at)$	$\frac{1}{ a }X\left(\frac{s}{a}\right)$	$aR$
Conjugation	$x^*(t)$	$X^*(s^*)$	$R$
Time-Domain Convolution	$x_1 * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$sX(s)$	At least $R$
Laplace-Domain Differentiation	$-tx(t)$	$\frac{d}{ds}X(s)$	$R$
Time-Domain Integration	$\int_{-\infty}^t x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R \cap \{\text{Re}(s) > 0\}$

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 Property
 

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Initial Value Theorem  $x(0^+) = \lim_{s \rightarrow \infty} sX(s)$

Final Value Theorem  $\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s)$ 


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Table 7.2: Transform pairs for the (bilateral) Laplace transform

Pair	$x(t)$	$X(s)$	ROC
1	$\delta(t)$	1	All $s$
2	$u(t)$	$\frac{1}{s}$	$\text{Re}(s) > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\text{Re}(s) < 0$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\text{Re}(s) > 0$
5	$-t^n u(-t)$	$\frac{n!}{s^{n+1}}$	$\text{Re}(s) < 0$
6	$e^{-at} u(t)$	$\frac{1}{s+a}$	$\text{Re}(s) > -a$
7	$-e^{-at} u(-t)$	$\frac{1}{s+a}$	$\text{Re}(s) < -a$
8	$t^n e^{-at} u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\text{Re}(s) > -a$
9	$-t^n e^{-at} u(-t)$	$\frac{n!}{(s+a)^{n+1}}$	$\text{Re}(s) < -a$
10	$(\cos \omega_0 t) u(t)$	$\frac{s}{s^2 + \omega_0^2}$	$\text{Re}(s) > 0$
11	$(\sin \omega_0 t) u(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$	$\text{Re}(s) > 0$
12	$(e^{-at} \cos \omega_0 t) u(t)$	$\frac{s+a}{(s+a)^2 + \omega_0^2}$	$\text{Re}(s) > -a$
13	$(e^{-at} \sin \omega_0 t) u(t)$	$\frac{\omega_0}{(s+a)^2 + \omega_0^2}$	$\text{Re}(s) > -a$

Combining the above equations, we have

$$\begin{aligned} X(s) &= \frac{1}{3}V_2(s/3) \\ &= \frac{1}{3}e^{-7(s/3)}V_1(s/3) \\ &= \frac{1}{3}e^{-7s/3}V_1(s/3) \\ &= \frac{1}{3}e^{-7s/3} \frac{-10}{(s/3+5)(s/3-5)} \quad \text{and} \end{aligned}$$

$$\begin{aligned} R_X &= 3R_{V_2} \\ &= 3R_{V_1} \\ &= 3(-5 < \operatorname{Re}(s) < 5) \\ &= -15 < \operatorname{Re}(s) < 15. \end{aligned}$$

Thus, we have shown that

$$X(s) = \frac{1}{3}e^{-7s/3} \frac{-10}{(s/3+5)(s/3-5)} \quad \text{for } -15 < \operatorname{Re}(s) < 15. \quad \blacksquare$$

**Example 7.20.** Find the Laplace transform  $X$  of the function

$$x(t) = [e^{-t} + e^{-2t}]u(t).$$

*Solution.* We can calculate  $X$  as

$$\begin{aligned} X(s) &= \mathcal{L}\{[e^{-t} + e^{-2t}]u(t)\}(s) \\ &= \mathcal{L}\{e^{-t}u(t)\}(s) + \mathcal{L}\{e^{-2t}u(t)\}(s) \\ &= \frac{1}{s+1} + \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -1 \cap \operatorname{Re}(s) > -2 \\ &= \frac{s+2+s+1}{(s+1)(s+2)} \\ &= \frac{2s+3}{(s+1)(s+2)} \quad \text{for } \operatorname{Re}(s) > -1. \end{aligned}$$

Thus, we have that

$$X(s) = \frac{2s+3}{(s+1)(s+2)} \quad \text{for } \operatorname{Re}(s) > -1. \quad \blacksquare$$

**Example 7.21.** Find the Laplace transform  $X$  of the function

$$x(t) = [e^{-2t} + e^{-3t}]u(t-1).$$

*Solution.* To begin, we rewrite  $x$  as

$$x(t) = [e^{-2t} + e^{-3t}]v_1(t)$$

where

$$v_1(t) = u(t-1).$$

Taking the Laplace transform of the preceding equations, we have

$$\begin{aligned} V_1(s) &= \mathcal{L}\{u(t-1)\}(s) \\ &= e^{-s} \left(\frac{1}{s}\right) \quad \text{for } \operatorname{Re}(s) > 0. \end{aligned}$$

and

$$\begin{aligned} X(s) &= \mathcal{L}\{[e^{-2t} + e^{-3t}]v_1(t)\}(s) \\ &= \mathcal{L}\{e^{-2t}v_1(t)\}(s) + \mathcal{L}\{e^{-3t}v_1(t)\}(s). \end{aligned}$$

Now, we focus on simplifying the preceding expression for  $X$ . Let  $R_{V_1}$  denote the ROC of  $V_1$ . Then, we have

$$\begin{aligned} \mathcal{L}\{e^{-2t}v_1(t)\}(s) &= V_1(s+2) \quad \text{for } s \in R_{V_1} - 2 \quad \text{and} \\ \mathcal{L}\{e^{-3t}v_1(t)\}(s) &= V_1(s+3) \quad \text{for } s \in R_{V_1} - 3. \end{aligned}$$

Thus, we have

$$\begin{aligned} X(s) &= \mathcal{L}\{e^{-2t}v_1(t)\}(s) + \mathcal{L}\{e^{-3t}v_1(t)\}(s) \\ &= V_1(s+2) + V_1(s+3) \quad \text{for } (R_{V_1} - 2) \cap (R_{V_1} - 3). \end{aligned}$$

Substituting the earlier expression for  $V_1$  into the above equation yields

$$\begin{aligned} X(s) &= e^{-(s+2)} \frac{1}{s+2} + e^{-(s+3)} \frac{1}{s+3} \quad \text{for } (\operatorname{Re}(s) > -2) \cap (\operatorname{Re}(s) > -3) \\ &= e^{-(s+2)} \frac{1}{s+2} + e^{-(s+3)} \frac{1}{s+3} \quad \text{for } \operatorname{Re}(s) > -2. \end{aligned}$$

Thus, we have that

$$X(s) = e^{-(s+2)} \frac{1}{s+2} + e^{-(s+3)} \frac{1}{s+3} \quad \text{for } \operatorname{Re}(s) > -2. \quad \blacksquare$$

**Example 7.22.** Find the Laplace transform  $X$  of the function

$$x(t) = \delta(t) + u(t).$$

*Solution.* Taking the Laplace transform of  $x$ , we have

$$\begin{aligned} X(s) &= \mathcal{L}\{\delta + u\}(s) \\ &= \mathcal{L}\delta(s) + \mathcal{L}u(s). \end{aligned}$$

From Table 7.2, we have

$$\begin{aligned} \delta(t) &\xleftrightarrow{\text{LT}} 1 \quad \text{for all } s \quad \text{and} \\ u(t) &\xleftrightarrow{\text{LT}} \frac{1}{s} \quad \text{for } \operatorname{Re}(s) > 0. \end{aligned}$$

Substituting the above results, we obtain

$$\begin{aligned} X(s) &= 1 + \frac{1}{s} \quad \text{for } \operatorname{Re}(s) > 0 \\ &= \frac{s+1}{s}. \end{aligned}$$

Thus, we have that

$$X(s) = \frac{s+1}{s} \quad \text{for } \operatorname{Re}(s) > 0. \quad \blacksquare$$

**Example 7.23.** Find the Laplace transform  $X$  of the function

$$x(t) = te^{-3|t|}.$$

*Solution.* To begin, we rewrite  $x$  as

$$x(t) = te^{3t}u(-t) + te^{-3t}u(t).$$

Taking the Laplace transform of  $x$ , we have

$$\begin{aligned} X(s) &= \mathcal{L}\{te^{3t}u(-t) + te^{-3t}u(t)\}(s) \\ &= \mathcal{L}\{te^{3t}u(-t)\}(s) + \mathcal{L}\{te^{-3t}u(t)\}(s) \\ &= \frac{-(1!)}{(s-3)^2} + \frac{1!}{(s+3)^2} \quad \text{for } \operatorname{Re}(s) > -3 \cap \operatorname{Re}(s) < 3 \\ &= \frac{-(s+3)^2 + (s-3)^2}{(s+3)^2(s-3)^2} \\ &= \frac{-(s^2 + 6s + 9) + s^2 - 6s + 9}{(s+3)^2(s-3)^2} \\ &= \frac{-12s}{(s+3)^2(s-3)^2}. \end{aligned}$$

Thus, we have that

$$X(s) = \frac{-12s}{(s+3)^2(s-3)^2} \quad \text{for } -3 < \operatorname{Re}(s) < 3. \quad \blacksquare$$

**Example 7.24.** Consider the function

$$y(t) = e^{-t}[x * x(t)].$$

Let  $X$  and  $Y$  denote the Laplace transforms of  $x$  and  $y$ , respectively. Find an expression for  $Y$  in terms of  $X$ .

*Solution.* To begin, we rewrite  $y$  as

$$y(t) = e^{-t}v_1(t)$$

where

$$v_1(t) = x * x(t).$$

Let  $V_1$  denote the Laplace transform of  $v$ . Let  $R_X$ ,  $R_Y$ , and  $R_V$  denote the ROCs of  $X$ ,  $Y$ , and  $V$ , respectively. Taking the Laplace transform of the above equations yields

$$\begin{aligned} V_1(s) &= \mathcal{L}v_1(s) \\ &= \mathcal{L}\{x * x\}(s) \\ &= \mathcal{L}x(s)\mathcal{L}x(s) \\ &= X^2(s) \quad \text{for } s \in R_X \quad \text{and} \\ Y(s) &= \mathcal{L}\{e^{-t}v_1(t)\}(s) \\ &= V_1(s+1) \quad \text{for } s \in R_{V_1} - 1. \end{aligned}$$

Substituting the above expression for  $V_1$  into the above formula for  $Y$ , we have

$$\begin{aligned} Y(s) &= V_1(s+1) \\ &= X^2(s+1) \quad \text{for } s \in R_X - 1. \quad \blacksquare \end{aligned}$$

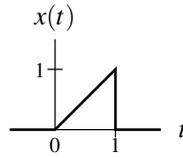


Figure 7.13: Function for the Laplace transform example.

**Example 7.25.** Using a Laplace transform table and properties of the Laplace transform, find the Laplace transform  $X$  of the function  $x$  shown in Figure 7.13.

*First solution (which incurs more work due to differentiation).* First, we express  $x$  using unit-step functions. We have

$$\begin{aligned} x(t) &= t[u(t) - u(t-1)] \\ &= tu(t) - tu(t-1). \end{aligned}$$

Taking the Laplace transform of both sides of the preceding equation, we obtain

$$X(s) = \mathcal{L}\{tu(t)\}(s) - \mathcal{L}\{tu(t-1)\}(s).$$

We have

$$\mathcal{L}\{tu(t)\}(s) = \frac{1}{s^2} \quad \text{and}$$

$$\begin{aligned} \mathcal{L}\{tu(t-1)\}(s) &= -\mathcal{L}\{-tu(t-1)\}(s) = -\frac{d}{ds} \left( \frac{e^{-s}}{s} \right) \\ &= -\left( \frac{d}{ds} e^{-s} s^{-1} \right) \\ &= -(-e^{-s} s^{-1} - s^{-2} e^{-s}) \\ &= \frac{e^{-s}}{s} + \frac{e^{-s}}{s^2}. \end{aligned}$$

Combining the above results, we have

$$\begin{aligned} X(s) &= \frac{1}{s^2} - \left[ \frac{e^{-s}}{s} + \frac{e^{-s}}{s^2} \right] \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} \\ &= \frac{1 - se^{-s} - e^{-s}}{s^2}. \end{aligned}$$

Since  $x$  is finite duration, the ROC of  $X$  is the entire complex plane.

*Second solution (which incurs less work by avoiding differentiation).* First, we express  $x$  using unit-step functions. We have

$$\begin{aligned} x(t) &= t[u(t) - u(t-1)] = tu(t) - tu(t-1) \\ &= tu(t) - (t-1)u(t-1) - u(t-1). \end{aligned}$$

Taking the Laplace transform of both sides of the preceding equation, we obtain

$$X(s) = \mathcal{L}\{tu(t)\}(s) - \mathcal{L}\{(t-1)u(t-1)\}(s) - \mathcal{L}\{u(t-1)\}(s).$$

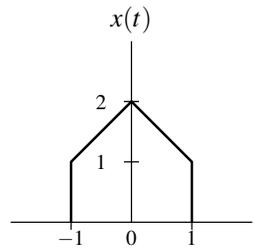


Figure 7.14: Function for the Laplace transform example.

We have

$$\begin{aligned}\mathcal{L}\{tu(t)\}(s) &= \frac{1}{s^2}, \\ \mathcal{L}\{(t-1)u(t-1)\}(s) &= e^{-s}\mathcal{L}\{tu(t)\}(s) = e^{-s}\frac{1}{s^2} = \frac{e^{-s}}{s^2}, \quad \text{and} \\ \mathcal{L}\{u(t-1)\}(s) &= \frac{e^{-s}}{s}.\end{aligned}$$

Combining the above results, we have

$$\begin{aligned}X(s) &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} \\ &= \frac{1 - e^{-s} - se^{-s}}{s^2}.\end{aligned}$$

Since  $x$  is finite duration, the ROC of  $X$  is the entire complex plane. ■

**Example 7.26.** Find the Laplace transform  $X$  of the function  $x$  shown in Figure 7.14.

*First solution (which incurs more work due to differentiation).* First, we express  $x$  using unit-step functions. We have

$$\begin{aligned}x(t) &= (t+2)[u(t+1) - u(t)] + (-t+2)[u(t) - u(t-1)] \\ &= tu(t+1) - tu(t) + 2u(t+1) - 2u(t) - tu(t) + tu(t-1) + 2u(t) - 2u(t-1) \\ &= tu(t+1) + 2u(t+1) - 2tu(t) + tu(t-1) - 2u(t-1).\end{aligned}$$

Taking the Laplace transform of both sides of the preceding equation, we obtain

$$\begin{aligned}X(s) &= -\mathcal{L}\{-tu(t+1)\}(s) + 2\mathcal{L}\{u(t+1)\}(s) + 2\mathcal{L}\{-tu(t)\}(s) - \mathcal{L}\{-tu(t-1)\}(s) - 2\mathcal{L}\{u(t-1)\}(s) \\ &= -\frac{d}{ds}\left(\frac{e^s}{s}\right) + 2\left(\frac{e^s}{s}\right) + 2\frac{d}{ds}\left(\frac{1}{s}\right) - \frac{d}{ds}\left(\frac{e^{-s}}{s}\right) - 2\left(\frac{e^{-s}}{s}\right) \\ &= -\left(\frac{se^s - e^s}{s^2}\right) + 2\left(\frac{e^s}{s}\right) + 2\left(-\frac{1}{s^2}\right) - \left(\frac{s(-e^{-s}) - e^{-s}}{s^2}\right) - 2\left(\frac{e^{-s}}{s}\right) \\ &= \frac{-se^s + e^s + se^{-s} + e^{-s} - 2 + 2se^s - 2e^{-s}}{s^2} \\ &= \frac{se^s + e^s - 2 - se^{-s} + e^{-s}}{s^2}.\end{aligned}$$

Since  $x$  is (bounded and) finite duration, the ROC of  $X$  must be the entire complex plane. Thus, we have that

$$X(s) = \frac{se^s + e^s - 2 - se^{-s} + e^{-s}}{s^2} \quad \text{for all } s.$$

*Second solution (which incurs less work by avoiding differentiation).* Alternatively, we can rearrange our above expression for  $x$  to obtain

$$\begin{aligned} x(t) &= [tu(t+1) + u(t+1) - u(t+1)] + 2u(t+1) - 2tu(t) + [tu(t-1) - u(t-1) + u(t-1)] - 2u(t-1) \\ &= (t+1)u(t+1) + u(t+1) - 2tu(t) + (t-1)u(t-1) - u(t-1). \end{aligned}$$

Taking the Laplace transform of both sides of the preceding equation, we have

$$\begin{aligned} X(s) &= e^s \mathcal{L}\{tu(t)\}(s) + \mathcal{L}\{u(t+1)\}(s) - 2\mathcal{L}\{tu(t)\}(s) + e^{-s} \mathcal{L}\{tu(t)\}(s) - \mathcal{L}\{u(t-1)\}(s) \\ &= e^s \frac{1}{s^2} + e^s \frac{1}{s} - 2 \frac{1}{s^2} + e^{-s} \frac{1}{s^2} - e^{-s} \frac{1}{s} \\ &= \frac{e^s + se^s - 2 + e^{-s} - se^{-s}}{s^2}. \end{aligned}$$

In the case of this alternate solution, the expression for  $X$  is considerably easier to simplify. ■

## 7.10 Determination of the Inverse Laplace Transform

As suggested earlier, in practice, we rarely use (7.3) directly in order to compute the inverse Laplace transform. This formula requires a contour integration, which is not usually very easy to compute. Instead, we employ a partial fraction expansion of the function. In so doing, we obtain a number of simpler functions for which we can usually find the inverse Laplace transform in a table (e.g., such as Table 7.2). In what follows, we assume that the reader is already familiar with partial fraction expansions. A tutorial on partial fraction expansions is provided in Appendix B for those who might not be acquainted with such expansions.

**Example 7.27.** Find the inverse Laplace transform  $x$  of

$$X(s) = \frac{2}{s^2 - s - 2} \quad \text{for } -1 < \operatorname{Re}(s) < 2.$$

*Solution.* We begin by rewriting  $X$  in the factored form

$$X(s) = \frac{2}{(s+1)(s-2)}.$$

Then, we find a partial fraction expansion of  $X$ . We know that  $X$  has an expansion of the form

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s-2}.$$

Calculating the coefficients of the expansion, we obtain

$$\begin{aligned} A_1 &= (s+1)X(s)|_{s=-1} \\ &= \left. \frac{2}{s-2} \right|_{s=-1} \\ &= -\frac{2}{3} \quad \text{and} \\ A_2 &= (s-2)X(s)|_{s=2} \\ &= \left. \frac{2}{s+1} \right|_{s=2} \\ &= \frac{2}{3}. \end{aligned}$$

So,  $X$  has the expansion

$$X(s) = \frac{2}{3} \left( \frac{1}{s-2} \right) - \frac{2}{3} \left( \frac{1}{s+1} \right).$$

Taking the inverse Laplace transform of both sides of this equation, we have

$$x(t) = \frac{2}{3} \mathcal{L}^{-1} \left\{ \frac{1}{s-2} \right\} (t) - \frac{2}{3} \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\} (t). \quad (7.6)$$

Using Table 7.2 and the given ROC, we have

$$\begin{aligned} -e^{2t}u(-t) &\stackrel{\text{LT}}{\longleftrightarrow} \frac{1}{s-2} \quad \text{for } \operatorname{Re}(s) < 2 \quad \text{and} \\ e^{-t}u(t) &\stackrel{\text{LT}}{\longleftrightarrow} \frac{1}{s+1} \quad \text{for } \operatorname{Re}(s) > -1. \end{aligned}$$

Substituting these results into (7.6), we obtain

$$\begin{aligned} x(t) &= \frac{2}{3} [-e^{2t}u(-t)] - \frac{2}{3} [e^{-t}u(t)] \\ &= -\frac{2}{3} e^{2t}u(-t) - \frac{2}{3} e^{-t}u(t). \end{aligned} \quad \blacksquare$$

**Example 7.28** (Rational function with a repeated pole). Find the inverse Laplace transform  $x$  of

$$X(s) = \frac{2s+1}{(s+1)^2(s+2)} \quad \text{for } \operatorname{Re}(s) > -1.$$

*Solution.* To begin, we find a partial fraction expansion of  $X$ . We know that  $X$  has an expansion of the form

$$X(s) = \frac{A_{11}}{s+1} + \frac{A_{12}}{(s+1)^2} + \frac{A_{21}}{s+2}.$$

Calculating the coefficients of the expansion, we obtain

$$\begin{aligned} A_{11} &= \frac{1}{(2-1)!} \left[ \left( \frac{d}{ds} \right)^{2-1} [(s+1)^2 X(s)] \right] \Big|_{s=-1} \\ &= \frac{1}{1!} \left[ \frac{d}{ds} [(s+1)^2 X(s)] \right] \Big|_{s=-1} \\ &= \left[ \frac{d}{ds} \left( \frac{2s+1}{s+2} \right) \right] \Big|_{s=-1} \\ &= \left[ \frac{(s+2)(2) - (2s+1)(1)}{(s+2)^2} \right] \Big|_{s=-1} \\ &= \left[ \frac{2s+4-2s-1}{(s+2)^2} \right] \Big|_{s=-1} \\ &= \left[ \frac{3}{(s+2)^2} \right] \Big|_{s=-1} \\ &= 3, \\ A_{12} &= \frac{1}{(2-2)!} \left[ \left( \frac{d}{ds} \right)^{2-2} [(s+1)^2 X(s)] \right] \Big|_{s=-1} \\ &= \frac{1}{0!} [(s+1)^2 X(s)] \Big|_{s=-1} \\ &= \frac{2s+1}{s+2} \Big|_{s=-1} \\ &= \frac{-1}{1} \\ &= -1, \quad \text{and} \end{aligned}$$

$$\begin{aligned}
 A_{21} &= (s+2)X(s)|_{s=-2} \\
 &= \frac{2s+1}{(s+1)^2} \Big|_{s=-2} \\
 &= \frac{-3}{1} \\
 &= -3.
 \end{aligned}$$

Thus,  $X$  has the expansion

$$X(s) = \frac{3}{s+1} - \frac{1}{(s+1)^2} - \frac{3}{s+2}.$$

Taking the inverse Laplace transform of both sides of this equation yields

$$x(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}(t) - 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t). \quad (7.7)$$

From Table 7.2 and the given ROC, we have

$$\begin{aligned}
 e^{-t}u(t) &\xleftrightarrow{\text{LT}} \frac{1}{s+1} \quad \text{for } \text{Re}(s) > -1, \\
 te^{-t}u(t) &\xleftrightarrow{\text{LT}} \frac{1}{(s+1)^2} \quad \text{for } \text{Re}(s) > -1, \quad \text{and} \\
 e^{-2t}u(t) &\xleftrightarrow{\text{LT}} \frac{1}{s+2} \quad \text{for } \text{Re}(s) > -2.
 \end{aligned}$$

Substituting these results into (7.7), we obtain

$$\begin{aligned}
 x(t) &= 3e^{-t}u(t) - te^{-t}u(t) - 3e^{-2t}u(t) \\
 &= (3e^{-t} - te^{-t} - 3e^{-2t})u(t).
 \end{aligned}$$

■

**Example 7.29** (Inverse Laplace transform of improper rational function). Find the inverse Laplace transform  $x$  of

$$X(s) = \frac{2s^2 + 4s + 5}{(s+1)(s+2)} \quad \text{for } \text{Re}(s) > -1.$$

*Solution.* We begin by noting that  $X$  is not a strictly-proper rational function. So, we need to perform polynomial long division in order to express  $X$  as the sum of a polynomial and a strictly-proper rational function. By long division, we have

$$\begin{array}{r}
 2 \\
 s^2 + 3s + 2 \overline{) 2s^2 + 4s + 5} \\
 \underline{2s^2 + 6s + 4} \phantom{0} \\
 -2s + 1
 \end{array}$$

In other words, we have

$$X(s) = 2 + \frac{-2s+1}{s^2+3s+2}.$$

For convenience, we define

$$V(s) = \frac{-2s+1}{(s+1)(s+2)}$$

so that

$$X(s) = 2 + V(s).$$

Observe that  $V$  is a strictly-proper rational function. So, we can find a partial fraction expansion of  $V$ . Now, we find this expansion. We know that such an expansion is of the form

$$V(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}.$$

Calculating the expansion coefficients, we have

$$\begin{aligned} A_1 &= (s+1)V(s)|_{s=-1} \\ &= \left. \frac{-2s+1}{s+2} \right|_{s=-1} \\ &= 3 \quad \text{and} \\ A_2 &= (s+2)V(s)|_{s=-2} \\ &= \left. \frac{-2s+1}{s+1} \right|_{s=-2} \\ &= -5 \end{aligned}$$

So, we have

$$\begin{aligned} X(s) &= 2 + V(s) \\ &= 2 + \frac{3}{s+1} - \frac{5}{s+2}. \end{aligned}$$

Taking the inverse Laplace transform, we obtain

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}X(t) \\ &= 2\mathcal{L}^{-1}\{1\}(t) + 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) - 5\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t). \end{aligned}$$

Considering the ROC of  $X$ , we can obtain the following from Table 7.2:

$$\begin{aligned} \delta(t) &\xleftrightarrow{\text{LT}} 1, \\ e^{-t}u(t) &\xleftrightarrow{\text{LT}} \frac{1}{s+1} \quad \text{for } \text{Re}(s) > -1, \quad \text{and} \\ e^{-2t}u(t) &\xleftrightarrow{\text{LT}} \frac{1}{s+2} \quad \text{for } \text{Re}(s) > -2. \end{aligned}$$

Finally, we can write

$$\begin{aligned} x(t) &= 2\delta(t) + 3e^{-t}u(t) - 5e^{-2t}u(t) \\ &= 2\delta(t) + (3e^{-t} - 5e^{-2t})u(t). \end{aligned} \quad \blacksquare$$

**Example 7.30.** Find all possible inverse Laplace transforms of

$$X(s) = \frac{1}{s^2 + 3s + 2}. \quad (7.8)$$

*Solution.* We begin by rewriting  $X$  in factored form as

$$X(s) = \frac{1}{(s+1)(s+2)}.$$

Then, we find the partial fraction expansion of  $X$ . We know that such an expansion has the form

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}.$$

Calculating the coefficients of the expansion, we have

$$A_1 = (s+1)X(s)|_{s=-1} = \frac{1}{s+2} \Big|_{s=-1} = 1 \quad \text{and}$$

$$A_2 = (s+2)X(s)|_{s=-2} = \frac{1}{s+1} \Big|_{s=-2} = -1.$$

So, we have

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}.$$

Taking the inverse Laplace transform of both sides of this equation yields

$$x(t) = \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\} (t) - \mathcal{L}^{-1} \left\{ \frac{1}{s+2} \right\} (t). \quad (7.9)$$

For the Laplace transform  $X$ , three possible ROCs exist:

- i)  $\text{Re}(s) < -2$ ,
- ii)  $-2 < \text{Re}(s) < -1$ , and
- iii)  $\text{Re}(s) > -1$ .

Thus, three possible inverse Laplace transforms exist for  $X$ , depending on the choice of ROC.

First, let us consider the case of the ROC  $\text{Re}(s) < -2$ . From Table 7.2, we have

$$-e^{-t}u(-t) \xleftrightarrow{\text{LT}} \frac{1}{s+1} \quad \text{for } \text{Re}(s) < -1 \quad \text{and}$$

$$-e^{-2t}u(-t) \xleftrightarrow{\text{LT}} \frac{1}{s+2} \quad \text{for } \text{Re}(s) < -2.$$

Substituting these results into (7.9), we have

$$x(t) = -e^{-t}u(-t) + e^{-2t}u(-t)$$

$$= [-e^{-t} + e^{-2t}]u(-t).$$

Second, let us consider the case of the ROC  $-2 < \text{Re}(s) < -1$ . From Table 7.2, we have

$$-e^{-t}u(-t) \xleftrightarrow{\text{LT}} \frac{1}{s+1} \quad \text{for } \text{Re}(s) < -1 \quad \text{and}$$

$$e^{-2t}u(t) \xleftrightarrow{\text{LT}} \frac{1}{s+2} \quad \text{for } \text{Re}(s) > -2.$$

Substituting these results into (7.9), we have

$$x(t) = -e^{-t}u(-t) - e^{-2t}u(t).$$

Third, let us consider the case of the ROC  $\text{Re}(s) > -1$ . From Table 7.2, we have

$$e^{-t}u(t) \xleftrightarrow{\text{LT}} \frac{1}{s+1} \quad \text{for } \text{Re}(s) > -1 \quad \text{and}$$

$$e^{-2t}u(t) \xleftrightarrow{\text{LT}} \frac{1}{s+2} \quad \text{for } \text{Re}(s) > -2.$$

Substituting these results into (7.9), we have

$$x(t) = e^{-t}u(t) - e^{-2t}u(t)$$

$$= [e^{-t} - e^{-2t}]u(t). \quad \blacksquare$$

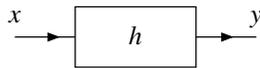


Figure 7.15: Time-domain view of a LTI system with input  $x$ , output  $y$ , and impulse response  $h$ .

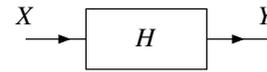


Figure 7.16: Laplace-domain view of a LTI system with input Laplace transform  $X$ , output Laplace transform  $Y$ , and system function  $H$ .

## 7.11 Characterizing LTI Systems Using the Laplace Transform

Consider a LTI system with input  $x$ , output  $y$ , and impulse response  $h$ , as depicted in Figure 7.15. Such a system is characterized by the equation

$$y(t) = x * h(t).$$

Let  $X$ ,  $Y$ , and  $H$  denote the Laplace transforms of  $x$ ,  $y$ , and  $h$ , respectively. Taking the Laplace transform of both sides of the above equation and using the time-domain convolution property of the Laplace transform, we have

$$Y(s) = H(s)X(s).$$

The quantity  $H$  is known as the **system function** or **transfer function** of the system. If the ROC of  $H$  includes the imaginary axis, then  $H(j\omega)$  is the frequency response of the system. The system can be represented with a block diagram labelled in the Laplace domain as shown in Figure 7.16, where the system is labelled by its system function  $H$ .

## 7.12 Interconnection of LTI Systems

From the properties of the Laplace transform and the definition of the system function, we can derive a number of equivalences involving the system function and series- and parallel-interconnected systems.

Suppose that we have two LTI systems  $\mathcal{H}_1$  and  $\mathcal{H}_2$  with system functions  $H_1$  and  $H_2$ , respectively, that are connected in a series configuration as shown in the left-hand side of Figure 7.17(a). Let  $h_1$  and  $h_2$  denote the impulse responses of  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively. The impulse response  $h$  of the overall system is given by

$$h(t) = h_1 * h_2(t).$$

Taking the Laplace transform of both sides of this equation yields

$$\begin{aligned} H(s) &= \mathcal{L}\{h_1 * h_2\}(s) \\ &= \mathcal{L}h_1(s)\mathcal{L}h_2(s) \\ &= H_1(s)H_2(s). \end{aligned}$$

Thus, we have the equivalence shown in Figure 7.17(a). Also, since multiplication commutes, we also have the equivalence shown in Figure 7.17(b).

Suppose that we have two LTI systems  $\mathcal{H}_1$  and  $\mathcal{H}_2$  with system functions  $H_1$  and  $H_2$  that are connected in a parallel configuration as shown on the left-hand side of Figure 7.18. Let  $h_1$  and  $h_2$  denote the impulse responses of  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively. The impulse response  $h$  of the overall system is given by

$$h(t) = h_1(t) + h_2(t).$$

Taking the Laplace transform of both sides of the equation yields

$$\begin{aligned} H(s) &= \mathcal{L}\{h_1 + h_2\}(s) \\ &= \mathcal{L}h_1(s) + \mathcal{L}h_2(s) \\ &= H_1(s) + H_2(s). \end{aligned}$$

Thus, we have the equivalence shown in Figure 7.18.

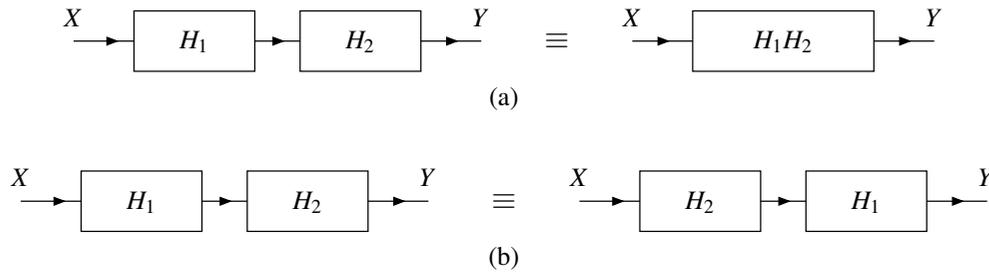


Figure 7.17: Equivalences involving system functions and the series interconnection of LTI systems. The (a) first and (b) second equivalences.

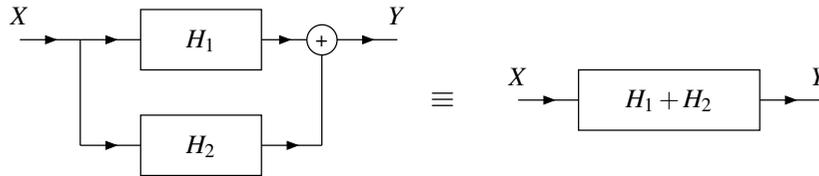


Figure 7.18: Equivalence involving system functions and the parallel interconnection of LTI systems.

## 7.13 System Function and System Properties

Many properties of a system can be readily determined from the characteristics of its system function, as we shall elaborate upon in the sections that follow.

### 7.13.1 Causality

From Theorem 4.8, we know that, a LTI system is causal if its impulse response is causal. A causal function, however, is inherently right sided. Consequently, from the properties of the ROC of the Laplace transform discussed in Section 7.7, we have the theorem below.

**Theorem 7.12.** *The ROC associated with the system function of a causal LTI system is a right-half plane or the entire complex plane.*

*Proof.* Let  $h$  and  $H$  denote the impulse response and system function of a causal LTI system  $\mathcal{H}$ . The system  $\mathcal{H}$  is causal if and only if  $h$  is causal. Since a causal signal is always right sided and a right-sided signal always has a Laplace transform with a right-half plane ROC, the ROC of  $H$  must be a right-half plane. ■

In general, the converse of the above theorem is not necessarily true. That is, it is not always true that a right-half plane ROC is associated with a causal system function. If  $H$  is rational, however, we have that the converse does hold, as indicated by the theorem below.

**Theorem 7.13.** *For a LTI system with a rational system function  $H$ , causality of the system is equivalent to the ROC of  $H$  being the right-half plane to the right of the rightmost pole or, if  $H$  has no poles, the entire complex plane.*

*Proof.* From Theorem 7.12, we know that the ROC of  $H$  is a right-half plane. Furthermore, since  $H$  is rational, we also know from the properties of the ROC that the ROC of  $H$  must be the region to the right of the rightmost pole of  $H$ . ■

**Example 7.31.** For the LTI system with each system function  $H$  below, determine whether the system is causal.

- (a)  $H(s) = \frac{1}{s+1}$  for  $\text{Re}(s) > -1$ ;
- (b)  $H(s) = \frac{1}{s^2-1}$  for  $-1 < \text{Re}(s) < 1$ ;
- (c)  $H(s) = \frac{e^s}{s+1}$  for  $\text{Re}(s) < -1$ ; and
- (d)  $H(s) = \frac{e^s}{s+1}$  for  $\text{Re}(s) > -1$ .

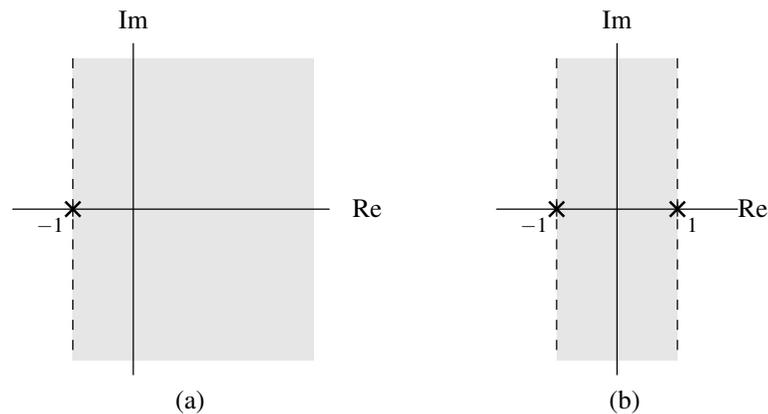


Figure 7.19: Pole and ROCs of the rational system functions in the causality example. The cases of the (a) first (b) second system functions.

*Solution.* (a) The poles of  $H$  are plotted in Figure 7.19(a) and the ROC is indicated by the shaded area. The system function  $H$  is rational and the ROC is the right-half plane to the right of the rightmost pole. Therefore, the system is causal.

(b) The poles of  $H$  are plotted in Figure 7.19(b) and the ROC is indicated by the shaded area. The system function is rational but the ROC is not a right-half plane. Therefore, the system is not causal.

(c) The system function  $H$  has a left-half plane ROC. Therefore,  $h$  is a left-sided signal. Thus, the system is not causal.

(d) The system function  $H$  has a right-half plane ROC but is not rational. Thus, we cannot make any conclusion directly from the system function. Instead, we draw our conclusion from the impulse response  $h$ . Taking the inverse Laplace transform of  $H$ , we obtain

$$h(t) = e^{-(t+1)}u(t+1).$$

Thus, the impulse response  $h$  is not causal. Therefore, the system is not causal. ■

### 7.13.2 BIBO Stability

In this section, we consider the relationship between the system function and BIBO stability. The first important result is given by the theorem below.

**Theorem 7.14.** *A LTI system is BIBO stable if and only if the ROC of its system function  $H$  contains the imaginary axis (i.e.,  $\text{Re}(s) = 0$ ).*

*Proof.* To begin, we observe that the ROC of  $H$  containing the imaginary axis is equivalent to the condition

$$H(j\omega) \text{ converges for all } \omega. \quad (7.10)$$

Let  $h$  denote the inverse Laplace transform of  $H$  (i.e.,  $h$  is the impulse response of the system). From earlier in Theorem 4.11, we know that the system is BIBO stable if and only if

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty \quad (7.11)$$

(i.e.,  $h$  is absolutely integrable). Thus, the proof of the theorem is equivalent to showing that (7.11) holds if and only if (7.10) holds.

First, we show that (7.11) being satisfied implies that (7.10) holds. So, we assume that (7.11) holds. From the definition of  $H$  (evaluated on the imaginary axis), we have

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} dt.$$

Now, we note the fact that an integral is convergent if it is absolutely convergent. In particular, for any function  $f$ ,

$$\int_{-\infty}^{\infty} f(t) dt \text{ converges if } \int_{-\infty}^{\infty} |f(t)| dt \text{ converges.}$$

Thus,  $H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} dt$  converges if  $\int_{-\infty}^{\infty} |h(t)e^{-j\omega t}| dt$  converges. We have, however, that

$$\int_{-\infty}^{\infty} |h(t)e^{-j\omega t}| dt = \int_{-\infty}^{\infty} |h(t)| |e^{-j\omega t}| dt = \int_{-\infty}^{\infty} |h(t)| dt.$$

So,  $H(j\omega)$  converges if (7.11) holds. Thus, (7.11) being satisfied implies that (7.10) holds. The proof that (7.10) being satisfied implies that (7.11) holds is left as an exercise for the reader. ■

In the case that the system is causal, a more specific result can be derived. This result is given by the theorem below.

**Theorem 7.15.** *A causal LTI system with a (proper) rational system function  $H$  is BIBO stable if and only if all of the poles of  $H$  lie in the left half of the plane (i.e., all of the poles have negative real parts).*

*Proof.* From Theorem 7.14, we know that the system is stable if and only if the ROC of  $H$  contains the imaginary axis. Since the system is causal and  $H$  is rational, the ROC of  $H$  is to the right of its rightmost pole. Therefore, the ROC contains the imaginary axis if and only if the rightmost pole is to the left of the imaginary axis. All of the poles of  $H$  lying to the left of the imaginary axis is equivalent to all of these poles having negative real parts. ■

Observe from the preceding two theorems (i.e., Theorems 7.14 and 7.15) that, in the case of a LTI system, the characterization of the BIBO stability property is much simpler in the Laplace domain (via the system function) than the time domain (via the impulse response). For this reason, analyzing the stability of LTI systems is typically performed using the Laplace transform.

**Example 7.32.** A LTI system has the system function

$$H(s) = \frac{1}{(s+1)(s+2)}.$$

Given that the system is BIBO stable, determine the ROC of  $H$ .

*Solution.* Clearly, the system function  $H$  is rational with poles at  $-1$  and  $-2$ . Therefore, only three possibilities exist for the ROC:

- i)  $\text{Re}(s) < -2$ ,
- ii)  $-2 < \text{Re}(s) < -1$ , and
- iii)  $\text{Re}(s) > -1$ .

In order for the system to be stable, however, the ROC of  $H$  must include the entire imaginary axis. Therefore, the ROC must be  $\text{Re}(s) > -1$ . This ROC is illustrated in Figure 7.20. ■

**Example 7.33.** A LTI system is causal and has the system function

$$H(s) = \frac{1}{(s+2)(s^2+2s+2)}.$$

Determine whether this system is BIBO stable.

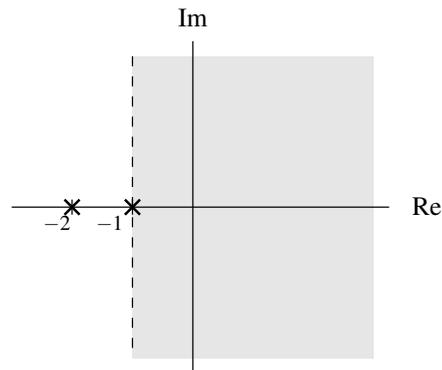


Figure 7.20: ROC for example.

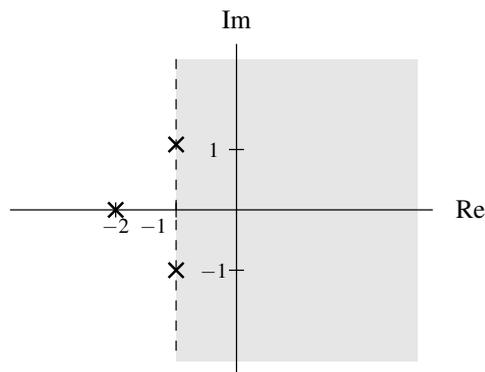


Figure 7.21: Poles of the system function.

*Solution.* We begin by factoring  $H$  to obtain

$$H(s) = \frac{1}{(s+2)(s+1-j)(s+1+j)}.$$

(Using the quadratic formula, one can confirm that  $s^2 + 2s + 2 = 0$  has roots at  $s = -1 \pm j$ .) Thus,  $H$  has poles at  $-2$ ,  $-1 + j$ , and  $-1 - j$ . The poles are plotted in Figure 7.21. Since the system is causal and all of the poles of  $H$  are in the left half of the plane, the system is stable. ■

**Example 7.34.** For each LTI system with system function  $H$  given below, determine the ROC of  $H$  that corresponds to a BIBO stable system.

(a)  $H(s) = \frac{s(s-1)}{(s+2)(s+1+j)(s+1-j)}$ ;

(b)  $H(s) = \frac{s}{(s+1)(s-1)(s-1-j)(s-1+j)}$ ;

(c)  $H(s) = \frac{(s+j)(s-j)}{(s+2-j)(s+2+j)}$ ; and

(d)  $H(s) = \frac{s-1}{s}$ .

*Solution.* (a) The function  $H$  has poles at  $-2$ ,  $-1 + j$ , and  $-1 - j$ . The poles are shown in Figure 7.22(a). Since  $H$  is rational, the ROC must be bounded by poles or extend to infinity. Consequently, only three distinct ROCs are possible:

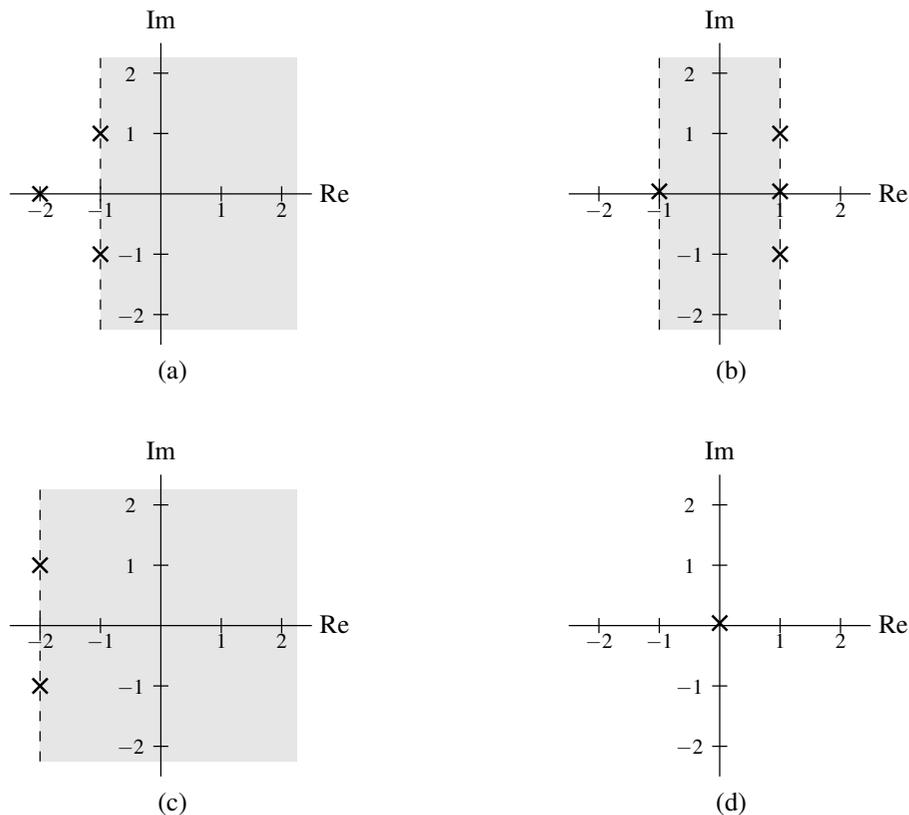


Figure 7.22: Poles and ROCs of the system function  $H$  in the (a) first, (b) second, (c) third, and (d) fourth parts of the example.

- i)  $\text{Re}(s) < -2$ ,
- ii)  $-2 < \text{Re}(s) < -1$ , and
- iii)  $\text{Re}(s) > -1$ .

Since we want a stable system, the ROC must include the entire imaginary axis. Therefore, the ROC must be  $\text{Re}(s) > -1$ . This is the shaded region in the Figure 7.22(a).

(b) The function  $H$  has poles at  $-1$ ,  $1$ ,  $1 + j$ , and  $1 - j$ . The poles are shown in Figure 7.22(b). Since  $H$  is rational, the ROC must be bounded by poles or extend to infinity. Consequently, only three distinct ROCs are possible:

- i)  $\text{Re}(s) < -1$ ,
- ii)  $-1 < \text{Re}(s) < 1$ , and
- iii)  $\text{Re}(s) > 1$ .

Since we want a stable system, the ROC must include the entire imaginary axis. Therefore, the ROC must be  $-1 < \text{Re}(s) < 1$ . This is the shaded region in Figure 7.22(b).

(c) The function  $H$  has poles at  $-2 + j$  and  $-2 - j$ . The poles are shown in Figure 7.22(c). Since  $H$  is rational, the ROC must be bounded by poles or extend to infinity. Consequently, only two distinct ROCs are possible:

- i)  $\text{Re}(s) < -2$  and
- ii)  $\text{Re}(s) > -2$ .

Since we want a stable system, the ROC must include the entire imaginary axis. Therefore, the ROC must be  $\text{Re}(s) > -2$ . This is the shaded region in Figure 7.22(c).

(d) The function  $H$  has a pole at  $0$ . The pole is shown in Figure 7.22(d). Since  $H$  is rational, it cannot converge at  $0$  (which is a pole of  $H$ ). Consequently, the ROC can never include the entire imaginary axis. Therefore, the system function  $H$  can never be associated with a stable system. ■

### 7.13.3 Invertibility

In this section, we consider the relationship between the system function and invertibility. The first important result is given by the theorem below.

**Theorem 7.16** (Inverse of LTI system). *Let  $\mathcal{H}$  be a LTI system with system function  $H$ . If the inverse  $\mathcal{H}^{-1}$  of  $\mathcal{H}$  exists,  $\mathcal{H}^{-1}$  is LTI and has a system function  $H_{\text{inv}}$  that satisfies*

$$H(s)H_{\text{inv}}(s) = 1. \quad (7.12)$$

*Proof.* Let  $h$  denote the inverse Laplace transform of  $H$ . From Theorem 4.9, we know that the system  $\mathcal{H}$  is invertible if and only if there exists another LTI system with impulse response  $h_{\text{inv}}$  satisfying

$$h * h_{\text{inv}} = \delta.$$

Let  $H_{\text{inv}}$  denote the Laplace transform of  $h_{\text{inv}}$ . Taking the Laplace transform of both sides of the above equation, we have

$$\mathcal{L}\{h * h_{\text{inv}}\} = \mathcal{L}\delta.$$

From the time-domain convolution property of the Laplace transform and Table 7.2 (i.e.,  $\mathcal{L}\delta(s) = 1$ ), we obtain

$$H(s)H_{\text{inv}}(s) = 1. \quad \blacksquare$$

From the preceding theorem, we have the following result:

**Theorem 7.17** (Invertibility of LTI system). *A LTI system  $\mathcal{H}$  with system function  $H$  is invertible if and only if there exists a function  $H_{\text{inv}}$  satisfying*

$$H(s)H_{\text{inv}}(s) = 1.$$

*Proof.* The proof follows immediately from the result of Theorem 7.16 by simply observing that  $\mathcal{H}$  being invertible is equivalent to the existence of  $\mathcal{H}^{-1}$ .  $\blacksquare$

From the above theorems, we have that a LTI system  $\mathcal{H}$  with system function  $H$  has an inverse if and only if a solution for  $H^{\text{inv}}$  exists in (7.12). Furthermore, if an inverse system exists, its system function is given by

$$H_{\text{inv}}(s) = \frac{1}{H(s)}.$$

Since distinct systems can have identical system functions (but with differing ROCs), the inverse of a LTI system is not necessarily unique. In practice, however, we often desire a stable and/or causal system. So, although multiple inverse systems may exist, we are frequently only interested in one specific choice of inverse system (due to these additional constraints of stability and/or causality).

**Example 7.35.** Consider the LTI system with system function

$$H(s) = \frac{s+1}{s+2} \quad \text{for } \text{Re}(s) > -2.$$

Determine all possible inverses of this system. Comment on the stability of each of these inverse systems.

*Solution.* The system function  $H_{\text{inv}}$  of the inverse system is given by

$$H_{\text{inv}}(s) = \frac{1}{H(s)} = \frac{s+2}{s+1}.$$

Two ROCs are possible for  $H_{\text{inv}}$ :

- i)  $\text{Re}(s) < -1$  and
- ii)  $\text{Re}(s) > -1$ .

Each ROC is associated with a distinct inverse system. The first ROC is associated with an unstable system since this ROC does not include the imaginary axis. The second ROC is associated with a stable system, since this ROC includes the entire imaginary axis. ■

## 7.14 LTI Systems and Differential Equations

Many LTI systems of practical interest can be described by  $N$ th-order linear differential equations with constant coefficients. Such a system with input  $x$  and output  $y$  can be characterized by an equation of the form

$$\sum_{k=0}^N b_k \left(\frac{d}{dt}\right)^k y(t) = \sum_{k=0}^M a_k \left(\frac{d}{dt}\right)^k x(t), \quad (7.13)$$

where  $M \leq N$ . Let  $X$  and  $Y$  denote the Laplace transforms of  $x$  and  $y$ , respectively. Let  $H$  denote the system function of the system. Taking the Laplace transform of both sides of the above equation, we obtain

$$\mathcal{L} \left\{ \sum_{k=0}^N b_k \left(\frac{d}{dt}\right)^k y(t) \right\} (s) = \mathcal{L} \left\{ \sum_{k=0}^M a_k \left(\frac{d}{dt}\right)^k x(t) \right\} (s).$$

Using the linearity property of the Laplace transform, we can rewrite this equation as

$$\sum_{k=0}^N b_k \mathcal{L} \left\{ \left(\frac{d}{dt}\right)^k y(t) \right\} (s) = \sum_{k=0}^M a_k \mathcal{L} \left\{ \left(\frac{d}{dt}\right)^k x(t) \right\} (s).$$

Using the time differentiation property of the Laplace transform, we have

$$\sum_{k=0}^N b_k s^k Y(s) = \sum_{k=0}^M a_k s^k X(s).$$

Factoring, we have

$$Y(s) \sum_{k=0}^N b_k s^k = X(s) \sum_{k=0}^M a_k s^k.$$

Dividing both sides of this equation by  $X(s) \sum_{k=0}^N b_k s^k$ , we obtain

$$\frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^M a_k s^k}{\sum_{k=0}^N b_k s^k}.$$

Thus,  $H$  is given by

$$H(s) = \frac{\sum_{k=0}^M a_k s^k}{\sum_{k=0}^N b_k s^k}.$$

Observe that, for a system of the form considered above (i.e., a system characterized by an equation of the form of (7.13)), the system function is always rational. It is for this reason that rational functions are of particular interest.

**Example 7.36** (Differential equation to system function). A LTI system with input  $x$  and output  $y$  is characterized by the differential equation

$$y''(t) + \frac{D}{M} y'(t) + \frac{K}{M} y(t) = x(t),$$

where  $D$ ,  $K$ , and  $M$  are positive real constants, and the prime symbol is used to denote derivative. Find the system function  $H$  of this system.

*Solution.* Taking the Laplace transform of the given differential equation, we obtain

$$s^2Y(s) + \frac{D}{M}sY(s) + \frac{K}{M}Y(s) = X(s).$$

Rearranging the terms and factoring, we have

$$\left(s^2 + \frac{D}{M}s + \frac{K}{M}\right)Y(s) = X(s).$$

Dividing both sides by  $\left(s^2 + \frac{D}{M}s + \frac{K}{M}\right)X(s)$ , we obtain

$$\frac{Y(s)}{X(s)} = \frac{1}{s^2 + \frac{D}{M}s + \frac{K}{M}}.$$

Thus,  $H$  is given by

$$H(s) = \frac{1}{s^2 + \frac{D}{M}s + \frac{K}{M}}. \quad \blacksquare$$

**Example 7.37** (System function to differential equation). A LTI system with input  $x$  and output  $y$  has the system function

$$H(s) = \frac{s}{s + R/L},$$

where  $L$  and  $R$  are positive real constants. Find the differential equation that characterizes this system.

*Solution.* Let  $X$  and  $Y$  denote the Laplace transforms of  $x$  and  $y$ , respectively. To begin, we have

$$\begin{aligned} Y(s) &= H(s)X(s) \\ &= \left(\frac{s}{s + R/L}\right)X(s). \end{aligned}$$

Rearranging this equation, we obtain

$$\begin{aligned} \left(s + \frac{R}{L}\right)Y(s) &= sX(s) \\ \Rightarrow sY(s) + \frac{R}{L}Y(s) &= sX(s). \end{aligned}$$

Taking the inverse Laplace transform of both sides of this equation (by using the linearity and time-differentiation properties of the Laplace transform), we have

$$\begin{aligned} \mathcal{L}^{-1}\{sY(s)\}(t) + \frac{R}{L}\mathcal{L}^{-1}Y(t) &= \mathcal{L}^{-1}\{sX(s)\}(t) \\ \Rightarrow \frac{d}{dt}y(t) + \frac{R}{L}y(t) &= \frac{d}{dt}x(t). \end{aligned}$$

Therefore, the system is characterized by the differential equation

$$\frac{d}{dt}y(t) + \frac{R}{L}y(t) = \frac{d}{dt}x(t). \quad \blacksquare$$

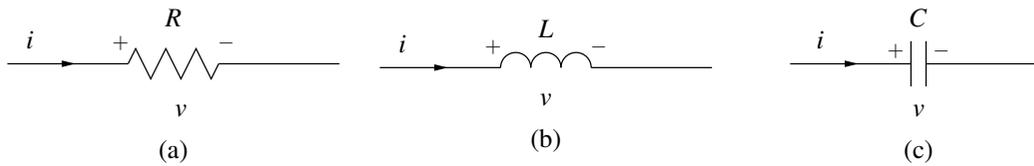


Figure 7.23: Basic electrical components. (a) Resistor, (b) inductor, and (c) capacitor.

## 7.15 Circuit Analysis

One application of the Laplace transform is circuit analysis. In this section, we consider this particular application. The basic building blocks of many electrical networks are resistors, inductors, and capacitors. In what follows, we briefly introduce each of these circuit elements.

A **resistor** is a circuit element that opposes the flow of electric current. The resistor, shown in schematic form in Figure 7.23(a), is governed by the relationship

$$v(t) = Ri(t) \quad (\text{or equivalently, } i(t) = \frac{1}{R}v(t)),$$

where  $R$ ,  $v$  and  $i$  denote the resistance of, voltage across, and current through the resistor, respectively. Note that the resistance  $R$  is a nonnegative quantity (i.e.,  $R \geq 0$ ). In the Laplace domain, the above relationship becomes

$$V(s) = RI(s) \quad (\text{or equivalently, } I(s) = \frac{1}{R}V(s)),$$

where  $V$  and  $I$  denote the Laplace transforms of  $v$  and  $i$ , respectively.

An **inductor** is a circuit element that converts an electric current into a magnetic field and vice versa. The inductor, shown in schematic form in Figure 7.23(b), is governed by the relationship

$$v(t) = L \frac{d}{dt} i(t) \quad \left( \text{or equivalently, } i(t) = \frac{1}{L} \int_{-\infty}^t v(\tau) d\tau \right),$$

where  $L$ ,  $v$ , and  $i$  denote the inductance of, voltage across, and current through the inductor, respectively. Note that the inductance  $L$  is a nonnegative quantity (i.e.,  $L \geq 0$ ). In the Laplace domain, the above relationship becomes

$$V(s) = sLI(s) \quad (\text{or equivalently, } I(s) = \frac{1}{sL}V(s)),$$

where  $V$  and  $I$  denote the Laplace transforms of  $v$  and  $i$ , respectively.

A **capacitor** is a circuit element that stores electric charge. The capacitor, shown in schematic form in Figure 7.23(c), is governed by the relationship

$$v(t) = \frac{1}{C} \int_{-\infty}^t i(\tau) d\tau \quad (\text{or equivalently, } i(t) = C \frac{d}{dt} v(t)),$$

where  $C$ ,  $v$ , and  $i$  denote the capacitance of, voltage across, and current through the capacitor, respectively. Note that the capacitance  $C$  is a nonnegative quantity (i.e.,  $C \geq 0$ ). In the Laplace domain, the above relationship becomes

$$V(s) = \frac{1}{sC} I(s) \quad (\text{or equivalently, } I(s) = sCV(s)),$$

where  $V$  and  $I$  denote the Laplace transforms of  $v$  and  $i$ , respectively.

Observe that the equations that characterize inductors and capacitors are arguably much simpler to express in the Laplace domain than in the time domain. Consequently, the use of the Laplace transform can greatly simplify the process of analyzing circuits containing inductors and capacitors.

**Example 7.38** (Simple RC network). Consider the resistor-capacitor (RC) network shown in Figure 7.24 with input  $v_1$  and output  $v_2$ . This system is LTI and can be characterized by a linear differential equation with constant coefficients. (a) Find the system function  $H$  of this system. (b) Determine whether the system is BIBO stable. (c) Determine the step response of the system.

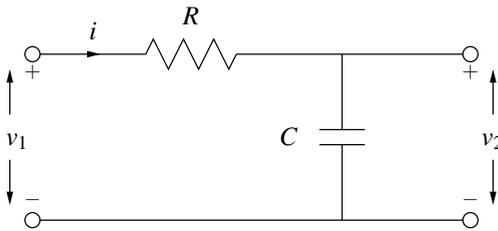


Figure 7.24: Simple RC network.

*Solution.* (a) From basic circuit analysis, we have

$$v_1(t) = Ri(t) + v_2(t) \quad \text{and} \quad (7.14a)$$

$$i(t) = C \frac{d}{dt} v_2(t). \quad (7.14b)$$

Taking the Laplace transform of (7.14) yields

$$V_1(s) = RI(s) + V_2(s) \quad \text{and} \quad (7.15a)$$

$$I(s) = CsV_2(s). \quad (7.15b)$$

Using these equations, we can write

$$\begin{aligned} V_1(s) &= R[CsV_2(s)] + V_2(s) \\ \Rightarrow V_1(s) &= RCsV_2(s) + V_2(s) \\ \Rightarrow V_1(s) &= [1 + RCs]V_2(s). \end{aligned}$$

Thus, we have that the system function  $H$  is given by

$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{1 + RCs} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} = \frac{\frac{1}{RC}}{s - (-\frac{1}{RC})}.$$

Since the system can be physically realized, it must be causal. Therefore,  $H$  must have the ROC  $\text{Re}(s) > -\frac{1}{RC}$ . So, we have

$$H(s) = \frac{1}{1 + RCs} \quad \text{for } \text{Re}(s) > -\frac{1}{RC}.$$

(b) Since resistance and capacitance are positive quantities,  $-\frac{1}{RC}$  is a negative value. Consequently, the ROC always includes the entire imaginary axis, and the system is stable.

(c) Now, let us calculate the step response of the system. We know that the system input-output behavior is characterized by the equation

$$\begin{aligned} V_2(s) &= H(s)V_1(s) \\ &= \left( \frac{1}{1 + RCs} \right) V_1(s). \end{aligned}$$

To compute the step response, we need to consider an input equal to the unit-step function. So,  $v_1 = u$ , implying that  $V_1(s) = \frac{1}{s}$ . Substituting this expression for  $V_1$  into the above expression for  $V_2$ , we have

$$\begin{aligned} V_2(s) &= \left( \frac{1}{1 + RCs} \right) \left( \frac{1}{s} \right) \\ &= \frac{\frac{1}{RC}}{s(s + \frac{1}{RC})}. \end{aligned}$$

Now, we need to compute the inverse Laplace transform of  $V_2$  in order to determine  $v_2$ . To simplify this task, we find the partial fraction expansion for  $V_2$ . We know that this expansion is of the form

$$V_2(s) = \frac{A_1}{s} + \frac{A_2}{s + \frac{1}{RC}}.$$

Solving for the coefficients of the expansion, we obtain

$$\begin{aligned} A_1 &= sV_2(s)|_{s=0} \\ &= 1 \quad \text{and} \\ A_2 &= (s + \frac{1}{RC})V_2(s)|_{s=-\frac{1}{RC}} \\ &= \frac{\frac{1}{RC}}{-\frac{1}{RC}} \\ &= -1. \end{aligned}$$

Thus, we have that  $V_2$  has the partial fraction expansion given by

$$V_2(s) = \frac{1}{s} - \frac{1}{s + \frac{1}{RC}}.$$

Taking the inverse Laplace transform of both sides of the equation, we obtain

$$v_2(t) = \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} (t) - \mathcal{L}^{-1} \left\{ \frac{1}{s + \frac{1}{RC}} \right\} (t).$$

Using Table 7.2 and the fact that the system is causal (which implies the necessary ROC), we obtain

$$\begin{aligned} v_2(t) &= u(t) - e^{-t/(RC)}u(t) \\ &= (1 - e^{-t/(RC)})u(t). \end{aligned} \quad \blacksquare$$

## 7.16 Stability Analysis

As mentioned earlier, since BIBO stability is more easily characterized for LTI systems in the Laplace domain than the time domain, the Laplace domain is often used to analyze system stability. In what follows, we will consider this application of the Laplace transform in more detail.

**Example 7.39** (Feedback system). Consider the system shown in Figure 7.25 that has input Laplace transform  $X$  and output Laplace transform  $Y$ , and is formed by the interconnection of two causal LTI systems labelled with their system functions  $H_1$  and  $H_2$ . The system functions  $H_1$  and  $H_2$  are given by

$$H_1(s) = \frac{1}{s^2 + as + (a-2)} \quad \text{and} \quad H_2(s) = -1,$$

where  $a$  is a real constant. (a) Find the system function  $H$  of the (overall) system. (b) Determine the values of the parameter  $a$  for which the system is BIBO stable.

*Solution.* (a) From the system diagram, we can write

$$\begin{aligned} V(s) &= X(s) + H_2(s)Y(s) \quad \text{and} \\ Y(s) &= H_1(s)V(s). \end{aligned}$$

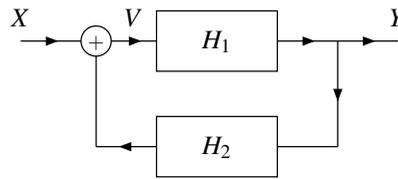


Figure 7.25: Feedback system.

Combining these two equations and simplifying, we obtain

$$\begin{aligned}
 Y(s) &= H_1(s)[X(s) + H_2(s)Y(s)] \\
 \Rightarrow Y(s) &= H_1(s)X(s) + H_1(s)H_2(s)Y(s) \\
 \Rightarrow Y(s)[1 - H_1(s)H_2(s)] &= H_1(s)X(s) \\
 \Rightarrow \frac{Y(s)}{X(s)} &= \frac{H_1(s)}{1 - H_1(s)H_2(s)}.
 \end{aligned}$$

Since  $H(s) = \frac{Y(s)}{X(s)}$ , we have

$$H(s) = \frac{H_1(s)}{1 - H_1(s)H_2(s)}.$$

Substituting the given expressions for  $H_1$  and  $H_2$ , and simplifying, we can write

$$\begin{aligned}
 H(s) &= \frac{\frac{1}{s^2 + as + (a-2)}}{1 + \left(\frac{1}{s^2 + as + (a-2)}\right)} \\
 &= \frac{1}{s^2 + as + (a-2) + 1} \\
 &= \frac{1}{s^2 + as + (a-1)}.
 \end{aligned}$$

(b) In order to assess the stability of the system, we need to know the poles of the system function  $H$ . So, we use the quadratic formula in order to factor the denominator of  $H$ . Solving for the roots  $s$  of the denominator of  $H$ , we obtain

$$\begin{aligned}
 s &= \frac{-a \pm \sqrt{a^2 - 4(a-1)}}{2} \\
 &= \frac{-a \pm \sqrt{a^2 - 4a + 4}}{2} \\
 &= \frac{-a \pm \sqrt{(a-2)^2}}{2} \\
 &= \frac{-a \pm (a-2)}{2} \\
 &= \{-1, 1-a\}.
 \end{aligned}$$

So,  $s^2 + as + (a-1) = (s+1)(s+a-1)$ . Thus, we have

$$H(s) = \frac{1}{(s+1)(s+a-1)}.$$

Since the system is causal, the system is stable if and only if all of the poles are strictly to the left of the imaginary axis. The system has two poles, one at  $-1$  and one at  $1-a$ . Thus, we know that

$$1 - a < 0 \Rightarrow a > 1.$$

Therefore, the system is stable if and only if  $a > 1$ . ■

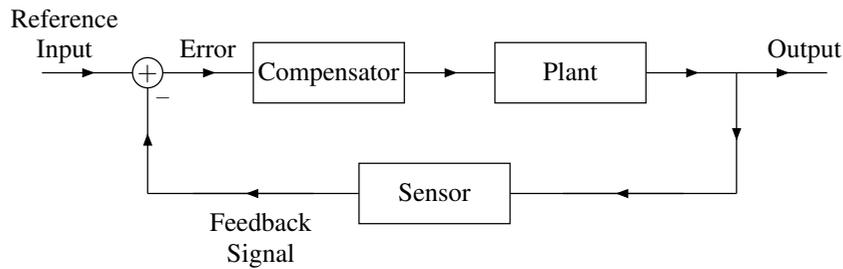


Figure 7.26: Feedback control system.

### 7.16.1 Feedback Control Systems

In control systems applications, we wish to have some physical quantity (such as a position or force) track a desired value over time. The input to the control system is the desired (i.e., reference) value for the quantity being controlled, and the output is the actual value of the quantity. The difference between the actual value and reference value constitutes an error. The goal of the control system is to force this error to be as close to zero as possible. When the error is zero, this corresponds to the actual value being equal to the desired value. For example, in a simple heating/cooling thermostat system, the reference input would be temperature setting on the thermostat wall control and the output would be the actual room temperature. In a flight control system, the reference input would be the desired values for the position and orientation of the plane specified by the pilot's flight controls and the output would be the plane's actual position and orientation.

A very commonly occurring configuration used for control system applications is known as a feedback control system. A feedback control system consists of three interconnected subsystems:

1. a plant, which is the system whose output corresponds to the quantity being controlled;
2. a sensor, which is used to measure the actual value of the quantity being controlled; and
3. a compensator (also called a controller), which is used to ensure that the overall system behaves in a manner that the output closely tracks the reference input in addition to possibly satisfying other criteria such as being stable.

The general structure of a feedback control system is shown in Figure 7.26. The reference input corresponds to the desired value for the quantity being controlled. The output corresponds to the actual value of the quantity being controlled (which is measured by a sensor). The adder output corresponds to error (i.e., the difference between the desired and actual values of the controlled quantity). Again, in control applications, the objective is to have the output track the reference input as closely as possible over time. In other words, we wish for the error to be as close to zero as possible. It is the responsibility of the compensator to ensure that this happens. If the compensator is well designed, the error will remain close to zero over time.

Consider a simple control system application corresponding to a robot arm. In this system, the reference input corresponds to the desired position and orientation of the robot-arm end effector (i.e., the device at the end of the robot arm), while the output corresponds to the actual position and orientation of the end effector. If the compensator is well designed, then the actual position and orientation of the end effector should track the desired value over time.

Having introduced the notion of feedback control systems, we now consider the application of stabilizing an unstable plant with a feedback control system.

**Example 7.40** (Stabilization of unstable plant). Consider the causal LTI system with input Laplace transform  $X$ , output Laplace transform  $Y$ , and system function

$$P(s) = \frac{10}{s-1},$$

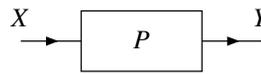


Figure 7.27: Plant.

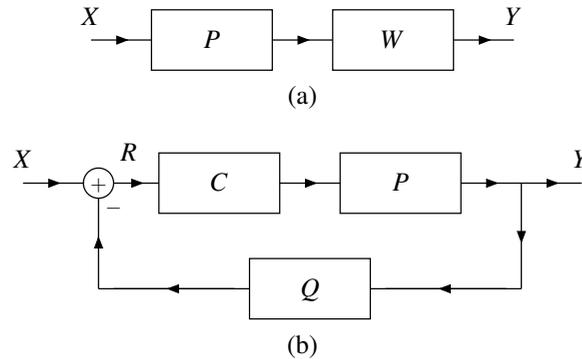


Figure 7.28: Two configurations for stabilizing the unstable plant. (a) Simple cascade system and (b) feedback control system.

as depicted in Figure 7.27. One can easily confirm that this system is not BIBO stable, due to the pole of  $P$  at 1. (Since the system is causal, the ROC of  $P$  is the RHP given by  $\text{Re}(s) > 1$ . Clearly, this ROC does not include the imaginary axis. Therefore, the system is not stable.) In what follows, we consider two different strategies for stabilizing this unstable system as well as their suitability for use in practice.

(a) STABILIZATION OF UNSTABLE PLANT VIA POLE-ZERO CANCELLATION. Suppose that the system in Figure 7.27 is connected in series with another causal LTI system with system function

$$W(s) = \frac{s-1}{10(s+1)},$$

in order to yield a new system with input Laplace transform  $X$  and output Laplace transform  $Y$ , as shown in Figure 7.28(a). Show that this new system is BIBO stable.

(b) STABILIZATION OF UNSTABLE PLANT VIA FEEDBACK. Suppose now that the system in Figure 7.27 is interconnected with two other causal LTI systems with system functions  $C$  and  $Q$ , as shown in Figure 7.28(b), in order to yield a new system with input Laplace transform  $X$ , output Laplace transform  $Y$ , and system function  $H$ . Moreover, suppose that

$$C(s) = \beta \quad \text{and} \quad Q(s) = 1,$$

where  $\beta$  is a real constant. Show that, with an appropriate choice of  $\beta$ , the resulting system is BIBO stable.

(c) PRACTICAL ISSUES. Parts (a) and (b) of this example consider two different schemes for stabilizing the unstable system in Figure 7.27. As it turns out, a scheme like the one in part (a) is not useful in practice. Identify the practical problems associated with this approach. Indicate whether the scheme in part (b) suffers from the same shortcomings.

*Solution.* (a) From the block diagram in Figure 7.28(a), the system function  $H$  of the overall system is

$$\begin{aligned} H(s) &= P(s)W(s) \\ &= \left( \frac{10}{s-1} \right) \left( \frac{s-1}{10(s+1)} \right) \\ &= \frac{1}{s+1}. \end{aligned}$$

Since the system is causal and  $H$  is rational, the ROC of  $H$  is  $\text{Re}(s) > -1$ . Since the ROC includes the imaginary axis, the system is BIBO stable.

Although our only objective in this example is to stabilize the unstable plant, we note that, as it turns out, the system also has a somewhat reasonable step response. Recall that, for a control system, the output should track the input. Since, in the case of the step response, the input is  $u$ , we would like the output to at least approximate  $u$ . The step response  $s$  is given by

$$\begin{aligned} s(t) &= \mathcal{L}^{-1} \{U(s)H(s)\}(t) \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{s(s+1)} \right\}(t) \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{1}{s+1} \right\}(t) \\ &= (1 - e^{-t})u(t). \end{aligned}$$

Evidently,  $s$  is a somewhat crude approximation of the desired response  $u$ .

(b) From the block diagram in Figure 7.28(b), we can write

$$\begin{aligned} R(s) &= X(s) - Q(s)Y(s) \quad \text{and} \\ Y(s) &= C(s)P(s)R(s). \end{aligned}$$

Combining these equations (by substituting the expression for  $R$  in the first equation into the second equation), we obtain

$$\begin{aligned} Y(s) &= C(s)P(s)[X(s) - Q(s)Y(s)] \\ \Rightarrow Y(s) &= C(s)P(s)X(s) - C(s)P(s)Q(s)Y(s) \\ \Rightarrow [1 + C(s)P(s)Q(s)]Y(s) &= C(s)P(s)X(s) \\ \Rightarrow \frac{Y(s)}{X(s)} &= \frac{C(s)P(s)}{1 + C(s)P(s)Q(s)}. \end{aligned}$$

Since  $H(s) = \frac{Y(s)}{X(s)}$ , we have

$$H(s) = \frac{C(s)P(s)}{1 + C(s)P(s)Q(s)}.$$

Substituting the given expressions for  $P$ ,  $C$ , and  $Q$ , we have

$$\begin{aligned} H(s) &= \frac{\beta \left( \frac{10}{s-1} \right)}{1 + \beta \left( \frac{10}{s-1} \right) (1)} \\ &= \frac{10\beta}{s-1 + 10\beta} \\ &= \frac{10\beta}{s - (1 - 10\beta)}. \end{aligned}$$

The system function  $H$  is rational and has a single pole at  $1 - 10\beta$ . Since the system is causal, the ROC must be the RHP given by  $\text{Re}(s) > 1 - 10\beta$ . For the system to be stable, we require that the ROC includes the imaginary axis. Thus, the system is stable if  $1 - 10\beta < 0$  which implies  $10\beta > 1$ , or equivalently  $\beta > \frac{1}{10}$ .

Although our only objective in this example is to stabilize the unstable plant, we note that, as it turns out, the system also has a reasonable step response. (This is not by chance, however. Some care had to be exercised in the choice of the form of the compensator system function  $C$ . The process involved in making this choice requires knowledge of control systems beyond the scope of this book, however.) Recall that, for a control system, the output

should track the input. Since, in the case of the step response, the input is  $u$ , we would like the output to at least approximate  $u$ . The step response  $s$  is given by

$$\begin{aligned} s(t) &= \mathcal{L}^{-1} \{U(s)H(s)\} (t) \\ &= \mathcal{L}^{-1} \left\{ \frac{10\beta}{s(s - [1 - 10\beta])} \right\} (t) \\ &= \mathcal{L}^{-1} \left\{ \frac{10\beta}{10\beta - 1} \left( \frac{1}{s} - \frac{1}{s - (1 - 10\beta)} \right) \right\} (t) \\ &= \frac{10\beta}{10\beta - 1} \left( 1 - e^{-(10\beta - 1)t} \right) u(t) \\ &\approx u(t) \quad \text{for large } \beta. \end{aligned}$$

Clearly, as  $\beta$  increases,  $s$  better approximates the desired response  $u$ .

(c) The scheme in part (a) for stabilizing the unstable plant relies on pole-zero cancellation. Unfortunately, in practice, it is not possible to achieve pole-zero cancellation. In short, the issue is one of approximation. Our analysis of systems is based on theoretical models specified in terms of equations. These theoretical models, however, are only approximations of real-world systems. This approximate nature is due to many factors, including (but not limited to) the following:

1. We cannot determine the system function of a system exactly, since this involves measurement, which always has some error.
2. We cannot build a system with such precision that it will have exactly some prescribed system function. The system function will only be close to the desired one.
3. The system function of most systems will vary at least slightly with changes in the physical environment (e.g., changes in temperature and pressure, or changes in gravitational forces due to changes in the phase of the moon, and so on).
4. Although a system may be represented by a LTI model, the likely reality is that the system is not exactly LTI, which introduces error.

For reasons such as these, the effective poles and zeros of the system function will only be approximately where we expect them to be. Pole-zero cancellation, however, requires a pole and zero to be placed at exactly the same location. So, any error will prevent the pole-zero cancellation from occurring. Since at least some small error is unavoidable in practice, the desired pole-zero cancellation will not be achieved.

The scheme in part (b) for stabilizing the unstable plant is based on feedback. With the feedback approach, the poles of the system function are not cancelled with zeros. Instead, the poles are completely changed/relocated. For this reason, we can place the poles such that, even if the poles are displaced slightly (due to approximation error), the stability of the system will not be compromised. Therefore, this second scheme does not suffer from the same practical problem that the first one does. ■

## 7.17 Unilateral Laplace Transform

As mentioned earlier, two different versions of the Laplace transform are commonly employed, namely, the bilateral and unilateral versions. So far, we have considered only the bilateral Laplace transform. Now, we turn our attention to the unilateral Laplace transform. The **unilateral Laplace transform** of the function  $x$  is denoted as  $\mathcal{L}_u x$  or  $X$  and is defined as

$$\mathcal{L}_u x(s) = X(s) = \int_{0^-}^{\infty} x(t)e^{-st} dt. \quad (7.16)$$

The **inverse unilateral Laplace transform** has the same definition as in the case of the bilateral transform, namely (7.3).

Comparing the definitions of the unilateral and bilateral Laplace transforms given by (7.16) and (7.2), respectively, we can see that these definitions only differ in the lower limit of integration. Due to the similarity in these definitions, an important relationship exists between these two transforms, as we shall now demonstrate. Consider the bilateral Laplace transform of the function  $xu$  for an arbitrary function  $x$ . We have

$$\begin{aligned}\mathcal{L}\{xu\}(s) &= \int_{-\infty}^{\infty} x(t)u(t)e^{-st} dt \\ &= \int_{0^-}^{\infty} x(t)e^{-st} dt \\ &= \mathcal{L}_u x(s).\end{aligned}$$

In other words, the unilateral Laplace transform of the function  $x$  is simply the bilateral Laplace transform of the function  $xu$ . Since  $\mathcal{L}_u x = \mathcal{L}\{xu\}$  and  $xu$  is always right sided, the ROC associated with  $\mathcal{L}_u x$  is always a right-half plane. For this reason, we often do not explicitly indicate the ROC when working with the unilateral Laplace transform.

From earlier in this chapter, we know that the bilateral Laplace transform is invertible. That is, if the function  $x$  has the bilateral Laplace transform  $X = \mathcal{L}x$ , then  $\mathcal{L}^{-1}X = x$ . Now, let us consider the invertibility of the unilateral Laplace transform. To do this, we must consider the quantity  $\mathcal{L}_u^{-1}\mathcal{L}_u x$ . Since  $\mathcal{L}_u x = \mathcal{L}\{xu\}$  and the inverse equations for the unilateral and bilateral Laplace transforms are identical, we can write

$$\begin{aligned}\mathcal{L}_u^{-1}\mathcal{L}_u x(t) &= \mathcal{L}_u^{-1}\{\mathcal{L}\{xu\}\}(t) \\ &= \mathcal{L}^{-1}\{\mathcal{L}\{xu\}\}(t) \\ &= x(t)u(t) \\ &= \begin{cases} x(t) & t \geq 0 \\ 0 & t < 0. \end{cases}\end{aligned}$$

Thus, we have that  $\mathcal{L}_u^{-1}\mathcal{L}_u x = x$  only if  $x$  is causal. In other words, the unilateral Laplace transform is invertible only for causal functions. For noncausal functions, we can only recover  $x(t)$  for  $t \geq 0$ . In essence, the unilateral Laplace transform discards all information about the value of the function  $x$  at  $t$  for  $t < 0$ . Since this information is discarded, it cannot be recovered by an inverse unilateral Laplace transform operation.

Due to the close relationship between the unilateral and bilateral Laplace transforms, these two transforms have some similarities in their properties. Since these two transforms are not identical, however, their properties differ in some cases, often in subtle ways. The properties of the unilateral Laplace transform are summarized in Table 7.3.

By comparing the properties of the unilateral and bilateral Laplace transform listed in Tables 7.3 and 7.1, respectively, we can see that the unilateral Laplace transform has some of the same properties as its bilateral counterpart, namely, the linearity, Laplace-domain shifting, conjugation, and Laplace-domain differentiation properties. The initial-value and final-value theorems also apply in the case of the unilateral Laplace transform.

Since the unilateral and bilateral Laplace transforms are defined differently, their properties also differ in some cases. These differences can be seen by comparing the bilateral Laplace transform properties listed in Table 7.1 with the unilateral Laplace transform properties listed in Table 7.3. In the unilateral case, we can see that:

1. the time-domain convolution property has the additional requirement that the functions being convolved must be causal;
2. the time/Laplace-domain scaling property has the additional constraint that the scaling factor must be positive;
3. the time-domain differentiation property has an extra term in the expression for  $\mathcal{L}_u\{\frac{d}{dt}x(t)\}$ ; and
4. the time-domain integration property has a different lower limit in the time-domain integral (namely,  $0^-$  instead of  $-\infty$ ).

Also, in the unilateral case, the time-domain shifting property does not hold (except in special circumstances).

Since  $\mathcal{L}_u x = \mathcal{L}\{xu\}$ , we can easily generate a table of unilateral Laplace transform pairs from a table of bilateral transform pairs. Using the bilateral Laplace transform pairs from Table 7.2, and the preceding relationship between the

Table 7.3: Properties of the unilateral Laplace transform

Property	Time Domain	Laplace Domain
Linearity	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(s) + a_2X_2(s)$
Laplace-Domain Shifting	$e^{s_0t}x(t)$	$X(s - s_0)$
Time/Laplace-Domain Scaling	$x(at), a > 0$	$\frac{1}{a}X\left(\frac{s}{a}\right)$
Conjugation	$x^*(t)$	$X^*(s^*)$
Time-Domain Convolution	$x_1(t) * x_2(t), x_1$ and $x_2$ are causal	$X_1(s)X_2(s)$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$sX(s) - x(0^-)$
Laplace-Domain Differentiation	$-tx(t)$	$\frac{d}{ds}X(s)$
Time-Domain Integration	$\int_{0^-}^t x(\tau)d\tau$	$\frac{1}{s}X(s)$

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 Property
 

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Initial Value Theorem  $x(0^+) = \lim_{s \rightarrow \infty} sX(s)$

Final Value Theorem  $\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s)$ 


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Table 7.4: Transform pairs for the unilateral Laplace transform

Pair	$x(t), t \geq 0$	$X(s)$
1	$\delta(t)$	1
2	1	$\frac{1}{s}$
3	$t^n$	$\frac{n!}{s^{n+1}}$
4	$e^{-at}$	$\frac{1}{s+a}$
5	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
6	$\cos \omega_0 t$	$\frac{s}{s^2 + \omega_0^2}$
7	$\sin \omega_0 t$	$\frac{\omega_0}{s^2 + \omega_0^2}$
8	$e^{-at} \cos \omega_0 t$	$\frac{s+a}{(s+a)^2 + \omega_0^2}$
9	$e^{-at} \sin \omega_0 t$	$\frac{\omega_0}{(s+a)^2 + \omega_0^2}$

unilateral and bilateral Laplace transforms, we can trivially deduce the unilateral Laplace transform pairs in Table 7.4. Since, in the unilateral case, the ROC is always a right-half plane, we do not explicitly indicate the ROC in the table. That is, the ROC is implicitly assumed to be a right-half plane.

The inverse unilateral Laplace transform is computed through the same means used in the bilateral case (e.g., partial fraction expansions). The only difference is that the ROC is always assumed to be a right-half plane.

## 7.18 Solving Differential Equations Using the Unilateral Laplace Transform

Many systems of interest in engineering applications can be characterized by constant-coefficient linear differential equations. As it turns out, a system that is described by such an equation need not be linear. In particular, the system will be linear only if the initial conditions for the differential equation are all zero. If one or more of the initial conditions is nonzero, then the system is what we refer to as **incrementally linear**. For our purposes here, incrementally linear systems can be thought of as a generalization of linear systems. The unilateral Laplace transform is sometimes quite useful due to its ability to easily handle nonzero initial conditions. For example, one common use of the unilateral Laplace transform is in solving constant-coefficient linear differential equations with nonzero initial conditions. In what follows, we consider some examples that exploit the unilateral Laplace transform to this end.

**Example 7.41.** Consider the causal incrementally-linear TI system with input  $x$  and output  $y$  characterized by the differential equation

$$y'(t) + 3y(t) = x(t),$$

where the prime symbol denotes derivative. If  $x(t) = e^{-t}u(t)$  and  $y(0^-) = 1$ , find  $y$ .

*Solution.* We begin by taking the unilateral Laplace transform of both sides of the given differential equation. This

yields

$$\begin{aligned}\mathcal{L}_u\{y'\}(s) + 3\mathcal{L}_u y(s) &= \mathcal{L}_u x(s) \\ \Rightarrow sY(s) - y(0^-) + 3Y(s) &= X(s) \\ \Rightarrow (s+3)Y(s) &= X(s) + y(0^-) \\ \Rightarrow Y(s) &= \frac{X(s) + y(0^-)}{s+3}.\end{aligned}$$

Since  $x(t) = e^{-t}u(t)$ , we have

$$X(s) = \mathcal{L}_u\{e^{-t}\}(s) = \frac{1}{s+1}.$$

Substituting this expression for  $X$  and the given initial conditions (i.e.,  $y(0^-) = 1$ ) into the above equation for  $Y$ , we obtain

$$Y(s) = \frac{\left(\frac{1}{s+1}\right) + 1}{s+3} = \frac{\left(\frac{s+2}{s+1}\right)}{s+3} = \frac{s+2}{(s+1)(s+3)}.$$

Now, we find a partial fraction expansion for  $Y$ . Such an expansion has the form

$$Y(s) = \frac{A_1}{s+1} + \frac{A_2}{s+3}.$$

Calculating the expansion coefficients, we obtain

$$\begin{aligned}A_1 &= (s+1)Y(s)|_{s=-1} \\ &= \left.\frac{s+2}{s+3}\right|_{s=-1} \\ &= \frac{1}{2} \quad \text{and} \\ A_2 &= (s+3)Y(s)|_{s=-3} \\ &= \left.\frac{s+2}{s+1}\right|_{s=-3} \\ &= \frac{1}{2}.\end{aligned}$$

So, we can rewrite  $Y$  as

$$Y(s) = \frac{1/2}{s+1} + \frac{1/2}{s+3}.$$

Taking the inverse unilateral Laplace transform of  $Y$ , we obtain

$$\begin{aligned}y(t) &= \mathcal{L}_u^{-1}Y(s) \\ &= \frac{1}{2}\mathcal{L}_u^{-1}\left\{\frac{1}{s+1}\right\}(t) + \frac{1}{2}\mathcal{L}_u^{-1}\left\{\frac{1}{s+3}\right\}(t) \\ &= \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} \quad \text{for } t \geq 0. \quad \blacksquare\end{aligned}$$

**Example 7.42** (Unilateral Laplace transform of second-order derivative). Find the unilateral Laplace transform  $Y$  of  $y$  in terms of the unilateral Laplace transform  $X$  of  $x$ , where

$$y(t) = x''(t)$$

and the prime symbol denotes derivative (e.g.,  $x''$  is the second derivative of  $x$ )

*Solution.* Define the function

$$v(t) = x'(t) \quad (7.17)$$

so that

$$y(t) = v'(t). \quad (7.18)$$

Let  $V$  denote the unilateral Laplace transform of  $v$ . Taking the unilateral Laplace transform of (7.17) (using the time-domain differentiation property), we have

$$\begin{aligned} V(s) &= \mathcal{L}_u \{x'\}(s) \\ &= sX(s) - x(0^-). \end{aligned} \quad (7.19)$$

Taking the unilateral Laplace transform of (7.18) (using the time-domain differentiation property), we have

$$\begin{aligned} Y(s) &= \mathcal{L}_u \{v'\}(s) \\ &= sV(s) - v(0^-). \end{aligned} \quad (7.20)$$

Substituting (7.19) into (7.20), we have

$$\begin{aligned} Y(s) &= s(sX(s) - x(0^-)) - v(0^-) \\ &= s^2X(s) - sx(0^-) - x'(0^-). \end{aligned}$$

Thus, we have that

$$Y(s) = s^2X(s) - sx(0^-) - x'(0^-). \quad \blacksquare$$

**Example 7.43.** Consider the causal incrementally-linear TI system with input  $x$  and output  $y$  characterized by the differential equation

$$y''(t) + 3y'(t) + 2y(t) = x(t),$$

where the prime symbol denotes derivative. If  $x(t) = 5u(t)$ ,  $y(0^-) = 1$ , and  $y'(0^-) = -1$ , find  $y$ .

*Solution.* We begin by taking the unilateral Laplace transform of both sides of the given differential equation. This yields

$$\begin{aligned} &\mathcal{L}_u \{y'' + 3y' + 2y\}(s) = \mathcal{L}_u x(s) \\ \Rightarrow &\mathcal{L}_u \{y''\}(s) + 3\mathcal{L}_u \{y'\}(s) + 2\mathcal{L}_u y(s) = \mathcal{L}_u x(s) \\ \Rightarrow &[s^2Y(s) - sy(0^-) - y'(0^-)] + 3[sY(s) - y(0^-)] + 2Y(s) = X(s) \\ \Rightarrow &s^2Y(s) - sy(0^-) - y'(0^-) + 3sY(s) - 3y(0^-) + 2Y(s) = X(s) \\ \Rightarrow &[s^2 + 3s + 2]Y(s) = X(s) + sy(0^-) + y'(0^-) + 3y(0^-) \\ \Rightarrow &Y(s) = \frac{X(s) + sy(0^-) + y'(0^-) + 3y(0^-)}{s^2 + 3s + 2}. \end{aligned}$$

Since  $x(t) = 5u(t)$ , we have

$$X(s) = \mathcal{L}_u \{5u(t)\}(s) = \frac{5}{s}.$$

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Substituting this expression for  $X$  and the given initial conditions into the above equation yields

$$Y(s) = \frac{\left(\frac{5}{s}\right) + s - 1 + 3}{s^2 + 3s + 2} = \frac{s^2 + 2s + 5}{s(s+1)(s+2)}.$$

Now, we must find a partial fraction expansion of  $Y$ . Such an expansion is of the form

$$Y(s) = \frac{A_1}{s} + \frac{A_2}{s+1} + \frac{A_3}{s+2}.$$

Calculating the expansion coefficients, we obtain

$$\begin{aligned} A_1 &= sY(s)\Big|_{s=0} \\ &= \frac{s^2 + 2s + 5}{(s+1)(s+2)} \Big|_{s=0} \\ &= \frac{5}{2}, \\ A_2 &= (s+1)Y(s)\Big|_{s=-1} \\ &= \frac{s^2 + 2s + 5}{s(s+2)} \Big|_{s=-1} \\ &= -4, \quad \text{and} \\ A_3 &= (s+2)Y(s)\Big|_{s=-2} \\ &= \frac{s^2 + 2s + 5}{s(s+1)} \Big|_{s=-2} \\ &= \frac{5}{2}. \end{aligned}$$

So, we can rewrite  $Y$  as

$$Y(s) = \frac{5/2}{s} - \frac{4}{s+1} + \frac{5/2}{s+2}.$$

Taking the inverse unilateral Laplace transform of  $Y$  yields

$$\begin{aligned} y(t) &= \mathcal{L}_u^{-1}Y(t) \\ &= \frac{5}{2}\mathcal{L}_u^{-1}\left\{\frac{1}{s}\right\}(t) - 4\mathcal{L}_u^{-1}\left\{\frac{1}{s+1}\right\}(t) + \frac{5}{2}\mathcal{L}_u^{-1}\left\{\frac{1}{s+2}\right\}(t) \\ &= \frac{5}{2} - 4e^{-t} + \frac{5}{2}e^{-2t} \quad \text{for } t \geq 0. \quad \blacksquare \end{aligned}$$

**Example 7.44** (RC network). Consider the resistor-capacitor (RC) network shown in Figure 7.29 with input  $v_0$  and output  $v_1$ . If  $R = 100$ ,  $C = \frac{1}{100}$ ,  $v_0(t) = 3e^{-2t}u(t)$ , and  $v_1(0^-) = 1$ , find  $v_1$ .

*Solution.* From basic circuit analysis, we have

$$\begin{aligned} v_0(t) &= Ri(t) + v_1(t) \quad \text{and} \\ i(t) &= C\frac{d}{dt}v_1(t). \end{aligned}$$

Combining the preceding two equations, we obtain

$$\begin{aligned} v_0(t) &= R\left[C\frac{d}{dt}v_1(t)\right] + v_1(t) \\ &= RC\frac{d}{dt}v_1(t) + v_1(t). \end{aligned}$$

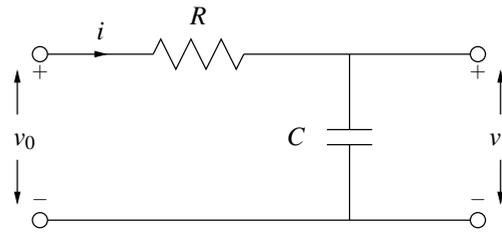


Figure 7.29: RC network.

Taking the unilateral Laplace transform of both sides of this equation yields

$$\begin{aligned}
 \mathcal{L}_u v_0(s) &= \mathcal{L}_u \left\{ RC \frac{d}{dt} v_1(t) + v_1(t) \right\} (s) \\
 \Rightarrow \mathcal{L}_u v_0(s) &= RC \mathcal{L}_u \left\{ \frac{d}{dt} v_1(t) \right\} (s) + \mathcal{L}_u v_1(s) \\
 \Rightarrow V_0(s) &= RC [sV_1(s) - v_1(0^-)] + V_1(s) \\
 \Rightarrow V_0(s) &= RCsV_1(s) - RCv_1(0^-) + V_1(s) \\
 \Rightarrow V_0(s) + RCv_1(0^-) &= RCsV_1(s) + V_1(s) \\
 \Rightarrow V_1(s) &= \frac{V_0(s) + RCv_1(0^-)}{RCs + 1}.
 \end{aligned}$$

Since  $v_0(t) = 3e^{-2t}u(t)$ , we have

$$V_0(s) = \frac{3}{s+2}.$$

Substituting this expression for  $V_0$  into the above equation for  $V_1$ , we obtain

$$\begin{aligned}
 V_1(s) &= \frac{\left(\frac{3}{s+2}\right) + 1}{s+1} \\
 &= \frac{3}{(s+1)(s+2)} + \frac{1}{s+1} \\
 &= \frac{s+5}{(s+1)(s+2)}.
 \end{aligned}$$

Now, we find a partial fraction expansion of  $V_1$ . Such an expansion is of the form

$$V_1(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}.$$

Calculating the expansion coefficients yields

$$\begin{aligned}
 A_1 &= (s+1)V_1(s) \Big|_{s=-1} \\
 &= \frac{s+5}{s+2} \Big|_{s=-1} \\
 &= 4 \quad \text{and} \\
 A_2 &= (s+2)V_1(s) \Big|_{s=-2} \\
 &= \frac{s+5}{s+1} \Big|_{s=-2} \\
 &= -3.
 \end{aligned}$$

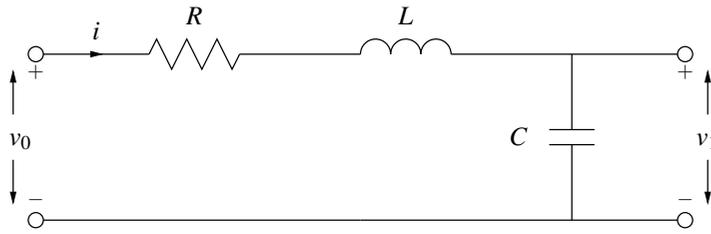


Figure 7.30: RLC network.

Thus, we can rewrite  $V_1$  as

$$V_1(s) = \frac{4}{s+1} - \frac{3}{s+2}.$$

Taking the inverse unilateral Laplace transform of  $V_1$ , we obtain

$$\begin{aligned} v_1(t) &= \mathcal{L}_u^{-1} V_1(t) \\ &= \mathcal{L}_u^{-1} \left\{ \frac{4}{s+1} - \frac{3}{s+2} \right\} (t) \\ &= 4\mathcal{L}_u^{-1} \left\{ \frac{1}{s+1} \right\} (t) - 3\mathcal{L}_u^{-1} \left\{ \frac{1}{s+2} \right\} (t) \\ &= 4e^{-t} - 3e^{-2t} \quad \text{for } t \geq 0. \end{aligned}$$

**Example 7.45** (RLC network). Consider the resistor-inductor-capacitor (RLC) network shown in Figure 7.30 with input  $v_0$  and output  $v_1$ . If  $R = 2$ ,  $L = 1$ ,  $C = 1$ ,  $v_0(t) = u(t)$ ,  $v_1(0^-) = 0$ , and  $v_1'(0^-) = 1$ , find  $v_1$ .

*Solution.* From basic circuit analysis, we can write

$$\begin{aligned} v_0(t) &= Ri(t) + L \frac{d}{dt} i(t) + v_1(t) \quad \text{and} \\ v_1(t) &= \frac{1}{C} \int_{-\infty}^t i(\tau) d\tau \quad \Rightarrow \quad i(t) = C \frac{d}{dt} v_1(t). \end{aligned}$$

Combining the above equations, we obtain

$$\begin{aligned} v_0(t) &= R \left[ C \frac{d}{dt} v_1(t) \right] + L \frac{d}{dt} \left[ C \frac{d}{dt} v_1(t) \right] + v_1(t) \\ &= RC \frac{d}{dt} v_1(t) + LC \frac{d^2}{dt^2} v_1(t) + v_1(t). \end{aligned}$$

Taking the unilateral Laplace transform of both sides of the preceding equation yields

$$\begin{aligned} \mathcal{L}_u v_0(s) &= RC \mathcal{L}_u \left\{ \frac{d}{dt} v_1(t) \right\} (s) + LC \mathcal{L}_u \left\{ \left( \frac{d}{dt} \right)^2 v_1(t) \right\} (s) + \mathcal{L}_u v_1(s) \\ \Rightarrow V_0(s) &= RC [sV_1(s) - v_1(0^-)] + LC [s^2 V_1(s) - sv_1(0^-) - v_1'(0^-)] + V_1(s) \\ \Rightarrow V_0(s) &= RCsV_1(s) - RCv_1(0^-) + LCs^2 V_1(s) - LCsv_1(0^-) - LCv_1'(0^-) + V_1(s) \\ \Rightarrow [LCs^2 + RCs + 1] V_1(s) &= V_0(s) + RCv_1(0^-) + LCsv_1(0^-) + LCv_1'(0^-) \\ \Rightarrow V_1(s) &= \frac{V_0(s) + [RC + LCs] v_1(0^-) + LCv_1'(0^-)}{LCs^2 + RCs + 1}. \end{aligned}$$

Since  $v_0 = u$ , we have  $V_0(s) = \frac{1}{s}$ . Substituting this expression for  $V_0$  into the preceding equation for  $V_1$ , we obtain

$$\begin{aligned} V_1(s) &= \frac{\left(\frac{1}{s}\right) + 1}{s^2 + 2s + 1} \\ &= \frac{\left(\frac{s+1}{s}\right)}{(s+1)^2} \\ &= \frac{1}{s(s+1)}. \end{aligned}$$

Now, we find a partial fraction expansion of  $V_1$ . Such an expansion is of the form

$$V_1(s) = \frac{A_1}{s} + \frac{A_2}{s+1}.$$

Solving for the expansion coefficients, we obtain

$$\begin{aligned} A_1 &= sV_1(s)\Big|_{s=0} \\ &= \frac{1}{s+1}\Big|_{s=0} \\ &= 1 \quad \text{and} \\ A_2 &= (s+1)V_1(s)\Big|_{s=-1} \\ &= \frac{1}{s}\Big|_{s=-1} \\ &= -1. \end{aligned}$$

Thus, we can rewrite  $V_1$  as

$$V_1(s) = \frac{1}{s} - \frac{1}{s+1}.$$

Taking the inverse unilateral Laplace transform of  $V_1$ , we obtain

$$\begin{aligned} v_1(t) &= \mathcal{L}_u^{-1} \left\{ \frac{1}{s} \right\} (t) - \mathcal{L}_u^{-1} \left\{ \frac{1}{s+1} \right\} (t) \\ &= 1 - e^{-t} \quad \text{for } t \geq 0. \end{aligned} \quad \blacksquare$$

## 7.19 Exercises

### 7.19.1 Exercises Without Answer Key

7.1 Using the definition of the Laplace transform, find the Laplace transform  $X$  of each of function  $x$  below.

(a)  $x(t) = e^{-at}u(t)$ ;

(b)  $x(t) = e^{-a|t|}$ ; and

(c)  $x(t) = [\cos \omega_0 t]u(t)$ . [Note:  $\int e^{ax} \cos bx dx = \frac{1}{a^2+b^2} (e^{ax}[a \cos bx + b \sin bx]) + C$ .]

7.2 Using properties of the Laplace transform and a table of Laplace transform pairs, find the Laplace transform  $X$  of each function  $x$  below.

(a)  $x(t) = e^{-2t}u(t)$ ;

(b)  $x(t) = 3e^{-2t}u(t) + 2e^{5t}u(-t)$ ;

(c)  $x(t) = e^{-2t}u(t+4)$ ;

(d)  $x(t) = \int_{-\infty}^t e^{-2\tau}u(\tau)d\tau$ ;

(e)  $x(t) = -e^{at}u(-t+b)$ , where  $a$  and  $b$  are real constants and  $a > 0$ ;

(f)  $x(t) = te^{-3t}u(t+1)$ ; and

(g)  $x(t) = tu(t+2)$ .

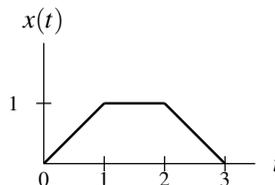
7.3 Using properties of the Laplace transform and a Laplace transform table, find the Laplace transform  $X$  of each function  $x$  below.

$$(a) x(t) = \begin{cases} t & 0 \leq t < 1 \\ t-2 & 1 \leq t < 2 \\ 0 & \text{otherwise;} \end{cases}$$

$$(b) x(t) = \begin{cases} 1+t & -1 \leq t < 0 \\ 1-t & 0 \leq t < 1 \\ 0 & \text{otherwise;} \end{cases} \quad \text{and}$$

$$(c) x(t) = \begin{cases} t & 0 \leq t < 1 \\ 1 & 1 \leq t < 2 \\ t-1 & 2 \leq t < 3 \\ 0 & \text{otherwise.} \end{cases}$$

7.4 Using properties of the Laplace transform and a Laplace transform table, find the Laplace transform  $X$  of the function  $x$  shown in the figure below.



7.5 For each case below, find the Laplace transform  $Y$  of the function  $y$  in terms of the Laplace transform  $X$  of the function  $x$ .

(a)  $y(t) = x(at-b)$  where  $a$  and  $b$  are real constants and  $a \neq 0$ ;

(b)  $y(t) = e^{-3t}[x * x(t-1)]$ ;

(c)  $y(t) = tx(3t-2)$ ;

(d)  $y(t) = \mathcal{D}x_1(t)$ , where  $x_1(t) = x^*(t-3)$  and  $\mathcal{D}$  denotes the derivative operator;

(e)  $y(t) = e^{-5t}x(3t+7)$ ; and

(f)  $y(t) = e^{-j5t}x(t+3)$ .

7.6 The function  $x$  has the Laplace transform

$$X(s) = \frac{(s + \frac{1}{2})(s - \frac{1}{2})}{s(s+1)(s-1)}.$$

Determine whether  $x$  is left sided, right sided, two sided, or finite duration for each ROC of  $X$  below.

- (a)  $\text{Re}(s) < -1$ ;
- (b)  $-1 < \text{Re}(s) < 0$ ;
- (c)  $0 < \text{Re}(s) < 1$ ; and
- (d)  $\text{Re}(s) > 1$ .

7.7 A function  $x$  has the Laplace transform

$$X(s) = \frac{s + \frac{1}{2}}{(s+1-j)(s+1+j)(s+2)}.$$

Plot the ROC of  $X$  if  $x$  is (a) left sided; (b) right sided; (c) two sided; (d) causal.

7.8 A function  $x$  has the Laplace transform

$$X(s) = \frac{s + \frac{1}{2}}{(s-1)(s+1-j)(s+1+j)(s+2)}.$$

Determine how many distinct possibilities exist for  $x$ . (It is not necessary to explicitly find all of them.)

7.9 A causal function  $x$  has the Laplace transform

$$X(s) = \frac{-2s}{s^2 + 3s + 2}.$$

- (a) Assuming that  $x$  has no singularities at 0, find  $x(0^+)$ .
- (b) Assuming that  $\lim_{t \rightarrow \infty} x(t)$  exists, find this limit.

7.10 Find the inverse Laplace transform  $x$  of each function  $X$  below.

- (a)  $X(s) = \frac{s-5}{s^2-1}$  for  $-1 < \text{Re}(s) < 1$ ;
- (b)  $X(s) = \frac{2s^2+4s+5}{(s+1)(s+2)}$  for  $\text{Re}(s) > -1$ ; and
- (c)  $X(s) = \frac{3s+1}{s^2+3s+2}$  for  $-2 < \text{Re}(s) < -1$ .

7.11 Find the causal inverse Laplace transform  $x$  of each function  $X$  below.

- (a)  $X(s) = \frac{s^2+4s+5}{s^2+2s+1}$ ; and
- (b)  $X(s) = \frac{-3s^2-6s-2}{(s+1)^2(s+2)}$ .

7.12 Find all possible inverse Laplace transforms of

$$H(s) = \frac{7s-1}{s^2-1} = \frac{4}{s+1} + \frac{3}{s-1}.$$

**7.13** Consider a LTI system with input  $x$ , output  $y$ , and system function  $H$ , where

$$H(s) = \frac{s+1}{s^2+2s+2}.$$

Find the differential equation that characterizes the behavior of the system.

**7.14** A causal LTI system with input  $x$  and output  $y$  is characterized by the differential equation

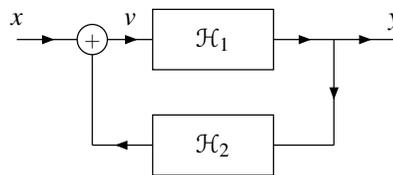
$$y''(t) + 4y'(t) + 3y(t) = 2x'(t) + x(t),$$

where the prime symbol denotes derivative. Find the system function  $H$  of the system.

**7.15** Consider the LTI system with input  $x$ , output  $y$ , and system function  $H$ , as shown in the figure below. Suppose that the systems  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are causal and LTI with the respective system functions

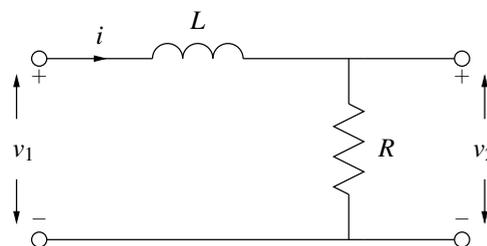
$$H_1(s) = \frac{1}{s-1} \quad \text{and} \quad H_2(s) = A,$$

where  $A$  is a real constant.



- Find an expression for  $H$  in terms of  $H_1$  and  $H_2$ .
- Determine for what values of  $A$  the system is BIBO stable.

**7.16** Consider the LTI resistor-inductor (RL) network with input  $v_1$  and output  $v_2$  shown in the figure below.



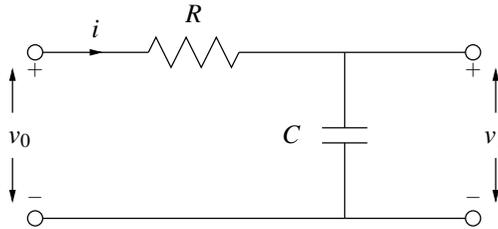
- Find the system function  $H$  of the system.
- Determine whether the system is BIBO stable.
- Find the step response  $s$  of the system.

**7.17** Consider the causal (incrementally-linear TI) system with input  $x$  and output  $y$  that is characterized by the differential equation

$$y''(t) + 7y'(t) + 12y(t) = x(t),$$

where the prime symbol denotes derivative. If  $y(0^-) = -1$ ,  $y'(0^-) = 0$ , and  $x(t) = u(t)$ , find  $y$ .

7.18 Consider the LTI resistor-capacitor (RC) network shown in the figure below, where  $R = 1000$  and  $C = \frac{1}{1000}$ .



- (a) Find the differential equation that characterizes the relationship between the input  $v_0$  and output  $v_1$ .  
 (b) If  $v_1(0^-) = 2$ , and  $v_0(t) = 2e^{-3t}$ , find  $v_1$ .

7.19 Consider a LTI system with the system function

$$H(s) = \frac{s^2 + 7s + 12}{s^2 + 3s + 12}.$$

Find all possible inverses of this system. For each inverse, identify its system function and the corresponding ROC. Also, indicate whether the inverse is causal and/or stable. (Note: You do not need to find the impulse responses of these inverse systems.)

7.20 Consider the causal LTI system with input  $x$  and output  $y$  that is characterized by the differential equation

$$y''(t) - y(t) = x'(t) + ax(t),$$

where  $a$  is a real constant and the prime symbol denotes derivative. Determine for what values of  $a$  the system is BIBO stable.

7.21 In wireless communication channels, the transmitted signal is propagated simultaneously along multiple paths of varying lengths. Consequently, the signal received from the channel is the sum of numerous delayed and amplified/attenuated versions of the original transmitted signal. In this way, the channel distorts the transmitted signal. This is commonly referred to as the multipath problem. In what follows, we examine a simple instance of this problem.

Consider a LTI communication channel with input  $x$  and output  $y$ . Suppose that the transmitted signal  $x$  propagates along two paths. Along the intended direct path, the channel has a delay of  $T$  and gain of one. Along a second (unintended indirect) path, the signal experiences a delay of  $T + \tau$  and gain of  $a$ . Thus, the received signal  $y$  is given by  $y(t) = x(t - T) + ax(t - T - \tau)$ . Find the transfer function  $H$  of a LTI system that can be connected in series with the output of the communication channel in order to recover the (delayed) signal  $x(t - T)$  without any distortion. Determine whether this system is physically realizable.

### 7.19.2 Exercises With Answer Key

7.22 Using the definition of the Laplace transform, find the Laplace transform  $X$  of each of function  $x$  below.

- (a)  $x(t) = (\sin at)u(t)$ , where  $a$  is a real constant; and  
 (b)  $x(t) = (\sinh at)u(t)$ , where  $\sinh t = \frac{1}{2}(e^t - e^{-t})$  and  $a$  is a real constant.  
 (c)  $x(t) = (\cosh at)u(t)$ , where  $\cosh t = \frac{1}{2}(e^t + e^{-t})$  and  $a$  is a real constant.

**Short Answer.** (a)  $X(s) = \frac{a}{s^2 + a^2}$  for  $\text{Re}(s) > 0$ ; (b)  $X(s) = \frac{a}{s^2 - a^2}$  for  $\text{Re}(s) > |a|$ ; (c)  $X(s) = \frac{s}{s^2 - a^2}$  for  $\text{Re}(s) > |a|$

**7.23** Using properties of the Laplace transform and a table of Laplace transform pairs, find the Laplace transform  $X$  of each function  $x$  given below.

- (a)  $x(t) = t^2 e^{-t} u(t-1)$ ;  
 (b)  $x(t) = t^2 u(t-1)$ ;  
 (c)  $x(t) = (t+1)u(t-1)$ ; and  
 (d)  $x(t) = u(t-1) - u(t-2)$ .

**Short Answer.** (a)  $X(s) = e^{-s-1} \left( \frac{s^2+4s+5}{(s+1)^3} \right)$  for  $\text{Re}(s) > -1$ ; (b)  $X(s) = e^{-s} \left( \frac{s^2+2s+2}{s^3} \right)$  for  $\text{Re}(s) > 0$ ;  
 (c)  $X(s) = e^{-s} \left( \frac{2s+1}{s^2} \right)$  for  $\text{Re}(s) > 0$ ; (d)  $X(s) = \frac{e^{-s}-e^{-2s}}{s}$  for all  $s$ .

**7.24** Find the inverse Laplace transform  $x$  of each function  $X$  given below.

- (a)  $X(s) = e^{-7s} \frac{6s+13}{(s+2)(s+3)}$  for  $\text{Re}(s) > -2$ ;  
 (b)  $X(s) = \frac{-3s+2}{(s+1)^2}$  for  $\text{Re}(s) > -1$ ;  
 (c)  $X(s) = \frac{7s^2+19s+17}{(s+1)^2(s+2)}$  for  $\text{Re}(s) > -1$ ;  
 (d)  $X(s) = \frac{s^2+s+2}{(s+1)^2}$  for  $\text{Re}(s) > -1$ ; and  
 (e)  $X(s) = \frac{s^2-s+1}{(s+3)^2(s+2)}$  for  $\text{Re}(s) > -2$ .

**Short Answer.** (a)  $x(t) = e^{-2(t-7)}u(t-7) + 5e^{-3(t-7)}u(t-7)$ ; (b)  $x(t) = 5te^{-t}u(t) - 3e^{-t}u(t)$ ; (c)  $x(t) = 7e^{-2t}u(t) + 5te^{-t}u(t)$ ; (d)  $x(t) = \delta(t) - e^{-t}u(t) + 2te^{-t}u(t)$ ; (e)  $x(t) = 7e^{-2t}u(t) - 6e^{-3t}u(t) - 13te^{-3t}u(t)$

**7.25** Find the inverse Laplace transform  $x$  of the function  $X(s) = \frac{-3}{(s+2)(s-1)}$  if the ROC of  $X$  is:

- (a)  $-2 < \text{Re}(s) < 1$ ;  
 (b)  $\text{Re}(s) > 1$ ; and  
 (c)  $\text{Re}(s) < -2$ .

**Short Answer.** (a)  $x(t) = e^{-2t}u(t) + e^t u(-t)$ ; (b)  $x(t) = [e^{-2t} - e^t]u(t)$ ; (c)  $x(t) = [-e^{-2t} + e^t]u(-t)$

**7.26** Let  $X$  denote the Laplace transform of  $x$ . Suppose that  $x$  is absolutely integrable and  $X$  is rational with a pole at 1 and the other poles unknown. Determine if  $x$  can be each of the following:

- (a) left sided;  
 (b) right sided;  
 (c) two sided;  
 (d) finite duration.

**Short Answer.** (a) yes; (b) no; (c) yes; (d) no

**7.27** Use the Laplace transform to compute the convolution  $y(t) = x_1 * x_2(t)$ , where

$$x_1(t) = e^{-at}u(t), \quad x_2(t) = e^{-bt}u(t),$$

and  $a$  and  $b$  are strictly positive real constants. [Note that there are two cases to consider, namely  $a = b$  and  $a \neq b$ .]

**Short Answer.** if  $a \neq b$ ,  $y(t) = \frac{1}{b-a}(e^{-at} - e^{-bt})u(t)$ ; if  $a = b$ ,  $y(t) = te^{-at}u(t)$

**7.28** For the causal LTI system with input  $x$  and output  $y$  that is characterized by each differential equation given below, find the system function  $H$  of the system.

- (a)  $y''(t) + 3y'(t) + 2y(t) = 5x'(t) + 7x(t)$ .

**Short Answer.** (a)  $H(s) = \frac{5s+7}{s^2+3s+2}$  for  $\text{Re}(s) > -1$

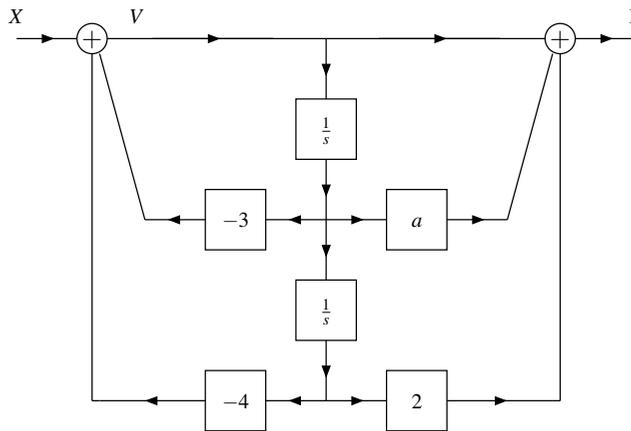
**7.29** For the LTI system with input  $x$  and output  $y$  and each system function  $H$  given below, find the differential equation that characterizes the system.

(a)  $H(s) = \frac{7s+3}{15s^2+4s+1}$ .

**Short Answer.** (a)  $15y''(t) + 4y'(t) + y(t) = 7x'(t) + 3x(t)$

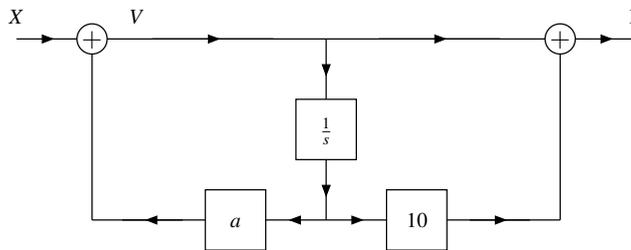
**7.30** Consider the system  $\mathcal{H}$  with input Laplace transform  $X$  and output Laplace transform  $Y$  as shown in the figure. In the figure, each subsystem is LTI and causal and labelled with its system function, and  $a$  is a real constant.

(a) Find the system function  $H$  of the system  $\mathcal{H}$ . (b) Determine whether the system  $\mathcal{H}$  is BIBO stable.



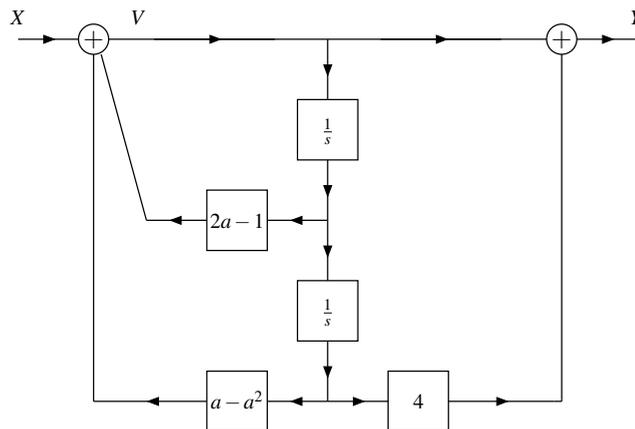
**Short Answer.** (a)  $H(s) = \frac{s^2+as+2}{s^2+3s+4}$  for  $\text{Re}(s) > -\frac{3}{2}$ ; (b) system is BIBO stable.

**7.31** Consider the system  $\mathcal{H}$  with input Laplace transform  $X$  and output Laplace transform  $Y$  as shown in the figure. In the figure, each subsystem is LTI and causal and labelled with its system function, and  $a$  is a real constant. Determine for what values of  $a$  the system  $\mathcal{H}$  is BIBO stable.



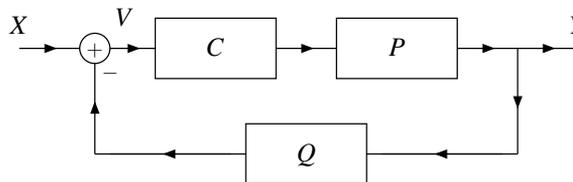
**Short Answer.**  $H(s) = \frac{s+10}{s-a}$  for  $\text{Re}(s) > a$ ; system is BIBO stable if  $a < 0$

**7.32** Consider the system  $\mathcal{H}$  with input Laplace transform  $X$  and output Laplace transform  $Y$  as shown in the figure. In the figure, each subsystem is LTI and causal and labelled with its system function, and  $a$  is a real constant. Determine for what values of  $a$  the system  $\mathcal{H}$  is BIBO stable.



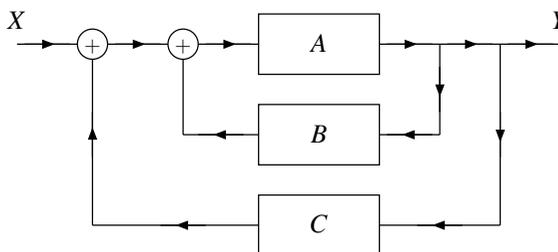
**Short Answer.**  $H(s) = \frac{s^2+4}{s^2+(1-2a)s+a^2-a}$  for  $\text{Re}(s) > a$ ; BIBO stable if  $a < 0$

- 7.33 Consider the system with input Laplace transform  $X$  and output Laplace transform  $Y$  as shown in the figure. In the figure, each subsystem is LTI and causal and labelled with its system function; and  $P(s) = 1/s$ ,  $C(s) = as + 3$ ,  $Q(s) = 1$ , and  $a$  is a real constant. (a) Find the system function  $H$  of the system  $\mathcal{H}$ . (b) Determine for what values of  $a$  the system  $\mathcal{H}$  is BIBO stable.



**Short Answer.** (a)  $H(s) = \frac{as+3}{s+as+3}$  for  $\text{Re}(s) > \frac{-3}{a+1}$ ; (b)  $a > -1$

- 7.34 Consider the LTI system  $\mathcal{H}$  shown in the figure below, where the Laplace transforms of the input and output are denoted as  $X$  and  $Y$ , respectively. Each of the subsystems in the diagram are LTI and labelled with their system functions. Find the system function  $H$  of the system  $\mathcal{H}$  in terms of  $A$ ,  $B$ , and  $C$ .



**Short Answer.**  $H(s) = \frac{A(s)}{1-A(s)[B(s)+C(s)]}$ .

- 7.35 Consider the design of a thermostat system for a room in a building, where the input  $x$  is the desired room temperature and the output  $y$  is the actual room temperature. For the purposes of this design, the system will be modelled as a LTI system with system function  $H$ . Determine which of the functions  $H_1$  and  $H_2$  given below would be a more desirable choice for  $H$ . Explain the rationale for your decision. [Hint: Consider the unit-step response of the system.]

- (a)  $H_1(s) = \frac{s^2}{s^2+9}$ ; and  
 (b)  $H_2(s) = 1 + \frac{1}{20} \left[ \frac{s(s+10)}{(s+10)^2+1} \right]$ .

Short Answer.  $H_2$

## 7.20 MATLAB Exercises

7.101 Consider a causal LTI system with the system function

$$H(s) = \frac{1}{-2s^7 - s^6 - 3s^5 + 2s^3 + s - 3}.$$

- (a) Use MATLAB to find and plot the poles of  $H$ .
- (b) Determine whether the system is BIBO stable.

7.102 Consider a LTI system with the system function

$$H(s) = \frac{1}{1.0000s^4 + 2.6131s^3 + 3.4142s^2 + 2.6131s + 1.0000}.$$

(This system corresponds to a fourth-order Butterworth lowpass filter with a cutoff frequency of 1 rad/s.) Plot the response  $y$  of the system to each input  $x$  given below. In each case, plot  $y$  over the interval  $[0, 20]$ .

- (a)  $x = \delta$ ; and
- (b)  $x = u$ .

(Hint: The `tf`, `impz`, and `step` functions may be helpful.)

## Appendix A

# Complex Analysis

### A.1 Introduction

Complex analysis is an essential tool in the study of signals and systems. For this reason, a brief review of complex analysis is provided in this appendix.

### A.2 Complex Numbers

A **complex number** is a number of the form

$$z = x + jy,$$

where  $x$  and  $y$  are real and  $j$  is the constant defined by

$$j^2 = -1$$

(i.e.,  $j = \sqrt{-1}$ ). The **real part**, **imaginary part**, **magnitude**, and **argument** of the complex number  $z$  are denoted as  $\operatorname{Re} z$  and  $\operatorname{Im} z$ ,  $|z|$ , and  $\arg z$ , respectively, and defined as

$$\begin{aligned} \operatorname{Re} z &= x, & \operatorname{Im} z &= y, \\ |z| &= \sqrt{x^2 + y^2}, & \arg z &= \operatorname{atan2}(y, x) + 2\pi k, \end{aligned}$$

where  $k$  is an arbitrary integer, and

$$\operatorname{atan2}(y, x) \triangleq \begin{cases} \arctan(y/x) & x > 0 \\ \pi/2 & x = 0 \text{ and } y > 0 \\ -\pi/2 & x = 0 \text{ and } y < 0 \\ \arctan(y/x) + \pi & x < 0 \text{ and } y \geq 0 \\ \arctan(y/x) - \pi & x < 0 \text{ and } y < 0. \end{cases} \quad (\text{A.1})$$

(The notation  $\angle z$  is sometimes also used to denote the quantity  $\arg z$ .) The complex number  $z$  can be represented by a point  $(x, y)$  in the complex plane, as illustrated in Figure A.1. This figure also shows the relationship between the real part, imaginary part, magnitude, and argument of a complex number.

For any given complex number  $z$ , the quantity  $\arg z$  is not unique. This follows from the fact that, for any integer  $k$ , the quantities  $\theta$  and  $\theta + 2\pi k$  physically represent the same overall angular displacement. The value  $\theta$  of  $\arg z$  that lies in the range  $-\pi < \theta \leq \pi$  is called the **principal argument** of  $z$  and is denoted as  $\operatorname{Arg} z$ . For a particular nonzero  $z$ , this quantity is uniquely specified. In particular,  $\operatorname{Arg} z = \operatorname{atan2}(y, x)$ .

As an aside, we note that the function  $\operatorname{atan2}(y, x)$  computes the angle that the directed line segment from  $(0, 0)$  to  $(x, y)$  forms with the real axis, and is defined such that  $-\pi < \operatorname{atan2}(y, x) \leq \pi$ .

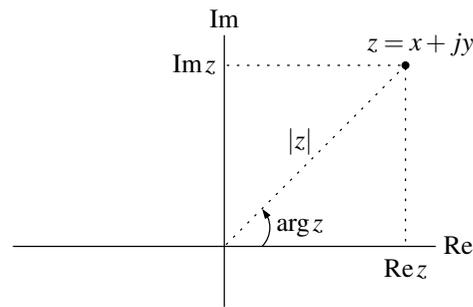


Figure A.1: Graphical representation of a complex number.

**Example A.1.** Compute the real part, imaginary part, magnitude, and principal argument of each complex number  $z$  below.

- (a)  $z = \frac{\sqrt{3}}{2} + j\frac{1}{2}$ ; and  
 (b)  $z = 1 - j$ .

*Solution.* (a) We have

$$\begin{aligned} \operatorname{Re} z &= \frac{\sqrt{3}}{2}, \quad \operatorname{Im} z = \frac{1}{2}, \\ |z| &= \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1, \quad \text{and} \\ \operatorname{Arg} z &= \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}. \end{aligned}$$

(b) We have

$$\begin{aligned} \operatorname{Re} z &= 1, \quad \operatorname{Im} z = -1, \\ |z| &= \sqrt{1^2 + (-1)^2} = \sqrt{2} \quad \text{and} \\ \operatorname{Arg} z &= \arctan\left(\frac{-1}{1}\right) = -\pi/4. \end{aligned} \quad \blacksquare$$

### A.3 Representations of Complex Numbers

Two different representations are commonly used for complex numbers, namely, the Cartesian and polar forms. The Cartesian form is also sometimes referred to as rectangular form. Depending on the particular situation, one form may be more convenient to use than the other.

In the case of the **Cartesian form**, a complex number  $z$  is represented as

$$z = x + jy,$$

where  $x$  and  $y$  are real. That is,  $z$  is expressed directly in terms of its real and imaginary parts. The quantity  $z$  can also be treated as a point  $(x, y)$  in a Cartesian coordinate system as shown in Figure A.2(a).

In the case of the **polar form**, a complex number  $z$  is represented as

$$z = r(\cos \theta + j \sin \theta),$$

where  $r$  and  $\theta$  are real and  $r \geq 0$ . One can show through simple geometry that  $r = |z|$  and  $\theta = \arg z$ . Thus, in the polar case, a complex number is expressed directly in terms of its magnitude and argument. In this way, we

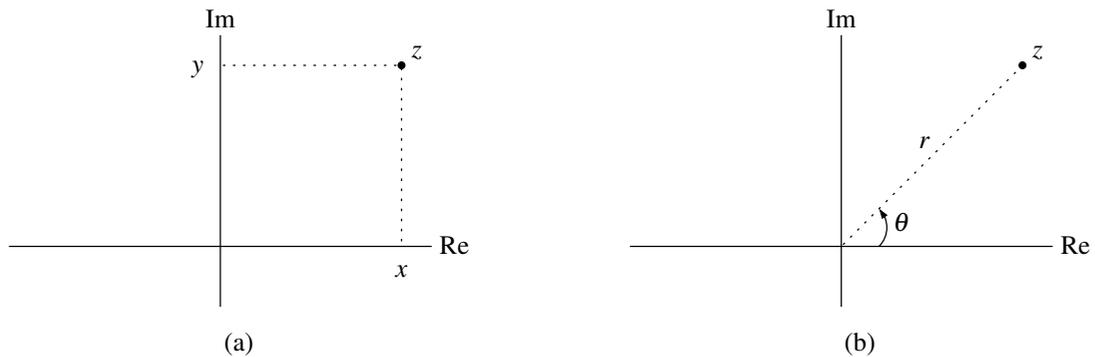


Figure A.2: Representations of complex numbers. The (a) Cartesian and (b) polar forms.

can treat the quantity  $z$  as a point  $(r, \theta)$  in a polar coordinate system, as shown in Figure A.2(b). As we note later,  $\cos \theta + j \sin \theta = e^{j\theta}$ . Therefore, the polar form can equivalently be expressed as

$$z = re^{j\theta}.$$

This exponential notation is often used, due to its relative compactness.

## A.4 Arithmetic Operations

In what follows, we consider a few basic arithmetic operations on complex numbers.

### A.4.1 Conjugation

The **conjugate** of the complex number  $z = x + jy$  is denoted as  $z^*$  and defined as

$$z^* = x - jy.$$

Geometrically, the conjugation operation reflects a point in the complex plane about the real axis, as illustrated in Figure A.3. One can easily verify that for any complex numbers  $z$ ,  $z_1$ , and  $z_2$ , the following identities hold:

$$\begin{aligned} zz^* &= |z|^2, \\ \operatorname{Re} z &= \frac{1}{2}(z + z^*), \\ \operatorname{Im} z &= \frac{1}{2j}(z - z^*), \\ (z_1 + z_2)^* &= z_1^* + z_2^*, \\ (z_1 z_2)^* &= z_1^* z_2^*, \text{ and} \\ (z_1/z_2)^* &= z_1^*/z_2^*. \end{aligned}$$

Trivially, we also have that

$$\begin{aligned} |z^*| &= |z| \quad \text{and} \\ \arg z^* &= -\arg z. \end{aligned}$$

### A.4.2 Addition

Consider the addition of the complex numbers  $z_1$  and  $z_2$ . Suppose that  $z_1$  and  $z_2$  are expressed in Cartesian form as

$$z_1 = x_1 + jy_1 \quad \text{and} \quad z_2 = x_2 + jy_2.$$

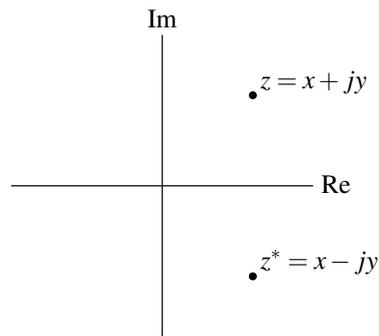


Figure A.3: Conjugate of complex number.

Then, the sum of  $z_1$  and  $z_2$  can be computed as

$$\begin{aligned} z_1 + z_2 &= (x_1 + jy_1) + (x_2 + jy_2) \\ &= (x_1 + x_2) + j(y_1 + y_2). \end{aligned}$$

Suppose that  $z_1$  and  $z_2$  are expressed in polar form as

$$z_1 = r_1 e^{j\theta_1} \quad \text{and} \quad z_2 = r_2 e^{j\theta_2}.$$

Then, the sum of  $z_1$  and  $z_2$  can be computed as

$$\begin{aligned} z_1 + z_2 &= r_1 e^{j\theta_1} + r_2 e^{j\theta_2} \\ &= (r_1 \cos \theta_1 + r_2 \cos \theta_2) + j(r_1 \sin \theta_1 + r_2 \sin \theta_2). \end{aligned}$$

**Example A.2** (Addition with the Cartesian form). Given that  $z_1 = 3 + j4$  and  $z_2 = -2 - j3$ , compute  $z_1 + z_2$ .

*Solution.* We have

$$z_1 + z_2 = (3 + j4) + (-2 - j3) = 1 + j. \quad \blacksquare$$

**Example A.3** (Addition with the polar form). Given that  $z_1 = 3 + j4$  and  $z_2 = \sqrt{2}e^{j5\pi/4}$ , compute  $z_1 + z_2$ .

*Solution.* In order to compute this sum, we first convert  $z_2$  to Cartesian form to obtain

$$\begin{aligned} \operatorname{Re} z_2 &= \sqrt{2} \cos \frac{5\pi}{4} = -1 \quad \text{and} \\ \operatorname{Im} z_2 &= \sqrt{2} \sin \frac{5\pi}{4} = -1. \end{aligned}$$

Then, we have

$$z_1 + z_2 = (3 + j4) + (-1 - j) = 2 + j3. \quad \blacksquare$$

### A.4.3 Multiplication

Consider the multiplication of the complex numbers  $z_1$  and  $z_2$ . Suppose that  $z_1$  and  $z_2$  are represented in Cartesian form as

$$z_1 = x_1 + jy_1 \quad \text{and} \quad z_2 = x_2 + jy_2.$$

Then, the product of  $z_1$  and  $z_2$  can be computed as

$$\begin{aligned} z_1 z_2 &= (x_1 + jy_1)(x_2 + jy_2) \\ &= x_1 x_2 + jx_1 y_2 + jx_2 y_1 - y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1). \end{aligned}$$

Suppose that  $z_1$  and  $z_2$  are represented in polar form as

$$z_1 = r_1 e^{j\theta_1} \quad \text{and} \quad z_2 = r_2 e^{j\theta_2}.$$

Then, the product of  $z_1$  and  $z_2$  can be computed as

$$\begin{aligned} z_1 z_2 &= (r_1 e^{j\theta_1}) (r_2 e^{j\theta_2}) \\ &= r_1 r_2 e^{j(\theta_1 + \theta_2)}. \end{aligned}$$

**Example A.4** (Multiplication with the Cartesian form). Given that  $z_1 = 2 + j3$  and  $z_2 = 3 - j4$ , compute  $z_1 z_2$ .

*Solution.* Using straightforward algebraic manipulation, we have

$$z_1 z_2 = (2 + j3)(3 - j4) = 6 - j8 + j9 + 12 = 18 + j. \quad \blacksquare$$

**Example A.5** (Multiplication with the polar form). Given that  $z_1 = \sqrt{2}e^{j\pi/4}$  and  $z_2 = 3e^{j\pi/6}$ , compute  $z_1 z_2$ .

*Solution.* Using straightforward algebraic manipulation, we obtain

$$z_1 z_2 = (\sqrt{2}e^{j\pi/4})(3e^{j\pi/6}) = 3\sqrt{2}e^{j5\pi/12}. \quad \blacksquare$$

#### A.4.4 Division

Consider the division of the complex numbers  $z_1$  and  $z_2$ . Suppose that  $z_1$  and  $z_2$  are represented in Cartesian form as

$$z_1 = x_1 + jy_1 \quad \text{and} \quad z_2 = x_2 + jy_2.$$

Then, the quotient of  $z_1$  and  $z_2$  can be computed as

$$\begin{aligned} \frac{z_1}{z_2} &= \left( \frac{z_1}{z_2} \right) \left( \frac{z_2^*}{z_2^*} \right) = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{z_1 z_2^*}{|z_2|^2} \\ &= \frac{(x_1 + jy_1)(x_2 - jy_2)}{x_2^2 + y_2^2} \\ &= \frac{x_1 x_2 - jx_1 y_2 + jx_2 y_1 + y_1 y_2}{x_2^2 + y_2^2} \\ &= \frac{x_1 x_2 + y_1 y_2 + j(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}. \end{aligned}$$

Suppose that  $z_1$  and  $z_2$  are represented in polar form as

$$z_1 = r_1 e^{j\theta_1} \quad \text{and} \quad z_2 = r_2 e^{j\theta_2}.$$

Then, the quotient of  $z_1$  and  $z_2$  can be computed as

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} \\ &= \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}. \end{aligned}$$

**Example A.6** (Division with the Cartesian form). Given that  $z_1 = 1 + j$  and  $z_2 = 2 - j$ , compute  $z_1/z_2$ .

*Solution.* Using straightforward algebraic manipulation, we have

$$\frac{z_1}{z_2} = \left( \frac{1+j}{2-j} \right) \left( \frac{2+j}{2+j} \right) = \frac{2+2j+j-1}{2^2+1^2} = \frac{1+3j}{5}. \quad \blacksquare$$

**Example A.7** (Division with the polar form). Given that  $z_1 = 2e^{j\pi/3}$  and  $z_2 = 3e^{j\pi/4}$ , compute  $z_1/z_2$ .

*Solution.* Using straightforward algebraic manipulation, we have

$$\frac{z_1}{z_2} = \frac{2e^{j\pi/3}}{3e^{j\pi/4}} = \frac{2}{3}e^{j\left(\frac{\pi}{3}-\frac{\pi}{4}\right)} = \frac{2}{3}e^{j\pi/12}. \quad \blacksquare$$

### A.4.5 Miscellany

For arbitrary complex numbers  $z_1$  and  $z_2$ , the following identities hold:

$$\begin{aligned} |z_1 z_2| &= |z_1| |z_2|, \\ \left| \frac{z_1}{z_2} \right| &= \frac{|z_1|}{|z_2|} \quad \text{for } z_2 \neq 0, \\ \arg z_1 z_2 &= \arg z_1 + \arg z_2, \quad \text{and} \\ \arg \left( \frac{z_1}{z_2} \right) &= \arg z_1 - \arg z_2 \quad \text{for } z_2 \neq 0. \end{aligned}$$

## A.5 Arithmetic Properties of Complex Numbers

In what follows, we consider some of the properties of arithmetic over the complex numbers.

### A.5.1 Commutative Property

For complex numbers, addition and multiplication are commutative. That is, for any two complex numbers  $z_1$  and  $z_2$ , the following identities hold:

$$\begin{aligned} z_1 + z_2 &= z_2 + z_1 \quad \text{and} \\ z_1 z_2 &= z_2 z_1. \end{aligned}$$

### A.5.2 Associative Property

For complex numbers, addition and multiplication are associative. That is, for any three complex numbers  $z_1$ ,  $z_2$ , and  $z_3$ , the following identities hold:

$$\begin{aligned} (z_1 + z_2) + z_3 &= z_1 + (z_2 + z_3) \quad \text{and} \\ (z_1 z_2) z_3 &= z_1 (z_2 z_3). \end{aligned}$$

### A.5.3 Distributive Property

The distributive property also holds for complex numbers. That is, for any three complex numbers  $z_1$ ,  $z_2$ , and  $z_3$ , the following identity holds:

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

## A.6 Roots of Complex Numbers

Every complex number  $z$  has  $n$  distinct  $n$ th roots in the complex plane. In particular, the  $n$ th roots of  $z = re^{j\theta}$ , where  $r = |z|$  and  $\theta = \arg z$ , are given by

$$\sqrt[n]{r}e^{j(\theta+2\pi k)/n} \quad \text{for } k = 0, 1, \dots, n-1. \quad (\text{A.2})$$

**Example A.8.** Find the four fourth roots of 2.

*Solution.* Let  $z_k$  for  $k = 0, 1, 2, 3$  denote each of the four fourth roots of 2. From (A.2), we have

$$z_k = \sqrt[4]{2}e^{j(0+2\pi k)/4} = \sqrt[4]{2}e^{jk\pi/2}.$$

Thus, we obtain

$$\begin{aligned} z_0 &= \sqrt[4]{2}e^{j0} = \sqrt[4]{2}, \\ z_1 &= \sqrt[4]{2}e^{j\pi/2} = j\sqrt[4]{2}, \\ z_2 &= \sqrt[4]{2}e^{j\pi} = -\sqrt[4]{2}, \quad \text{and} \\ z_3 &= \sqrt[4]{2}e^{j3\pi/2} = -j\sqrt[4]{2}. \end{aligned}$$

So, we conclude that the four fourth roots of 2 are

$$\sqrt[4]{2}, \quad j\sqrt[4]{2}, \quad -\sqrt[4]{2}, \quad \text{and} \quad -j\sqrt[4]{2}. \quad \blacksquare$$

## A.7 Euler's Relation and De Moivre's Theorem

An important relationship exists between exponentials and sinusoids as given by the theorem below.

**Theorem A.1** (Euler's relation). *For any real  $\theta$ , the following identity holds:*

$$e^{j\theta} = \cos \theta + j \sin \theta. \quad (\text{A.3})$$

*This identity is known as Euler's relation.*

*Proof.* The preceding theorem can be proven as follows. Recall that the Maclaurin series expansions of  $e^x$ ,  $\cos x$ , and  $\sin x$  are given by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \quad (\text{A.4})$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, \quad \text{and} \quad (\text{A.5})$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots. \quad (\text{A.6})$$

Using (A.4), we can write  $e^{j\theta}$  as the series

$$e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \dots.$$

By regrouping terms and using (A.5) and (A.6), we obtain

$$\begin{aligned} e^{j\theta} &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + j \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right) \\ &= \cos \theta + j \sin \theta. \quad \blacksquare \end{aligned}$$

From Euler's relation, we can deduce the following additional identities:

$$\cos \theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta}) \quad \text{and} \quad (\text{A.7a})$$

$$\sin \theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta}). \quad (\text{A.7b})$$

Another important result involving exponential functions is given by the theorem below.

**Theorem A.2** (De Moivre's theorem). *For all real  $\theta$  and all integer  $n$ , the following identity holds:*

$$e^{jn\theta} = (e^{j\theta})^n. \quad (\text{A.8})$$

This result is known as **De Moivre's theorem**.

*Proof.* The result of the above theorem can be proven by induction, and is left as an exercise for the reader. (See Exercise A.10.) ■

Note that, in the preceding theorem,  $n$  must be an integer. The identity (A.8) does not necessarily hold if  $n$  is not an integer. For example, consider  $\theta = -\pi$  and  $n = \frac{1}{2}$  (where  $n$  is clearly not an integer). We have that

$$e^{jn\theta} = e^{j(1/2)(-\pi)} = e^{-j\pi/2} = -j \quad \text{and}$$

$$(e^{j\theta})^n = (e^{-j\pi})^{1/2} = (-1)^{1/2} = j.$$

Clearly, in this case,  $e^{jn\theta} \neq (e^{j\theta})^n$ .

## A.8 Conversion Between Cartesian and Polar Form

Suppose that we have a complex number  $z = x + jy = re^{j\theta}$ . Using Euler's relation, we can derive the following expressions for converting from polar to Cartesian form:

$$x = r \cos \theta \quad \text{and} \quad (\text{A.9a})$$

$$y = r \sin \theta. \quad (\text{A.9b})$$

Similarly, we can deduce the following expressions for converting from Cartesian to polar form:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad (\text{A.10a})$$

$$\theta = \text{atan2}(y, x), \quad (\text{A.10b})$$

where the atan2 function is as defined in (A.1).

If we choose to use the arctan function directly in order to compute  $\theta$  (instead of using the atan2 function), we must be careful to consider the quadrant in which the point  $(x, y)$  lies. This complication is due to the fact that the arctan function is defined such that  $-\frac{\pi}{2} \leq \arctan \theta \leq \frac{\pi}{2}$ . Consequently, if the point does not lie in the first or fourth quadrant of the complex plane, the arctan function will not yield the desired angle.

**Example A.9.** Express each complex number  $z$  given below in polar form.

(a)  $z = 1 + j\sqrt{2}$ ; and

(b)  $z = -1 - j$ .

*Solution.* (a) The magnitude and argument of  $z$  are given by

$$|z| = \sqrt{1^2 + (\sqrt{2})^2} = \sqrt{3} \quad \text{and}$$

$$\arg z = \arctan\left(\frac{\sqrt{2}}{1}\right) = \arctan \sqrt{2}.$$

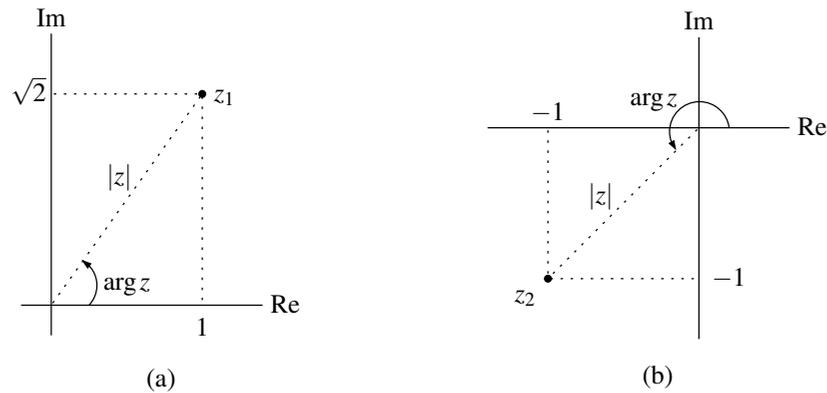


Figure A.4: Example of converting complex numbers from Cartesian to polar form. The case of the (a) first and (b) second part of the example.

Thus, we have

$$z = \sqrt{3}e^{j(\arctan \sqrt{2})}.$$

The result is plotted in Figure A.4(a).

(b) The magnitude and argument of  $z$  are given by

$$|z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2} \quad \text{and}$$

$$\arg z = \arctan\left(\frac{-1}{-1}\right) - \pi = \arctan 1 - \pi = -\frac{3\pi}{4}.$$

Thus, we have

$$z = \sqrt{2}e^{-j3\pi/4}.$$

The result is plotted in Figure A.4(b). ■

## A.9 Complex Functions

A **complex function** maps complex numbers to complex numbers. For example, the function

$$f(z) = z^2 + 2z + 1,$$

where  $z$  is complex, is a complex function.

A **complex polynomial function** is mapping of the form

$$f(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n,$$

where  $a_0, a_1, \dots, a_n$ , and  $z$  are complex. A **complex rational function** is a mapping of the form

$$f(z) = \frac{a_0 + a_1z + a_2z^2 + \cdots + a_nz^n}{b_0 + b_1z + b_2z^2 + \cdots + b_mz^m},$$

where  $a_0, a_1, \dots, a_n, b_0, b_1, \dots, b_m$  and  $z$  are complex. In the context of systems theory, polynomial and rational functions play a particularly important role.

Given any complex function  $f(z)$ , we can always write  $f(z) = \operatorname{Re}[f(z)] + j\operatorname{Im}[f(z)]$ . Writing  $z$  in Cartesian form as  $z = x + jy$ , we have that  $f(z) = \operatorname{Re}[f(x + jy)] + j\operatorname{Im}[f(x + jy)]$ . Now, we can express  $\operatorname{Re}[f(x + jy)]$  as a real-valued function  $v$  of the two real variables  $x$  and  $y$ . Similarly, we can express  $\operatorname{Im}[f(x + jy)]$  as a real-valued function  $w$  of the two real variables  $x$  and  $y$ . Thus, we can always express a complex function  $f$  in the form

$$f(z) = f(x + jy) = v(x, y) + jw(x, y), \quad (\text{A.11})$$

where  $v$  and  $w$  are each real-valued functions of the two real variables  $x$  and  $y$  (and  $z = x + jy$ ).

## A.10 Circles, Disks, and Annuli

A **circle** in the complex plane with center  $z_0$  and radius  $r$  is the set of points  $z$  satisfying

$$|z - z_0| = r,$$

where  $r$  is a strictly positive real constant. A plot of a circle is shown in Figure A.5.

A **disk** is the set of points inside of a circle, possibly including the points on the circle itself. If the points on the circle are not included in the set, the disk is said to be open; otherwise, the disk is said to be closed. More formally, an **open disk** with center  $z_0$  and radius  $r$  is the set of points  $z$  satisfying

$$|z - z_0| < r,$$

where  $r$  is a strictly positive real constant. A plot of an open disk is shown in Figure A.6. Similarly, a **closed disk** with center  $z_0$  and radius  $r$  is the set of points  $z$  satisfying

$$|z - z_0| \leq r,$$

where  $r$  is a strictly positive real constant. A plot of a closed disk is shown in Figure A.7.

An **annulus** (i.e., a ring) is the set of points between two concentric circles, possibly including the points on one or both circles. If an annulus does not include the points on its two defining circles, it is said to be open. More formally, an **open annulus** with center  $z_0$ , inner radius  $r_1$ , and outer radius  $r_2$  is the set of points  $z$  satisfying

$$r_1 < |z - z_0| < r_2,$$

where  $r_1$  and  $r_2$  are strictly positive real constants. A plot of an annulus is shown in Figure A.8.

## A.11 Limit

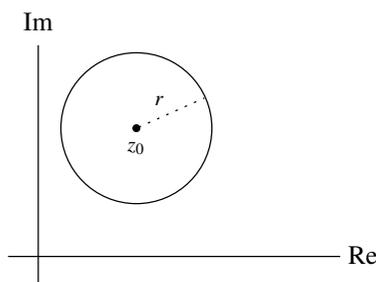
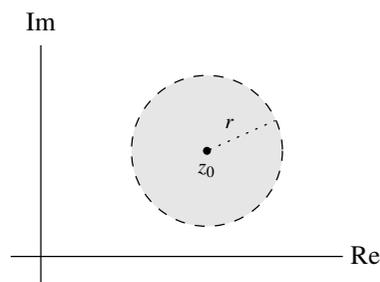
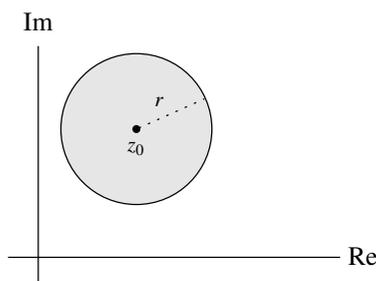
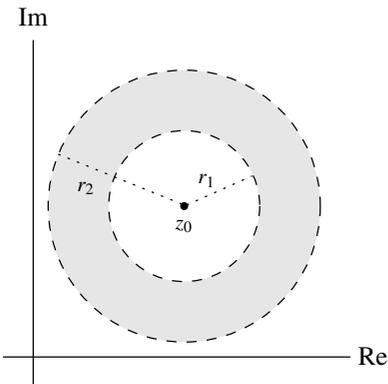
Let  $f$  be a complex function and  $z_0$  a complex number. We want to define the limit of  $f(z)$  as  $z$  approaches  $z_0$ . Unlike in the case of real functions, the value  $z_0$  can be approached from infinitely many directions in the complex plane. In order for the limit to be useful, however, we want it to be defined in such a way that it is independent of the direction from which  $z_0$  is approached. With this in mind, we define the notion of a limit below.

A function  $f$  evaluated at  $z$  is said to have the limit  $L$  as  $z$  approaches  $z_0$  if

1.  $f$  is defined in some open disk about  $z_0$ , except possibly at the point  $z_0$ ; and
2. for every positive real number  $\varepsilon$ , there exists a positive real number  $\delta$  such that  $|f(z) - L| < \varepsilon$  for all values of  $z$  in the disk  $|z - z_0| < \delta$  except  $z = z_0$ .

This limit can be expressed as

$$\lim_{z \rightarrow z_0} f(z) = L.$$

Figure A.5: Circle about  $z_0$  with radius  $r$ .Figure A.6: Open disk of radius  $r$  about  $z_0$ .Figure A.7: Closed disk of radius  $r$  about  $z_0$ .Figure A.8: Open annulus about  $z_0$  with inner radius  $r_1$  and outer radius  $r_2$ .

## A.12 Continuity

A function  $f$  is said to be **continuous at a point**  $z_0$  if  $f(z_0)$  is defined and given by

$$f(z_0) = \lim_{z \rightarrow z_0} f(z).$$

A function is said to be **continuous** if it is continuous at every point in its domain. Polynomial functions are continuous everywhere. For example, the function  $f(z) = 3z^3 + z^2 - z + 1$  is continuous for all complex  $z$ . Rational functions (i.e., quotients of polynomials) are continuous everywhere except at points where the denominator polynomial becomes zero. For example, the function  $f(z) = \frac{(z+j)(z-j)}{(z-1)(z+1)}$  is continuous for all complex  $z$  except  $z = 1$  and  $z = -1$ .

## A.13 Differentiability

A function  $f$  is said to be **differentiable at a point**  $z = z_0$  if the limit

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. This limit is called the **derivative** of  $f$  at the point  $z_0$ . A function is said to be **differentiable** if it is differentiable at every point in its domain.

In general, the rules for differentiating sums, products, and quotients are the same for complex functions as for real functions. Let  $f$  and  $g$  be function and let  $a$  be a scalar constant. Let the prime symbol denote a derivative. If  $f'(z_0)$  and  $g'(z_0)$  exist, then

1.  $(af)'(z_0) = af'(z_0)$  for any complex constant  $a$ ;

2.  $(f + g)'(z_0) = f'(z_0) + g'(z_0)$ ;
3.  $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$ ;
4.  $(f/g)'(z_0) = \frac{g(z_0)f'(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$ ; and
5. if  $z_0 = g(w_0)$  and  $g'(w_0)$  exists, then the derivative of  $f(g(z))$  at  $w_0$  is  $f'(z_0)g'(w_0)$  (i.e., the chain rule).

Polynomial functions are differentiable everywhere. Rational functions are differentiable everywhere except at points where the denominator polynomial becomes zero.

## A.14 Analyticity

A function  $f$  is said to be **analytic at a point**  $z_0$  if it is differentiable at every point in some open disk about  $z_0$ . A function  $f$  is said to be **analytic** if it is analytic at every point in its domain.

One extremely useful test for the analyticity of a function is given by the theorem below.

**Theorem A.3** (Cauchy-Riemann equations). *Let  $f$  be a complex function expressed in the form of (A.11). So, we have*

$$f(z) = v(x, y) + jw(x, y),$$

where  $z = x + jy$ . The function  $f$  is analytic in  $S$  if and only if  $v$  and  $w$  satisfy the following conditions at all points in  $S$ :

$$\frac{\partial v}{\partial x} = \frac{\partial w}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial x}.$$

These equations are known as the **Cauchy-Riemann equations**.

*Proof.* A proof of this theorem is somewhat tedious and therefore omitted here. ■

Polynomial functions are both continuous and differentiable everywhere. Therefore, such functions are analytic everywhere. Rational functions are both continuous and differentiable everywhere, except at points where the denominator polynomial becomes zero. Consequently, rational functions are analytic at all but these points.

**Example A.10.** Determine for what values of  $z$  the function  $f(z) = z^2$  is analytic.

*Solution.* First, we observe that  $f$  is a polynomial function. Then, we recall that polynomial functions are analytic everywhere. Therefore,  $f$  is analytic everywhere.

*Alternate Solution.* We can demonstrate the analyticity of  $f$  using Theorem A.3. We express  $z$  in Cartesian form as  $z = x + jy$ . We rewrite  $f$  in the form of  $f(x, y) = v(x, y) + jw(x, y)$  as follows:

$$f(z) = f(x + jy) = (x + jy)^2 = x^2 + j2xy - y^2 = (x^2 - y^2) + j(2xy).$$

Thus, we have that  $f(z) = v(x, y) + jw(x, y)$ , where

$$v(x, y) = x^2 - y^2 \quad \text{and} \quad w(x, y) = 2xy.$$

Now, computing the partial derivatives, we obtain

$$\frac{\partial v}{\partial x} = 2x, \quad \frac{\partial w}{\partial y} = 2x, \quad \frac{\partial v}{\partial y} = -2y, \quad \text{and} \quad \frac{\partial w}{\partial x} = 2y.$$

From this, we can see that

$$\frac{\partial v}{\partial x} = \frac{\partial w}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial x}.$$

Therefore, the Cauchy-Riemann equations are satisfied for all complex  $z = x + jy$ . Therefore,  $f$  is analytic everywhere. ■

**Example A.11.** Determine for what values of  $z$  the function  $f(z) = 1/z$  is analytic.

*Solution.* We can deduce the analyticity properties of  $f$  as follows. First, we observe that  $f$  is a rational function. Then, we recall that a rational function is analytic everywhere except at points where its denominator polynomial becomes zero. Since the denominator polynomial of  $f$  only becomes zero at 0,  $f$  is analytic everywhere except at 0.

*Alternate Solution.* To study the analyticity of  $f$ , we use Theorem A.3. We express  $z$  in Cartesian form as  $z = x + jy$ . We rewrite  $f$  in the form  $f(x, y) = v(x, y) + jw(x, y)$  as follows:

$$f(z) = f(x + jy) = \frac{1}{x + jy} = \left( \frac{1}{x + jy} \right) \left( \frac{x - jy}{x - jy} \right) = \frac{x - jy}{x^2 + y^2}.$$

Thus, we have that  $f(x, y) = v(x, y) + jw(x, y)$ , where

$$v(x, y) = \frac{x}{x^2 + y^2} = x(x^2 + y^2)^{-1} \quad \text{and}$$

$$w(x, y) = \frac{-y}{x^2 + y^2} = -y(x^2 + y^2)^{-1}.$$

Now, computing the partial derivatives, we obtain

$$\frac{\partial v}{\partial x} = (x^2 + y^2)^{-1} + (-1)(x^2 + y^2)^{-2}(2x^2) = \frac{-2x^2 + (x^2 + y^2)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

$$\frac{\partial w}{\partial y} = (-1)(x^2 + y^2)^{-1} + (-1)(x^2 + y^2)^{-2}(2y)(-y) = \frac{2y^2 - (x^2 + y^2)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

$$\frac{\partial v}{\partial y} = (-1)(x^2 + y^2)^{-2}(2y)x = \frac{-2xy}{(x^2 + y^2)^2}, \quad \text{and}$$

$$\frac{\partial w}{\partial x} = (-1)(x^2 + y^2)^{-2}(2x)(-y) = \frac{2xy}{(x^2 + y^2)^2}.$$

So, we have that, for  $z \neq 0$  (i.e.,  $x$  and  $y$  not both zero),

$$\frac{\partial v}{\partial x} = \frac{\partial w}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial x}$$

(i.e., the Cauchy-Riemann equations are satisfied). Therefore,  $f$  is analytic everywhere except at 0. ■

## A.15 Zeros and Singularities

If a function  $f$  is analytic in a domain  $D$  and is zero at a point  $z_0$  in  $D$ ,  $f$  is said to have a **zero** at  $z_0$ . If, in addition, the first  $n - 1$  derivatives of  $f$  are also zero at  $z_0$  (i.e.,  $f^{(1)}(z_0) = f^{(2)}(z_0) = \dots = f^{(n-1)}(z_0) = 0$ ),  $f$  is said to have an  **$n$ th-order zero** at  $z_0$ . An analytic function  $f$  is said to have an  $n$ th order zero at infinity if the function  $g(z) = f(1/z)$  has an  $n$ th order zero at 0.

A point at which a function fails to be analytic is called a **singularity**. A singularity may be isolated or nonisolated. If a function  $f$  is analytic for  $z$  in an annulus  $0 < |z - z_0| < r$  but not at  $z_0$ ,  $f$  is said to have an **isolated singularity** at  $z_0$ . There are three types of isolated singularities: a removable singularity, an essential singularity, and a pole.

Herein, we are often interested in polynomial and rational functions. Polynomial functions do not have singularities, since such functions are analytic everywhere. In contrast, rational functions can have singularities. In the case of rational functions, we are normally interested in poles (since rational functions cannot have essential singularities and removable singularities are not very interesting).

Consider a rational function  $f$ . We can always express such a function in factored form as

$$f(z) = \frac{K(z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2}\cdots(z-a_M)^{\alpha_M}}{(z-b_1)^{\beta_1}(z-b_2)^{\beta_2}\cdots(z-b_N)^{\beta_N}},$$

where  $K$  is complex,  $a_1, a_2, \dots, a_M, b_1, b_2, \dots, b_N$  are distinct complex constants, and  $\alpha_1, \alpha_2, \dots, \alpha_M$  and  $\beta_1, \beta_2, \dots, \beta_N$  are positive integers. One can show that  $f$  has poles at  $b_1, b_2, \dots, b_N$  and zeros at  $a_1, a_2, \dots, a_M$ . Furthermore, the  $k$ th pole (i.e.,  $b_k$ ) is of order  $\beta_k$ , and the  $k$ th zero (i.e.,  $a_k$ ) is of order  $\alpha_k$ . A pole of first order is said to be **simple**, whereas a pole of order two or greater is said to be **repeated**. A similar terminology can also be applied to zeros (i.e., **simple zero** and **repeated zero**).

**Example A.12** (Poles and zeros of a rational function). Find and plot the poles and (finite) zeros of the function

$$f(z) = \frac{z^2(z^2+1)(z-1)}{(z+1)(z^2+3z+2)(z^2+2z+2)}.$$

*Solution.* We observe that  $f$  is a rational function, so we can easily determine the poles and zeros of  $f$  from its factored form. We now proceed to factor  $f$ . First, we factor  $z^2+3z+2$ . To do this, we solve for the roots of  $z^2+3z+2=0$  to obtain

$$z = \frac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)} = -\frac{3}{2} \pm \frac{1}{2} = \{-1, -2\}.$$

(For additional information on how to find the roots of a quadratic equation, see Section A.16.) So, we have

$$z^2 + 3z + 2 = (z+1)(z+2).$$

Second, we factor  $z^2+2z+2$ . To do this, we solve for the roots of  $z^2+2z+2=0$  to obtain

$$z = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = -1 \pm j = \{-1+j, -1-j\}.$$

So, we have

$$z^2 + 2z + 2 = (z+1-j)(z+1+j).$$

Lastly, we factor  $z^2+1$ . Using the well-known factorization for a sum of squares, we obtain

$$z^2 + 1 = (z+j)(z-j).$$

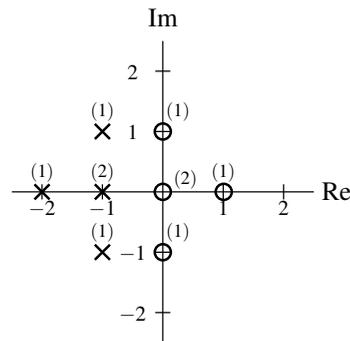
Combining the above results, we can rewrite  $f$  as

$$f(z) = \frac{z^2(z+j)(z-j)(z-1)}{(z+1)^2(z+2)(z+1-j)(z+1+j)}.$$

From this expression, we can trivially deduce that  $f$  has:

- first order zeros at  $1, j$ , and  $-j$ ,
- a second order zero at  $0$ ,
- first order poles at  $-1+j, -1-j, -2$ , and
- a second order pole at  $-1$ .

The zeros and poles of this function are plotted in Figure A.9. In such plots, the poles and zeros are typically denoted by the symbols “x” and “o”, respectively. ■

Figure A.9: Plot of the poles and zeros of  $f$  (with their orders indicated in parentheses).

**Example A.13.** Find the zeros and poles of the function

$$f(z) = \frac{z^3 - 2}{z^5 + 4}.$$

*Solution.* The zeros of  $f$  are given by the roots of  $z^3 - 2 = 0$ , which is equivalent to  $z^3 = 2 = 2e^{j0}$ . This equation has three distinct solutions (i.e., the three third roots of 2), namely,

$$\sqrt[3]{2}, \quad \sqrt[3]{2}e^{j2\pi/3}, \quad \text{and} \quad \sqrt[3]{2}e^{j4\pi/3}.$$

The poles of  $f$  are given by the roots of  $z^5 + 4 = 0$ , which is equivalent to  $z^5 = -4 = 4e^{-j\pi}$ . This equation has five distinct solutions (i.e., the five fifth roots of  $-4$ ), namely,

$$\begin{aligned} \sqrt[5]{4}e^{j(-\pi+0)/5} &= \sqrt[5]{4}e^{-j\pi/5}, \\ \sqrt[5]{4}e^{j(-\pi+2\pi)/5} &= \sqrt[5]{4}e^{j\pi/5}, \\ \sqrt[5]{4}e^{j(-\pi+4\pi)/5} &= \sqrt[5]{4}e^{j3\pi/5}, \\ \sqrt[5]{4}e^{j(-\pi+6\pi)/5} &= \sqrt[5]{4}e^{j\pi} = -\sqrt[5]{4}, \quad \text{and} \\ \sqrt[5]{4}e^{j(-\pi+8\pi)/5} &= \sqrt[5]{4}e^{j7\pi/5}. \end{aligned}$$

## A.16 Quadratic Formula

Consider the equation  $az^2 + bz + c = 0$ , where  $a$ ,  $b$ , and  $c$  are real,  $z$  is complex, and  $a \neq 0$ . The roots of this equation are given by

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (\text{A.12})$$

This formula is often useful in factoring quadratic polynomials with complex conjugate roots. For example, from the quadratic formula, we can conclude that the general quadratic  $az^2 + bz + c$  has the factorization

$$az^2 + bz + c = a(z - z_0)(z - z_1),$$

where

$$z_0 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad z_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

## A.17 Exercises

### A.17.1 Exercises Without Answer Key

**A.1** Express each of the following complex numbers in Cartesian form:

- (a)  $2e^{j2\pi/3}$ ;
- (b)  $\sqrt{2}e^{j\pi/4}$ ;
- (c)  $2e^{j7\pi/6}$ ; and
- (d)  $3e^{j\pi/2}$ .

**A.2** Express each complex number below in polar form. In each case, plot the value in the complex plane, clearly indicating its magnitude and argument. State the principal value for the argument.

- (a)  $-\sqrt{3} + j$ ;
- (b)  $-\frac{1}{2} - j\frac{\sqrt{3}}{2}$ ;
- (c)  $\sqrt{2} - j\sqrt{2}$ ;
- (d)  $1 + j\sqrt{3}$ ;
- (e)  $-1 - j\sqrt{3}$ ; and
- (f)  $-3 + 4j$ .

**A.3** Evaluate each of the expressions below, stating the final result in the specified form. When giving a final result in polar form, state the principal value of the argument.

- (a)  $2\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right) + j\left(\frac{1}{\sqrt{2}}e^{j(-3\pi/4)}\right)$  (in Cartesian form);
- (b)  $\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}e^{j(-3\pi/4)}\right)$  (in polar form);
- (c)  $\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right)/(1 + j)$  (in polar form);
- (d)  $e^{1+j\pi/4}$  (in Cartesian form);
- (e)  $\left(\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)^*\right)^8$  (in polar form);
- (f)  $(1 + j)^{10}$  (in Cartesian form); and
- (g)  $\frac{1 + j}{1 - j}$  (in polar form).

**A.4** Show that each of the identities below holds, where  $z$ ,  $z_1$ , and  $z_2$  are arbitrary complex numbers.

- (a)  $|z_1/z_2| = |z_1|/|z_2|$  for  $z_2 \neq 0$ ;
- (b)  $\arg(z_1/z_2) = \arg z_1 - \arg z_2$  for  $z_2 \neq 0$ ;
- (c)  $z + z^* = 2\operatorname{Re}\{z\}$ ;
- (d)  $zz^* = |z|^2$ ; and
- (e)  $(z_1z_2)^* = z_1^*z_2^*$ .

**A.5** Use Euler's relation to show that each of the identities below holds.

- (a)  $\cos \theta = \frac{1}{2}[e^{j\theta} + e^{-j\theta}]$ ;
- (b)  $\sin \theta = \frac{1}{2j}[e^{j\theta} - e^{-j\theta}]$ ; and
- (c)  $\cos^2 \theta = \frac{1}{2}[1 + \cos 2\theta]$ .

**A.6** For each rational function  $f$  of a complex variable given below, find the poles and zeros of  $f$  and their orders. Also, plot the poles and zeros of  $f$  in the complex plane.

- (a)  $f(z) = z^2 + jz + 3$ ;
- (b)  $f(z) = z + 3 + 2z^{-1}$ ;

$$(c) f(z) = \frac{(z^2 + 2z + 5)(z^2 + 1)}{(z^2 + 2z + 2)(z^2 + 3z + 2)};$$

$$(d) f(z) = \frac{z^3 - z}{z^2 - 4};$$

$$(e) f(z) = \frac{z + \frac{1}{2}}{(z^2 + 2z + 2)(z^2 - 1)}; \text{ and}$$

$$(f) f(z) = \frac{z^2(z^2 - 1)}{(z^2 + 4z + \frac{17}{4})^2(z^2 + 2z + 2)}.$$

**A.7** Determine the points at which each function  $f$  given below is: i) continuous, ii) differentiable, and iii) analytic. To deduce the answer, use your knowledge about polynomial and rational functions. Simply state the final answer along with a short justification (i.e., two or three sentences). (In other words, it is not necessary to use the Cauchy-Riemann equations for this problem.)

$$(a) f(z) = 3z^3 - jz^2 + z - \pi;$$

$$(b) f(z) = \frac{z - 1}{(z^2 + 3)(z^2 + z + 1)};$$

$$(c) f(z) = \frac{z}{z^4 - 16}; \text{ and}$$

$$(d) f(z) = z + 2 + z^{-1}.$$

**A.8** Use the Cauchy-Riemann equations to show that the function  $f(z) = e^{az}$  is analytic for all  $z$ , where  $a$  is a real constant and  $z$  is complex.

**A.9** For each function  $f$  of a real variable given below, find an expression for  $|f(\omega)|$  and  $\arg f(\omega)$ .

$$(a) f(\omega) = \frac{1}{(1 + j\omega)^{10}};$$

$$(b) f(\omega) = \frac{-2 - j\omega}{(3 + j\omega)^2};$$

$$(c) f(\omega) = \frac{2e^{j11\omega}}{(3 + j5\omega)^7};$$

$$(d) f(\omega) = \frac{-5}{(-1 - j\omega)^4};$$

$$(e) f(\omega) = \frac{j\omega^2}{(j\omega - 1)^{10}}; \text{ and}$$

$$(f) f(\omega) = \frac{j\omega - 1}{j\omega + 1}.$$

**A.10** Use induction to prove De Moivre's theorem (i.e., Theorem A.2), which states that  $(e^{j\theta})^n = e^{j\theta n}$  for all real  $\theta$  and all integer  $n$ .

**A.11** Show that each of the following identities hold:

$$(a) \sum_{n=\langle N \rangle} e^{j(2\pi/N)kn} = \begin{cases} N & \text{if } k/N \text{ is an integer} \\ 0 & \text{otherwise,} \end{cases}$$

where  $\sum_{n=\langle N \rangle}$  denotes a summation over a single period of the  $N$ -periodic summand (i.e., the expression being summed); and

$$(b) \int_T e^{j(2\pi/T)kt} = \begin{cases} T & k = 0 \\ 0 & \text{otherwise,} \end{cases}$$

where  $\int_T$  denotes integration over a single period of the  $T$ -periodic integrand (i.e., the expression being integrated).

### A.17.2 Exercises With Answer Key

**A.12** A rational function  $F$  has a first-order pole at  $-1$ , a second-order pole at  $-2$ , and a first-order zero at  $0$ . The function is known not to have any other poles or zeros. If  $F(1) = 1$ , find  $F$ .

**Short Answer.**  $F(s) = \frac{18s}{(s+1)(s+2)^2}$

**A.13** For each function  $F$  given below, find the poles and zeros of  $F$  and their orders.

(a)  $F(s) = \frac{s-3}{s^5+7s}$ .

**Short Answer.** (a) first-order zero at  $3$ ; first-order poles at  $0$ ,  $\sqrt[4]{7}e^{j\pi/4}$ ,  $\sqrt[4]{7}e^{j3\pi/4}$ ,  $\sqrt[4]{7}e^{-j3\pi/4}$ , and  $\sqrt[4]{7}e^{-j\pi/4}$ .

### A.18 MATLAB Exercises

**A.101** Consider the rational function

$$f(z) = \frac{64z^4 - 48z^3 - 36z^2 + 27z}{64z^6 - 128z^5 - 112z^4 + 320z^3 - 84z^2 - 72z + 27}.$$

Use the Symbolic Math Toolbox in order to find the poles and zeros of  $f$  and their orders. (Hint: Some of the following functions may be useful: `sym`, `solve`, `factor`, and `pretty`.)

**A.102** Use the `roots` function to find the poles and zeros of the rational function

$$f(z) = \frac{z^4 + 6z^3 + 10z^2 + 8z}{z^9 + 21z^8 + 199z^7 + 1111z^6 + 4007z^5 + 9639z^4 + 15401z^3 + 15689z^2 + 9192z + 2340}.$$

Plot these poles and zeros using the `plot` function.

## Appendix B

# Partial Fraction Expansions

Sometimes we find it beneficial to be able to express a rational function as a sum of lower-order rational functions. This type of decomposition is known as a partial fraction expansion. Partial fraction expansions are often useful in the calculation of inverse Laplace and inverse Fourier transforms.

Suppose that we have a rational function

$$f(z) = \frac{\alpha_m z^m + \alpha_{m-1} z^{m-1} + \dots + \alpha_1 z + \alpha_0}{\beta_n z^n + \beta_{n-1} z^{n-1} + \dots + \beta_1 z + \beta_0},$$

where  $m$  and  $n$  are nonnegative integers. Such a function is said to be **strictly proper** if  $m < n$  (i.e., the order of the numerator polynomial is strictly less than the order of the denominator polynomial). We can always write a rational function as the sum of a polynomial function and a strictly proper rational function. This can be accomplished through polynomial long division. In what follows, we consider partial fraction expansions of strictly proper rational functions.

Consider a rational function  $f$  of the form

$$f(z) = \frac{a_m z^m + a_{m-1} z^{m-1} + \dots + a_1 z + a_0}{z^n + b_{n-1} z^{n-1} + \dots + b_1 z + b_0},$$

where  $a_0, a_1, \dots, a_{m-1}, b_0, b_1, \dots, b_{n-1}$  are complex constants,  $m$  and  $n$  are nonnegative integers, and  $m < n$  (i.e.,  $f$  is strictly proper). Let us denote the polynomial in the denominator of the above expression for  $f$  as  $d$ . We can always factor  $d$  as

$$d(z) = (z - p_1)(z - p_2) \cdots (z - p_n),$$

where the  $p_k$  are the roots of  $d$ .

First, let us suppose that the roots of  $d$  are distinct (i.e., the  $p_k$  are distinct). In this case,  $f$  can be expanded as

$$f(z) = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \dots + \frac{A_{n-1}}{z - p_{n-1}} + \frac{A_n}{z - p_n},$$

where

$$A_k = (z - p_k)f(z)|_{z=p_k}.$$

To see why the preceding formula for  $A_k$  is correct, we simply evaluate  $(z - p_k)f(z)|_{z=p_k}$ . We have that

$$\begin{aligned} (z - p_k)f(z)|_{z=p_k} &= \left[ \frac{A_1(z - p_k)}{z - p_1} + \frac{A_2(z - p_k)}{z - p_2} + \dots + \frac{A_{k-1}(z - p_k)}{z - p_{k-1}} + A_k + \right. \\ &\quad \left. \frac{A_{k+1}(z - p_k)}{z - p_{k+1}} + \dots + \frac{A_n(z - p_k)}{z - p_n} \right] \Big|_{z=p_k} \\ &= A_k \end{aligned}$$

Now, let us suppose that the roots of  $d$  are not distinct. In this case, we can factor  $d$  as

$$d(z) = (z - p_1)^{q_1} (z - p_2)^{q_2} \cdots (z - p_P)^{q_P}.$$

One can show that, in this case,  $f$  has a partial fraction expansion of the form

$$\begin{aligned} f(z) = & \left[ \frac{A_{1,1}}{z - p_1} + \frac{A_{1,2}}{(z - p_1)^2} + \cdots + \frac{A_{1,q_1}}{(z - p_1)^{q_1}} \right] \\ & + \left[ \frac{A_{2,1}}{z - p_2} + \cdots + \frac{A_{2,q_2}}{(z - p_2)^{q_2}} \right] \\ & + \cdots + \left[ \frac{A_{P,1}}{z - p_P} + \cdots + \frac{A_{P,q_P}}{(z - p_P)^{q_P}} \right], \end{aligned}$$

where

$$A_{k,\ell} = \frac{1}{(q_k - \ell)!} \left[ \left[ \frac{d}{dz} \right]^{q_k - \ell} [(z - p_k)^{q_k} f(z)] \right] \Big|_{z=p_k}.$$

Note that the  $q_k$ th-order pole  $p_k$  contributes  $q_k$  terms to the partial fraction expansion.

**Example B.1** (Simple pole). Find the partial fraction expansion of the function

$$f(z) = \frac{3}{z^2 + 3z + 2}.$$

*Solution.* First, we rewrite  $f$  with the denominator polynomial factored to obtain

$$f(z) = \frac{3}{(z + 1)(z + 2)}.$$

From this, we know that  $f$  has a partial fraction expansion of the form

$$f(z) = \frac{A_1}{z + 1} + \frac{A_2}{z + 2},$$

where  $A_1$  and  $A_2$  are constants to be determined. Now, we calculate  $A_1$  and  $A_2$  as follows:

$$\begin{aligned} A_1 &= (z + 1)f(z) \Big|_{z=-1} \\ &= \frac{3}{z + 2} \Big|_{z=-1} \\ &= 3 \quad \text{and} \\ A_2 &= (z + 2)f(z) \Big|_{z=-2} \\ &= \frac{3}{z + 1} \Big|_{z=-2} \\ &= -3. \end{aligned}$$

Thus, the partial fraction expansion of  $f$  is given by

$$f(z) = \frac{3}{z + 1} - \frac{3}{z + 2}. \quad \blacksquare$$

**Example B.2** (Repeated pole). Find the partial fraction expansion of the function

$$f(z) = \frac{4z + 8}{(z + 1)^2(z + 3)}.$$

*Solution.* Since  $f$  has a repeated pole, we know that  $f$  has a partial fraction expansion of the form

$$f(z) = \frac{A_{1,1}}{z+1} + \frac{A_{1,2}}{(z+1)^2} + \frac{A_{2,1}}{z+3}.$$

where  $A_{1,1}$ ,  $A_{1,2}$ , and  $A_{2,1}$  are constants to be determined. To calculate these constants, we proceed as follows:

$$\begin{aligned} A_{1,1} &= \frac{1}{(2-1)!} \left[ \left( \frac{d}{dz} \right)^{2-1} [(z+1)^2 f(z)] \right] \Big|_{z=-1} \\ &= \frac{1}{1!} \left[ \frac{d}{dz} [(z+1)^2 f(z)] \right] \Big|_{z=-1} \\ &= \left[ \frac{d}{dz} \left( \frac{4z+8}{z+3} \right) \right] \Big|_{z=-1} \\ &= [4(z+3)^{-1} + (-1)(z+3)^{-2}(4z+8)] \Big|_{z=-1} \\ &= \left[ \frac{4}{(z+3)^2} \right] \Big|_{z=-1} \\ &= \frac{4}{4} \\ &= 1, \\ A_{1,2} &= \frac{1}{(2-2)!} \left[ \left( \frac{d}{dz} \right)^{2-2} [(z+1)^2 f(z)] \right] \Big|_{z=-1} \\ &= \frac{1}{0!} [(z+1)^2 f(z)] \Big|_{z=-1} \\ &= \left[ \frac{4z+8}{z+3} \right] \Big|_{z=-1} \\ &= \frac{4}{2} \\ &= 2, \quad \text{and} \\ A_{2,1} &= (z+3)f(z) \Big|_{z=-3} \\ &= \frac{4z+8}{(z+1)^2} \Big|_{z=-3} \\ &= \frac{-4}{4} \\ &= -1. \end{aligned}$$

Thus, the partial fraction expansion of  $f$  is given by

$$f(z) = \frac{1}{z+1} + \frac{2}{(z+1)^2} - \frac{1}{z+3}. \quad \blacksquare$$

**Example B.3** (Improper rational function). Find the partial fraction expansion of the function

$$f(z) = \frac{2z^3 + 9z^2 - z + 2}{z^2 + 3z + 2}.$$

*Solution.* Since  $f$  is not strictly proper, we must rewrite  $f$  as the sum of a polynomial function and a strictly proper rational function. Using polynomial long division, we have

$$\begin{array}{r} z^2 + 3z + 2 \overline{) 2z^3 + 9z^2 - z + 2} \\ \underline{2z^3 + 6z^2 + 4z} \phantom{+ 2} \\ 3z^2 - 5z + 2 \\ \underline{3z^2 + 9z + 6} \\ -14z - 4 \end{array}$$

Thus, we have

$$f(z) = 2z + 3 + g(z),$$

where

$$g(z) = \frac{-14z - 4}{z^2 + 3z + 2} = \frac{-14z - 4}{(z + 2)(z + 1)}.$$

Now, we find a partial fraction expansion of  $g$ . Such an expansion is of the form

$$g(z) = \frac{A_1}{z + 1} + \frac{A_2}{z + 2}.$$

Solving for the expansion coefficients, we have

$$\begin{aligned} A_1 &= (z + 1)g(z)|_{z=-1} \\ &= \left. \frac{-14z - 4}{z + 2} \right|_{z=-1} \\ &= 10 \quad \text{and} \\ A_2 &= (z + 2)g(z)|_{z=-2} \\ &= \left. \frac{-14z - 4}{z + 1} \right|_{z=-2} \\ &= -24. \end{aligned}$$

Thus,  $g$  has the expansion

$$g(z) = \frac{10}{z + 1} - \frac{24}{z + 2}.$$

Thus, we can decompose  $f$  using a partial fraction expansion as

$$f(z) = 2z + 3 + \frac{10}{z + 1} - \frac{24}{z + 2}. \quad \blacksquare$$

## B.1 Exercises

### B.1.1 Exercises Without Answer Key

**B.1** Find the partial fraction expansion of each function  $f$  given below.

$$(a) f(z) = \frac{-z^2 + 2z + 7}{4z^3 + 24z^2 + 44z + 24};$$

$$(b) f(z) = \frac{-16z - 10}{8z^2 + 6z + 1}; \text{ and}$$

$$(c) f(z) = \frac{7z + 26}{z^2 + 7z + 10}.$$

**B.2** Find the partial fraction expansion of each function  $f$  given below.

$$(a) f(z) = \frac{-2z^2 + 5}{z^3 + 4z^2 + 5z + 2};$$

$$(b) f(z) = \frac{2z^2 + 15z + 21}{z^2 + 4z + 3}; \text{ and}$$

$$(c) f(z) = \frac{4z^3 + 36z^2 + 103z + 95}{(z+1)(z+3)^3}.$$

### B.1.2 Exercises With Answer Key

**B.3** Find the partial fraction expansion of each function  $f$  given below.

$$(a) f(z) = \frac{2}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}; \text{ and}$$

$$(b) f(z) = \frac{\frac{1}{10}}{(1 - \frac{9}{10}z^{-1})(1 - z^{-1})}.$$

**Short Answer.** (a)  $f(z) = \frac{-2}{1 - \frac{1}{4}z^{-1}} + \frac{4}{1 - \frac{1}{2}z^{-1}}$ ; (b)  $f(z) = \frac{-9/10}{1 - \frac{9}{10}z^{-1}} + \frac{1}{1 - z^{-1}}$

## B.2 MATLAB Exercises

**B.101** Use MATLAB to find a partial fraction expansion for each function  $f$  given below. [Hint: The `partfrac` function may be helpful.]

$$(a) f(z) = \frac{1}{z^2 + 3z + 1}.$$



## Appendix C

# Solution of Constant-Coefficient Linear Differential Equations

### C.1 Overview

Many systems of practical interest can be represented using linear differential equations with constant coefficients. For this reason, we are interested in solution techniques for such equations. This appendix briefly introduces time-domain methods for solving constant-coefficient linear differential equations.

### C.2 Constant-Coefficient Linear Differential Equations

An  $N$ th-order linear differential equation with constant coefficients has the general form

$$\sum_{k=0}^N b_k \left( \frac{d}{dt} \right)^k y(t) = \sum_{k=0}^M a_k \left( \frac{d}{dt} \right)^k x(t),$$

where the  $a_k$  and  $b_k$  are constants. If the right-hand side of the above equation is identically equal to zero, the equation is said to be **homogeneous**. Otherwise, the equation is said to be **nonhomogeneous**. Depending on whether the above equation is homogeneous or nonhomogeneous, the solution method differs slightly.

### C.3 Solution of Homogeneous Equations

First, we consider the solution of homogeneous equations. In this case, we have an equation of the form

$$\sum_{k=0}^N b_k \left( \frac{d}{dt} \right)^k y(t) = 0. \quad (\text{C.1})$$

Let us define the quantity

$$\phi(s) \triangleq \sum_{k=0}^N b_k s^k.$$

Then, we refer to

$$\phi(s) = 0$$

as the characteristic (or auxiliary) equation of (C.1). The solution of (C.1) depends on the roots of the characteristic equation, as specified by the theorem below.

**Theorem C.1.** Suppose that  $\phi(s) = 0$  is the characteristic equation associated with the homogeneous linear differential equation

$$\sum_{k=0}^N b_k \left(\frac{d}{dt}\right)^k y(t) = 0.$$

If  $\phi(s) = 0$  has a real root  $p$  of multiplicity  $k$ , then a solution of the differential equation is

$$(a_0 + a_1 t + \dots + a_{k-1} t^{k-1}) e^{pt}.$$

If  $\phi(s) = 0$  has a pair of complex conjugate roots  $\sigma \pm j\omega$ , each root being of multiplicity  $k$ , then a solution of the differential equation is

$$e^{\sigma t} \left[ (a_0 + a_1 t + \dots + a_{k-1} t^{k-1}) \cos \omega t + (b_0 + b_1 t + \dots + b_{k-1} t^{k-1}) \sin \omega t \right].$$

A general solution of the differential equation is obtained by taking a linear combination of the solutions obtained by considering all roots of  $\phi(s) = 0$ .

From the above theorem, we can see that, in order to solve an equation of the form (C.1), we begin by finding the roots of the corresponding characteristic equation. Then, we find a solution associated with each distinct root (or pair of complex conjugate roots). Finally, the general solution is formed by taking a linear combination of all of these individual solutions.

**Example C.1.** Find the general solution to the differential equation

$$y''(t) + 4y'(t) + 5y(t) = 0.$$

*Solution.* The given differential equation has the characteristic equation

$$s^2 + 4s + 5 = 0.$$

Solving for the roots of the characteristic equation yields

$$\begin{aligned} s &= \frac{-4 \pm \sqrt{4^2 - 4(5)}}{2} \\ &= -2 \pm \frac{1}{2} \sqrt{-4} \\ &= -2 \pm j. \end{aligned}$$

Thus, we have one pair of complex conjugate roots (i.e.,  $-2 \pm j$ ), each root being of multiplicity 1. Therefore, the general solution to the given equation is of the form

$$y(t) = e^{-2t} (a_1 \cos t + b_1 \sin t). \quad \blacksquare$$

**Example C.2.** Find the general solution to the differential equation

$$y''(t) + 5y'(t) + 6y(t) = 0.$$

*Solution.* The given differential equation has the characteristic equation

$$s^2 + 5s + 6 = 0,$$

which can be factored as

$$(s + 2)(s + 3) = 0.$$

Clearly, the characteristic equation has the roots  $-2$  and  $-3$ , each of multiplicity 1. Therefore, the general solution of the given equation is of the form

$$y(t) = a_1 e^{-3t} + a_2 e^{-2t}. \quad \blacksquare$$

**Example C.3.** Find the general solution to the differential equation

$$y''(t) + 2y'(t) + y(t) = 0.$$

*Solution.* The given differential equation has the characteristic equation

$$s^2 + 2s + 1 = 0,$$

which can be factored as

$$(s + 1)^2 = 0.$$

Clearly, the characteristic equation has the root  $-1$  of multiplicity 2. Therefore, the general solution to the given equation is of the form

$$y(t) = (a_0 + a_1t)e^{-t}. \quad \blacksquare$$

## C.4 Particular Solution of Nonhomogeneous Equations

So far, we have only considered the solution of homogeneous equations. Now, we consider the nonhomogeneous case. In the nonhomogeneous case, we have an equation of the form

$$\sum_{k=0}^N b_k \left( \frac{d}{dt} \right)^k y(t) = f(t). \quad (\text{C.2})$$

As it turns out, in order to find a general solution to the above equation, we must first find a particular solution.

To find a particular solution, we must consider the form of the function  $f$ . Suppose that a particular solution to (C.2) is given by the function  $y_p$ . Since  $y_p$  is a solution to (C.2), when we substitute  $y_p$  into (C.2), the left-hand side of (C.2) must equal  $f$ . Therefore,  $y_p$  and its derivatives must be comprised of terms that resemble the terms of  $f$ . Thus, by examining  $f$ , we can deduce a general expression for  $y_p$  containing one or more unknown coefficients. We, then, solve for these unknown coefficients. This solution technique is sometimes referred to as the **method of undetermined coefficients**.

Table C.1 shows the general form of  $y_p$  that should be nominally used in the case of several frequently encountered forms for  $f$ . There is, however, one caveat. The general expression chosen for  $y_p$  must not include any terms that are linearly dependent on terms in the solution to the corresponding complementary equation. If linearly dependent terms exist in our preliminary choice of  $y_p$ , we must replace each term  $v(t)$  with  $t^m v(t)$  where  $m$  is the smallest positive integer eliminating this linear dependence. To illustrate how this solution process works in more detail, we will now consider a few example problems.

**Example C.4.** Find a particular solution of the differential equation

$$y''(t) + 4y'(t) + 5y(t) = 5t^2 + 3t + 8.$$

*Solution.* We begin by considering the form of the function on the right-hand side of the given differential equation. Since terms in  $t^2$ ,  $t^1$ , and  $t^0$  yield terms in  $t^2$ ,  $t^1$ , and  $t^0$  when substituted into the left-hand side of the given equation, we deduce that a particular solution is of the form

$$y_p(t) = a_2t^2 + a_1t + a_0. \quad (\text{C.3})$$

Table C.1: Forms for the particular solution

$f(t)$	$y_p(t)$
$c_0 + c_1t + \dots + c_nt^n$	$p_0 + p_1t + \dots + p_nt^n$
$ce^{at}$	$pe^{at}$
$c \sin bt$ or $c \cos bt$	$p \sin bt + q \cos bt$
$(c_0 + c_1t + \dots + c_nt^n)e^{at}$	$(p_0 + p_1t + \dots + p_nt^n)e^{at}$
$(c_0 + c_1t + \dots + c_nt^n) \cos bt$ or $(c_0 + c_1t + \dots + c_nt^n) \sin bt$	$(p_0 + p_1t + \dots + p_nt^n) \cos bt + (q_0 + q_1t + \dots + q_nt^n) \sin bt$
$(c_0 + c_1t + \dots + c_nt^n)e^{at} \cos bt$ or $(c_0 + c_1t + \dots + c_nt^n)e^{at} \sin bt$	$(p_0 + p_1t + \dots + p_nt^n)e^{at} \cos bt + (q_0 + q_1t + \dots + q_nt^n)e^{at} \sin bt$

Differentiating  $y_p$ , we obtain

$$y_p'(t) = 2a_2t + a_1 \quad \text{and}$$

$$y_p''(t) = 2a_2.$$

Substituting  $y_p$  and its derivatives into the given differential equation yields

$$2a_2 + 4[2a_2t + a_1] + 5[a_2t^2 + a_1t + a_0] = 5t^2 + 3t + 8$$

$$\Rightarrow 2a_2 + 8a_2t + 4a_1 + 5a_2t^2 + 5a_1t + 5a_0 = 5t^2 + 3t + 8$$

$$\Rightarrow [5a_2]t^2 + [8a_2 + 5a_1]t + [2a_2 + 4a_1 + 5a_0] = 5t^2 + 3t + 8.$$

Comparing the left- and right-hand sides of the above equation, we see that

$$5a_2 = 5,$$

$$8a_2 + 5a_1 = 3, \quad \text{and}$$

$$2a_2 + 4a_1 + 5a_0 = 8.$$

Solving this system of equations yields  $a_0 = 2$ ,  $a_1 = -1$ , and  $a_2 = 1$ . Therefore, from (C.3), the given differential equation has the particular solution

$$y_p(t) = t^2 - t + 2. \quad \blacksquare$$

**Example C.5.** Find a particular solution of the differential equation

$$y''(t) + 4y'(t) + 5y(t) = e^{-2t}.$$

*Solution.* We begin by considering the form of the function on the right-hand side of the given differential equation. Since terms in  $e^{-2t}$  yield terms in  $e^{-2t}$  when substituted into the left-hand side of the equation, we deduce that the particular solution  $y_p$  is of the form

$$y_p(t) = ae^{-2t}.$$

Differentiating  $y_p$ , we obtain

$$y_p'(t) = -2ae^{-2t} \quad \text{and}$$

$$y_p''(t) = 4ae^{-2t}.$$

Substituting  $y_p$  and its derivatives into the given differential equation yields

$$\begin{aligned} & 4ae^{-2t} + 4[-2ae^{-2t}] + 5[ae^{-2t}] = e^{-2t} \\ \Rightarrow & 4ae^{-2t} - 8ae^{-2t} + 5ae^{-2t} = e^{-2t} \\ \Rightarrow & ae^{-2t} = e^{-2t}. \end{aligned}$$

Comparing the left- and right-hand sides of the above equation, we have that  $a = 1$ . Therefore, the given differential equation has the particular solution

$$y_p(t) = e^{-2t}. \quad \blacksquare$$

**Example C.6.** Find a particular solution of the differential equation

$$y''(t) + 4y'(t) + 5y(t) = \sin t.$$

*Solution.* To begin, we examine the form of the function of the right-hand side of the given differential equation. Since terms in  $\sin t$  yield terms in  $\sin t$  and  $\cos t$  when substituted into the left-hand side of the given equation, we deduce that the particular solution  $y_p$  is of the form

$$y_p(t) = a_1 \cos t + a_2 \sin t.$$

Differentiating  $y_p$ , we obtain

$$\begin{aligned} y_p'(t) &= -a_1 \sin t + a_2 \cos t \quad \text{and} \\ y_p''(t) &= -a_1 \cos t - a_2 \sin t. \end{aligned}$$

Substituting  $y_p$  and its derivatives into the given differential equation yields

$$\begin{aligned} & [-a_1 \cos t - a_2 \sin t] + 4[-a_1 \sin t + a_2 \cos t] + 5[a_1 \cos t + a_2 \sin t] = \sin t \\ \Rightarrow & [-a_1 + 4a_2 + 5a_1] \cos t + [-a_2 - 4a_1 + 5a_2] \sin t = \sin t \\ \Rightarrow & [4a_1 + 4a_2] \cos t + [4a_2 - 4a_1] \sin t = \sin t. \end{aligned}$$

Comparing the left- and right-hand sides of the above equation, we have that

$$\begin{aligned} 4a_1 + 4a_2 &= 0 \quad \text{and} \\ 4a_2 - 4a_1 &= 1. \end{aligned}$$

Solving this system of equations yields  $a_1 = -\frac{1}{8}$  and  $a_2 = \frac{1}{8}$ . Therefore, the given differential equation has the particular solution

$$y_p(t) = -\frac{1}{8} \cos t + \frac{1}{8} \sin t. \quad \blacksquare$$

## C.5 General Solution of Nonhomogeneous Equations

With every nonhomogeneous constant-coefficient linear differential equation

$$\sum_{k=0}^N b_k \left(\frac{d}{dt}\right)^k y(t) = \sum_{k=0}^M a_k \left(\frac{d}{dt}\right)^k x(t),$$

we can associate a homogeneous equation

$$\sum_{k=0}^N b_k \left(\frac{d}{dt}\right)^k y(t) = 0$$

called the **complementary equation**. The complementary equation is formed by simply setting the function  $x$  (and its derivatives) to zero in the original equation.

As it turns out, in order to find the solution of a nonhomogeneous equation, we must find a particular solution to the given equation and also a general solution to its complementary equation. This process is more precisely specified by the theorem below.

**Theorem C.2.** *A general solution of the linear differential equation*

$$\sum_{k=0}^N b_k \left(\frac{d}{dt}\right)^k y(t) = \sum_{k=0}^M a_k \left(\frac{d}{dt}\right)^k x(t)$$

has the form

$$y(t) = y_c(t) + y_p(t),$$

where  $y_c$  is a general solution of the associated complementary equation and  $y_p$  is any particular solution of the given equation.

**Example C.7.** Consider the differential equation

$$y''(t) + 2y'(t) + 2y(t) = -2t + 4.$$

- (a) Find the general solution of this equation.  
 (b) Find the solution if  $y(0) = 1$  and  $y'(0) = 0$ .

*Solution.* (a) First, we need to find the general solution  $y_c$  of the corresponding complementary equation

$$y''(t) + 2y'(t) + 2y(t) = 0.$$

This equation has the characteristic equation

$$s^2 + 2s + 2 = 0.$$

Solving for the roots of this equation, we have

$$\begin{aligned} s &= \frac{-2 \pm \sqrt{2^2 - 4(2)}}{2} \\ &= -1 \pm j. \end{aligned}$$

Therefore, the characteristic equation has a pair of complex conjugate roots  $-1 \pm j$ , each root being of multiplicity 1. From this, we know that the complementary equation has a general solution of the form

$$y_c(t) = e^{-t}(a_1 \cos t + b_1 \sin t). \quad (\text{C.4})$$

Now, we must find a particular solution  $y_p$  of the given differential equation. We consider the form of the function on the right-hand side of the given equation. Since terms in  $t^1$  and  $t^0$  yield terms in  $t^1$  and  $t^0$  when substituted into the left-hand side of the equation, we deduce that  $y_p$  is of the form

$$y_p(t) = c_1 t + c_0.$$

Differentiating  $y_p$ , we obtain

$$\begin{aligned}y_p'(t) &= c_1 \quad \text{and} \\y_p''(t) &= 0.\end{aligned}$$

Substituting  $y_p$  and its derivatives into the given differential equation yields

$$\begin{aligned}2c_1 + 2[c_1t + c_0] &= -2t + 4 \\ \Rightarrow [2c_1]t + [2c_1 + 2c_0] &= -2t + 4.\end{aligned}$$

Comparing the left- and right-hand sides of the above equation, we have

$$\begin{aligned}2c_1 &= -2 \quad \text{and} \\ 2c_1 + 2c_0 &= 4.\end{aligned}$$

Solving this system of equations yields  $c_0 = 3$  and  $c_1 = -1$ . Therefore, the particular solution is given by

$$y_p(t) = -t + 3. \tag{C.5}$$

Combining the results of (C.4) and (C.5), we conclude that the given equation has the general solution

$$\begin{aligned}y(t) &= y_c(t) + y_p(t) \\ &= e^{-t}(a_1 \cos t + b_1 \sin t) - t + 3.\end{aligned}$$

(b) We compute the derivative of  $y$  as

$$y'(t) = -e^{-t}(a_1 \cos t + b_1 \sin t) + e^{-t}(-a_1 \sin t + b_1 \cos t) - 1.$$

From the given initial conditions, we have

$$\begin{aligned}0 &= -a_1 + b_1 - 1 \quad \text{and} \\ 1 &= a_1 + 3.\end{aligned}$$

Solving for  $a_1$  and  $b_1$  yields  $a_1 = -2$  and  $b_1 = -1$ . Therefore, we have

$$\begin{aligned}y(t) &= e^{-t}(-2 \cos t - \sin t) - t + 3 \\ &= -e^{-t}(2 \cos t + \sin t) - t + 3.\end{aligned}$$

■

**Example C.8.** Consider the differential equation

$$y''(t) + 3y'(t) + 2y(t) = e^{-t}.$$

(a) Find the general solution of this equation.

(b) Determine the solution if  $y(0) = -1$  and  $y'(0) = 1$ .

*Solution.* First, we need to find the general solution  $y_c$  of the corresponding complementary equation

$$y''(t) + 3y'(t) + 2y(t) = 0.$$

This equation has the characteristic equation

$$s^2 + 3s + 2 = 0,$$

which can be factored as

$$(s + 2)(s + 1) = 0.$$

Thus, the characteristic equation has roots at  $-2$  and  $-1$ , each of multiplicity 1. From this, we can deduce that

$$y_c(t) = a_1 e^{-t} + a_2 e^{-2t}. \quad (\text{C.6})$$

Now, we need to find a particular solution  $y_p$  of the given differential equation. Since  $y_c$  contains a term with  $e^{-t}$ , we deduce that  $y_p$  is of the form

$$y_p(t) = cte^{-t}.$$

Differentiating  $y_p$ , we obtain

$$\begin{aligned} y_p'(t) &= ce^{-t} - cte^{-t} \quad \text{and} \\ y_p''(t) &= -ce^{-t} - c(e^{-t} - te^{-t}) = -2ce^{-t} + cte^{-t}. \end{aligned}$$

Substituting  $y_p$  and its derivatives into the given differential equation yields

$$\begin{aligned} &(-2ce^{-t} + cte^{-t}) + 3(ce^{-t} - cte^{-t}) + 2cte^{-t} = e^{-t} \\ \Rightarrow &(-2c + 3c)e^{-t} + (c - 3c + 2c)te^{-t} = e^{-t} \\ \Rightarrow &ce^{-t} = e^{-t}. \end{aligned}$$

Comparing the left- and right-hand sides of this equation, we conclude  $c = 1$ . Therefore, we have

$$y_p(t) = te^{-t}. \quad (\text{C.7})$$

Combining the results from (C.6) and (C.7), we have

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= a_1 e^{-t} + a_2 e^{-2t} + te^{-t}. \end{aligned}$$

(b) We compute the derivative of  $y$  as

$$y'(t) = -a_1 e^{-t} - 2a_2 e^{-2t} + e^{-t} - te^{-t}.$$

From the given initial conditions, we have

$$\begin{aligned} 1 &= -a_1 - 2a_2 + 1 \quad \text{and} \\ -1 &= a_1 + a_2. \end{aligned}$$

Solving for  $a_1$  and  $a_2$  yields

$$\begin{aligned} a_1 &= -2 \\ a_2 &= 1. \end{aligned}$$

Therefore, we have that

$$y(t) = -2e^{-t} + e^{-2t} + te^{-t}. \quad \blacksquare$$

## C.6 Exercises

### C.6.1 Exercises Without Answer Key

**C.1** Find the general solution to each of the differential equations below.

- (a)  $8y''(t) + 6y'(t) + y(t) = 0$ ;
- (b)  $y'''(t) + 5y''(t) + 17y'(t) + 13y(t) = 0$ ; [Hint: One root of the characteristic equation is  $-1$ .]; and
- (c)  $y''(t) + 9y'(t) + 20y(t) = 0$ ;
- (d)  $y''(t) + 2y'(t) + y(t) = 0$ .

**C.2** Find a particular solution to each of the differential equations below. In each case, the solution  $y_c$  of the corresponding complementary equation is given.

- (a)  $y''(t) + 3y'(t) + 2y(t) = t^2$  with  $y_c(t) = c_1e^{-t} + c_2e^{-2t}$ ;
- (b)  $y''(t) + 3y'(t) + 2y(t) = e^{-3t} + t$  with  $y_c(t) = c_1e^{-t} + c_2e^{-2t}$ ;
- (c)  $y''(t) + 4y'(t) + 3y(t) = e^{-t}$  with  $y_c(t) = c_1e^{-t} + c_2e^{-3t}$ ; and
- (d)  $y''(t) + 2y'(t) + y(t) = \sin t$  with  $y_c(t) = c_0e^{-t} + c_1te^{-t}$ .

**C.3** Consider the differential equation

$$y''(t) + 3y'(t) + 2y(t) = t + 1.$$

- (a) Find the general solution to this equation.
- (b) Determine the solution in the case that  $y(0) = -\frac{1}{4}$  and  $y'(0) = -\frac{1}{2}$ .

**C.4** Consider the differential equation

$$y''(t) + 5y'(t) + 6y(t) = 2e^{-3t}.$$

- (a) Find the general solution of this equation.
- (b) Determine the solution in the case that  $y(0) = 0$  and  $y'(0) = 1$ .

**C.5** Find the general solution to each of the differential equations below.

- (a)  $y''(t) + 7y'(t) + 12y(t) = 6t^2 - 5t + 18$ ;
- (b)  $y''(t) + 7y'(t) + 12y(t) = e^{-3t}$ ;
- (c)  $y''(t) + 4y'(t) + 8y(t) = e^{-t}$ ; and
- (d)  $y''(t) + 2y'(y) + 5y(t) = 1 + e^{-t}$ .

### C.6.2 Exercises With Answer Key

Currently, there are no exercises available with an answer key.

## C.7 MATLAB Exercises

**C.101** Use the `dsolve` function in MATLAB to solve each of the differential equations in Exercise [C.1](#).

**C.102** Use the `dsolve` function in MATLAB to solve Exercise [C.4](#).



## Appendix D

# Miscellaneous Information

### D.1 Integrals

$$\int x^n dx = \begin{cases} \frac{1}{n+1}x^{n+1} + C & n \neq -1 \\ \ln|x| + C & n = -1 \end{cases}$$

$$\int \cos x dx = \sin x + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int xe^x dx = e^x(x-1) + C$$

$$\int xe^{ax} dx = \frac{1}{a^2}e^{ax}(ax-1) + C$$

$$\int x^2 e^{ax} dx = \frac{1}{a^3}e^{ax}(a^2x^2 - 2ax + 2) + C$$

$$\int e^{ax} \cos bxdx = \frac{e^{ax}[a \cos bx + b \sin bx]}{a^2 + b^2} + C$$

$$\int e^{ax} \sin bxdx = \frac{e^{ax}[a \sin bx - b \cos bx]}{a^2 + b^2} + C$$

$$\int x \cos x dx = \cos x + x \sin x + C$$

$$\int x \sin x dx = \sin x - x \cos x + C$$

$$\int u dv = uv - \int v du$$

### D.2 Derivatives

$$\frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\begin{aligned}\frac{d}{dx} \cos x &= -\sin x \\ \frac{d}{dx} \sin x &= \cos x\end{aligned}$$

### D.3 Arithmetic and Geometric Series

The sum of the arithmetic sequence  $a, a + d, a + 2d, \dots, a + (n - 1)d$  is given by

$$\sum_{k=0}^{n-1} (a + kd) = \frac{n[2a + d(n - 1)]}{2}. \quad (\text{D.1})$$

The sum of the geometric sequence  $a, ra, r^2a, \dots, r^{n-1}a$  is given by

$$\sum_{k=0}^{n-1} r^k a = a \frac{r^n - 1}{r - 1} \quad \text{for } r \neq 1. \quad (\text{D.2})$$

The sum of the infinite geometric sequence  $a, ra, r^2a, \dots$  is given by

$$\sum_{k=0}^{\infty} r^k a = \frac{a}{1 - r} \quad \text{for } |r| < 1. \quad (\text{D.3})$$

---

# Appendix E

## MATLAB

### E.1 Introduction

MATLAB is a software tool that is useful for solving a wide variety of problems arising in engineering applications. The MATLAB software is a product of a company called The MathWorks. Extensive information on this software (including detailed guides and manuals) is available, free of charge, from the company's web site (<http://www.mathworks.com>). A number of helpful books on MATLAB are also available [1, 2]. In this appendix, a reasonably detailed introduction to MATLAB is also provided.

### E.2 Octave

Although MATLAB is very powerful, it is a commercial software product. Therefore, MATLAB is not free. Fortunately, an open-source MATLAB-like software package is available called Octave. Octave is available for download from its official web site <http://www.octave.org>. This software is included in several major Linux distributions (e.g., Fedora and Ubuntu). As well, Octave is also available for the Cygwin environment under Microsoft Windows. (For more details about Cygwin, see <http://www.cygwin.org>.)

### E.3 Invoking MATLAB

On most UNIX systems, the MATLAB software is started by invoking the `matlab` command.

#### E.3.1 UNIX

The MATLAB software is invoked using a command line of the form:

```
matlab [ options ]
```

The `matlab` command supports a number of options including the following:

`-help` or `-h`

Display information on MATLAB options.

`-nodisplay`

Disable all graphical output. The MATLAB desktop will not be started.

`-nojvm`

Disable all Java support by not starting the Java virtual machine. In particular, the MATLAB desktop will not be started.

Table E.1: Keys for command-line editing

Key	Action
Up arrow	Recall previous line
Down arrow	Recall next line
Left arrow	Move left one character
Right arrow	Move right one character
Home	Move to beginning of line
End	Move to end of line
Ctrl-C	Cancel current line

`-nodesktop`

Disable the MATLAB desktop. Use the current terminal for commands.

`-display displayname`

Specify the X display to use for graphics output.

Like most UNIX programs, MATLAB uses the X Windows System for rendering graphics output. The `DISPLAY` environment variable provides the default display setting for graphics output. If necessary, one can override the default display setting by explicitly specifying the display to use via the `-display` option.

When running MATLAB remotely, it may be necessary to disable the desktop (with the `-nodesktop` option). This is due to the fact that, when the desktop is enabled, MATLAB tends to require a relatively large amount of network bandwidth, which can be problematic over lower-speed network connections.

### E.3.2 Microsoft Windows

Unfortunately, the author has not used MATLAB under Microsoft Windows. So, he cannot comment on the specifics of running MATLAB under this operating system.

## E.4 Command Line Editor

In MATLAB, several keys are quite useful for editing purposes, as listed in Table E.1. For example, the arrow keys can be used to perform editing in the usual way.

## E.5 MATLAB Basics

Arguably, one of the most helpful commands in MATLAB is the `help` command. This command can be used to obtain information on many of the operators, functions, and commands available in MATLAB. For example, to find information on the `help` command, one can type:

```
help help
```

In a similar vein, the `doc` command can be used to obtain detailed documentation on many of the functions and commands in MATLAB. For example, to display documentation on the `doc` command, one can type:

```
doc doc
```

### E.5.1 Identifiers

Identifiers (i.e., variable/function names) are case sensitive and may consist of uppercase and lowercase letters, underscores, and digits, but the first character cannot be a digit or an underscore. Although an identifier can be arbitrarily long, only the first  $n$  characters are significant, where  $n$  depends on the particular version of MATLAB being used. Any characters after the first  $n$  are simply ignored. (The `namelengthmax` function can be used to query the precise

Table E.2: Predefined variables

Variable	Description
pi	$\pi$
i	$\sqrt{-1}$
j	$\sqrt{-1}$
nan	not-a-number (NaN)
inf	infinity
ans	last expression evaluated that was not assigned to variable
date	date
clock	wall clock
realmin	smallest usable positive real number
realmax	largest usable positive real number
bitmax	largest usable positive integer

Table E.3: Operators

Symbol	Description
+	addition
-	subtraction
*	multiplication
/	right division
\	left division
^	exponentiation
'	conjugate transpose
.*	element-wise multiplication
./	element-wise division
.^	element-wise exponentiation
.'	transpose

value of  $n$ .) Several variables are automatically predefined by MATLAB as listed in Table E.2. You can assign a new value to a predefined variable, but its original value will be lost.

## E.5.2 Basic Functionality

In MATLAB, comments begin with a percent sign character and continue to the end of line. Some of the operators supported by MATLAB are listed in Table E.3. Some math functions provided by MATLAB are listed in Tables E.4, E.5, E.6, E.7, E.8, and E.9. Note that the `sinc` function in MATLAB does not compute the sinc function as defined (by (3.24)) herein. Instead, this MATLAB function computes the normalized sinc function (as defined by (3.25)).

**Example E.1.** Some examples of very basic calculations performed using MATLAB are as follows:

```
a = [1 2 3; 4 5 6; 7 8 9] % 3 x 3 array
b = [1 2 3
     4 5 6
     7 8 9] % 3 x 3 array
a - b
x = [1; 3; -1] % 3-dimensional column vector
y = x .* x + 3 * x + 2
y = a * x

t = 5;
s = t ^ 2 + 3 * t - 7;
```

Table E.4: Elementary math functions

Name	Description
abs	magnitude of complex number
angle	principal argument of complex number
imag	imaginary part of complex number
real	real part of complex number
conj	conjugate of complex number
round	round to nearest integer
fix	round towards zero
floor	round towards $-\infty$
ceil	round towards $\infty$
sign	signum function
rem	remainder (with same sign as dividend)
mod	remainder (with same sign as divisor)

Table E.5: Other math-related functions

Name	Description
min	minimum value
max	maximum value
mean	mean value
std	standard deviation
median	median value
sum	sum of elements
prod	product of elements
cumsum	cumulative sum of elements
cumprod	cumulative product of elements
polyval	evaluate polynomial
cart2pol	Cartesian-to-polar coordinate conversion
pol2cart	polar-to-Cartesian coordinate conversion

Table E.6: Exponential and logarithmic functions

Name	Description
exp	exponential function
log	natural logarithmic function
log10	base-10 logarithmic function
sqrt	square root function

Table E.7: Trigonometric functions

Name	Description
sin	sine function
cos	cosine function
tan	tangent function
asin	arcsine function
acos	arccosine function
atan	arctangent function
atan2	two-argument form of arctangent function

Table E.8: Other math functions

Name	Description
sinc	normalized sinc function (as defined in (3.25))

Table E.9: Radix conversion functions

Name	Description
dec2bin	convert decimal to binary
bin2dec	convert binary to decimal
dec2hex	convert decimal to hexadecimal
hex2dec	convert hexadecimal to decimal
dec2base	convert decimal to arbitrary radix
base2dec	convert arbitrary radix to decimal

Table E.10: Array size functions

Name	Description
size	query array dimensions
length	query vector/array dimension

```
z = 3 + 4 * j; % complex number in Cartesian form
z = 20 * exp(j * 10); % complex number in polar form
```

The `disp` function prints a single string. For example, the following code fragment prints “Hello, world” (followed by a newline character):

```
disp('Hello, world');
```

The `sprintf` function provides very sophisticated string formatting capabilities, and is often useful in conjunction with the `disp` function. The use of the `sprintf` function is illustrated by the following code fragment:

```
name = 'Jane Doe';
id = '06020997';
mark = 91.345678912;
disp(sprintf('The student %s (ID %s) received a grade of %4.2f%%.', ...
    name, id, mark));
```

The `sprintf` function is very similar in spirit to the function of the same name used in the C programming language.

## E.6 Arrays

Frequently, it is necessary to determine the dimensions of an array (i.e., matrix or vector). For this purpose, MATLAB provides two very useful functions as listed in Table E.10. The function `size` can be used to determine the number of rows and/or columns in an array:

- `size(a)` returns a row vector with the number of rows and columns in `a` as elements (in that order);
- `size(a, 1)` returns the number of rows in `a`; and
- `size(a, 2)` returns the number of columns in `a`.

The function `length` is used to find the maximum of the two array dimensions. That is, `length(a)` is equivalent to `max(size(a))`. Usually, the `length` function is used in conjunction with arrays that are known to be row/column vectors.

**Example E.2.** Suppose that `a = [1 2 3 4; 5 6 7 8]`. Then, `size(a)` returns `[2 4]`, `size(a, 1)` returns 2, `size(a, 2)` returns 4, and `length(a)` returns 4. ■

Table E.11: Examples of abbreviated forms of vectors

Abbreviated Form	Long Form
1 : 4	[1 2 3 4]
0 : 0.2 : 1	[0 0.2 0.4 0.6 0.8 1]
1 : -1 : -2	[1 0 -1 -2]
0 : 10 : 25	[0 10 20]
-1.5 : -1 : -4	[-1.5 -2.5 -3.5]

Table E.12: Array subscripting examples

Expression	Meaning
a(1, :)	first row of a
a(:, 1)	first column of a
a(1 : 50)	first 50 elements of a arranged in a row vector
a(1 : 10)	first 10 elements of a arranged in a row vector (i.e., the first column of a)
a(1 : 2 : 10, :)	odd-indexed rows of a
a(:, 2 : 2 : 10)	even-indexed columns of a
a(1 : 5, :)	rows 1 to 5 of a
a(:, 6 : 10)	columns 6 to 10 of a
a(1 : 2, 9 : 10)	submatrix consisting of elements that are in rows 1,2 and also in columns 9,10
x(1 : 3)	first three elements of x (arranged as a row or column vector to match x)

### E.6.1 Arrays with Equally-Spaced Elements

Often, it is necessary to specify a vector with equally-spaced elements. As a result, MATLAB provides a compact means for specifying such a vector. In particular, an expression of the following form is employed:

$$start : step : end$$

The above expression is equivalent to a row vector with its first element equal to *start* and each of the subsequent elements increasing in value by *step* until the value would exceed *end*. Note that *step* may be negative.

**Example E.3.** In Table E.11, some examples of abbreviated forms of vectors are given. ■

### E.6.2 Array Subscripting

Suppose that we have an array *a*. We can access elements of the array by specifying the rows and columns in which the elements are contained. In particular, *a(rowspec, colspec)* is the array consisting of the elements of *a* that are in the rows specified by *rowspec* and columns specified by *colspec*. Here, *rowspec* is either a vector containing row indices or the special value “:” which means “all rows”. Similarly, *colspec* is either a vector containing column indices or the special value “:” which means “all columns”. We can also access elements of the array *a* by specifying a 1-dimensional element index, where elements in the array are numbered in column-major order. That is, *a(indspec)* is the vector of elements of *a* that have the indices specified by *indspec*. Here, *indspec* is either a vector containing element indices or the special value “:” which means “all elements”.

**Example E.4.** Suppose that *a* is a  $10 \times 10$  matrix and *x* is  $10 \times 1$  vector. Some examples of array subscripting involving *a* and *x* are given in Table E.12. ■

### E.6.3 Other Array Functions

Certain types of matrices tend to be used frequently in code. For this reason, MATLAB provides functions for generating some common forms of matrices. These functions are listed in Table E.13.

Table E.13: Special matrix/vector functions

Name	Description
eye	identity matrix
ones	matrix of ones
zeros	matrix of zeros
diag	diagonal matrix
rand	random matrix
linspace	vector with linearly spaced elements
logspace	vector with logarithmically spaced elements

Table E.14: Basic array manipulation functions

Name	Description
rot90	rotate array by 90 degrees
fliplr	flip array horizontally
flipud	flip array vertically
reshape	change array dimensions

MATLAB provides functions for performing some common operations to matrices. These are listed in Table E.14.

## E.7 Scripts

Instead of interactively entering MATLAB code for immediate execution, code can be placed in a file and then executed. Normally, MATLAB code is placed in what are called M-files. The term “M file” originates from the fact that these files use a name ending in the suffix “.m”. To create an M-file script, one simply creates a file with a name ending in the suffix “.m”. Then, the code in the M-file can be invoked by using a command with the same name as the M-file but without the “.m” extension. For reasons that will become apparent shortly, the base name of the M-file (i.e., the name without the “.m” extension) must be a valid MATLAB identifier. For example, `2foobar.m` is not a valid M-file name since “2foobar” is not a valid MATLAB identifier. (Recall that MATLAB identifiers cannot start with a digit such as “2”.) Also, in order for MATLAB to find an M-file, the file must be in one of the directories listed in the MATLAB search path. We will explain how to query and change the MATLAB search path later in this section. Before doing so, however, we provide a few examples of M-file scripts below.

**Example E.5.** Suppose that we have an M-file script called `hello.m` with the following contents:

```
% Print a greeting.
disp('Hello, World.');
```

Then, the code in this file can be executed by simply typing the following in MATLAB:

```
hello
```

That is, we invoke the code in the M-file by using the base name of the file. (It is tacitly assumed that the file `hello.m` has been placed in one of the directories listed in the MATLAB search path.) ■

**Example E.6.** In order to save some typing, we can create a file called `main.m` containing the following:

```
a = [
    0.9501 0.8913 0.8214 0.9218;
    0.2311 0.7621 0.4447 0.7382;
    0.6068 0.4565 0.6154 0.1763;
    0.4860 0.0185 0.7919 0.4057;
];
y0 = a * [1 2 3 4].';
y1 = a * [-1 2.5 3 4].';
```

```
y3 = a * [41 -22 3 4].'
```

Then, to invoke the code in the above file, we simply type the following in our MATLAB session:

```
main
```

(Again, it is tacitly assumed that the file `main.m` has been placed in one of the directories listed in the MATLAB search path.) ■

Generally, one should avoid giving a script file a name that is associated with a previously defined variable or function, as this leads to the potential for naming conflicts. For example, it would be a bad idea to name a script file as `sin.m`, since the `sin` function is already built into MATLAB.

Clearly, MATLAB needs a means to locate M-file scripts, since there are usually many directories in which a user might choose to place a script. For this purpose, the MATLAB search path is used. The MATLAB search path is a list of directories in which MATLAB looks for M-files. In order for the code in an M-file to be successfully located by MATLAB and executed, the M-file must be stored in a directory listed in the MATLAB search path. The MATLAB search path can be queried with the `path` command:

```
path
```

This command will output all of the directories in the MATLAB search path (i.e., all of the directories in which MATLAB will look for M-file scripts).

A new directory can be added to the MATLAB search path with the `addpath` command:

```
addpath dirname
```

This adds the directory `dirname` to the MATLAB search path.

A few other commands are also sometimes useful in the context of M-file scripts. These commands are described below.

The working directory for MATLAB can be changed using the `cd` command:

```
cd dirname
```

The working directory is changed to the directory named `dirname`.

The current working directory can be queried with the `pwd` command:

```
pwd
```

This command will display the current working directory.

## E.8 Relational and Logical Operators

The relational and logical operators provided by MATLAB are listed in Tables E.16 and E.17, respectively. Some functions that are also quite useful in relational and logical expressions are listed in Table E.18. As far as boolean expressions are concerned, MATLAB considers any nonzero number to be true and zero to be false.

**Example E.7.** Suppose that `a = [1 2 3 4 5]` and `b = [5 4 3 2 1]`. Some examples of expressions using `a` and `b` in conjunction with relational operators are given in Table E.15. ■

**Example E.8.** The following code fragment illustrates how one might use some of the relational/logical functions such as `all`, `any`, and `find`:

```
a = [1 2 3; 4 5 6; 7 8 9];
if all(a > 0)
    disp('All matrix elements are positive.');
```

**end**

```
if all(a < 0)
    disp('All matrix elements are negative.');
```

Table E.15: Examples of expressions involving relational operators

Expression	Value
<code>a &gt; b</code>	<code>[0 0 0 1 1]</code>
<code>a == b</code>	<code>[0 0 1 0 0]</code>
<code>a &lt; b</code>	<code>[1 1 0 0 0]</code>
<code>a &gt;= 2 &amp; a &lt;= 3</code>	<code>[0 1 1 0 0]</code>
<code>a &lt; 2   a &gt; 4</code>	<code>[1 0 0 0 1]</code>
<code>~a</code>	<code>[0 0 0 0 0]</code>

Table E.16: Relational operators

Symbol	Description
<code>&lt;</code>	less than
<code>&lt;=</code>	less than or equal
<code>&gt;</code>	greater than
<code>&gt;=</code>	greater than or equal
<code>==</code>	equal
<code>~=</code>	not equal

Table E.17: Logical operators

Symbol	Description
<code>&amp;</code>	element-wise and
<code> </code>	element-wise or
<code>~</code>	not
<code>&amp;&amp;</code>	short-circuit and
<code>  </code>	short-circuit or

Table E.18: Relational and logical functions

Name	Description
<code>any</code>	any element nonzero
<code>all</code>	all elements nonzero
<code>find</code>	find nonzero elements
<code>exist</code>	check if variables exist
<code>isfinite</code>	detect finite values
<code>isinf</code>	detect infinities
<code>isnan</code>	detect NaNs
<code>isempty</code>	detect empty matrices
<code>isstr</code>	detect strings
<code>strcmp</code>	compare strings

Table E.19: Operator precedence

Operators	Precedence Level
( )	highest
.' ' .^ ^	
+ - ~	
.* * ./ .\ / \	
+ -	
:	
< <= > >= == ~=	
&	
&&	
	lowest

Unary + and - have higher precedence than the binary forms.  
In older versions of MATLAB, & and | have the same precedence.

```

end
if ~any(a == 0)
    disp('All matrix elements are nonzero.');
```

**end**

```

if all(real(a) == a)
    disp('All matrix elements are real.');
```

**end**

```

i = find(a >= 8);
disp('The following matrix elements are greater than or equal to 8:');
disp(a(i));
```

■

## E.9 Operator Precedence

The operator precedence is shown in Table E.19.

## E.10 Control Flow

In the sections that follow, we introduce the conditional execution and looping constructs available in MATLAB.

### E.10.1 If-Elseif-Else

The **if-elseif-else** construct allows groups of statements to be conditionally executed, and has a number of variants. The simplest variant (i.e., the **if** variant) has the form:

```

if expression
    statements
end
```

If the expression *expression* is true, then the statements *statements* are executed. The next simplest variant (i.e., the **if-else** variant) has the form:

```

if expression1
    statements1
else
    statements2
end
```

If the expression  $expression_1$  is true, then the statements  $statements_1$  are executed. Otherwise, the statements  $statements_2$  are executed. Finally, the most general variant has the form:

```

if  $expression_1$ 
     $statements_1$ 
elseif  $expression_2$ 
     $statements_2$ 
    :
elseif  $expression_{n-1}$ 
     $statements_{n-1}$ 
else
     $statements_n$ 
end

```

Note that the **elseif** and **else** clauses are optional.

**Example E.9.** The following code fragment tests the sign of the variable  $x$  and prints an appropriate message:

```

if  $x > 0$ 
    disp('x is positive');
elseif  $x < 0$ 
    disp('x is negative');
else
    disp('x is neither positive nor negative');
end

```

## E.10.2 Switch

The **switch** construct provides another means to conditionally execute groups of statements. The general form of this construct is as follows:

```

switch  $expression$ 
case  $test\_expression_1$ 
     $statements_1$ 
case  $test\_expression_2$ 
     $statements_2$ 
    :
case  $test\_expression_{n-1}$ 
     $statements_{n-1}$ 
otherwise
     $statements_n$ 
end

```

The switch expression  $expression$  is compared to each of the test expressions  $test\_expression_1$ ,  $test\_expression_2$ , ...,  $test\_expression_{n-1}$  in order. The first test expression, say  $test\_expression_k$ , matching the expression  $expression$  has its corresponding statements  $statements_k$  executed. If none of the test expressions match the switch expression and an **otherwise** clause is present, the statements  $statements_n$  in this clause are executed. The switch expression must be either a scalar or string.

**Example E.10.** The following code fragment examines the real variable  $n$  and prints some information concerning its value:

```

n = 5;
switch mod(n, 2)
case 0

```

```

        disp('number is even integer');
    case 1
        disp('number is odd integer');
    case {0.5, 1.5}
        disp('number is half integer');
    otherwise
        disp('number is not an integer');
end

```

■

**Example E.11.** The following code fragment converts a mass specified in a variety of units to kilograms:

```

x = 100; % input mass
units = 'lb'; % units for input mass
switch units
case {'g'}
    y = 0.001 * x;
case {'kg'}
    y = x;
case {'lb'}
    y = 0.4536 * x;
otherwise
    error('unknown units');
end
disp(sprintf('%f %s converts to %f kg', x, units, y));

```

■

### E.10.3 For

The **for** construct allows a group of statements to be repeated a fixed number of times. This construct has the general form:

```

for variable = array
    statements
end

```

The statements *statements* are executed once for each value in the array *array*, where the variable *variable* is set to the corresponding array value each time.

**Example E.12** (Degree to radian conversion). The following code fragment outputs a table for converting angles from units of degrees to radians:

```

disp('Degrees      Radians');
for theta_degrees = -5 : 0.5 : 5
    theta_radians = theta_degrees * pi / 180;
    disp(sprintf('%7.1f      %7.4f', theta_degrees, theta_radians));
end

```

■

**Example E.13.** The following code fragment applies a linear transformation (represented by the matrix *a*) to each column of the matrix  $[0\ 2\ 4\ 6; 1\ 3\ 5\ 7]$ :

```

a = [1 0; 1 -1];
for v = [0 2 4 6; 1 3 5 7]
    disp(a * v);
end

```

■

### E.10.4 While

The **while** construct allows a group of statements to be executed an indefinite number of times. This construct has the form:

```
while expression
    statements
end
```

The statements *statements* are executed repeatedly as long as the condition *expression* is true.

**Example E.14.** Suppose that we would like to compute the smallest machine-representable positive real number that, when added to one, is still greater than one. This quantity is sometimes referred to as **machine epsilon**. Due to finite-precision effects, there is a lower bound on this quantity. Although machine epsilon is available via the built-in eps variable in MATLAB, we can compute its value explicitly using the following code:

```
epsilon = 1;
while (1 + epsilon / 2) > 1
    epsilon = epsilon / 2;
end
disp(epsilon);
```

### E.10.5 Break and Continue

Sometimes, it may necessary to prematurely break out of a loop or prematurely continue with its next iteration. This is accomplished via **break** and **continue**.

**Example E.15.** The following two code fragments are equivalent, where the first employs a **break** statement and the second does not:

```
% Code fragment 1
done = 0;
while 1
    if done
        break % Terminate (i.e., break out of) loop.
    end
    % Do something here.
    % If we are finished, set done to one.
end

% Code fragment 2
done = 0;
while ~done
    % Do something here.
    % If we are finished, set done to one.
end
```

**Example E.16.** The following code fragment gives an example of the use of the **continue** statement:

```
a = [1 0 3 2 0];
for i = a
    if i == 0
        % Skip over the processing of a zero element in the array.
        continue
    end
    % Process the nonzero array element.
    disp(i);
end
```

The above code will print only the nonzero elements of the array a.

## E.11 Functions

MATLAB supports user-defined functions. To create a user-defined function, the code for the function is placed in an M-file. In this sense, user-defined functions are very similar to script files. For this reason, most of the material on script files in Section E.7 is also applicable here. There is, however, one key difference between a script and function file. A function file must include a **function** directive (whereas a script file must not). This directive is primarily used to indicate the number of input and output arguments for a function.

The first (non-comment) line in function file must contain the **function** directive. This directive has the form:

```
function [ argout1, argout2, ..., argoutn ] = funcname (argin1, argin2, ..., arginm)
```

This indicates that the function has the name *funcname*, the *m* input arguments *argin*<sub>1</sub>, *argin*<sub>2</sub>, ..., *argin*<sub>*m*</sub>, and *n* output arguments *argout*<sub>1</sub>, *argout*<sub>2</sub>, ..., *argout*<sub>*n*</sub>. The function name *funcname* should always be the same as the base name of the file in which the function is stored (i.e., the file name without the “.m” suffix). Immediately following the line containing the **function** directive, one should provide comments to be printed in response to a `help` inquiry for the function. The body of a function extends from the **function** directive to a corresponding **end** directive or, if no such **end** directive is present, the end of the file. The code in a function executes until either the end of the function is reached or a **return** statement is encountered.

In MATLAB all input arguments to a function are passed by value. For this reason, changes to the input arguments made inside of a function will not propagate to the caller. Also, any variables accessed/manipulated inside of a function are local in scope to that function.

**Example E.17** (Sinc function). As noted earlier, the `sinc` function provided by MATLAB computes the normalized sinc function (as defined by (3.25)), not the sinc function (as defined by (3.24)). In Listing E.1, we define a function `mysinc` that computes the sinc function. The command `help mysinc` will result in MATLAB printing the first block of comments from the above function file.

Listing E.1: `mysinc.m`

```
1 function y = mysinc(x)
2     % mysinc - Compute the sinc function.
3     % mysinc(x) returns a matrix whose elements are the sinc of the
4     % elements of x
5
6     % Initialize the output array to all ones.
7     y = ones(size(x));
8     % Determine the indices of all nonzero elements in the input array.
9     i = find(x);
10    % Compute the sinc function for all nonzero elements.
11    % The zero elements are already covered, since the output
12    % array was initialized to all ones above.
13    y(i) = sin(x(i)) ./ (x(i));
14 end ■
```

**Example E.18** (Factorial function). Suppose that we would like to write a function called `myfactorial` that takes a single integer argument *n* and returns *n!*. We can achieve this functionality with the code in Listing E.2, which is placed in a file named `myfactorial.m`. The code is invoked by calling the `myfactorial` function. For example, `myfactorial(4)` returns the value 24.

Listing E.2: `myfactorial.m`

```
1 function y = myfactorial(x)
2     % myfactorial - compute factorial
3
4     y = 1;
5     for n = 2 : x
```

Table E.20: Special predefined function variables

Name	Description
nargin	number of input arguments
nargout	number of output arguments
varargin	variable-length input argument
varargout	variable-length output argument

```

6         y = y * n;
7     end
8 end

```

In MATLAB, functions may take a variable number of input arguments and may return a variable number of output arguments. In order for a function to determine the number of input and output arguments and access these arguments, several variables are automatically defined upon entry to a function. These variables are listed in Table E.20. In what follows, we give some examples of functions that take a variable number of input arguments.

**Example E.19** (Function with variable number of input arguments). Suppose that we would like to write a function called `mysum` that takes one, two, or three arguments and returns their sum. We can achieve this functionality with the code in Listing E.3, which is placed in a file named `mysum.m`. The code is invoked by calling the `mysum` function. For example, `mysum(1)` returns the value 1, `mysum(1, 2)` returns the value 3, and `mysum(1, 2, 3)` returns the value 6.

Listing E.3: `mysum.m`

```

1 function y = mysum(a, b, c)
2     % mysum - calculate the sum (of one to three quantities)
3
4     if nargin == 1
5         % function called with one argument
6         y = a;
7     elseif nargin == 2
8         % function called with two arguments
9         y = a + b;
10    elseif nargin == 3
11        % function called with three arguments
12        y = a + b + c;
13    else
14        error('invalid number of arguments');
15    end
16 end

```

Perhaps, we would like to write a function similar to the one in the previous example, except we would like to be able to handle an arbitrary number of input arguments (possibly many more than three). This can be accomplished by using the special predefined `varargin` variable.

**Example E.20** (Variable number of input arguments). Suppose that we would like to write a function that returns the sum of its input arguments, but allows the number of input arguments to be arbitrary. We can achieve this functionality with the code in Listing E.4, which is placed in a file named `mysum2.m`. The code is invoked by calling the `mysum2` function. For example, `mysum2(1)` returns the value 1, `mysum2(1, 2, 3)` returns the value 6, and `mysum2(1, 1, 1, 1, 1, 1, 1, 1, 1)` returns the value 8.

Listing E.4: `mysum2.m`

```

1 function y = mysum2(varargin)
2     % mysum2 - Compute the sum of the input arguments
3

```

Table E.21: Basic plotting functions

Name	Description
plot	linear x-y plot
loglog	log log x-y plot
semilogx	semi-log x-y plot (x-axis logarithmic)
semilogy	semi-log x-y plot (y-axis logarithmic)
polar	polar plot
bar	bar chart
stem	stem plot
pcolor	pseudocolor (checkerboard) plot

Table E.22: Other graphing functions/commands

Name	Description
axis	control axis scaling and appearance
hold	hold current plot
subplot	multiple axes in single figure
figure	create figure

```

4     if nargin == 0
5         y = [];
6         return
7     end
8     y = varargin{1};
9     for i = 2 : nargin
10        y = y + varargin{i};
11    end
12 end

```

■

## E.12 Graphing

MATLAB has a very rich set of graphing capabilities. Herein, we will try to illustrate some of these capabilities by way of examples.

Some of the basic plotting functions are listed in Table E.21 and several other graphing-related functions/commands are given in Table E.22.

When generating plots, it is sometimes desirable to be able to specify line styles, line colors, and marker styles. The supported line styles, line colors, and marker styles are listed in Tables E.23, E.24, and E.25, respectively.

**Example E.21** (Simple plot). Suppose that we want to plot the function  $\sin t$  for  $t$  in the interval  $[-4\pi, 4\pi]$ . This can be accomplished with the following code:

```

t = linspace(-4 * pi, 4 * pi, 500);
y = sin(t);
plot(t, y);

```

The resulting plot is shown in Figure E.1.

■

Often, we need to add annotations to plots (e.g., a title, axis labels, etc.). This is accomplished using the functions listed in Table E.26. Sometimes, we wish to use special symbols (such as Greek letters) in annotations. Fortunately, numerous symbols are available in MATLAB as listed in Table E.27.

**Example E.22** (Annotated plot). Suppose that we would like to plot the function  $\alpha(\omega) = |\omega|^2 \sin \omega$ , using special symbols in the axis labels to display Greek letters. This can be accomplished with the following code:

Table E.23: Line styles

Symbol	Line Style
-	solid
:	dotted
-.	dash dot
--	dashed

Table E.24: Line colors

Symbol	Line Color
b	blue
g	green
r	red
c	cyan
m	magenta
y	yellow
k	black
w	white

Table E.25: Marker styles

Symbol	Marker Style
.	point
o	circle
x	cross
+	plus sign
*	asterisk
s	square
d	diamond
v	triangle (down)
^	triangle (up)
<	triangle (left)
>	triangle (right)
p	pentagram
h	hexagram

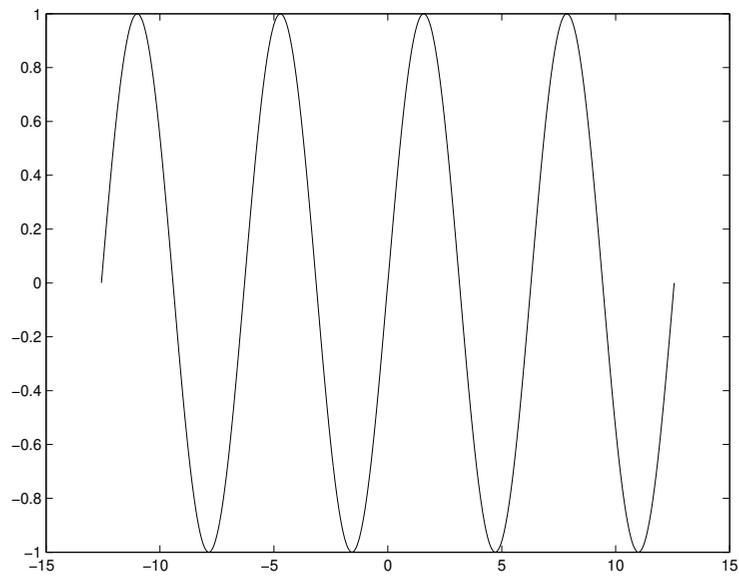


Figure E.1: Plot from example.

Table E.26: Graph annotation functions

Name	Description
title	graph title
xlabel	x-axis label
ylabel	y-axis label
grid	grid lines
text	arbitrarily-positioned text
gtext	mouse-positioned text

Table E.27: Special symbols for annotation text

String	Symbol
<code>\alpha</code>	$\alpha$
<code>\beta</code>	$\beta$
<code>\delta</code>	$\delta$
<code>\gamma</code>	$\gamma$
<code>\omega</code>	$\omega$
<code>\theta</code>	$\theta$
<code>\Delta</code>	$\Delta$
<code>\Omega</code>	$\Omega$

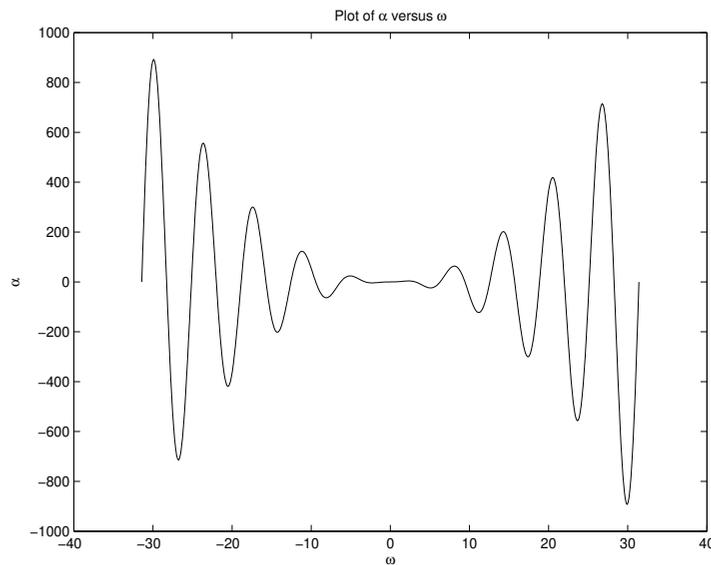


Figure E.2: Plot from example.

```
w = linspace(-10 * pi, 10 * pi, 500);
a = abs(w) .^ 2 .* sin(w);
plot(w, a);
xlabel('\omega');
ylabel('\alpha');
title('Plot of \alpha versus \omega');
```

The resulting plot is shown in Figure E.2. ■

**Example E.23** (Multiple plots on single axes). Suppose that we want to plot both  $\sin t$  and  $\cos t$  on the same axes for  $t$  in the interval  $[-2\pi, 2\pi]$ . This can be accomplished with the following code:

```
t = linspace(-2 * pi, 2 * pi, 500);
plot(t, sin(t), ':'); % plot using dotted line
hold on % cause the second plot not to erase the first
plot(t, cos(t), '-'); % plot using solid line
hold off % allow the next plot to erase previous plots
```

The resulting plot is shown in Figure E.3. ■

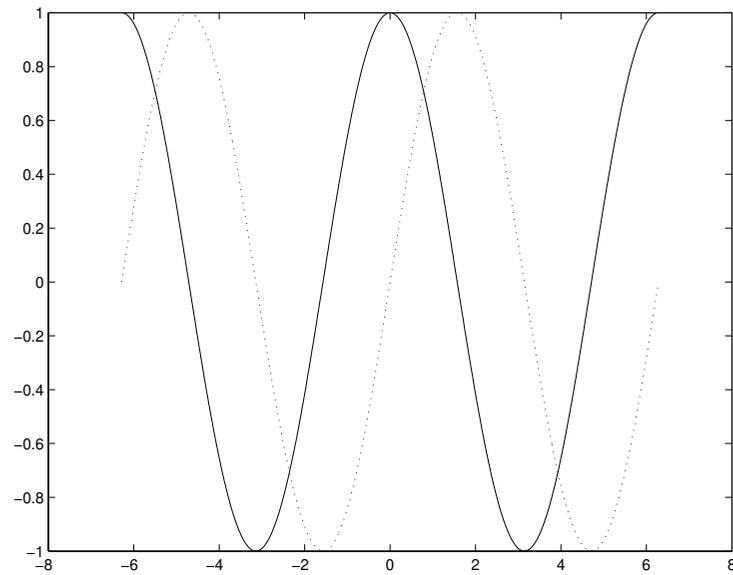


Figure E.3: Plot from example.

**Example E.24** (Multiple axes on same figure). Suppose that we would like to plot the functions  $\cos t$ ,  $\sin t$ ,  $\arccos t$ , and  $\arcsin t$  using four separate plots in the same figure. This can be done using the following code:

```
t = linspace(-pi, pi, 500);
subplot(2, 2, 1); % select first subplot in 2-by-2 layout
plot(t, cos(t));
title('cos(t)')
subplot(2, 2, 2); % select second subplot in 2-by-2 layout
plot(t, sin(t));
title('sin(t)');
t = linspace(-1, 1, 500);
subplot(2, 2, 3); % select third subplot in 2-by-2 layout
plot(t, acos(t));
title('acos(t)')
subplot(2, 2, 4); % select fourth subplot in 2-by-2 layout
plot(t, asin(t));
title('asin(t)')
```

The resulting graphs are shown in Figure E.4. ■

## E.13 Printing

To print copies of figures, the `print` command is employed. This command has various forms, but one of the simplest is as follows:

```
print -ddevice filename
```

Here, *device* specifies the output format and *filename* is the name of the file in which to save the output. For example, for a PostScript printer, the `ps` format should be used. For more details on additional options, type `help print`.

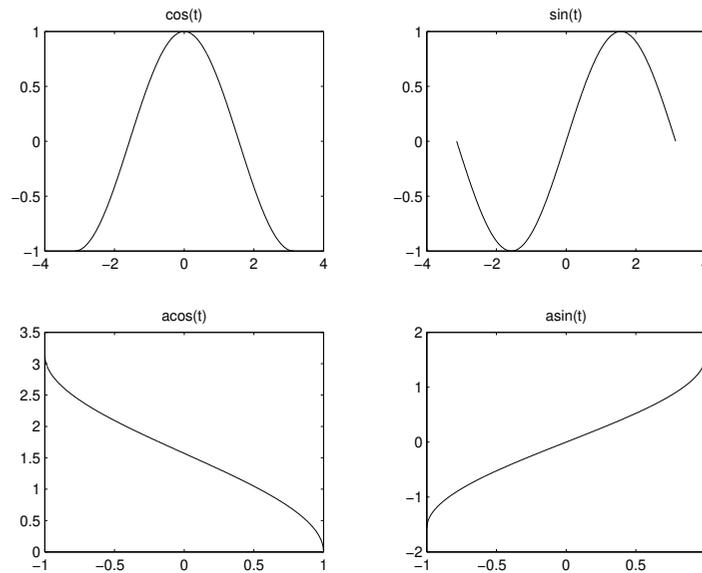


Figure E.4: Plot from example.

## E.14 Symbolic Math Toolbox

Symbolic computation is sometimes quite helpful in solving engineering problems. For example, a very complicated formula or equation involving several variables might need to be simplified without assuming specific values for the variables in the formula/equation. The Symbolic Math Toolbox provides MATLAB with such symbolic computation capabilities.

### E.14.1 Symbolic Objects

The Symbolic Math Toolbox defines a new data type called a **symbolic object**. The toolbox uses symbolic objects to represent symbolic variables, constants, and expressions. In essence, a symbolic object can have as its value any valid mathematical expression.

Symbolic objects can be used in many of the same ways as non-symbolic objects. One must, however, keep in mind that performing a computation symbolically is quite different than performing it non-symbolically. Generally speaking, symbolic computation is much slower than non-symbolic computation.

### E.14.2 Creating Symbolic Objects

A symbolic variable is created using either the `sym` function or `syms` directive. For example, a symbolic variable `x` (whose value is simply itself) can be created using the `sym` function with the code:

```
x = sym('x');
```

The same result can be achieved in a less verbose manner using the `syms` directive as follows:

```
syms x;
```

Also, the `syms` directive allows multiple variables to be specified. So, a single invocation of this directive can be used to create multiple variables. For example, symbolic variables named `x`, `y`, and `z` can be created with the code:

```
syms x y z;
```

The `sym` function can also be used to create symbolic constants. For example, we can create a symbolic constant `p` that has the value  $\pi$  with the code:

```
p = sym('pi');
```

Note that the character string argument passed to the `sym` function can only contain a variable name or constant. It cannot be an arbitrary expression. For example, code like the following is not allowed:

```
f = sym('a * x ^ 2 + b * x + c');
% ERROR: cannot pass arbitrary expression to sym function
```

From symbolic variables and constants, more complicated symbolic expressions can be constructed. For example, a symbolic expression `f` with the value  $a * x^2 + b * x + c$  can be created with the code:

```
syms a b c x;
f = a * x ^ 2 + b * x + c;
```

As another example, a symbolic expression `f` with the value  $\pi * r^2$  can be created with the code:

```
syms r;
f = sym('pi') * r ^ 2;
```

### E.14.3 Manipulating Symbolic Objects

Symbolic objects can often be used in the same way as non-symbolic objects. For example, we can do things like:

```
syms t;
f = t + 1;
g0 = f ^ 3 - 2 * f - 21;
g1 = cos(f) * sin(f / 3);
```

Symbolic objects are quite distinctive from other types of objects in MATLAB. For example, the following two lines of code have very different effects:

```
x = pi;
x = sym('pi');
```

It is important to understand the difference in what these two lines of code do.

To substitute some expression/variable for another variable, use the `subs` function. For example, to substitute  $t + 1$  for  $t$  in the expression  $t^2$ , we can use the following code:

```
syms t;
f = t ^ 2;
g = subs(f, t, t + 1)
```

After executing the preceding code, `g` is associated with the expression  $(t + 1)^2$ .

To factor a symbolic expression, use the `factor` function. For example, suppose that we want to factor the polynomial  $t^2 + 3t + 2$ . This could be accomplished with the following code:

```
syms t;
f = t ^ 2 + 3 * t + 2;
g = factor(f)
```

After executing the preceding code, `g` is associated with the (row-vector) expression  $[t + 2, t + 1]$ . Note that the `factor` function will only produce factors with real roots.

To simplify a symbolic expression, use the `simplify` function. For example, suppose that we want to substitute  $2 * t + 1$  for  $t$  in the expression  $t^2 - 1$  and simplify the result. This can be accomplished with the following code:

```
syms t;
f = t ^ 2 - 1;
g = simplify(subs(f, t, 2 * t + 1))
```

After executing the preceding code,  $g$  is associated with the expression  $(2 * t + 1) ^ 2 - 1$ .

To expand an expression, use the `expand` function. For example, to compute  $(t + 1)^5$ , we can use the following code:

```
syms t;
f = (t + 1) ^ 5;
g = expand(f)
```

After executing the preceding code,  $g$  is associated with the expression:

$$t ^ 5 + 5 * t ^ 4 + 10 * t ^ 3 + 10 * t ^ 2 + 5 * t + 1$$

To display an expression in a human-friendly format, use the `pretty` function. For example, to compute  $[\frac{1}{2}t^2 + \frac{1}{3}(t + 1)]^{-4}$  in an expanded and beautified format, we can use the following code:

```
syms t;
f = ((1/2) * t^2 + (1/3) * (t+1)) ^ (-4);
pretty(expand(f))
```

The output of the `pretty` function in this case might look something like:

```

              1
-----
      8      7      6      5      4      3      2
t      t      t      11 t      65 t      22 t      4 t      4 t      1
-- + -- + -- + ----- + ----- + ----- + ----- + --- + ---
16   6   3   27      162      81      27      81   81
```

**Example E.25** (Sum of an arithmetic sequence). Consider a sum of the form

$$S(a, d, n) \triangleq \sum_{k=0}^{n-1} (a + kd)$$

(i.e., the sum of an arithmetic sequence). Suppose that we did not happen to know that this sum can be calculated as

$$S(a, d, n) = \frac{1}{2}n(2a + d(n - 1)).$$

We could determine a general formula for the sum using the following code:

```
syms a d k n;
simplify(symsum(a + k * d, k, 0, n - 1))
```

The code yields the result  $a * n + (d * n * (n - 1)) / 2$ . Clearly, this result is equivalent to the known expression for  $S(a, d, n)$  given above. ■

**Example E.26** (Derivatives). Use MATLAB to compute each of the derivatives given below.

(a)  $\frac{d}{dt}(e^t \cos 3t)$ ; and

(b)  $\frac{d}{d\omega} \left( \frac{\cos \omega}{\omega^2 + 1} \right)$ .

*Solution.* (a) The derivative  $\frac{d}{dt}(e^t \cos 3t)$  can be evaluated with the code:

```
clear
syms t
pretty(simplify(diff(exp(t) * cos(3 * t), t)))
```

The result produced should resemble:

$$\exp(t) (\cos(3 t) - \sin(3 t) 3)$$

(b) The derivative  $\frac{d}{d\omega} \left( \frac{\cos \omega}{\omega^2 + 1} \right)$  can be evaluated with the code:

```
clear
syms omega
pretty(simplify(diff(cos(omega) / (omega ^ 2 + 1), omega)))
```

The result produced should resemble:

$$\frac{\sin(\omega)}{\omega^2 + 1} - \frac{2 \omega \cos(\omega)}{(\omega^2 + 1)^2}$$

**Example E.27** (Definite and indefinite integrals). Use MATLAB to compute each of the integrals given below.

(a)  $\int_{t-1}^{t+1} (\tau+1)^2 d\tau$ ;

(b)  $\int t \cos 2t dt$ ;

(c)  $\int_0^{\pi/3} \cos^2(t) dt$ ; and

(d)  $\int (t^{-1} + t) dt$ .

*Solution.* (a) The integral  $\int_{t-1}^{t+1} (\tau+1)^2 d\tau$  can be evaluated with the code:

```
clear
syms t tau
pretty(simplify(int((tau + 1) ^ 2, tau, [(t - 1) (t + 1)])))
```

The result produced should resemble:

$$2 t^2 + 4 t + \frac{8}{3}$$

(b) The integral  $\int t \cos 2t dt$  can be evaluated with the code:

```
clear
syms t
pretty(simplify(int(t * cos(2 * t), t)))
```

The result produced should resemble:

$$\frac{\cos(2 t)}{4} + \frac{t \sin(2 t)}{2}$$

(c) The integral  $\int_0^{\pi/3} \cos^2(t) dt$  can be evaluated with the code:

```
clear
syms t pi
pretty(simplify(int(cos(t) ^ 2, t, [0 (pi / 3)])))
```

The result produced should resemble:

$$\frac{\pi}{6} + \frac{\sqrt{3}}{8}$$

(d) The integral  $\int (t^{-1} + t) dt$  can be evaluated with the code:

Table E.28: Functions related to signal processing

Name	Description
besself	Bessel analog filter design
bode	bode plot
butter	Butterworth analog/digital filter design
freqs	frequency response of analog filter
impz	compute impulse response
lsim	simulate system
step	compute step response
tf	generate system model from transfer function coefficients

```
clear
syms t
pretty(simplify(int(t ^ (-1) + t, t)))
```

The result produced should resemble:

$$\log(t) + \frac{t^2}{2}$$


#### E.14.4 Plotting Symbolic Expressions

To plot a symbolic expression in one variable, the `ezplot` function can be used. For example, to plot the function  $f(t) = 3t^2 - 4t + 2$ , we can use the code:

```
syms t;
ezplot(3 * t ^ 2 - 4 * t + 2);
```

The range of the independent variable may optionally be specified. For example, to plot the expression for  $t$  in the interval  $[-1, 1]$ , we can use the code:

```
syms t;
ezplot(3 * t ^ 2 - 4 * t + 2, [-1 1]);
```

### E.15 Signal Processing

MATLAB provides many functions that are helpful in the analysis of signals and systems. Some of the more useful functions are listed in Table E.28. In what follows, we provide several examples of how some of these functions can be used.

#### E.15.1 Frequency Responses

In signal processing, we often need to evaluate the frequency responses of LTI systems. Usually such systems have a frequency response that is rational. That is, the frequency response  $H_F$  is of the form

$$H_F(\omega) = \frac{\sum_{k=0}^{M-1} \alpha_k \omega^k}{\sum_{k=0}^{N-1} \beta_k \omega^k},$$

where  $\alpha_k$  and  $\beta_k$  are complex constants. Fortunately, a function of this form can be evaluated quite easily in MATLAB using the `polyval` function.

Often, it is more convenient to characterize a LTI system by its transfer function rather than its frequency response. Most systems of interest have a transfer function that is rational. That is, the transfer function  $H_{TF}$  is of the form

$$H_{TF}(s) = \frac{\sum_{k=0}^{M-1} a_k s^k}{\sum_{k=0}^{N-1} b_k s^k},$$

where  $a_k$  and  $b_k$  are complex constants. Although we could compute the frequency response by using `polyval` to evaluate the transfer function at points on the imaginary axis, there is an easier way to accomplish this task. In particular, we can simply use the `freqs` function provided by MATLAB.

**Example E.28** (Computing and plotting frequency responses with `freqs`). Consider the LTI system with transfer function

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}.$$

This system is a second-order Butterworth lowpass filter with a cutoff frequency of 1 rad/s. Suppose that we would like to evaluate the frequency response of this system at various values of  $\omega$ . This is equivalent to evaluating the transfer function  $H$  at points on the imaginary axis. To this end, we can employ the `freqs` function in MATLAB. More specifically, we can calculate and plot the magnitude and phase responses of the above system with the following code:

```
% Initialize the numerator and denominator coefficients of the transfer
% function.
tfnum = [0 0 1];
tfdenom = [1 sqrt(2) 1];

% Plot the magnitude and phase responses.
freqs(tfnum, tfdenom, linspace(0, 2, 500));
```

**Example E.29** (Plotting frequency responses). Suppose that we would like to have a function that behaves in a similar way to the MATLAB `freqs` function, but with a few differences in how plotting is performed. In particular, we would like the plots generated with the magnitude response as a unitless quantity and the phase response in unwrapped form. This can be accomplished with the code given in Listing E.5.

Listing E.5: `myfreqs.m`

```
1 function [freqresp, omega] = myfreqs(tfnum, tfdenom, omega)
2     % The myfreqs function has essentially the same interface as the
3     % MATLAB freqs function, but performs plotting slightly differently.
4     % The magnitude response is plotted as a unitless quantity (not in
5     % decibels).
6     % The phase response is plotted with the phase unwrapped.
7
8     % If the frequencies have been specified as an input argument, then simply
9     % pass them through to the real freqs function.
10    if nargin >= 3
11        [freqresp, omega] = freqs(tfnum, tfdenom, omega);
12    else
13        [freqresp, omega] = freqs(tfnum, tfdenom);
14    end
15
16    % If no output arguments were specified, plot the frequency response.
17    if nargin == 0
18
19        % Compute the magnitude response as a unitless quantity.
```

```

20     magresp = abs(freqresp);
21
22     % Compute the phase response with the phase unwrapped.
23     phaseresp = unwrap(angle(freqresp));
24
25     % On the first of two graphs, plot the magnitude response.
26     subplot(2, 1, 1);
27     plot(omega, magresp);
28     title('Magnitude Response');
29     xlabel('Frequency (rad/s)');
30     ylabel('Magnitude (unitless)');
31
32     % On the second of two graphs, plot the phase response.
33     subplot(2, 1, 2);
34     plot(omega, phaseresp);
35     title('Phase Response');
36     xlabel('Frequency (rad/s)');
37     ylabel('Angle (rad)');
38
39     end
40 end

```

### E.15.2 Impulse and Step Responses

Sometimes, we need to determine the response of a LTI system to a specific input. Two inputs of particular interest are the unit-impulse function  $\delta$  and unit-step function  $u$ . Fortunately, it is quite easy to compute impulse and step responses using the `impz` and `stepz` functions in MATLAB, as illustrated by the example below.

**Example E.30** (Computing impulse and step responses). Consider the LTI system with the transfer function

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}.$$

Suppose that we wish to calculate and plot the impulse and step responses of this system. This can be accomplished with the code given in Listing E.6. Executing this code produces the plots shown in Figure E.5.

Listing E.6: Computing and plotting the impulse and step responses

```

1  % Initialize the numerator and denominator coefficients of the transfer
2  % function.
3  tfnum = [0 0 1];
4  tfdenom = [1 sqrt(2) 1];
5
6  % Determine the system model associated with the given transfer function.
7  sys = tf(tfnum, tfdenom);
8
9  % Plot the impulse response.
10 subplot(2, 1, 1);
11 impulse(sys);
12
13 % Plot the step response.
14 subplot(2, 1, 2);
15 step(sys);

```

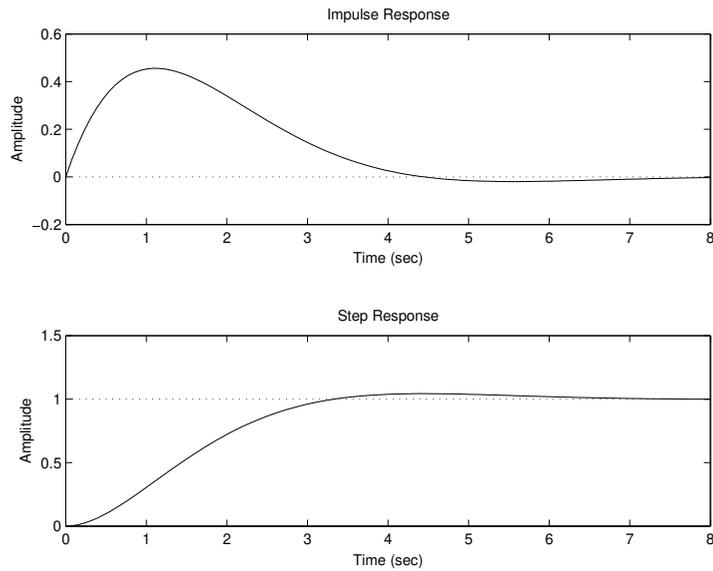


Figure E.5: Plot from example.

### E.15.3 Filter Design

A number of functions are provided in MATLAB to assist in the design of various types of filters. In what follows, we consider a few examples of using such functions.

**Example E.31** (Butterworth lowpass filter design). Suppose that we want to design a tenth-order Butterworth lowpass filter with a cutoff frequency of 100 rad/s. We can design such a filter as well as plot its frequency response using the code given in Listing E.7 below. The frequency response obtained for the filter is shown in Figure E.6.

Listing E.7: Butterworth lowpass filter design

```

1 % Calculate the transfer function coefficients for a tenth-order Butterworth
2 % lowpass filter with a cutoff frequency of 100 rad/s.
3 [tfnum, tfdenom] = butter(10, 100, 's');
4
5 % Plot the frequency response of the filter.
6 freqs(tfnum, tfdenom);
```

**Example E.32** (Bessel lowpass filter design). Suppose that we want to design a tenth-order Bessel lowpass filter with a cutoff frequency of 100 rad/s. We can design such a filter as well as plot its frequency response using the code given in Listing E.8 below. The frequency response obtained for the filter is shown in Figure E.7.

Listing E.8: Bessel lowpass filter design

```

1 % Calculate the transfer function coefficients for a tenth-order Bessel
2 % lowpass filter with a cutoff frequency of 100 rad/s.
3 [tfnum, tfdenom] = besself(10, 100);
4
5 % Plot the frequency response of the filter.
6 freqs(tfnum, tfdenom);
```

## E.16 Miscellany

Some other functions that might be useful are listed in Table E.29.

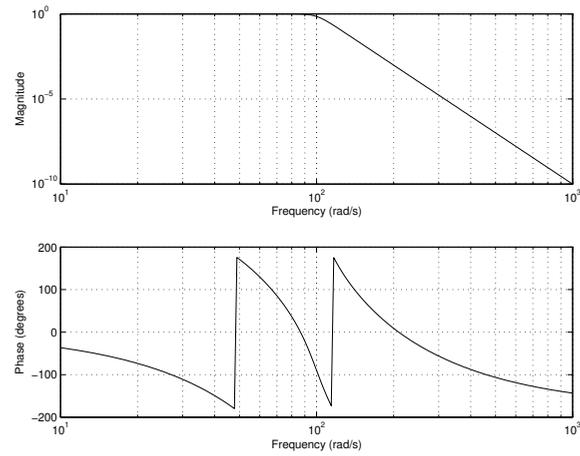


Figure E.6: Frequency response of the Butterworth lowpass filter.

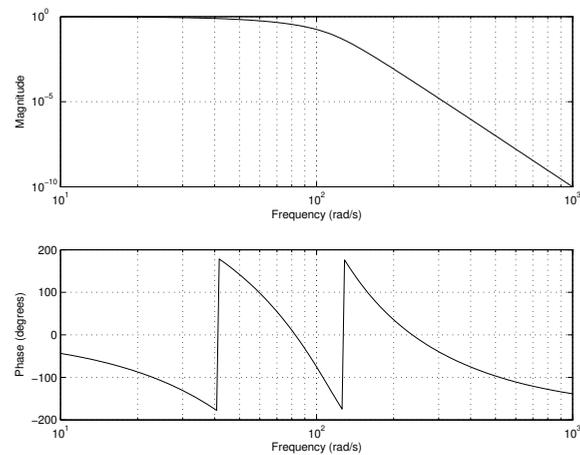


Figure E.7: Frequency response of the Bessel lowpass filter.

Table E.29: Miscellaneous functions/commands

Name	Description
roots	find roots of polynomial
clear	clear a variable
diary	log MATLAB session
echo	echo commands on execution (for debugging)
quit	quit MATLAB
format	output format for numbers

## E.17 Exercises

**E.101** Indicate whether each of the following is a valid MATLAB identifier (i.e., variable/function name):

- (a) 4ever
- (b) \$rich\$
- (c) foobar
- (d) foo\_bar
- (e) \_foobar

**E.102** Let  $T_C$ ,  $T_F$ , and  $T_K$  denote the temperature measured in units of Celsius, Fahrenheit, and Kelvin, respectively. Then, these quantities are related by

$$T_F = \frac{9}{5}T_C + 32 \quad \text{and}$$

$$T_K = T_C + 273.15.$$

Write a program that generates a temperature conversion table. The first column of the table should contain the temperature in Celsius. The second and third columns should contain the corresponding temperatures in units of Fahrenheit and Kelvin, respectively. The table should have entries for temperatures in Celsius from  $-50$  to  $50$  in steps of  $10$ .

**E.103** Let  $F$  denote the complex-valued function of a real variable given by

$$F(\omega) = \frac{1}{j\omega + 1}.$$

Write a program to plot  $|F(\omega)|$  and  $\arg F(\omega)$  for  $\omega$  in the interval  $[-10, 10]$ . Use `subplot` to place both plots on the same figure.

**E.104** In what follows, we consider a simple algorithm for generating a set of  $2^d$  points in the complex plane, where  $d$  is a positive integer. For  $n = 0, 1, \dots, 2^d - 1$ , this algorithm computes the  $n$ th point  $p_n$  as

$$p_n = \sum_{k=0}^{d-1} a_k \beta^{k+1},$$

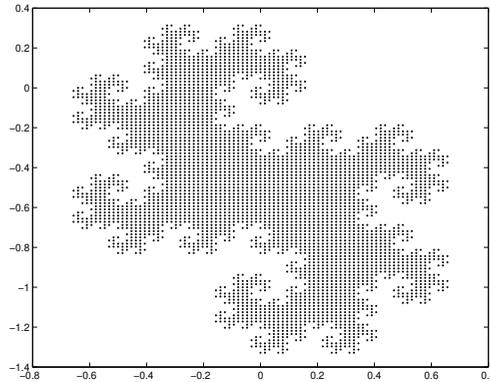
where  $\beta = \frac{1}{\sqrt{2}}e^{-j\pi/4}$ ,  $a_k \in \{0, 1\}$ , and  $n = \sum_{k=0}^{d-1} a_k 2^k$ . Note that the binary sequence  $a_{d-1}, \dots, a_1, a_0$  is simply the  $d$ -bit binary representation of the integer  $n$ , where  $a_0$  corresponds to the least-significant bit. For example, if  $d = 3$ , then the relationship between  $n$  and  $a_2, a_1, a_0$  is as shown in the following table:

$n$	$a_2$	$a_1$	$a_0$
0	0	0	0
1	0	0	1
2	0	1	0
3	0	1	1
4	1	0	0
5	1	0	1
6	1	1	0
7	1	1	1

(a) Write a function called `twindragon` that calculates and plots the point set obtained by using the above algorithm for a specified value of the parameter  $d$ . The value of  $d$  should be passed as an input argument to the function. [Hint: It is possible to implement this function in about 15 to 20 lines of code. The `polyval` and

dec2bin functions should be quite helpful.]

(b) Using the function developed in part (a), plot the set of points obtained with the parameter  $d$  equal to 12. In the limit as  $d$  approaches infinity, the resulting set of points converges to the well-known twin-dragon fractal set. Although choosing  $d = 12$  should be sufficient to see the approximate shape of the point set in the limiting case, larger values of  $d$  will yield better approximations, albeit at the expense of significantly more computation. You should obtain a plot resembling that shown below.



**E.105** In what follows, let  $\min(a, b)$  denote the minimum of  $a$  and  $b$ . For a complex number  $z$ , we define an iterative process that generates the complex-number sequence  $v_z(0), v_z(1), v_z(2), \dots$ , where

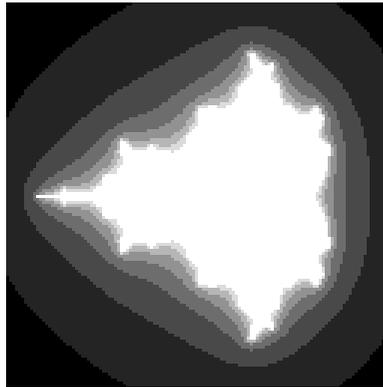
$$v_z(n) = \begin{cases} v_z^2(n-1) + z & n \geq 1 \\ 0 & n = 0. \end{cases}$$

Let  $g(z)$  denote the smallest value of  $n$  for which  $|v_z(n)| > 10$ . In the case that  $|v_z(n)| > 10$  is not satisfied for any  $n$ ,  $g(z)$  is simply defined to be  $\infty$ . Let  $f(z) = \min(g(z), 10)$ . For example, we have the following:

$z$	First few elements of $v_z$	$g$	$f$
0	0, 0, 0, 0	$\infty$	10
$j$	0, $j$ , $-1 + j$ , $-j$ , $-1 + j$ , $-j$	$\infty$	10
2	0, 2, 6, 38	3	3
$2j$	0, $2j$ , $-4 + 2j$ , $12 - 14j$	3	3

Write a function called `mandelbrotfunc` to compute the value  $f(z)$  for any (complex) value of  $z$ . Using the function `mandelbrotfunc`, evaluate  $f$  on an evenly-spaced rectangular grid (in the complex plane) having width 128, height 128, and with the bottom-left and top-right corners of the grid corresponding to the points  $-2.25 - 1.5j$  and  $1 + 1.5j$ , respectively. Store these computed values into a  $128 \times 128$  matrix. Then, using the `pcolor` function in MATLAB, plot the contents of this matrix. After the call to `pcolor`, use the command “`shading interp`” to eliminate the grid lines from the plot (which causes the plot to look somewhat less attractive).

A complete solution to this problem requires less than 25 lines of code (excluding comments). A correct solution should yield a plot resembling the one shown below. Incidentally, the innermost region in the plot is an approximation to the famous Mandelbrot (fractal) set.



**E.106** Consider the vector  $v$  defined by the following line of code:

```
v = [0 1 2 3 4 5]
```

Write an expression in terms of  $v$  that yields a new vector of the same dimensions as  $v$ , where each element  $t$  of the original vector  $v$  has been replaced by the given quantity below. In each case, the expression should be as short as possible.

- (a)  $2t - 3$ ;
- (b)  $1/(t + 1)$ ;
- (c)  $t^5 - 3$ ; and
- (d)  $|t| + t^4$ .

**E.107** (a) Write a function called `unitstep` that takes a single real argument  $t$  and returns  $u(t)$ , where

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(b) Modify the function from part (a) so that it takes a single vector argument  $t = [t_1 \ t_2 \ \dots \ t_n]^T$  (where  $n \geq 1$  and  $t_1, t_2, \dots, t_n$  are real) and returns the vector  $[u(t_1) \ u(t_2) \ \dots \ u(t_n)]^T$ . Your solution must employ a looping construct (e.g., a `for` loop).

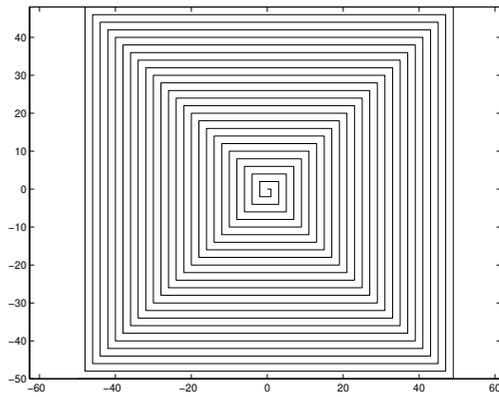
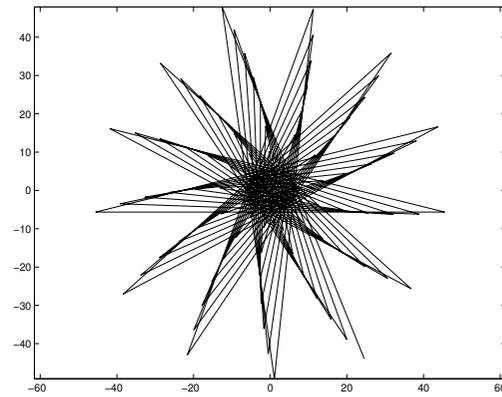
(c) With some ingenuity, part (b) of this problem can be solved using only two lines of code, without the need for any looping construct. Find such a solution. [Hint: In MATLAB, to what value does an expression like `[-2 -1 0 1 2] >= 0` evaluate?]

**E.108** In this problem, we consider an algorithm for generating a sequence  $p$  of  $n$  points in the plane (i.e.,  $p_0, p_1, \dots, p_{n-1}$ ). The first point  $p_0$  is chosen as the origin (i.e.,  $p_0 = [0 \ 0]^T$ ), with the remaining points being given by the formula

$$p_i = p_{i-1} + \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}^{i-1} \begin{bmatrix} i \\ 0 \end{bmatrix}.$$

(a) Using MATLAB, write a function called `drawpattern` that takes  $n$  and  $\theta$  as input arguments (in that order) with  $\theta$  being specified in degrees, and then computes and plots the points  $p_0, p_1, \dots, p_{n-1}$  connected by straight lines (i.e., draw a line from  $p_0$  to  $p_1$ ,  $p_1$  to  $p_2$ ,  $p_2$  to  $p_3$ , and so on). When performing the plotting, be sure to use `axis('equal')` in order to maintain the correct aspect ratio for the plot. For illustrative purposes, the plots produced for two sets of  $\theta$  and  $n$  values are shown in Figures (a) and (b) below.

(b) Generate the plots obtained by invoking `drawpattern` with  $n = 100$  and  $\theta$  set to each of the following values:  $89^\circ$ ,  $144^\circ$ , and  $154^\circ$ . [Note: In MATLAB, the `sin` and `cos` functions take values in radians, not degrees.]

(a)  $\theta = 90^\circ$  and  $n = 100$ (b)  $\theta = 166^\circ$  and  $n = 100$

## Appendix F

# Additional Exercises

### F.1 Overview

This appendix contains numerous additional exercises, which may be useful for practice purposes. In the case of each question where a short answer is possible, an answer key is provided.

### F.2 Exercises

- F.1** A communication channel heavily distorts high frequencies but does not significantly affect very low frequencies. Determine which of the functions  $x_1$ ,  $x_2$ ,  $x_3$ , and  $x_4$  would be least distorted by the communication channel, where

$$x_1(t) = \delta(t), \quad x_2(t) = 5, \quad x_3(t) = 10e^{j1000t}, \quad \text{and} \quad x_4(t) = 1/t.$$

**Short Answer.**  $x_2$

- F.2** Consider a system consisting of a communication channel with input  $x$  and output  $y$ . Since the channel is not ideal,  $y$  is typically a distorted version of  $x$ . Suppose that the channel can be modelled as a causal LTI system with impulse response  $h(t) = e^{-t}u(t) + \delta(t)$ . Determine whether we can devise a physically-realizable stable system that recovers  $x$  from  $y$ . If such a system exists, find its impulse response  $g$ .

**Short Answer.**  $g(t) = \delta(t) - e^{-2t}u(t)$

- F.3** A causal LTI system has impulse response  $h$ , system function  $H$ , and the following characteristics: 1)  $H$  is rational with one pole at  $-2$  and no zeros; and 2)  $h(0^+) = 4$ . Find  $h$ .

**Short Answer.**  $h(t) = 4e^{-2t}u(t)$

- F.4** A causal LTI system with input  $x$  and output  $y$  is characterized by the differential equation

$$y'(t) + 3y(t) = 2x(t).$$

Find the impulse response  $h$  of the system.

**Short Answer.**  $h(t) = 2e^{-3t}u(t)$

**F.5** A causal LTI system  $\mathcal{H}$  is such that  $y = \mathcal{H}x$ , where  $x(t) = e^{-2t}u(t)$  and  $y(t) = e^{-3t}u(t)$ . Find the unit-step response  $s$  of the system.

**Short Answer.**  $s(t) = \frac{2}{3}u(t) + \frac{1}{3}e^{-3t}u(t)$

**F.6** A causal stable LTI system  $\mathcal{H}$  has impulse response  $h$  and system function  $H$ . The function  $H$  is rational, contains a pole at  $-2$  and does not have a zero at the origin. The other poles and zeros of  $H$  are unknown. Determine whether each of the statements below is true, false, or uncertain (i.e., insufficient information to determine).

- (a)  $\mathcal{F}\{e^{3t}h(t)\}$  converges;
- (b)  $\int_{-\infty}^{\infty} h(t)dt = 0$ ;
- (c)  $th(t)$  is the impulse response of a causal and stable system;
- (d)  $\mathcal{D}h$  has at least one pole in its Laplace transform expression, where  $\mathcal{D}$  denotes the derivative operator.

**Short Answer.** (a) false; (b) false; (c) true; (d) true

**F.7** A communication channel can be well approximated by a LTI system with impulse response  $h(t) = \frac{1000}{\pi} \text{sinc}(1000t)$ . Determine which of the functions  $x_1$ ,  $x_2$ , and  $x_3$  would be least distorted by this channel, where

$$x_1(t) = \delta(t), \quad x_2(t) = u(t), \quad \text{and} \quad x_3(t) = \cos 100t.$$

**Short Answer.**  $x_3$

**F.8** A common problem in real-world instrumentation systems is electromagnetic interference caused by 60 Hz power lines. In particular, 60 Hz power lines can often introduce a significant amount of interference (i.e., noise) at 60 Hz and its higher harmonics (i.e., 120 Hz, 180 Hz, 240 Hz, and so on). Consider a causal system with an impulse response of the form  $h(t) = a[u(t) - u(t - b)]$ , where  $a$  and  $b$  are nonzero real constants. With an appropriate choice of  $a$  and  $b$ , such a system can be made to reject interference at 60 Hz and all of its higher harmonics. Find  $a$  and  $b$ .

**Short Answer.**  $a \neq 0$  and  $b = \frac{1}{60}$

**F.9** For the causal LTI system with input  $x$ , output  $y$ , and impulse response  $h$  that is characterized by each differential equation given below, find  $h$ .

(a)  $4y''(t) = 2x(t) - x'(t)$ .

**Short Answer.** (a)  $(\frac{1}{2}t - \frac{1}{4})u(t)$

---

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