Smoothed $\ell_p$-$\ell_2$ Solvers for Signal Denoising

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Introduction

The basis pursuit denoising (BPDN) refers to the solution of a non-smooth convex $\ell_1$-$\ell_2$ constrained minimization problem. Several authors have investigated a nonconvex extension, namely the $\ell_p$-$\ell_2$ problem

$$\min F(s) = \lambda \|s\|_p^p + \frac{1}{2} \|y - \Theta s\|_2^2$$

where $p \in (0, 1)$. We study problem (1) for signal denoising and propose two fast solvers where $\Theta$ represents either an orthogonal basis or an overcomplete dictionary.

Global Solution Discontinuity

With an orthogonal $\Theta$, the objective function in (1) becomes

$$F(s) = \lambda \|s\|_p^p + \|\Theta(s-\Theta^Ty)\|_2^2 = \lambda \|s\|_p^p + \|s-c\|_2^2$$

so that the problem of global minimization of $F(s)$ amounts to global minimization of the single-variable function $u(s;\lambda) = \lambda |s|^p + (s-c)^2$.

Smoothed $\ell_p$-$\ell_2$ Solver

The Case for Orthogonal $\Theta$

The minimizers of $u(s;\lambda)$ only occur in $[0, c]$. Over this interval, a local minimum inside the interval satisfies $u'(s;\lambda) = 0$ with $u''(s;\lambda) > 0$. In addition, a notch at $s = 0$ which is either a local or a global minimizer. In effect at a value $\lambda > 0$ the two minimizers are equal and both are global minimizers. By solving $u'(s;\lambda) = 0$ and $u(s;\lambda) = u(0,\lambda)$ simultaneously, we obtain

$$s = \frac{2(1-p)c}{2-p}, \quad \lambda = \frac{2}{1-p} \left(\frac{2(1-p)c}{2-p}\right)^{2-p}$$

The global minimizer jumps between the origin and the interior point $s$ as $\lambda$ varies across $\lambda$.

(a) If $\lambda \notin \{\lambda_L, \lambda_M\}$, solution jump will not occur; hence if $\lambda < \lambda_L$, set $s = 0$; if $\lambda > \lambda_M$, take the minimizer inside $[0, c]$ as the solution $s^\dagger$. This solution can be efficiently identified using a one-dimensional search technique.

(b) If $\lambda \in \{\lambda_L, \lambda_M\}$, to prevent solution jump, we take the unique global solution of $u(s;\lambda)$ with $p = 1$ as $s^\dagger$, which is simply computed by a soft-shrinkage operation as $s^\dagger = \text{sgn}(c) \cdot \max\{|c| - \lambda, 0|\}$.

Fast implementation which solves the $N$ single-variable $\ell_p$-$\ell_2$ problems in parallel are applied.

Smoothed $\ell_p$-$\ell_2$ Solver: The Case for Overcomplete $\Theta$

We develop an iterative technique in spirit similar to a proximal-point method where the iterate $s_k$ in the $k$th iteration is updated to

$$s_k = \arg\min \left\{ \lambda |s|^p + \frac{L}{2} \|s - s_{k-1}\|_2^2 \right\}$$

with $c_k = b_k = \frac{L}{2} \Theta^T(\Theta b_k - y)$. The FISTA type of iteration is carried out as

$$s_{k+1} = 1 + \sqrt{1 + 4t_k^2}/2, \quad b_{k+1} = s_k + \frac{t_k-1}{t_k+1} (s_k - s_{k-1})$$

Smoothed solution of problem (2) can be found using our algorithm introduced previously. Incorporating FISTA type of iteration accelerates convergence of the algorithm without substantial increase in computational complexity.

Performance Evaluation

Heavisine Denoising

Two smoothed $\ell_p$-$\ell_2$ solvers with $p \in (0, 1)$ for signal spaces with orthogonal basis or overcomplete dictionary have been proposed. By applying them to signal denoising problems, the proposed solvers are demonstrated to outperform their $\ell_1$-$\ell_2$ counterparts.

Conclusion

The work was supported by the Natural Sciences and Engineering Research Council of Canada.

References


Acknowledgements

The work was supported by the Natural Sciences and Engineering Research Council of Canada.