Global Design of Perfect-Reconstruction Orthogonal Cosine-Modulated Filter Banks

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An orthogonal cosine-modulated (OCM) filter bank

\[ h_k(n) = 2h(n) \cos \left( \frac{\pi}{M} \left( k + \frac{1}{2} \right) \left( n - \frac{D}{2} \right) + (-1)^k \frac{\pi}{4} \right) \]

\[ f_k(n) = 2h(n) \cos \left( \frac{\pi}{M} \left( k + \frac{1}{2} \right) \left( n - \frac{D}{2} \right) - (-1)^k \frac{\pi}{4} \right) \]

for \( 0 \leq k \leq M - 1 \) and \( 0 \leq n \leq N - 1 \)
An $M$-channel OCM filter bank is uniquely characterized by its prototype filter (PF)

$h = [h_0 \ h_1 \ \cdots \ h_{N-1}]^T$ denotes the coefficients of the prototype filter

An OCM filter bank satisfies the perfect reconstruction (PR) property

$$a_{l,n}(h) = h^T Q_{l,n} h - c_n = 0$$

for $0 \leq n \leq m - 1$ and $0 \leq l \leq M/2 - 1$
The “closeness” to the PR property can also be measured in frequency domain by means of

1) amplitude distortion

\[ e_m(\omega) = 1 - |T_0(e^{j\omega})|, \quad \text{for } \omega \in [0, \pi] \]

2) worst case aliasing error

\[ e_a(\omega) = \max_{1 \leq l \leq M-1} |T_l(e^{j\omega})|, \quad \text{for } \omega \in [0, \pi] \]

where \( T_0(e^{j\omega}) \) and \( T_l(e^{j\omega}) \) are the distortion transfer function and the alias transfer function, respectively.
The design of the PF of an OCM filter bank can be formulated as

\[
\text{minimize} \quad \int_{\omega_s}^{\pi} |H_0(e^{j\omega})|^2 d\omega
\]

subject to: PR condition

As the PF of an OCM filter bank has linear phase, \( h \) is symmetrical. The design problem can be reduced to

\[
\text{minimize} \quad e_2(\hat{h}) = \hat{h}^T \hat{P} \hat{h}
\]

subject to:

\[
a_{l,n}(\hat{h}) = \hat{h}^T \hat{Q}_{l,n} \hat{h} - c_n = 0
\]

for \( 0 \leq n \leq m - 1 \) and \( 0 \leq l \leq M/2 - 1 \)

where the design variables are reduced by half to

\[
\hat{h} = [h_0 \ h_1 \ \cdots \ h_{N/2-1}]^T.
\]
Global Design Method at a Glance

- Multiple local solutions exist for a nonconvex problem
- Algorithms in finding a locally optimal solution are available
- Start the local design algorithm from a good initial point

How do we secure such a good initial point?
We have formulated the design of the prototype filter (PF) of an orthogonal cosine-modulated (OCM) filter bank as

\[
\text{minimize} \quad e_2(\hat{h}) = \hat{h}^T \hat{P} \hat{h}
\]

subject to:

\[
a_{l,n}(\hat{h}) = \hat{h}^T \hat{Q}_{l,n} \hat{h} - c_n = 0
\]

for \(0 \leq n \leq m - 1\) and \(0 \leq l \leq M/2 - 1\)
Suppose we are in the $k$th iteration to compute $\delta$ so that $\hat{h}_{k+1} = \hat{h}_k + \delta$ reduces the PF’s stopband energy and better satisfies the PR conditions. Then,

$$
\hat{h}_{k+1}^T \hat{P} \hat{h}_{k+1} = \delta^T \hat{P} \delta + 2 \hat{h}_k^T \hat{P} \hat{h}_k + \hat{h}_k^T \hat{P} \hat{h}_k
$$

and

$$
a_{l,n}(\hat{h}_k + \delta) \approx a_{l,n}(\hat{h}_k) + g_{l,n}^T(\hat{h}_k)\delta = 0
$$

for $0 \leq n \leq m - 1$ and $0 \leq l \leq M/2 - 1$

And the $k$th iteration assumes the form

minimize $\delta^T \hat{P} \delta + \delta^T \hat{b}_k$

subject to: $G_k \delta = -a_k$

$\|\delta\|$ is small
The equality constraint can be eliminated via SVD of $G_k = U\Sigma V^T$ as

$$\delta = V_e \phi + \delta_s$$

(5)

Thus, the problem can be cast as

$$\begin{align*}
\text{minimize} & \quad \phi^T \tilde{P}_k \phi + \phi^T \tilde{b}_k \\
\text{subject to:} & \quad ||\phi|| \text{ is small}
\end{align*}$$

The Gauss-Newton technique with adaptively controlled weights is used as a post-processing step to achieve convergence at a small tolerance.
The design problem is a polynomial optimization problem (POP)

Two recent breakthroughs in solving POPs

- Global solutions of POPs are made available by Lasserre’s method
- Sparse SDP relaxation is proposed for global solutions of POPs of relatively larger scales

MATLAB toolbox SparsePOP and GloptiPoly can be used to find global solutions of POPs, but only for POPs of limited sizes
Example: Design a globally optimal OCM filter bank with $M = 2$, $m = 1$ and $\rho = 1$

- GloptiPoly and SparsePOP can be used to produce the globally optimal solution

$$h^{(2,1)} = \begin{bmatrix}
0.235923416966353 \\
0.440840267366581 \\
0.440840267366581 \\
0.235923416966353
\end{bmatrix}$$

- The software was found to work only for the following cases:
  a) $M = 2$, $1 \leq m \leq 5$;
  b) $M = 4$, $1 \leq m \leq 3$;
  c) $M = 6$, $m = 1$;
  d) $M = 8$, $m = 1$. 
Global Design of High-Order PFs

Two observations:

1. For a fixed $M$, the impulse responses with different $m$ exhibit a similar pattern and are close to each other.
2. For $m = 1$, the impulse responses with different $M$ also exhibit a similar shape.
Effect of zero-padding when $M = 4$

Effect of linear interpolation when $m = 1$
An improvement in initial point when $m = 1$, by downshifting $h_0^{\text{int}}$ by a constant value $d$ computed using the Gauss-Newton method with adaptively controlled weights.
An order-recursive algorithm in brief

1. Obtaining a low-order global design;
2. Using zero-padding/linear interpolation in conjunction of the G-N method with adaptively controlled weights of the impulse response to produce a desirable initial point for PF of slightly increased order, and carrying out the design by a locally optimal method;
3. Repeating step 2 until the filter order reaches the targeted value.
4. SIMULATION RESULTS

The design strategy was applied to design low-order PFs for the cases:

a) \( M = 2, 1 \leq m \leq 5; \)
b) \( M = 4, 1 \leq m \leq 3; \)
c) \( M = 6, m = 1; \)
d) \( M = 8, m = 1. \)

The impulse responses obtained were found to be practically identical to those generated from GloptiPoly.
Design of an OCM filter bank with $m = 20$, $M = 4$ and $\rho = 1$. Shown below are the impulse responses of the PF from global and local design, respectively.
Performance comparison for OCM filter banks with $m = 20$, $M = 4$ and $\rho = 1$

<table>
<thead>
<tr>
<th></th>
<th>Global design</th>
<th>Local design</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy in stopband</td>
<td>8.226e-13</td>
<td>6.585e-10</td>
</tr>
<tr>
<td>Largest eq. error</td>
<td>1.839e-15</td>
<td>2.297e-10</td>
</tr>
<tr>
<td>$\max(</td>
<td>e_m(\omega)</td>
<td>)$</td>
</tr>
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<td>$\max(</td>
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5. CONCLUSIONS

- Local design of OCM filter banks
- We have proposed a strategy for the global design of OCM filter banks based on
  - Recent progress in global polynomial optimization
  - The local design method introduced
  - Critical observations on the globally optimal low-order impulse responses
- Superiority of the proposed algorithm is demonstrated